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Research Article

Investigating a Class of Generalized Caputo-Type Fractional Integro-Differential Equations

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In this article, we prove some new uniqueness and Ulam-Hyers stability results of a nonlinear generalized fractional integro-differential equation in the frame of Caputo derivative involving a new kernel in terms of another function ψ. Our approach is based on Babenko's technique, Banach's fixed point theorem, and Banach's space of absolutely continuous functions. The obtained results are demonstrated by constructing numerical examples.

1. Introduction

It is notable that fractional calculus was and still is a new tool that uses fractional differential and integral equations to construct more modern mathematical models that can precisely describe complex frameworks. There are many definitions of fractional integrals (FIs) and fractional derivatives (FDs) accessible in the literature, for instance, the Riemann-Liouville and Caputo definitions that assumed a significant part in the advancement of the theory of fractional analysis. Referring to all books and papers in this field will be extremely many. In this regard, here, we refer to the most important of main references, e.g., Samko et al. [1] gave a broad comprehensive mathematical handling of fractional derivatives and integrals. Podlubny [2] and Kilbas et al. [3] have been introduced many useful results related to fractional differential equations (FDEs). Several applications have been implemented recently by a wide range of works on this subject, see [4–8].

However, the currently common operator is the generalized FD regarding another function, see [1, 3]. Agrawal [7] studied further various properties for generalized fractional derivatives and integrals. More recently, Almeida [9] inspired an idea of that generalization by projecting this generalization onto the definition of the Caputo fractional derivative with respect to another function, so-called ψ-Caputo, and introduced many interesting properties, which are more general than the classical Caputo FD. Jarad and Abdeljawad [10] provided interesting properties for generalized FDs and Laplace transform. Specifically, ψ-Caputo type FDEs with initial, boundary, and nonlocal conditions have been investigated by many researchers using fixed-point theories, see Almeida et al. [11, 12], Abdo et al. [13], and Wahash et al. [14]. A recent survey on ψ-Caputo type FDEs can be found in [15–18]. For more results in this direction, we refer to interesting works provided by Zhang et al. [19], Zhao et al. [20], Baitiche et al. [21], Benchohra et al. [22], Ravichandran et al. [23], Trujillo et al. [24], and Furati et al. [25].

Li in [17, 18] investigated some interesting results of the integral equations and the integro-differential equations involving Hadamard-type. In this work, our goal is to intend to address a general extension of these studies. Precisely, we consider the following ψ-Caputo type fractional integro-differential equation (FIDE)

$$\begin{cases} {}^C\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1\,{}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2\, I_a^{\psi,\varrho_2}\varphi(x) = \int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right) d\zeta, \\ \varphi(a) = 0, \end{cases} \tag{1}$$

where

(i) $0 < \varrho_1 < \varrho < 1, \varrho_2 > 0$, and $a_1, a_2 \in \mathbb{C}$

(ii) The symbol ${}^C\mathbb{D}_a^{\psi,\sigma}$ denotes the generalized Caputo FD of order $\sigma \in \{\varrho, \varrho_1\}$

(iii) The notation I_a^{ψ,ϱ_2} means the generalized Riemann-Liouville FI of order ϱ_2

(iv) $\mathbb{G} : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and $0 \le a < b < \infty$

(v) $\varphi \in AC_0[a, b]$ such that I_a^{ψ,ϱ_2} and $\mathbb{D}_a^{\psi,\sigma}$ exist and are both continuous in $[a, b]$

Observe that the considered system (1) covers the previous standard cases of nonlinear FIDEs by defining the kernel, i.e., if $\psi(x) = x$, $\psi(x) = \log(x)$, and $\psi(x) = x^\rho$, then, the problem (1) reduces to the Caputo type FIDE, Caputo-Hadamard type FIDE, and Caputo-Katugampola type FIDE, respectively.

The aim of this work is to develop the nonlinear FIDEs. In particular, we investigate the uniqueness and Ulam-Hyers stability of solution for the problem (1) by Banach's fixed point theorem and Babenko's technique [26]. Note that the presentation and structuring of the arguments for our problem are new, and our results generalize and cover some of the known results in the literature. In addition, the obtained results here are valid when the left hand side of the considered problem (1) involves many FDs and FIs. For more details, see Remark 22.

The remainder of this paper is organized as follows: in Section 2, we present some important tools related the fractional calculus and the functional spaces, in which we aim to determine our analysis strategies. Section 3 gives the main results and their illustrative examples. Finally, our brief conclusion is included in Section 4.

2. Preliminaries

In this section, we present some properties, lemmas, definitions, and important estimations needed in the proof of our result.

Defining the Banach space as

$$AC_0[a, b] = \left\{\varphi : \varphi \in AC[a, b] \text{ with } \varphi(a) = 0 \text{ and } \|\varphi\|_0 = \int_a^b |\varphi'(\zeta)| d\zeta < \infty\right\}. \tag{2}$$

Next, we present some important definitions and properties of advanced fractional calculus.

Definition 1 [3, 9]. The ψ-Riemann-Liouville FI and ψ-Caputo FD are defined by

$$\begin{aligned} I_a^{\psi,\varrho}\omega(x) &= \frac{1}{\Gamma(\varrho)} \int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1} \psi'(\zeta)\omega(\zeta) d\zeta, \varrho > 0, \\ {}^C\mathbb{D}_a^{\psi,\varrho}\omega(x) &= I_a^{\psi,n-\varrho}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\omega(x) \\ &= \frac{1}{\Gamma(n-\varrho)} \int_a^x (\psi(x) - \psi(\zeta))^{n-\varrho-1} \psi' \\ &\quad \cdot (\zeta)\omega_\psi^{[n]}(\zeta) d\zeta, \varrho > 0, \end{aligned} \tag{3}$$

respectively, where

$$n = -[-\varrho], \omega_\psi^{[n]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \omega(x). \tag{4}$$

Definition 2 [27]. The incomplete gamma function is represented by

$$\gamma(\varrho, \zeta) = \int_0^\zeta u^{\varrho-1} e^{-u} du = \zeta^\varrho \Gamma(\varrho) e^{-\zeta} \sum_{i=0}^\infty \frac{\zeta^i}{\Gamma(\varrho + i + 1)}, \varrho > 0, \zeta \ge 0. \tag{5}$$

Property 3 [3, 9]. Let $\varrho \ge 0$, and $\kappa > 0$. Then

$$I_a^{\psi,\varrho}(\psi(x) - \psi(a))^\kappa = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + \varrho + 1)} (\psi(x) - \psi(a))^{\kappa+\varrho}, x > a, \tag{6}$$

$${}^C\mathbb{D}_a^{\psi,\varrho}(\psi(x) - \psi(a))^\kappa = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \varrho + 1)} (\psi(x) - \psi(a))^{\kappa-\varrho}, x > a. \tag{7}$$

Property 4 [3, 9]. Let $\varrho, \kappa > 0$, and $\omega \in AC_0[a, b]$. Then

$${}^C\mathbb{D}_a^{\psi,\varrho} I_a^{\psi,\varrho}\omega(\zeta) = \omega(\zeta), \zeta > a, \tag{8}$$

$$I_a^{\psi,\varrho} I_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho+\kappa}\omega(\zeta), \zeta > a, \tag{9}$$

$${}^C\mathbb{D}_a^{\psi,\kappa} I_a^{\psi,\varrho}\omega(\zeta) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a, \tag{10}$$

$$I_a^{\psi,\varrho}\omega(a) = 0. \tag{11}$$

In the following, some very significant lemmas will be given.

Lemma 5. *Let $\varrho, \kappa \in [0, 1]$. If $\omega \in AC_0[a, b]$, then*

$$I_a^{\psi,\varrho}\,{}^C\mathbb{D}_a^{\psi,\varrho}\omega(\zeta) = \omega(\zeta), \zeta > a, \tag{12}$$

$$I_a^{\psi,\varrho}\,{}^C\mathbb{D}_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a. \tag{13}$$

Proof. Let $\omega \in AC_0[a,b]$. Then by Definition 1 and Property 4, we get

$$I_a^{\psi,\varrho C}\mathbb{D}_a^{\psi,\varrho}\omega(\zeta) = I_a^{\psi,\varrho} I_a^{\psi,1-\varrho}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,1}\omega_\psi^{[1]}(\zeta) = \omega(\zeta) - \omega(a) = \omega(\zeta), \zeta > a, \tag{14}$$

$$I_a^{\psi,\varrho C}\mathbb{D}_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho} I_a^{\psi,1-\kappa}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,\varrho-\kappa} I_a^{\psi,1}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,\varrho-\kappa}(\omega(\zeta) - \omega(a)) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a. \tag{15}$$

□

Lemma 7. *Let $\varrho > 0$. Then, $I_a^{\psi,\varrho}$ is bounded from $AC_0[a,b]$ into itself, and*

$$\|I_a^{\psi,\varrho}\omega\|_0 \le \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho \|\omega\|_0. \tag{16}$$

Proof. Let $\omega \in AC_0[a,b]$. Then

$$\omega(\zeta) = \int_a^\zeta \omega'(s)ds = \int_a^\zeta \mathfrak{z}(s)ds, \text{ where } \mathfrak{z}(\zeta) = \omega'(\zeta) \text{ and } \omega(a) = 0. \tag{17}$$

By virtue of Definition 1, we get

$$I_a^{\psi,\varrho}\omega(x) = I_a^{\psi,\varrho}\left(\int_a^\zeta \mathfrak{z}(s)ds\right)(x) = \frac{1}{\Gamma(\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)\int_a^\zeta \mathfrak{z}(s)dsd\zeta. \tag{18}$$

Taking advantage of the Dirichlet's formula, we have

$$\begin{aligned} I_a^{\psi,\varrho}\omega(x) &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\int_s^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)d\zeta ds \\ &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\left[-\frac{(\psi(x) - \psi(\zeta))^\varrho}{\varrho}\right]_{\zeta=s}^x ds \\ &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\left[\frac{(\psi(x) - \psi(s))^\varrho}{\varrho}\right] ds \\ &\le \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho\int_a^x |\mathfrak{z}(s)|ds \\ &= \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho\int_a^x |\omega'(s)|ds \\ &= \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho\|\omega\|_0. \end{aligned} \tag{19}$$

Now, we will provide and prove the next lemma: □

Lemma 9. *If $\varrho \ge 0$, then*

$$I_a^{\psi,\varrho}e^{\psi(x)} = e^{\psi(a)}(\psi(x) - \psi(a))^\varrho \sum_{i=0}^\infty \frac{(\psi(x) - \psi(a))^i}{\Gamma(\varrho+i+1)}. \tag{20}$$

Proof. Using Definition 1, we have

$$I_a^{\psi,\varrho}e^{\psi(x)} = \frac{1}{\Gamma(\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)e^{\psi(\zeta)}d\zeta. \tag{21}$$

Performing the substitution $s = \psi(x) - \psi(\zeta)$, we get

$$\begin{aligned} I_a^{\psi,\varrho}e^{\psi(x)} &= \frac{1}{\Gamma(\varrho)}\int_0^{\psi(x)-\psi(a)} s^{\varrho-1}e^{\psi(x)-s}ds \\ &= \frac{e^{\psi(x)}}{\Gamma(\varrho)}\int_0^{\psi(x)-\psi(a)} s^{\varrho-1}e^{-s}ds. \end{aligned} \tag{22}$$

From Definition 2, we obtain

$$I_a^{\psi,\varrho}e^{\psi(x)} = \gamma(\varrho, \psi(x) - \psi(a))\frac{e^{\psi(x)}}{\Gamma(\varrho)} = e^{\psi(a)}(\psi(x) - \psi(a))^\varrho \sum_{i=0}^\infty \frac{(\psi(x) - \psi(a))^i}{\Gamma(\varrho+i+1)}. \tag{23}$$

□

3. Main Results

Theorem 11. *Let $a_i \in \mathbb{C}(i = 1, 2)$, $0 < \varrho_1 < \varrho < 1$, and $\varrho_2 > 0$. If $h \in AC_0[a,b]$, then, the following linear problem*

$$\begin{cases} {}^C\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1\, {}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2\, I_a^{\psi,\varrho_2}\varphi(x) = h(x), \\ \varphi(a) = 0, \end{cases} \tag{24}$$

has a solution in the space $AC_0[a,b]$, that is

$$\varphi(x) = \sum_{\ell=0}^\infty (-1)^\ell \sum_{\ell_1+\ell_2=\ell} \binom{\ell}{\ell_1, \ell_2} \times a_1^{\ell_1} a_2^{\ell_2}\, I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}h(x). \tag{25}$$

Proof. Applying the operator $I_a^{\psi,\varrho}$ to both sides of Eq. (24), we obtain

$$I_a^{\psi,\varrho C}\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1 I_a^{\psi,\varrho C}\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2 I_a^{\psi,\varrho} I_a^{\psi,\varrho_2}\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{26}$$

According to Lemma 5, we find that

$$\varphi(x) + a_1 I_a^{\psi,\varrho-\varrho_1}\varphi(x) + a_2 I_a^{\psi,\varrho+\varrho_2}\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{27}$$

Observe that $\varphi(a) = 0$ and $0 < \varrho_1 < \varrho < 1$. It follows that

$$(1 + a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2})\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{28}$$

In view of Babenko approach, we have

$$\varphi(x) = (1 + a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2})^{-1}\, I_a^{\psi,\varrho}\mathrm{h}(x). \tag{29}$$

By using the multinomial theorem and Property 4, we obtain

$$\begin{aligned}\varphi(x) &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\left(a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2}\right)^{\ell} I_a^{\psi,\varrho}\mathrm{h}(x)\\ &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\left(a_1 I_a^{\psi,\varrho-\varrho_1}\right)^{\ell_1}\left(a_2 I_a^{\psi,\varrho+\varrho_2}\right)^{\ell_2} I_a^{\psi,\varrho}\mathrm{h}(x)\\ &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2} a_1^{\ell_1}a_2^{\ell_2} I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x).\end{aligned} \tag{30}$$

As $x \longrightarrow a$, we get $\varphi(a)=0$. Now, we need to prove the series is absolutely continuous on $[a,b]$ and converges in the space $AC_0[a,b]$. Indeed, by Lemma 7, we have

$$\left\| I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right\|_0 \le \eta\|\mathrm{h}\|_0, \tag{31}$$

where

$$\eta = \frac{(\psi(b)-\psi(a))^{\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}. \tag{32}$$

It follows that

$$\begin{aligned}\|\varphi\|_0 &\le \eta\sum_{\ell=0}^{\infty}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\left|a_1^{\ell_1}\right|\left|a_2^{\ell_2}\right|\frac{(\psi(b)-\psi(a))^{\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}\|\mathrm{h}\|_0\\ &= \eta\sum_{\ell=0}^{\infty}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\\ &\quad\times\frac{\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}\right)^{\ell_1}\left(|a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}\right)^{\ell_2}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}\|\mathrm{h}\|_0\\ &= \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b)\right.\\ &\qquad\left.-\psi(a))^{\varrho+\varrho_2}\right)\|\mathrm{h}\|_0,\end{aligned} \tag{33}$$

where

$$E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}\right) < \infty, \tag{34}$$

which is the value at $\nu_1 = |a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}$, $\nu_2 = |a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}$ of the multivariate Mittag-Leffler function $E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}(\nu_1,\nu_2)$ given in [3]. So, we conclude that the series to the right of Eq. (25) is convergent. Obviously, $\varphi(x)\in AC[a,b]$ due to $\mathrm{h}\in AC[a,b]$. To affirm that the obtained series could be a solution, we must see that it fulfills Eq. (24), i.e.,

$$\begin{aligned}&{}^{C}\mathbb{D}_a^{\psi,\varrho}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &\quad+a_1{}^{C}\mathbb{D}_a^{\psi,\varrho_1}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &\quad+a_2I_a^{\psi,\varrho_2}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &={}^{C}\mathbb{D}_a^{\psi,\varrho}\left(I_a^{\psi,\rho}\mathrm{h}(x)+\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\right.\\ &\quad\left.\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\rho-\rho_1)+\ell_2(\rho+\rho_2)+\rho}\mathrm{h}(x)\right)\\ &\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_2}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_0^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)\\ &=\mathrm{h}(x)+\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\rho-\rho_1)+\ell_2(\rho+\rho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_2}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)=\mathrm{h}(x),\end{aligned} \tag{35}$$

by the cancellation. Notice that each series is absolutely convergent and also the term arrangements are possibly cancellated. In fact,

$$\begin{aligned}&-\sum_{\ell_1+\ell_2=1}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell_1+\ell_2=0}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_n}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell_1+\ell_2=0}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)=0.\end{aligned} \tag{36}$$

The remainder terms cancel each other similarly. Plainly, the uniqueness follows promptly from the fact that the FIDE

$${}^{C}\mathbb{D}_a^{\psi,\varrho}\varphi(x)+a_1{}^{C}\mathbb{D}_a^{\psi,\varrho_1}\varphi(x)+a_2I_a^{\psi,\varrho_2}\varphi(x)=0, \tag{37}$$

only has solution zero due to the Babenko approach. □

Remark 13. Notice that the solution of Eq. (24) in $AC_0[a,b]$ is stable, if for all $\varepsilon>0$, there exists $\delta>0$ such that $\|\varphi\|_0<\varepsilon$

with $\|h\|_0 < \delta$. Taking advantage of the following inequality

$$\|\varphi\|_0 \le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b) - \psi(a))^{\varrho+\varrho_2}\right)\|\mathrm{h}\|_0, \tag{38}$$

we conclude that φ is stable.

Example 1. The following ψ-Caputo type FIDE

$${}^C\mathbb{D}_a^{\psi,0.9}\varphi(x) + 2\,{}^C\mathbb{D}_a^{\psi,0.7}\varphi(x) - I_a^{\psi,0.4}\varphi(x) = (\psi(x)-\psi(a))^\kappa, \tag{39}$$

has the solution in $AC_0[a,b]$, that is

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+0.2\ell_1+1.3\ell_2+1.9)}(\psi(x)-\psi(a))^{\kappa+0.2\ell_1+1.3\ell_2+0.9}. \tag{40}$$

So, according to Theorem 11, we have

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times I_a^{\psi,0.2\ell_1+1.3\ell_2+0.9}(\psi(x)-\psi(a))^\kappa. \tag{41}$$

From Property 3, we get

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+0.2\ell_1+1.3\ell_2+1.9)}(\psi(x)-\psi(a))^{\kappa+0.2\ell_1+1.3\ell_2+0.9}. \tag{42}$$

Example 2. The following ψ-Caputo type FIDE

$${}^C\mathbb{D}_a^{\psi,0.8}\varphi(x) + {}^C\mathbb{D}_a^{\psi,0.7}\varphi(x) - 3I_a^{\psi,0.2}\varphi(x) = e^{\psi(x)}, \tag{43}$$

has the solution in $AC_0[a,b]$ described as

$$\varphi(x) = e^{\psi(a)}\sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(-3)^{\ell_2}(\psi(x) - \psi(a))^{0.1\ell_1+\ell_2+0.8} \times \sum_{i=0}^{\infty}\frac{(\psi(x)-\psi(a))^i}{\Gamma(0.1\ell_1+\ell_2+1.8+i)}. \tag{44}$$

So, as stated by Theorem 11, we obtain

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(1)^{\ell_1}(-3)^{\ell_2} I_a^{\psi,0.1\ell_1+\ell_2+0.8}e^{\psi(x)}. \tag{45}$$

By virtue of Lemma 9, we obtain

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(-3)^{\ell_2}e^{\psi(a)}(\psi(x) - \psi(a))^{0.1\ell_1+\ell_2+0.8} \times \sum_{i=0}^{\infty}\frac{(\psi(x)-\psi(a))^i}{\Gamma(0.1\ell_1+\ell_2+1.8+i)}. \tag{46}$$

The uniqueness result of Eq. (1) will be proved through the following theorem.

Theorem 14. *Let $\mathbb{G}:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function, and assume that there exists a constant C such that*

$$|\mathbb{G}(x,\varphi_1)-\mathbb{G}(x,\varphi_2)| \le C|\varphi_1-\varphi_2|, x\in[a,b], \varphi_1,\varphi_2\in\mathbb{R}. \tag{47}$$

If

$$C\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right) < 1, \tag{48}$$

then, the problem (1) has a unique solution in $AC_0[a,b]$.

Proof. Consider the operator $\mathfrak{I}$ on $AC_0[a,b]$ defined by

$$\mathfrak{I}(\varphi) = \sum_{\ell=0}^{\infty}(-1)^\ell \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2} \times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta. \tag{49}$$

For $\varphi\in AC_0[a,b]$, we have $\int_a^x\mathbb{G}(\zeta,\varphi'(\zeta))d\zeta\in AC_0[a,b]$, since $\varphi'(\zeta)\in L(a,b)$ and $\mathbb{G}(\zeta,\varphi'(\zeta))\in L(a,b)$.

Hence,

$$\left\|\int_a^x\mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0 = \int_a^b\left|\mathbb{G}\left(x,\varphi'(x)\right)\right|dx \le \int_a^b\left|\mathbb{G}\left(x,\varphi'(x)\right) - \mathbb{G}(x,0)\right|dx + \int_a^b|\mathbb{G}(x,0)|dx \le C\int_a^b\left|\varphi'(x)\right|dx + \int_a^b|\mathbb{G}(x,0)|dx < \infty. \tag{50}$$

Using the inequality (38), we obtain

$$\|\mathfrak{I}(\varphi)\|_0 < \infty \text{ and } \mathfrak{I}(\varphi)(a) = 0. \tag{51}$$

Besides, $\mathfrak{I}(\varphi)$ is absolutely continuous on $[a,b]$ via Theorem 11. So, $\mathfrak{I}: AC_0[a,b]\longrightarrow AC_0[a,b]$. Now, we just have to show that $\mathfrak{J}$ is a contraction mapping. Let $\varphi, \varphi^* \in AC_0[a, b]$. Then

$$\begin{aligned}\|\mathfrak{J}(\varphi)-\mathfrak{J}(\varphi^*)\|_0 &\le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta-\int_a^x \mathbb{G}\left(\zeta,\varphi^{*'}(\zeta)\right)d\zeta\right\|_0.\end{aligned} \tag{52}$$

Since

$$\begin{aligned}&\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta-\int_a^x \mathbb{G}\left(\zeta,\varphi^{*'}(\zeta)\right)d\zeta\right\|_0\\ &\quad=\int_a^b\left|\mathbb{G}\left(x,\varphi'(x)\right)-\mathbb{G}\left(x,\varphi^{*'}(x)\right)\right|dx\\ &\quad\le C\int_a^b\left|\varphi'-\varphi^{*'}\right|dx=C\|\varphi-\varphi^*\|_0,\end{aligned} \tag{53}$$

we obtain

$$\begin{aligned}&\|\mathfrak{J}(\varphi)-\mathfrak{J}(\varphi^*)\|_0\\ &\quad\le C\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\quad\times\|\varphi-\varphi^*\|_0.\end{aligned} \tag{54}$$

Inequality (48) leads us to that $\mathfrak{J}$ is contraction mapping. □

3.1. Ulam-Hyers Stability (UHS). The first results about this type of stability emerged in 1940 by Ulam [28, 29]. From that point forward, the UHS is studied via several researchers. With the vast development of fractional calculus, the studying of stability for FDEs also attracted the numerous authors, see [30–32].

In this regard, we investigate some recent results on the UHS and generalized UHS of (1). For $\varepsilon > 0, x \in [a, b]$, and $\varphi_1 \in AC_0[a, b]$, the following inequality

$$\left|{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x)+a_1{}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x)+a_2I_a^{\psi,\varrho_2}\varphi_1(x)-\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right|\le\varepsilon, \tag{55}$$

is satisfied.

Remark 16. Let $\varepsilon > 0$. Then, $\varphi_1 \in AC_0[a, b]$ satisfies (55) iff there exists $\xi(x) \in AC_0[a, b]$ with $\xi(0) = 0$ such that

(i) $\|\xi\|_0 = \int_a^x|\xi'(\zeta)|d\zeta \le \varepsilon$, for $x \in [a, b]$

(ii) for $x \in [a, b]$

$$\begin{aligned}{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x)+a_1{}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x)+a_2I_a^{\psi,\varrho_2}\varphi_1(x)&=\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta\\ &+\int_a^x|\xi'(\zeta)|d\zeta.\end{aligned} \tag{56}$$

Lemma 17. *The solution of the problem (56) with $\varphi_1(0) = 0$ satisfies the following inequality*

$$\begin{aligned}\|\varphi_1-Z_{\mathbb{G}}\|_0&\le\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon,\end{aligned} \tag{57}$$

where

$$\begin{aligned}Z_{\mathbb{G}}(x)&:=\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}\\ &\times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta,\end{aligned} \tag{58}$$

and η is defined by (32).

Proof. By virtue of Lemma 8, the solution of Eq. (56) is described as

$$\begin{aligned}\varphi_1(x)&=\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}\\ &\times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\left[\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta+\int_a^x|\xi'(\zeta)|d\zeta\right].\end{aligned} \tag{59}$$

It follows from Eq. (59), Remark 16, and Eq. (38) that

$$\begin{aligned}\|\varphi_1-Z_{\mathbb{G}}\|_0&\le\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta+\int_a^x|\xi'(\zeta)|d\zeta-\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta\right\|_0\\ &\le\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\|\xi\|_0\\ &\le\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon.\end{aligned} \tag{60}$$

□

Theorem 19 (UHS). *Suppose that the hypotheses of Theorem 14 with Eq. (55) are satisfied. Then, Eq. (1) is UH stable.*

Proof. Assume that $\varepsilon > 0$ and $\varphi_1 \in AC_0[a, b]$ satisfy Eq. (55), and let $\varphi \in AC_0[a, b]$ be a unique solution of

$$\begin{cases}{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x)+a_1{}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x)+a_2I_a^{\psi,\varrho_2}\varphi_1(x)=\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta,\\ \varphi(a)=\varphi_1(a)=0,\end{cases} \tag{61}$$

that is

$$\varphi(x)=\varphi(a)+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\left[\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right]. \tag{62}$$

Since $\varphi(a)=\varphi_1(a)=0$, we obtain

$$\varphi(x)=\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\left[\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right]. \tag{63}$$

According to Lemma 17 and (38), we get

$$\begin{aligned}\|\varphi_1-\varphi\|_0 &\le \|\varphi_1-Z_{\mathbb{G}}\|_0+\|Z_{\mathbb{G}}-\varphi\|_0\le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon\\ &+\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times\left\|\int_a^x\mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta-\int_a^x\mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0.\end{aligned} \tag{64}$$

Using the assumption of Theorem 14, we have

$$\left\|\int_a^x\mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta-\int_a^x\mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0\le C\|\varphi_1-\varphi\|_0. \tag{65}$$

So,

$$\begin{aligned}\|\varphi_1-\varphi\|_0 &\le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon\\ &+\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times C\|\varphi_1-\varphi\|_0.\end{aligned} \tag{66}$$

From the inequality (48), we find that

$$\|\varphi_1-\varphi\|_0\le C_{\mathbb{G}}\varepsilon, \tag{67}$$

where $C_{\mathbb{G}}\coloneqq \mathfrak{R}/1-\mathfrak{R}C$ and

$$\mathfrak{R}\coloneqq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right). \tag{68}$$

□

Corollary 21. *Under assumptions of Theorem 19, if we put $\Phi(\varepsilon)=C_{\mathbb{G}}\varepsilon$ along with $\Phi(0)=0$, then Eq. (1) is a generalized UH stable.*

Example 3. Let $a=1$ and $b=\psi^{-1}(1+\psi(1))$. Then, there exists a unique solution for the following nonlinear ψ-Caputo-type FIDE

$$\begin{aligned}&{}^{C}\mathbb{D}_a^{\psi,0.9}\varphi(x)+I_a^{\psi,0.6}\varphi(x)\\ &=\int_a^x\left(\frac{e^{\zeta^2}}{C\left(3+e^{\zeta^2}\right)}\sin\varphi'(\zeta)+e^{\cos\zeta}+\ln\left(1+\sqrt{\zeta}\right)\right)d\zeta,\end{aligned} \tag{69}$$

where the constant C is to be determined. It is clear that

$$\mathbb{G}(x,z)=\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z+e^{\cos x}+\ln\left(1+\sqrt{x}\right), \tag{70}$$

is continuous from $[1,\psi^{-1}(1+\psi(1))]\times\mathbb{R}$ to $\mathbb{R}$ and satisfies

$$\begin{aligned}&|\mathbb{G}(x,z_1)-\mathbb{G}(x,z_2)|\\ &=\left|\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z_1-\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z_1\right|\\ &\le\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}|\sin z_1-\sin z_1|\le\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}|z_1-z_1|\\ &\le\frac{1}{C}|z_1-z_1|.\end{aligned} \tag{71}$$

Obviously, $\psi(b)-\psi(a)=1$ and

$$\sum_{\ell=0}^{\infty}\sum_{\ell_2=\ell}\binom{\ell}{\ell_2}\left(|1|(\psi(b)-\psi(a))^{1.5}\right)^{\ell_2}\frac{1}{\Gamma(1.5\ell_2+1.9)}=\sum_{\ell=0}^{\infty}\frac{1}{\Gamma(1.5\ell+1.9)}. \tag{72}$$

For $\ell\ge1$, we have

$$\ell+1\le1.5\ell+1.9, \tag{73}$$

$$\frac{1}{\Gamma(1.5\ell+1.9)}\le\frac{1}{\Gamma(\ell+1)}=\frac{1}{\ell!}. \tag{74}$$

Therefore,

$$\begin{aligned}\sum_{\ell=0}^{\infty}\sum_{\ell_2=\ell}\binom{\ell}{\ell_2}\frac{1}{\Gamma(1.5\ell_2+1.9)}&\le\frac{1}{\Gamma(1.9)}+\sum_{\ell=1}^{\infty}\frac{1}{k!}\\ &=\frac{1}{\Gamma(1.9)}-1+\sum_{\ell=0}^{\infty}\frac{1}{k!}\le0.04+e.\end{aligned} \tag{75}$$

Let us choose a positive C such that

$$C < \frac{1}{0.04 + e}. \tag{76}$$

It follows from Theorem 14 that Eq. (69) has a unique solution.

Moreover, by Theorem 19, and for any solution $\varphi_1(x) \in AC_0[a, b]$ of the inequality

$$\left| {}^{C}\mathbb{D}_a^{\psi,0.9}\varphi(x) + I_a^{\psi,0.6}\varphi(x) - \int_a^x \left(\frac{e^{\zeta^2}}{C\left(3 + e^{\zeta^2}\right)} \sin \varphi_1'(\zeta) + e^{\cos \zeta} + \ln\left(1 + \sqrt{\zeta}\right) \right) d\zeta \right| \le \varepsilon, \text{for } x \in [a, b], \tag{77}$$

there exists a unique solution $\varphi(x) \in AC_0[a, b]$ of Eq. (69) such that

$$\|\varphi_1 - \varphi\|_0 \le C_{\mathbb{G}}\varepsilon, \tag{78}$$

where $C_{\mathbb{G}} := \mathfrak{R}/1 - \mathfrak{R}C > 0$, $\mathfrak{R} = \eta/0.04 + e$ and $\eta = 1/\Gamma(\ell_1 0.9 + 1.5\ell_2 + 1.9)$. Consequently, Eq. (69) is UH stable.

Remark 22. All previous results can be generalized in which the left-hand side of (1) may contain several FDs and FIs. For example, Theorem 11 can be generalized as follows:

Theorem 23. *Suppose* $a_i \in \mathbb{C}(i = 1, \cdots, n), b_{i+n} \in \mathbb{C}$ $(i = 1, \cdots, m - n)$ *with* $0 < \alpha_1 < \cdots < \alpha_n < \alpha < 1$ *and* $0 \le \kappa_{n+1} < \kappa_{n+2} < \cdots < \kappa_m \in \mathbb{R}$. *If* $h \in AC_0[a, b]$, *then the following linear problem*

$$\begin{cases} {}^{C}\mathbb{D}_a^{\psi,\alpha}\varphi(x) + a_1 {}^{C}\mathbb{D}_a^{\psi,\alpha_1}\varphi(x) + \dots + a_n {}^{C}\mathbb{D}_a^{\psi,\alpha_n}\varphi(x) + b_{n+1} I_a^{\psi,k_{n+1}}\varphi(x) + b_{n+2} I_a^{\psi,k_{n+2}}\varphi(x) + \dots + b_m I_a^{\psi,k_m}\varphi(x) = \mathrm{h}(x), \\ \varphi(a) = 0, \end{cases} \tag{79}$$

has a solution

$$\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1+\cdots+\ell_m=\ell} \binom{\ell}{\ell_1, \ell_2, \cdots, \ell_m} \times a_1^{\ell_1} \cdots b_m^{\ell_m} I_a^{\psi,\ell_1(\alpha-\alpha_1)+\cdots+\ell_m(\alpha+\kappa_m)+\alpha} h(x). \tag{80}$$

where ${}^{C}\mathbb{D}_a^{\psi,\delta}$ *is the* ψ*-Caputo FD of order* $\delta(>0) \in \{\alpha, \alpha_i ; i = 1, \cdots, n\}$ *and* $I_a^{\psi,\sigma}$ *is generalized FI of order* $\sigma(>0) \in \{\kappa_j ; j = n + 1, \cdots, m\}$.

4. Conclusions

ψ-Caputo FD, a general fractional operator, is of great use because of its wide freedom to cover many classical fractional operators. In this work, we have studied the uniqueness of solution for the nonlinear ψ-Caputo type FIDE (1) by using the Banach space $AC_0[a, b]$, Banach's fixed point theorem, and Babenko's method. Moreover, the UH stability results to the proposed problem have been discussed. Also, some pertinent examples have been provided to justify the main results. The obtained results in this study extended and developed the current results introduced by [17, 18]. We have already concluded that our results are valid when the left-hand side of the considered problem (1) involves many FDs and IDs as shown in Remark 22. Furthermore, problem (1) covers previous standard cases of nonlinear FDEs and FIDEs by selecting the suitable standard kernel in the studied problem. More specifically, our results generalize some known results in literature like those that include Hadamard and Katugampola FDs.

For future research, we will consider a class of nonlinear FIDEs with the fuzzy initial conditions in a fractional case. It would also be interesting to study the same results for our current problem under the ψ-Hilfer operator [33] or Atangana-Baleanu operator [8].

Data Availability

The data of this study were used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors made equal contributions and read and supported the last manuscript.

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