Hindawi
Journal of Function Spaces
Volume 2022, Article ID 8079707, 7 pages
https://doi.org/10.1155/2022/8079707

Research Article

A Stability Result for a Swelling Porous System with Nonlinear Boundary Dampings

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Received 26 February 2022; Accepted 11 May 2022; Published 9 June 2022

Academic Editor: Gisele Mophou

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In this work, we consider a swelling porous system where the damping terms are on the boundary. We establish an explicit and general decay result, without imposing restrictive growth assumption near the origin on the damping terms. Our result allows a larger class of damping terms, and the usual exponential and polynomial decay estimates are only special cases. We also give some illustrative examples.

1. Introduction

Soil swelling is one of the problems which has been studied under the porous media theory. The soils with highly swelling index have high swelling properties. In the swelling soils, clay minerals are commonly found there, and they attract and absorb the water which may lead to increase in pressure. When the swelling soil is exposed to the water, the water molecules are attracted into the interstices of the soil matrix. With the water drawn is getting increasing, the soil plates are forced apart due to the rise in pressure within the soil pores leading to swelling or heaving of the soils. So, swelling soils are considered to be one of the sources of problems in foundation design and construction. For more information in the soil swelling problems, we refer the reader to [1–4] and the references therein. The mathematical model of linear theory of swelling porous elastic soils was established by Ieşan [5] and simplified by Quintanilla [6] as follows:

$$\begin{cases} \rho_1 \varphi_{tt} = P_{1x} - G_1 + F_1, \\ \rho_2 \psi_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1}$$

where the functions (P_1, G_1, F_1) represent the partial tension, internal body forces, and external forces acting on the displacement, respectively. Similar definitions hold for (P_2, G_2, F_2) but acting on the elastic solid. The constituent φ represents the displacement of the fluid and $\rho_1 > 0$ is the coefficient density of φ. Here, ψ is the elastic solid material and its coefficient density is $\rho_2 > 0$. Moreover, the constitutive equations of partial tensions are given by

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{A} \begin{bmatrix} \varphi_x \\ \psi_x \end{bmatrix}, \tag{2}$$

where a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number. The matrix A is positive definite in the sense that $a_1 a_3 \geq a_2^2$.

Quintanilla [6] investigated (1) by taking

$$G_1 = G_2 = \beta(\varphi_t - \psi_t), F_1 = a_3 \varphi_{xxt}, \ F_2 = 0, \tag{3}$$

and established an exponential stability result where $\beta > 0$. Wang and Guo [7] considered (1) by taking

$$G_1 = G_2 = 0, F_1 = -\rho_1 \xi(x)\varphi_t, \ F_2 = 0, \quad (4)$$

and they establish an exponential stability result by using spectral method, where $\xi(x)$ is an internal viscous damping function with a positive mean. Ramos et al. [8] studied (1), with a damping acting on the domain; that is,

$$G_1 = G_2 = 0, F_1 = 0, \ F_2 = -\gamma(t)g(\psi_t). \quad (5)$$

They established an exponential decay rate. Recently, Apalara [9] looked into (1), with viscoelastic damping acting on the domain. So, he took

$$G_1 = G_2 = 0, F_1 = 0, \ F_2 = -\int_0^t g(t-s)\psi_{xx}(s)ds \quad (6)$$

and established a general decay rate irrespective of the wave speed of the system. Very recently, Al-Mahdi et al. [10] also considered (1) with

$$G_1 = G_2 = 0, \quad F_1 = 0, \ F_2 = -\int_0^\infty g(s)\psi_{xx}(t-s)ds, \quad (7)$$

and established explicit and general decay results under a wider class of relaxation functions, and they also performed several numerical tests to illustrate their theoretical results. The reader can consult [11–20] and the references therein for some other interesting related results.

In this paper, we are concerned with the following swelling system

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} = 0, \ \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} = 0, \ \text{in } (0,1)\times(0,\infty), \end{cases} \quad (8)$$

together with initial and boundary conditions:

$$\begin{cases} \psi(0,t) = \varphi(0,t) = 0, \\ a_2\varphi_x(1,t) + a_3\psi_x(1,t) = -h_1(\psi_t(1,t)), \\ a_1\varphi_x(1,t) + a_2\psi_x(1,t) = -h_2(\varphi_t(1,t)), \end{cases} \quad (9)$$

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \end{cases} \quad (10)$$

where h_1 and h_2 are specific functions, a_1, a_3 are positive constants, and $a_2 \neq 0$ is a real number.

The use of dampings at the boundary is well known in mechanical structure. It is employed to stabilize motions and absorb shock. This can be achieved by adding some controllers at the boundary (see [21, 22]). In this paper, we aim to establish an explicit and general decay rate result for system (8). We obtain our result without imposing any restrictive growth assumption near the origin on the damping functions. The result in this paper allows a larger class of functions h_1 and h_2, from which the energy decay rates are not necessarily of exponential or polynomial types (see the examples in Section 3).

The proofs of our results are carried out, using the multiplier method and benefit from [22, 23] with necessary modifications dictated by the nature of our problem. To the best of our knowledge, this is the first work of this nature. For more works used the frictional damping acting in a part/whole domain or in the boundary, we point out to the work of [24–27].

The rest of the paper is organized as follows: in Section 2, we present some hypotheses and material needed for our work. Some essential lemmas and the statement with the proof of the decay result are given in Section 3.

2. Preliminaries

In this section, we present some material needed for the proofs of our main results.

In the sequel, we assume that system (8) has a unique solution

$$\psi, \varphi \in L^\infty\left(\mathbb{R}_+ ; H^2(0,1) \cap \mathscr{H}\right) \cap W^{1,\infty}(\mathbb{R}_+ ; \mathscr{H}) \cap W^{2,\infty}\left(\mathbb{R}_+ ; L^2(0,1)\right), \quad (11)$$

where $\mathscr{H} = \{f \in H^1(0,1) : f(0) = 0\}$. This result can be proved, for initial data in suitable function spaces, using standard arguments such as the Galerkin method.

We consider the following hypotheses:

A1. $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ (for $i = 1, 2$) is a nondecreasing C^1 function such that

$$\begin{aligned} &\Lambda_i(|s|) \le |h_i(s)| \le \Lambda_i^{-1}(|s|) \text{ for all } |s| \le r, i = 1, 2, \\ &c_1|s| \le |h_i(s)| \le c_2|s| \quad \text{for all } |s| \ge r, \end{aligned} \quad (12)$$

where Λ_1 and Λ_2 are strictly increasing C^1 functions on $[0, +\infty)$, $\Lambda_1(0) = \Lambda_2(0) = 0$, and r, c_1, c_2 are positive constants.

A2. The coefficients a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number such that

$$a_1 a_3 - a_2^2 > 0. \quad (13)$$

Remark 1. Hypothesis A1 implies that $sh_i(s) > 0$, for all $s \neq 0, i = 1, 2$.

Remark 2. Hypothesis A1 was introduced by Lasiecka and Tataru [28].

The following lemma will be of essential use in establishing the main result.

Lemma 3 (see [23]). *Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nonincreasing function and $\beta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a strictly increasing C^1 function, with $\beta(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. Assume that there exist $m, n \ge 0$ and $c > 0$ such that*

$$\int_S^\infty \beta'(t)F(t)^{1+m}dt \le cF(S)^{1+m} + \frac{cF(S)}{\beta^n}, \quad 1 \le S < +\infty. \tag{14}$$

Then, there exist two positive constants λ_1 and λ_2 such that

$$\begin{aligned} &E(t) \le \lambda_1 e^{-\lambda_2 \beta(t)} \forall t \ge 1, \text{ if } m = n = 0, \\ &E(t) \le \frac{\lambda_1}{\beta(t)^{1+n/m}} \quad \forall t \ge 1, \quad \text{if } m > 0. \end{aligned} \tag{15}$$

The energy functional associated with problem (8) is

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + a_1\varphi_x^2 + \rho_2\psi_t^2 + a_3\psi_x^2 + 2a_2\psi_x\varphi_x\right)dx. \tag{16}$$

Lemma 4. *The energy functional satisfies, along the solution of (8),*

$$E'(t) = -\psi_t h_1(\psi_t(1,t)) - \varphi_t h_2(\varphi_t(1,t)) \le 0. \tag{17}$$

Proof. By multiplying the two equations of (8) by φ_t and ψ_t, respectively, and then integrating over $(0,1)$ with using both boundary and initial conditions (9) and (10), the estimate (17) is established. □

Throughout this paper, we will use c to denote a generic positive constant.

3. Stability

In this section, we state and prove our main stability result which reads as follows:

Theorem 5. *Let (ψ, φ) be the solution of (8) and assume that A1 and A2 hold. Then, there exists a constant $\bar{c} > 0$ such that, for t large, the solution of (8) satisfies*

$$E(t) \le \bar{c}\left(G^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{18}$$

where

$$G(s) = s\left(\Lambda_1^{-1} + \Lambda_2^{-1}\right)^{-1}(s). \tag{19}$$

Proof of Theorem 8 will be carried out through several lemmas.

Lemma 6. *Let (ψ, φ) be the solution of (8); assume that A1 and A2 hold, and $\beta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a concave nondecreasing C^2 function. Then, for $T \ge S \ge 0$ and some positive constant c, the energy functional satisfies*

$$\int_S^T \beta'(t)E^2(t)dt \le cE^2(S) + c\int_S^T \beta'(t)E(t)\left[h_1^2(\psi_t(1,t))dt + h_2^2(\varphi_t(1,t))\right]dt. \tag{20}$$

Proof. Multiplying the first equation in (8) by $[3x\varphi_x - \varphi]\beta' E$ and the second by $[3x\psi_x + \psi]\beta' E$, integrating over $(0,1) \times (S,T)$, and adding the results and recalling (16), we get

$$\begin{aligned} \int_S^T \beta'(t)E^2(t)dt = &\int_S^T \beta'(t)E(t)\left[\frac{3}{2}\int_0^1 \rho_2\psi_t^2 dx \right. \\ &\left. - \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 dx + a_3\varphi_x^2\right)dx - 2\int_0^1 a_2\psi_x\varphi_x\right]dt \\ &- \left[\beta'(t)E(t)\int_0^1 [3(\rho_1 x\varphi_x\varphi_t + \rho_2 x\psi_x\psi_t) \right. \\ &\left. + (\rho_2\psi\psi_t - \rho_1\varphi\varphi_t)]dx\right]_S^T + \int_S^T \left(\beta''E + \beta'E'\right)(t) \\ &\cdot \left(\int_0^1 [3(\rho_1 x\varphi_x\varphi_t + \rho_2\psi_x\psi_t) + (\rho_2\psi\psi_t - \rho_1\varphi\varphi_t)]dx\right)dt \\ &+ \int_S^T \beta'(t)E(t)[\varphi(1,t)h_2(\varphi_t(1,t))]dt \\ &- \frac{N}{2}\int_S^T \beta'(t)E(t)[\psi(1,t)h_1(\psi_t(1,t))]dt \\ &- 3\int_S^T \beta'(t)E(t)[\varphi_x(1,t)h_2(\varphi_t(1,t))]dt \\ &- \frac{N}{2}\int_S^T \beta'(t)E(t)[\psi_x(1,t)h_1(\psi_t(1,t))]dt \\ &+ \frac{a_2}{2}\int_S^T \beta'(t)E(t)[\psi_x(1,t)\varphi_x(1,t)]dt. \end{aligned} \tag{21}$$

Using Young's inequality, (16), and the properties of E and β', we have

$$\begin{aligned} &\int_S^T \beta' E\left[\frac{3}{2}\int_0^1 \rho_2\psi_t^2 dx - \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 dx + a_3\psi_x^2\right)dx - 2\int_0^1 a_2\psi_x\varphi_x\right]dt \\ &\quad \le c\beta'(S)E^2(S). \end{aligned} \tag{22}$$

Exploiting Young's and Poincaré's inequalities and (16), we obtain

$$\begin{aligned} &\int_0^1 [3(\rho_1 x\varphi_x\varphi_t + \rho_2 x\psi_x\psi_t) + (\rho_2\psi\psi_t - \rho_1\varphi\varphi_t)]dx \\ &\quad \le C\int_0^1 \left(\varphi_x^2 + \varphi_t^2 + \psi_x^2 + \psi_t^2\right)dx \le cE(t). \end{aligned} \tag{23}$$

Using the properties of E and β', we conclude

$$\begin{aligned} &-\left[\beta' E\int_0^1 \left[3(\rho_1 x\varphi_x\varphi_t + \rho_2 x\psi_x\psi_t) + \frac{N}{2}(\rho_2\psi\psi_t - \rho_1\varphi\varphi_t)\right]dx\right]_S^T \\ &\quad \le C\left|\left[\beta'(t)E^2(t)\right]_S^T\right| \le c\beta'(S)E^2(S). \end{aligned} \tag{24}$$

As in the above calculations, we handle the third term in the right hand side of (21) as follows:

$$\begin{aligned}&\int_S^T\left(\beta''E+\beta'E'\right)\left(\int_0^1[(N+1)(\rho_1x\varphi_x\varphi_t+\rho_2\psi_x\psi_t)+(\rho_2\psi\psi_t-\rho_1\varphi\varphi_t)]dx\right)dt\\&\leq c\left|\int_S^T\beta''E^2dt\right|+c\left|\int_S^T\beta'E'Edt\right|\\&\leq CE^2(S)\left|\int_S^T\beta''dt\right|+c\beta'(S)\left|\int_S^T E'Edt\right|\\&\leq c\beta'(S)E^2(S).\end{aligned} \tag{25}$$

Using Cauchy-Schwarz's inequality, we conclude that

$$\psi(1,t)^2=\left(\int_0^1\psi_x dx\right)^2\leq\int_0^1\psi_x^2dx. \tag{26}$$

Similarly, we have

$$\varphi(1,t)^2\leq\int_0^1\varphi_x^2dx. \tag{27}$$

Now, using (26) and Young's inequality, we find that

$$\begin{aligned}\int_S^T\beta'E[\psi(1,t)h_1(\psi_t(1,t))]dt&\leq c\varepsilon\int_S^T\beta'E\int_0^1\psi_x^2dxdt\\&+c_\varepsilon\int_S^T\beta'Eh_1^2(\psi_t(1,t))dt.\end{aligned} \tag{28}$$

Similarly, using (27) and Young's inequality, we see that

$$\begin{aligned}\int_S^T\beta'E[\varphi(1,t)h_2(\varphi_t(1,t))]dt&\leq c\varepsilon\int_S^T\beta'E\int_0^1\varphi_x^2dxdt\\&+c_\varepsilon\int_S^T\beta'Eh_2^2(\varphi_t(1,t))dt.\end{aligned} \tag{29}$$

From (9), we have

$$\begin{bmatrix}a_1&a_2\\a_2&a_3\end{bmatrix}\begin{bmatrix}\varphi_x(1,t)\\\psi_x(1,t)\end{bmatrix}=\begin{bmatrix}-h_1(\psi_t(1,t))\\-h_2(\varphi_t(1,t))\end{bmatrix}. \tag{30}$$

Therefore, we get

$$\begin{aligned}\psi_x&=\frac{a_2h_2(\varphi_t(1,t))-a_1h_1(\psi_t(1,t))}{a_1a_3-a_2^2},\\\varphi_x&=\frac{a_2h_1(\psi_t(1,t))-a_3h_2(\varphi_t(1,t))}{a_1a_3-a_2^2},\end{aligned} \tag{31}$$

where $a_1a_3-a_2^2>0$ (as assumed in A2). Using (31) and Young's inequality, we have

$$\begin{aligned}&-3\int_S^T\beta'E[\varphi_x(1,t)h_2(\varphi_t(1,t))]dt-\int_S^T\beta'E[\psi_x(1,t)h_1(\psi_t(1,t))]dt\\&+\frac{a_2}{2}\int_S^T\beta'E\psi_x(1,t)\varphi_x(1,t)dt\leq c\int_S^T\beta'E\left[h_1^2(\psi_t(1,t))\right.\\&\left.+h_2^2(\varphi_t(1,t))\right]dt.\end{aligned} \tag{32}$$

We use (16), (28), (29), and (32), to estimate the last five terms in (21) as follows:

$$\begin{aligned}&\int_S^T\beta'E[\varphi(1,t)h_2(\varphi_t(1,t))]dt-\int_S^T\beta'E[\psi(1,t)h_1(\psi_t(1,t))]dt\\&-3\int_S^T\beta'E[\varphi_x(1,t)h_2(\varphi_t(1,t))]dt-\int_S^T\beta'E[\psi_x(1,t)h_1(\psi_t(1,t))]dt\\&+\frac{a_2}{2}\int_S^T\beta'E\psi_x(1,t)\varphi_x(1,t)dt\leq c\beta'(S)E^2(S)\\&+c\int_S^T\beta'E\left[h_1^2(\psi_t(1,t))+h_2^2(\varphi_t(1,t))\right]dt.\end{aligned} \tag{33}$$

Combining (21)–(25) and (28)–(33), then (20) is established. □

Lemma 7. *Let (ψ,φ) be the solution of (8); assume that A1 and A2 hold. Then, the energy functional satisfies*

$$E(t)\leq\frac{c}{\beta_0(t)^2}\quad\forall t\geq t_1, \tag{34}$$

where $\beta_0:R_+\longrightarrow R_+$ is a strictly increasing C^1 function, with $\beta_0(t)\longrightarrow+\infty$ as $t\longrightarrow+\infty$, $\beta_0'(t)=\Lambda_0(1/(\beta_0(t)))$, and $\Lambda_0:=(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}$.

Proof. First, we define the following function:

$$\chi(t):=1+\int_1^t\frac{1}{\Lambda_0(1/s)}ds,t\geq t_0, \tag{35}$$

for some $t_0>\max\{1,1/r\}$. Then,

$$\chi'(t)=\frac{1}{\Lambda_0(1/t)}>0\forall t\geq t_0,\ \chi'(t)\longrightarrow+\infty\text{ as }t\longrightarrow+\infty, \tag{36}$$

and $\chi'(t)$ is strictly increasing. Thus, χ is a convex and strictly increasing C^2 function, with $\chi(t)\longrightarrow+\infty$ as $t\longrightarrow+\infty$. If we let

$$\beta_0:=\chi^{-1},t\geq t_0, \tag{37}$$

then it is easy to check that β_0 is strictly increasing and $\beta_0'(t)=\Lambda_0(1/\beta_0(t))$ is strictly decreasing. So, β_0 is a concave C^2 function, with $\beta_0(t)\longrightarrow+\infty$ as $t\longrightarrow+\infty$. We use this particular function β_0 and take $t_1\geq t_0$ such that

$\beta_0'(t_1) < r$, to estimate the last integral in the right hand side of (20), for $T \ge S \ge t_1$.

To estimate $\int_S^T \beta'_0 E h_1^2(\psi_t(1,t))dt$, we consider the following cases:

$$C1 = \{t \in (S,T): |\psi_t(1,t)| > r\},$$
$$C2 = \left\{t \in (S,T): |\psi_t(1,t)| \le r \text{ and } \psi_t(1,t) \le \beta_0'(t)\right\},$$
$$C3 = \left\{t \in (S,T): |\psi_t(1,t)| \le r \text{ and } \psi_t(1,t) > \beta_0'(t)\right\}. \tag{38}$$

Using A1, Remark 2, (17), and the first case in (38), we have

$$\begin{aligned} h_1^2(\psi_t(1,t)) &= h_1(\psi_t(1,t))h_1(\psi_t(1,t)) \\ &\le c_2\psi_t(1,t)h_1(\psi_t(1,t)) \\ &\le c\left(-E'\right)(t), \end{aligned} \tag{39}$$

which gives

$$\begin{aligned} \int_{(S,T)\cap C1} \beta_0' E h_1^2(\psi_t(1,t))dt &\le \int_{(S,T)\cap C1} c\beta_0' E\left(-E'\right)(t) \\ &\le \int_S^T c\beta_0' E\left(-E'\right)(t)dt \le c\beta_0'(S)E^2(S). \end{aligned} \tag{40}$$

Now, using A1, the properties of Λ_1^{-1} and Λ_0^{-1}, and the second case in (38), we get

$$\begin{aligned} \int_{(S,T)\cap C2} \beta_0' E h_1^2(\psi_t(1,t))dt &\le c\int_{(S,T)\cap C2} \beta_0' E\left(\Lambda_1^{-1}(\psi_t(1,t))\right)^2 dt \\ &\le c\int_{(S,T)\cap C2} \beta_0' E\left(\Lambda_1^{-1}\left(\beta_0'(t)\right)\right)^2 dt \\ &\le c\int_{(S,T)\cap C2} \beta_0' E\left(\Lambda_0^{-1}\left(\beta_0'(t)\right)\right)^2 dt \\ &\le \int_S^T \beta_0' E\left(\Lambda_0^{-1}\left(\beta_0'(t)\right)\right)^2 dt. \end{aligned} \tag{41}$$

Finally, using A1, (17), the properties of Λ_1^{-1}, and the last case in (38), we obtain

$$\begin{aligned} \int_{(S,T)\cap C3} \beta_0' E h_1^2(\psi_t(1,t))dt &= \int_{(S,T)\cap C3} \beta_0' E h_1(\psi_t(1,t))h_1(\psi_t(1,t))dt \\ &\le \int_{(S,T)\cap C3} E\psi_t(1,t)\Lambda_1^{-1}(\psi_t(1,t))h_1(\psi_t(1,t))dt \\ &\le \Lambda_1^{-1}(r)\int_{(S,T)\cap C3} E\psi_t(1,t)h_1(\psi_t(1,t))dt \\ &\le c\int_{(S,T)\cap C3} E\psi_t(1,t)h_1(\psi_t(1,t))dt \le c\int_{(S,T)\cap C3} E\left(-E'\right)dt \\ &\le \int_S^T E\left(-E'\right)dt \le c\left(E^2(S) - E^2(T)\right) \le cE^2(S). \end{aligned} \tag{42}$$

Combining (40)–(42), we get

$$\int_S^T \beta_0' E h_1^2(\psi_t(1,t))dt \le cE^2(S) + c\int_S^T \beta_0' E\left(H_0^{-1}\left(\beta_0'(t)\right)\right)^2 dt. \tag{43}$$

Repeating the same above calculations, we also find

$$\int_S^T \beta_0' E h_2^2(\psi_t(1,t))dt \le cE^2(S) + c\int_S^T \beta_0' E\left(\Lambda_0^{-1}\left(\beta_0'(t)\right)\right)^2 dt. \tag{44}$$

Then, (20) becomes

$$\int_S^T \beta_0'(t)E^2(t)dt \le cE(S)^2 + c\int_S^T \beta_0' E\left(\Lambda_0^{-1}\left(\beta_0'(t)\right)\right)^2 dt. \tag{45}$$

Recalling $\beta_{0'}(t) = \Lambda_0(1/\beta_0(t))$ and letting $\beta_0(t) = s$, then for $T \ge S \ge t_1$, we have

$$\begin{aligned} \int_S^\infty \beta_0'(t)E^2(t)dt &\le cE(S)^2 + cE(S)\int_S^\infty \beta_0'(t)\left(\Lambda_0^{-1}(\beta_{0'}(t))\right)^2 dt \\ &= cE(S)^2 + cE(S)\int_{\beta_0(S)}^\infty \left(\Lambda_0^{-1}\left(\Lambda_0\left(\frac{1}{s}\right)\right)\right)^2 ds \\ &= cE(S)^2 + \frac{cE(S)}{\beta_0(S)}. \end{aligned} \tag{46}$$

Using (46) and Lemma 3 with $m = n = 1$, $F = E$, and $\beta = \beta_0$, we obtain

$$E(t) \le \frac{c}{\beta_0(t)^2}, \forall t \ge t_1. \tag{47}$$

Hence, (34) is established. □

Proof of Theorem 8. To prove (18), let us define $G(s) = s\Lambda_0(s)$ and take $s_0 > t_0$ such that $\Lambda_0(1/s_0) \le 1$.

Since Λ_0 is increasing, then we have $\Lambda_0(1/s) \le \Lambda_0(1/s_0) \le 1$, $\forall s \ge s_0$. Therefore, we obtain

$$\begin{aligned} \beta_0^{-1}(s) &\le 1 + (s-1)\frac{1}{\Lambda_0(1/s)} = \frac{\Lambda_0(1/s) + s - 1}{\Lambda_0(1/s)} \\ &\le \frac{s}{\Lambda_0(1/s)} = \frac{1}{G(1/s)}, \ \forall s \ge s_0. \end{aligned} \tag{48}$$

So, letting $t = 1/(G(1/s))$, we can see that

$$G\left(\frac{1}{s}\right) = \frac{1}{t} \Rightarrow \frac{1}{s} = G^{-1}\left(\frac{1}{t}\right) \Rightarrow s = \frac{1}{G^{-1}(1/t)}. \tag{49}$$

Then, using (48) and (49), we see that

$$\beta_0^{-1}(s) \le \frac{1}{G\left(G^{-1}(1/t)\right)} = \frac{1}{1/t} = t. \tag{50}$$

Hence,

$$s = \frac{1}{G^{-1}(1/t)} \leq \beta_0(t). \tag{51}$$

Now, it is easy to see that

$$\frac{1}{\beta_0(t)} \leq G^{-1}\left(\frac{1}{t}\right) \quad \forall t \geq t_0. \tag{52}$$

Therefore, using (34), estimate (18) is established. □

Example 9. As in [22], we consider the following examples to illustrate our decay result:

(1) Let $\Lambda_1(s) = \Lambda_2(s) = e^{-(\ln s)^2}$ near zero. Then, (18) gives

$$E(t) \leq ce^{-2(\ln t)^{1/2}} \tag{53}$$

(2) Let $\Lambda_1(s) = \Lambda_2(s) = e^{-1/s}$ near zero. Then, (18) implies

$$E(t) \leq \frac{c}{(\ln (t))^2} \tag{54}$$

(3) If $\Lambda_1(s) = \Lambda_2(s) = e^{-e^{1/s}}$ near zero, then, using (18), we have

$$E(t) \leq \frac{c}{(\ln (\ln (t)))^2} \tag{55}$$

(4) If $\Lambda_1(s) = \Lambda_2(s) = s$, then, using (18), we have

$$E(t) \leq \frac{c}{t} \tag{56}$$

(5) If $\Lambda_1(s) = s^{3/2}$, $\Lambda_2(s) = s^3$ near zero. Then,

$$G(s) = s\left(\frac{-1 + \sqrt{1 + 4s}}{2}\right)^3, \tag{57}$$

then,

$$\left(\frac{G(s)}{s}\right)^{1/3} = \frac{-1 + \sqrt{1 + 4s}}{2} \tag{58}$$

By approximating $\sqrt{1 + 4s}$ near $s = 0$, we have

$$\sqrt{1 + 4s} \approx 1 + 2s. \tag{59}$$

Hence,

$$\left(\frac{G(s)}{s}\right)^{1/3} \approx s, \text{near} s = 0, \tag{60}$$

which implies that $G^{-1}(s) \approx s^{1/4}$. Then, using (18), we have

$$E(t) \leq \frac{1}{\sqrt{t}}. \tag{61}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Acknowledgments

The authors thank King Fahd University of Petroleum and Minerals (KFUPM) and University of Sharjah for their continuous supports. The authors also thank the referees for their very careful reading and valuable comments. This work was initialed and finished during the visits of the second author to KFUPM in February 2022. This work is funded by KFUPM, Grant No. SB201005.

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