

Research Article

Characterizations of Probability Distribution by Some Sequential Relative Reliability Measures: An Application of Completeness in a Hilbert Space

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In this paper, by means of the concept of completeness in functional analysis, we characterize probability distributions using several relative measures of residual life of an item at independent sequences of random times. These sequences are the order statistics and the record statistics of a random sequence of times. The reliability quantities of the mean residual life, the vitality function, and the survival function are considered to construct the main characteristics of distributions.

1. Introduction

A characterization relation is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated probability distribution. The characterization properties derived in literature are mostly based on random samples from common univariate discrete and continuous distributions, and some multivariate continuous distributions are also considered. The main approach in such characterizations use the properties of sample moments, order statistics, record statistics, and reliability properties. The literature in the context of characterizations of probability distributions is extensive. We refer the reader to several review articles, monographs (e.g., Kagan et al. [1]), Rao and Shanbhag [2], Galambos and Kotz [3], Arnold et al. [4], and Galambos and Kotz [5]) and encyclopedic books on distributions by Johnson, Kotz, and their coauthors (Johnson et al. [6] and Kotz et al. [7]) have appeared to be excellent filters in the subject.

There are some characterization properties of standard distributions using the properties of order statistics (see, e.g., Pfeifer [8], Bairamov and Ozkal [9], Huang and Su [10], Beg et al. [11], Kayid and Izadkhah [12], Akbari et al. [13], Hu and Lin [14], and Betsch and Ebner [15]). Further-

more, there are several characterization results of reputable probability distributions derived by the properties of record statistics (see, e.g., Witte [16], Kamps [17], Franco and Ruiz [18], Wesolowski and Ahsanullah [19], Nagaraja and Barlevy [20], Balakrishnan and Stepanov [21], Baratpour et al. [22], and Ahmadi [23]).

There may be independent characteristics of a single distribution in different contexts so that when they are given, the underlying distribution of the associated random variable is characterized uniquely. For instance, in the context of reliability and survival analysis, the survival function (s.f.), the hazard rate (h.r.) function, the mean residual life (m.r.l.) function, and the vitality function (v.f.) determine the underlying distribution uniquely. In terms of these unique quantities, by holding particular relations involving them, one may also characterize a probability distribution (see, for example, Gupta and Keating [24], Navarro et al. [25], Nair and Sankaran [26], Szymkowiak [27], and Szymkowiak [28]).

The rest of the paper is organized as follows. In Section 2, we state some preliminary definitions and introduce some concepts in reliability theory and functional analysis. In Section 3, fundamental characterizations of distributions using weighted residual lives at random times which are order

statistics of a partial subset of a random sequence of times are obtained. In Section 4, the characterization properties of distributions are developed using weighted residual lives at record values of the random sequence of times. Finally, in Section 5, we conclude the paper with further detailed remarks and outline the results that are of potential interest in future.

2. Preliminaries

Suppose that X is a lifetime random variable with c.d.f. F and s.f. $\bar{F} := 1 - F$. Let us take X as the life length of a lifespan or any other item that has a lifetime. The residual lifetime of that lifespan or the item after time point t until which it has been alive, is denoted by X_t defined by $X_t := (X - t | X > t)$ for $t : F(t) < 1$. The random variable X_t , as a time-dependent conditional random variable, has s.f. $\bar{F}_t(x) = \bar{F}(t + x)/\bar{F}(t)$, for all $t, x \geq 0$. Being dependent on time, the stochastic residual lifetime process $\{X_t : t \geq 0\}$ may be of some interest in different contexts (see, e.g., Chung [29] and Pekalp [30]). Gathering data in accordance with appearing X_t in time as it goes by provides a more insightful and also further informative method to infer about target population. The mean residual lifetime function is defined as

$$m_X(t) := E(X_t) = \frac{\int_t^{+\infty} \bar{F}(x) dx}{\bar{F}(t)}. \quad (1)$$

Define the truncated random variable $X(t) := (X | X > t)$ according which the vitality function of X is defined as

$$v_X(t) := E(X(t)) = \frac{\int_0^{+\infty} \bar{F}(x \vee t) dx}{\bar{F}(t)}, \quad (2)$$

where $x \vee t = \max\{x, t\}$. It is notable here and also it leads to some basic conclusions in the sequel that the mean residual lifetime function and the vitality function each is a characteristic of the underlying distribution (cf. Oakes and Dasu [31] and Ruiz and Navarro [32]). Let Y be another lifetime random variable with s.f. \bar{F}_Y , the m.r.l. function m_Y and the v.f. v_Y . Then, to compare the residual live of X relative to the residual live of Y in different times, the following measures can be set:

$$\begin{aligned} R_1(X, Y; t) &:= \left(\frac{X - t}{m_Y(t)} | X > t \right), \\ R_2(X, Y; t) &:= \left(\frac{X}{v_Y(t)} | X > t \right), \\ R_3(X, Y; t) &:= \frac{I[x > t]}{\bar{F}_Y(t)}, \end{aligned} \quad (3)$$

where $I[x > t] = 1$ whenever $x > t$ and $I[x > t] = 0$ elsewhere. Due to the fact that each of the m.r.l., v.f., and s.f. determines the parent distribution uniquely, we can readily observe that X and Y are equal in distribution iff $E(R_i(X, Y; t)) = 1$, for all $t \geq 0$, for any $i = 1, 2, 3$. The random variables $R_i(X, Y; t)$, $i = 1, 2, 3$ can be considered a three relative stochastic process.

The residual life at random time has been determined to be an important measure in reliability, survival analysis, and life testing (Yue and Cao [33], Li and Zuo [34], Misra et al. [35], Cai and Zheng [36], and Dewan and Khaledi [37]). Let X and T be two lifetime random variables with c.d.f.s F and F_T and probability density functions (p.d.f.s) f and f_T , respectively. Denote by $X_T = {}^d[X - T | X > T]$, the residual life of X at the random time T with distribution function F_{X_T} is obtained. It has been shown that the idle time of the server in a $GI/G/1$ queuing system can be stated as a residual life at random time (cf. Dewan and Khaledi [37]). In the context of reliability, when X is the total random life of a warm stand by unit with a random age of T , the random variable X_T then stands as the actual working time of the stand by unit (cf. Yue and Cao [33]). If X and T are statistically independent having a common support, then

$$F_{X_T}(x) = \frac{\int_0^{+\infty} [F(x+t) - F(t)] f_T(t) dt}{P(X > T)}. \quad (4)$$

For ease of reference, before stating the main fundamental characterization properties, we provide some useful notions in functional analysis.

Definition 1. The sequence ψ_1, ψ_2, \dots is complete in a Hilbert space \mathcal{H} if the only element of H which is orthogonal to every ψ_n is the null element, that is,

$$\langle f, \psi_n \rangle = 0, \quad \forall n \geq 1 \Rightarrow f = 0, \quad (5)$$

where 0 stands for the zero element of H .

We recall that $\langle \cdot, \cdot \rangle$ signifies an inner product of H . In the present paper, we utilize the Hilbert space $L^2[a, b]$, whose inner product is determined by

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x) dx, \quad (6)$$

in which f_1 and f_2 are two real-valued square integrable functions defined on $[a, b]$. Notice that if ψ_1, ψ_2, \dots is a complete sequence in the Hilbert space H , then $\sum c_n \phi_n$ with $c_n = \langle f, \phi_n \rangle$ converges in H provided that $\sum |c_n|^2 < \infty$, and the limit equal to f . See, for instance, Higgins [38] for more detailed discussions about the theory of completeness. The following result which is a key concept development shows the characterization results in this paper.

Lemma 2 (see [39]). *Let ψ be an absolutely continuous function defined on $[a, b]$ with $\psi(a)\psi(b) \geq 0$, and let its derivative satisfy $\psi'(x) \neq 0$ a.e. on (a, b) . Then, under the assumption*

$$\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty, \text{ where } 1 \leq \lambda_1 < \lambda_2 < \dots, \quad (7)$$

the sequence $\psi^{\lambda_1}, \psi^{\lambda_2}, \dots$ is complete on (a, b) iff the function ψ is monotone on (a, b) .

In the continuing part of this paper, we consider the sequence of random variables $\{T_n, n \in \mathbb{N}\}$ constituting independent and identically distributed (i.i.d.) nonnegative random variables with p.d.f. f_T , c.d.f. F_T , and s.f. \bar{F}_T . The element $T_i, i = 1, \dots, n$ in the sequence may denote the time of occurrence of a certain event. Suppose that X denotes a random lifetime which is independent of the sequence $\{T_n, n \in \mathbb{N}\}$, and therefore,

$$\begin{aligned} P(X > \psi(\omega(T_1, \dots, T_n), x) | \omega(T_1, \dots, T_n) = t) \\ = P(X > \psi(\omega, x)), \text{ for all } n = 1, 2, \dots, \end{aligned} \tag{8}$$

where ψ is an arbitrary nonnegative function and the equality in the event $\{\omega(T_1, \dots, T_n) = \omega\}$ is not strict.

3. Reliability Measures on a Time Scale of Order Statistics

We assume that $T_{1:n} \leq T_{2:n} \leq \dots \leq T_{n:n}$ be the order statistics of the partial set $\{T_1, T_2, \dots, T_n\}$. The p.d.f. of the i th-order statistic $T_{i:n}$, for $i = 1, 2, \dots, n$, is

$$f_{T_{i:n}}(t) = \frac{n!}{(i-1)!(n-i)!} F_T^{i-1}(t) \bar{F}_T^{n-i}(t) f_T(t), \text{ for all } t > 0. \tag{9}$$

Fix $i \in \mathbb{N}$ and consider an integer $n = i, i + 1, \dots$. Let us suppose that $T_{i:n}$ represents the time of occurrence of the i th event among a series of n events which are occurred consecutively at the ordered times $T_{1:n} < T_{2:n} < \dots < T_{n:n}$. For instance, these times may denote the times at which n external shocks are absorbed by a system. It is then realized that the residual life of X at the random time $T_{i:n}$, denoted by $X_{T_{i:n}} = (X - T_{i:n} | X > T_{i:n})$, represents the residual lifetime of a device with lifetime X after the time at which the i th event occurs.

Let X have s.f. \bar{F} . We assume throughout the paper that X , which is used to denote the lifetime of a device, has a finite mean. For any $n = i, i + 1, \dots$ it is also assumed that $X_{T_{i:n}}$ has a finite mean. It follows that X is independent of $T_{i:n}$. Hence, the s.f. of $X_{T_{i:n}}$ is derived as

$$\begin{aligned} P(X_{T_{i:n}} > x) &= \frac{P(X > T_{i:n} + x)}{P(X > T_{i:n})} = \frac{\int_0^{+\infty} \bar{F}(t+x) f_{T_{i:n}}(t) dt}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \\ &= \int_0^{+\infty} \frac{\bar{F}(t+x)}{\bar{F}(t)} \left(\frac{\bar{F}(t) f_{T_{i:n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \right) dt \\ &= E \left(\frac{\bar{F}(T_{i:n}^* + x)}{\bar{F}(T_{i:n}^*)} \right), x \geq 0, \end{aligned} \tag{10}$$

in which $T_{i:n}^*$ is a nonnegative random variable having p.d.f.

$$f_{T_{i:n}^*}(t) = \frac{\bar{F}(t) f_{T_{i:n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} = \frac{\bar{F}(t) F_T^{i-1}(t) \bar{F}_T^{n-i}(t) f_T(t)}{\int_0^{+\infty} \bar{F}(t) F_T^{i-1}(t) \bar{F}_T^{n-i}(t) f_T(t) dt} \tag{11}$$

To derive a relative measure on random residual lives, suppose that Y is random variable denoting the lifetime of another device. Let Y have s.f. \bar{F}_Y and m.r.l. function m_Y , given by $m_Y(t) = \int_t^{+\infty} \bar{F}_Y(y) dy / \bar{F}_Y(t)$. Then, $X_{T_{i:n}} / m_Y(T_{i:n})$ has s.f.

$$\begin{aligned} P \left(\frac{X_{T_{i:n}}}{m_Y(T_{i:n})} > x \right) &= P \left(\frac{X - T_{i:n}}{m_Y(T_{i:n})} > x | X > T_{i:n} \right) \\ &= \frac{P(X > T_{i:n} + x m_Y(T_{i:n}))}{P(X > T_{i:n})} \\ &= \frac{\int_0^{+\infty} \bar{F}(t + x m_Y(t)) f_{T_{i:n}}(t) dt}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \\ &= \int_0^{+\infty} \frac{\bar{F}(t + x m_Y(t))}{\bar{F}(t)} \left(\frac{\bar{F}(t) f_{T_{i:n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \right) dt \\ &= E \left(\frac{\bar{F}(T_{i:n}^* + x m_Y(T_{i:n}^*))}{\bar{F}(T_{i:n}^*)} \right), x \geq 0, \end{aligned} \tag{12}$$

where $T_{i:n}^*$ is a nonnegative random variable with p.d.f. (10). Let Y have vitality function v_Y , given by $v_Y(t) = \int_0^{+\infty} \bar{F}_Y(y \vee t) dy / \bar{F}_Y(t)$. To obtain another relative measure by means of the vitality function, we derive

$$\begin{aligned} P \left(\frac{X(T_{i:n})}{v_Y(T_{i:n})} > x \right) &= P \left(\frac{X}{v_Y(T_{i:n})} > x | X > T_{i:n} \right) \\ &= \frac{P(X > T_{i:n} \vee x v_Y(T_{i:n}))}{P(X > T_{i:n})} \\ &= \frac{\int_0^{+\infty} \bar{F}(x v_Y(t) \vee t) f_{T_{i:n}}(t) dt}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \\ &= \int_0^{+\infty} \frac{\bar{F}(x v_Y(t) \vee t)}{\bar{F}(t)} \left(\frac{\bar{F}(t) f_{T_{i:n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{i:n}}(t) dt} \right) dt \\ &= E \left(\frac{\bar{F}(x v_Y(T_{i:n}^*) \vee T_{i:n}^*)}{\bar{F}(T_{i:n}^*)} \right), x \geq 0, \end{aligned} \tag{13}$$

where $T_{i:n}^*$ follows the p.d.f. given in (10). In the following, using several measures on X evaluated at the random time $T_{i:n}$, we present some characterization property. The notation $=^{st}$ is used to show equality in distribution.

Theorem 3. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. nonnegative absolutely continues random variables with s.f. \bar{F}_T . Let X be a random lifetime which is independent of T_n , for all $n = 1, 2, \dots$. Let Y be another random lifetime. Then,

- (i) $X =^{st} Y$ iff $E(X_{T_{i:n}}/m_Y(T_{i:n})) = 1$, for all $n = i, i + 1, \dots$, for some $i \in \mathbb{N}$ where m_Y is the m.r.l. function of Y
- (ii) $X =^{st} Y$ iff $E(X(T_{i:n})/v_Y(T_{i:n})) = 1$, for all $n = i, i + 1, \dots$, for some $i \in \mathbb{N}$ where v_Y is the vitality function of Y
- (iii) $X =^{st} Y$ iff $E((I[X > T_{i:n}])/\bar{F}_Y(T_{i:n})) = 1$, for all $n = i, i + 1, \dots$, for some $i \in \mathbb{N}$ where \bar{F}_Y is the s.f. of Y

Proof. To prove (i), first note that $T_{i:n}^*$ has p.d.f. (10). Since $E(X_{T_{i:n}}/m_Y(T_{i:n})) < +\infty$, we can write

$$\begin{aligned} E\left(\frac{X_{T_{i:n}}}{m_Y(T_{i:n})}\right) &= \int_0^{+\infty} P\left(\frac{X_{T_{i:n}}}{m_Y(T_{i:n})} > x\right) dx \\ &= \int_0^{+\infty} E\left(\frac{\bar{F}(T_{i:n}^* + xm_Y(T_{i:n}^*))}{\bar{F}(T_{i,n}^*)}\right) dx \\ &= E\left(\int_0^{+\infty} \frac{\bar{F}(T_{i,n}^* + xm_Y(T_{i,n}^*))}{\bar{F}(T_{i,n}^*)} dx\right) \quad (14) \\ &= E\left(\int_{T_{i,n}^*}^{+\infty} \frac{\bar{F}(x)}{m_Y(T_{i,n}^*)\bar{F}(T_{i,n}^*)} dx\right) \\ &= E\left(\frac{m_X(T_{i,n}^*)}{m_Y(T_{i,n}^*)}\right), \end{aligned}$$

where the change in the order of expectation and integral is due to the well-known Fubini theorem. To prove the “only if” part of the theorem, assume that X and Y have equal distributions, thus $m_X(t) = m_Y(t)$, for all $t \geq 0$, i.e., in view of (13), $E(X_{T_{i:n}}/m_Y(T_{i:n})) = 1$, for all $n = i, i + 1, \dots$, for any given $i \in \mathbb{N}$. To prove the “if” part, we utilize the concept of complexness in functional analysis. It can be observed that, for a fixed $i \in \mathbb{N}$ and for all $n = i, i + 1, \dots$,

$$\begin{aligned} E\left(\frac{X_{T_{i:n}}}{m_Y(T_{i:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} f_{T_{i:n}^*}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \\ &\quad \cdot \left(\frac{\bar{F}(t)F_T^{i-1}(t)\bar{F}_T^{n-i}(t)f_T(t)}{\int_0^{+\infty} \bar{F}(t)F_T^{i-1}(t)\bar{F}_T^{n-i}(t)f_T(t)dt} \right) dt. \end{aligned} \quad (15)$$

Let us set $m = n - i + 1$. It is then plain to see $E(X_{T_{i:n}}/m_Y(T_{i:n})) - 1 = 0$, for all $n \in \{i, i + 1, \dots\}$, if and only if,

$$\int_0^{+\infty} \phi(t)\bar{F}_T^m(t)dt = 0, \text{ for all } m \in \mathbb{N}, \quad (16)$$

where

$$\phi(t) = \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \frac{\bar{F}(t)f_T(t)F_T^{i-1}(t)}{\bar{F}_T(t)}, \text{ for all } t \geq 0. \quad (17)$$

Now, if we select $\psi(x) = \bar{F}_T(x)$ in Lemma 2, it concludes that the sequence $\{\bar{F}_T^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ . From completeness property, it follows from (15) that $\phi(t) = 0$, for all $t \geq 0$, which means that $m_X(t)/m_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., $m_X(t) = m_Y(t)$, for all $t \geq 0$. This is equivalent to saying that X and Y are identical in distribution. We prove the assertion (ii) now. We assume that $E(X(T_{i:n})/v_Y(T_{i:n})) < +\infty$. We have

$$\begin{aligned} E\left(\frac{X(T_{i:n})}{v_Y(T_{i:n})}\right) &= \int_0^{+\infty} P\left(\frac{X(T_{i:n})}{v_Y(T_{i:n})} > x\right) dx \\ &= \int_0^{+\infty} E\left(\frac{\bar{F}(xv_Y(T_{i,n}^*)\vee T_{i,n}^*)}{\bar{F}(T_{i,n}^*)}\right) dx \\ &= E\left(\int_0^{+\infty} \frac{\bar{F}(xv_Y(T_{i,n}^*)\vee T_{i,n}^*)}{\bar{F}(T_{i,n}^*)} dx\right) \quad (18) \\ &= E\left(\int_0^{+\infty} \frac{\bar{F}(x\vee T_{i,n}^*)}{v_Y(T_{i,n}^*)\bar{F}(T_{i,n}^*)} dx\right) \\ &= E\left(\frac{v_X(T_{i,n}^*)}{v_Y(T_{i,n}^*)}\right), \end{aligned}$$

where the expectations are with respect to $T_{i,n}^*$ which follows the p.d.f. (11). To prove the “only if” part, we suppose that X and Y have identical distributions and, therefore, $v_X(t) = v_Y(t)$, for all $t \geq 0$, i.e., in spirit of (17), $E(X(T_{i:n})/v_Y(T_{i:n})) = 1$, for all $n = i, i + 1, \dots$, for any given $i \in \mathbb{N}$. To establish the “if” part, we proceed as in the proof of assertion (i). It can be seen that, for a predetermined $i \in \mathbb{N}$ and for all $n = i, i + 1, \dots$,

$$\begin{aligned} E\left(\frac{X(T_{i:n})}{v_Y(T_{i:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} f_{T_{i,n}^*}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \\ &\quad \cdot \left(\frac{\bar{F}(t)F_T^{i-1}(t)\bar{F}_T^{n-i}(t)f_T(t)}{\int_0^{+\infty} \bar{F}(t)F_T^{i-1}(t)\bar{F}_T^{n-i}(t)f_T(t)dt} \right) dt. \end{aligned} \quad (19)$$

Take $m = n - i + 1$ and observe that $E(X(T_{i:n})/v_Y(T_{i:n})) - 1 = 0$, for all $n \in \{i, i + 1, \dots\}$, if and only if,

$$\int_0^{+\infty} \phi^*(t)\bar{F}_T^m(t)dt = 0, \text{ for all } m \in \mathbb{N}, \quad (20)$$

where

$$\phi^*(t) = \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \frac{\bar{F}(t)f_T(t)F_T^{i-1}(t)}{\bar{F}_T(t)}, \text{ for all } t \geq 0. \quad (21)$$

By considering $\psi(x) = \bar{F}_T(x)$ in Lemma 2, it follows that the sequence $\{\bar{F}_T^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ , according which Equation (19) implies that $\phi^*(t) = 0$, for all $t \geq 0$, which means that $v_X(t)/v_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., $v_X(t) = v_Y(t)$, for all $t \geq 0$. This further implies that X and Y have the same distribution. We finally prove the assertion (iii). It is implied by the iterated expectation rule and also using the assumption that X and T_n are independent for all $n \in \mathbb{N}$ that

$$E\left(\frac{I[X > T_{i:n}]}{\bar{F}_Y(T_{i:n})}\right) = E\left(E\left(\frac{I[X > T_{i:n}]}{\bar{F}_Y(T_{i:n})} \middle| T_{i:n}\right)\right) = E\left(\frac{\bar{F}_X(T_{i:n})}{\bar{F}_Y(T_{i:n})}\right). \tag{22}$$

The proof of the ‘‘only if’’ part is trivial from (21). To prove the ‘‘if’’ part, observe that for a fixed $i \in \mathbb{N}$ and for all $n = i, i + 1, \dots$,

$$\begin{aligned} E\left(\frac{I[X > T_{i:n}]}{\bar{F}_Y(T_{i:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} f_{T_{i:n}}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} \left(\frac{F_T^{i-1}(t) \bar{F}_T^{n-i}(t) f_T(t)}{\int_0^{+\infty} F_T^{i-1}(t) \bar{F}_T^{n-i}(t) f_T(t) dt} \right) dt. \end{aligned} \tag{23}$$

If we take $m = n - i + 1$ we get $E((I[X > T_{i:n}])/\bar{F}_Y(T_{i:n})) - 1 = 0$, for all $n \in \{i, i + 1, \dots\}$, if and only if,

$$\int_0^{+\infty} \phi^{**}(t) \bar{F}_T^m(t) dt = 0, \text{ for all } m \in \mathbb{N}, \tag{24}$$

where

$$\phi^{**}(t) = \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} \frac{f_T(t) F_T^{i-1}(t)}{\bar{F}_T(t)}, \text{ for all } t \geq 0. \tag{25}$$

By a choice of $\psi(x) = \bar{F}_T(x)$ in Lemma 2, it is realized that the sequence $\{\bar{F}_T^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ , by which Equation (23) guarantees that $\phi^{**}(t) = 0$, for all $t \geq 0$. This means that $\bar{F}_X(t)/\bar{F}_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are equally distributed. The proof of the theorem is complete. \square

In Theorem 3, the times have been scaled in the following order $T_{i:i} \leq T_{i:i+1} \leq \dots \leq T_{i:n} \leq T_{i:n+1} \leq \dots$, in which i is a fixed natural number and $n = i, i + 1, \dots$. Therefore, the result of Theorem 3 can be applied on the first-order statistic $T_{1:n}$, so that if $E(R_i(X, Y, T_{1:n})) = 1$ for all $n = 1, 2, \dots$ then X and Y are equal in distribution, where

$$\begin{aligned} R_1(X, Y; T_{1:n}) &= \left(\frac{X - T_{1:n}}{m_Y(T_{1:n})} \middle| X > T_{1:n} \right), \\ R_2(X, Y; T_{1:n}) &= \left(\frac{X}{v_Y(T_{1:n})} \middle| X > T_{1:n} \right), \\ R_3(X, Y; T_{1:n}) &= \left(\frac{I[X > T_{1:n}]}{\bar{F}_Y(T_{1:n})} \right). \end{aligned} \tag{26}$$

However, when the time is scaled in the order $T_{1:1} \leq$

$T_{2:2} \leq \dots \leq T_{n:n} \leq T_{n+1:n+1} \leq \dots$, the result of Theorem 3 is not applicable. Note that $T_{n:n} = \max\{T_1, T_2, \dots, T_n\}$. In this modified setting, we build the characterization properties. Note that the probability for survival of $X_{T_{n:n}}/m_Y(T_{n:n})$ after $x > 0$ is

$$P\left(\frac{X_{T_{n:n}}}{m_Y(T_{n:n})} > x\right) = E\left(\frac{\bar{F}(T_{n,n}^* + xm_Y(T_{n,n}^*))}{\bar{F}(T_{n,n}^*)}\right), \tag{27}$$

where $T_{n,n}^*$ is a nonnegative random variable with the following p.d.f. (see Equation (10)):

$$f_{T_{n,n}^*}(t) = \frac{\bar{F}(t) F_T^{n-1}(t) f_T(t)}{\int_0^{+\infty} \bar{F}(t) F_T^{n-1}(t) f_T(t) dt}. \tag{28}$$

In a similar manner, the probability for $X(T_{n:n})/v_Y(T_{n:n})$ being greater than x is

$$P\left(\frac{X(T_{n:n})}{v_Y(T_{n:n})} > x\right) = E\left(\frac{\bar{F}(xv_Y(T_{n,n}^*) \vee T_{n,n}^*)}{\bar{F}(T_{n,n}^*)}\right). \tag{29}$$

Next, we present the relevant characterization properties. The proof being similar to that of Theorem 3 has been shortened by omitting repeated steps.

Theorem 4. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. nonnegative absolutely continues random variables with s.f. \bar{F}_T . Let X be a random lifetime which is independent of T_n , for all $n = 1, 2, \dots$. Let Y be another random lifetime. Then,

- (i) $X =^{st} Y$ iff $E(X_{T_{n:n}}/m_Y(T_{n:n})) = 1$, for all $n \in \mathbb{N}$ where m_Y is the m.r.l. function of Y
- (ii) $X =^{st} Y$ iff $E(X(T_{n:n})/v_Y(T_{n:n})) = 1$, for all $n \in \mathbb{N}$ where v_Y is the vitality function of Y
- (iii) $X =^{st} Y$ iff $E((I[X > T_{n:n}])/\bar{F}_Y(T_{n:n})) = 1$, for all $n \in \mathbb{N}$ where \bar{F}_Y is the s.f. of Y

Proof. We prove assertion (i). Notice note that $T_{n,n}^*$ has p.d.f. (27) and we assume that $E(X_{T_{n:n}}/m_Y(T_{n:n})) < +\infty$. It can be seen in a similar way as in the proof of Theorem 3 that

$$E\left(\frac{X_{T_{n:n}}}{m_Y(T_{n:n})}\right) = E\left(\frac{m_X(T_{n,n}^*)}{m_Y(T_{n,n}^*)}\right), \tag{30}$$

The ‘‘only if’’ part is trivial. To prove the ‘‘if’’ part, we observe that

$$\begin{aligned} E\left(\frac{X_{T_{n:n}}}{m_Y(T_{n:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} f_{T_{n,n}^*}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \left(\frac{\bar{F}(t) F_T^{n-1}(t) f_T(t)}{\int_0^{+\infty} \bar{F}(t) F_T^{n-1}(t) f_T(t) dt} \right) dt. \end{aligned} \tag{31}$$

By assumption $E(X_{T_{n:n}}/m_Y(T_{n:n})) - 1 = 0$, for all $n \in \mathbb{N}$, which holds iff

$$\int_0^{+\infty} \phi_1(t) F_T^n(t) dt = 0, \text{ for all } n \in \mathbb{N}, \quad (32)$$

where

$$\phi_1(t) = \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \frac{\bar{F}(t) f_T(t)}{F_T(t)}, \text{ for all } t \geq 0. \quad (33)$$

Let us take $\psi(x) = F_T(x)$ in Lemma 2. We know that the sequence $\{F_T^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ . Hence, it follows from (31) that $\phi_1(t) = 0$, for all $t \geq 0$, which further implies that $m_X(t)/m_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are identical in their distribution. To establish the assertion (ii) we suppose that $E(X(T_{n:n})/v_Y(T_{n:n})) < +\infty$. Thus, analogously as in the proof of Theorem 3 (ii), we get

$$E\left(\frac{X(T_{n:n})}{v_Y(T_{n:n})}\right) = E\left(\frac{v_X(T_{n:n}^*)}{v_Y(T_{n:n}^*)}\right). \quad (34)$$

The proof for the ‘‘only if’’ part is straightforward. To establish the ‘‘if’’ part, we write for all $n = 1, 2, \dots$,

$$\begin{aligned} E\left(\frac{X(T_{n:n})}{v_Y(T_{n:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} f_{T_{n:n}^*}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \left(\frac{\bar{F}(t) F_T^{n-1}(t) f_T(t)}{\int_0^{+\infty} \bar{F}(t) F_T^{n-1}(t) f_T(t) dt} \right) dt. \end{aligned} \quad (35)$$

From assumption $E(X(T_{n:n})/v_Y(T_{n:n})) - 1 = 0$, for all $n \in \mathbb{N}$, if and only if,

$$\int_0^{+\infty} \phi_1^*(t) F_T^n(t) dt = 0, \text{ for all } n \in \mathbb{N}, \quad (36)$$

in which

$$\phi_1^*(t) = \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \frac{\bar{F}(t) f_T(t)}{F_T(t)}, \text{ for all } t \geq 0. \quad (37)$$

By choosing $\psi(x) = F_T(x)$ in Lemma 2, it is found that the sequence $\{F_T^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ , by which Equation (35) implies that $\phi_1^*(t) = 0$, for all $t \geq 0$, which means that $v_X(t)/v_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are identical in distribution. We now prove the assertion (iii). We have

$$E\left(\frac{I[X > T_{n:n}]}{\bar{F}_Y(T_{n:n})}\right) = E\left(\frac{\bar{F}_X(T_{n:n})}{\bar{F}_Y(T_{n:n})}\right). \quad (38)$$

The proof of the ‘‘only if’’ part is obvious from (37). To prove the ‘‘if’’ part, observe that for all $n = 1, 2, \dots$,

$$\begin{aligned} E\left(\frac{I[X > T_{n:n}]}{\bar{F}_Y(T_{n:n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} f_{T_{n:n}}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} \left(\frac{F_T^{n-1}(t) f_T(t)}{\int_0^{+\infty} F_T^{n-1}(t) f_T(t) dt} \right) dt. \end{aligned} \quad (39)$$

By assumption, it holds that $E((I[X > T_{n:n}])/\bar{F}_Y(T_{n:n})) - 1 = 0$, for all $n \in \mathbb{N}$ which stands true iff

$$\int_0^{+\infty} \phi_1^{**}(t) \bar{F}_T^n(t) dt = 0, \text{ for all } n \in \mathbb{N}, \quad (40)$$

where

$$\phi_1^{**}(t) = \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} \frac{f_T(t)}{F_T(t)}, \text{ for all } t \geq 0. \quad (41)$$

When we take $\psi(x) = \bar{F}_T(x)$, the Lemma 2 is applicable by which Equation (39) yields $\phi_1^{**}(t) = 0$, for all $t \geq 0$. This means that $\bar{F}_X(t)/\bar{F}_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y have equal distributions and hence the proof. \square

4. Reliability Measures on a Time Scale of Record Values

The observation T_i is called an upper record, if the value it takes exceeds that of all previous observations. Thus, T_j is an upper record if $T_j > T_i$ for every $i < j$. Analogously, the observation T_i is called a lower record, if the value it takes falls behind all previous observations. The times at which the upper record values appear are given by the random variables U_i , which are called upper record times and are defined by U_0 with probability 1 and, for $j \geq 1$, $U_j = \min \{i : T_i > T_{U_{j-1}}\}$. Then, the upper record value sequence $\{T_{U_n} : n = 0, 1, 2, \dots\}$ is considered. Similarly, the times at which the lower record values appear are the random variables L_i , which are called lower record times defined by L_0 with probability 1 and, for $j \geq 1$, $L_j = \min \{i : T_i < T_{L_{j-1}}\}$. Then, the lower record value sequence $\{T_{L_n} : n = 0, 1, 2, \dots\}$ is considered. Then, the random variable T_{U_n} , as the n th upper record, has p.d.f.

$$f_{T_{U_n}}(t) = \frac{(-\log(\bar{F}_T(t)))^n}{n!} f_T(t), \quad (42)$$

and the random variable T_{L_n} , as the n th lower record has p.d.f.

$$f_{T_{L_n}}(t) = \frac{(-\log(F_T(t)))^n}{n!} f_T(t). \quad (43)$$

It may be of interest to evaluate and measure the excess amount of future records of observations. Let us consider the r.v. X which denotes the lifetime of a unit which is independent of T_n , $n = 1, 2, \dots$, and hence, it is also independent of T_{U_n} and T_{L_n} . The random variable $X_{T_{U_n}} = (X$

$-T_{U_n}|X > T_{U_n})$ measures the excess amount of observations on X which are greater than the n th upper record of T_i 's. The random variable $X_{T_{L_n}} = (X - T_{L_n}|X > T_{L_n})$ measures the excess amount of observations on X which are greater than the n th lower record of T_i 's.

We give here other relative measures on random residual live. Let Y be the lifetime of a device to be compared with another device with lifetime X . Then, $X_{T_{U_n}}/m_Y(T_{U_n})$ has s.f.

$$\begin{aligned} P\left(\frac{X_{T_{U_n}}}{m_Y(T_{U_n})} > x\right) &= P\left(\frac{X - T_{U_n}}{m_Y(T_{U_n})} > x | X > T_{U_n}\right) \\ &= \frac{P(X > T_{U_n} + xm_Y(T_{U_n}))}{P(X > T_{U_n})} \\ &= \frac{\int_0^{+\infty} \bar{F}(t + xm_Y(t)) f_{T_{U_n}}(t) dt}{\int_0^{+\infty} \bar{F}(t) f_{T_{U_n}}(t) dt} \\ &= \int_0^{+\infty} \frac{\bar{F}(t + xm_Y(t))}{\bar{F}(t)} \left(\frac{\bar{F}(t) f_{T_{U_n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{U_n}}(t) dt} \right) dt \\ &= E\left(\frac{\bar{F}(T_{n,n}^{***} + xm_Y(T_{n,n}^{***}))}{\bar{F}(T_{n,n}^{***})}\right), \end{aligned} \quad (44)$$

where $T_{n,n}^{***}$ is a nonnegative random variable with the following p.d.f.:

$$f_{T_{n,n}^{***}}(t) = \frac{\bar{F}(t) f_{T_{U_n}}(t)}{\int_0^{+\infty} \bar{F}(t) f_{T_{U_n}}(t) dt} = \frac{\bar{F}(t) (-\log(\bar{F}_T(t)))^n f_T(t)}{\int_0^{+\infty} \bar{F}(t) (-\log(\bar{F}_T(t)))^n f_T(t) dt}. \quad (45)$$

Correspondingly, the relative measure $X_{T_{L_n}}/m_Y(T_{L_n})$ has s.f.

$$P\left(\frac{X_{T_{L_n}}}{m_Y(T_{L_n})} > x\right) = E\left(\frac{\bar{F}(T_{n,n}^{***} + xm_Y(T_{n,n}^{***}))}{\bar{F}(T_{n,n}^{***})}\right), \quad (46)$$

where $T_{n,n}^{***}$ follows the p.d.f.

$$f_{T_{n,n}^{***}}(t) = \frac{\bar{F}(t) (-\log(F_T(t)))^n f_T(t)}{\int_0^{+\infty} \bar{F}(t) (-\log(F_T(t)))^n f_T(t) dt}. \quad (47)$$

By an analogous method, the relative measure $X(T_{U_n})/v_Y(T_{U_n})$ is greater than x with probability

$$P\left(\frac{X(T_{U_n})}{v_Y(T_{U_n})} > x\right) = E\left(\frac{\bar{F}(xv_Y(T_{n,n}^{***}) \vee T_{n,n}^{***})}{\bar{F}(T_{n,n}^{***})}\right), \quad (48)$$

and also the relative measure $X(T_{L_n})/v_Y(T_{L_n})$ is greater than x with probability

$$P\left(\frac{X(T_{L_n})}{v_Y(T_{L_n})} > x\right) = E\left(\frac{\bar{F}(xv_Y(T_{n,n}^{***}) \vee T_{n,n}^{***})}{\bar{F}(T_{n,n}^{***})}\right). \quad (49)$$

Now, other characterization properties based on random residual life after subsequent (upper and lower) records are given.

Theorem 5. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. nonnegative absolutely continuous random variables with s.f. \bar{F}_T . Let X be a random lifetime which is independent of T_n , for all $n = 1, 2, \dots$. Let Y be another random lifetime. Then,

- (i) $X =^{st} Y$ iff $E(X_{T_{U_n}}/m_Y(T_{U_n})) = 1$, for all $n \in \mathbb{N}$ where m_Y is the m.r.l. function of Y
- (ii) $X =^{st} Y$ iff $E(X(T_{U_n})/v_Y(T_{U_n})) = 1$, for all $n \in \mathbb{N}$ where v_Y is the vitality function of Y
- (iii) $X =^{st} Y$ iff $E(I[X > T_{U_n}]/\bar{F}_Y(T_{U_n})) = 1$, for all $n \in \mathbb{N}$ where \bar{F}_Y is the s.f. of Y

Proof. We give the proof of assertion (i). Note that $T_{n,n}^{**}$ has p.d.f. (44) and assume that $E(X_{T_{U_n}}/m_Y(T_{U_n})) < +\infty$. As in the proof of Theorem 3, we have

$$E\left(\frac{X_{T_{U_n}}}{m_Y(T_{U_n})}\right) = E\left(\frac{m_X(T_{n,n}^{**})}{m_Y(T_{n,n}^{**})}\right), \quad (50)$$

The ‘‘only if’’ part is trivial. To prove the ‘‘if’’ part, we observe that

$$\begin{aligned} E\left(\frac{X_{T_{U_n}}}{m_Y(T_{U_n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} f_{T_{n,n}^{**}}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \left(\frac{\bar{F}(t) (-\log(\bar{F}_T(t)))^n f_T(t)}{\int_0^{+\infty} \bar{F}(t) (-\log(\bar{F}_T(t)))^n f_T(t) dt} \right) dt. \end{aligned} \quad (51)$$

In view of assumption $E(X_{T_{U_n}}/m_Y(T_{U_n})) - 1 = 0$, for all $n \in \mathbb{N}$, which is satisfied iff

$$\int_0^{+\infty} \phi_2(t) (-\log(\bar{F}_T(t)))^n dt = 0, \text{ for all } n \in \mathbb{N}, \quad (52)$$

where

$$\phi_2(t) = \left\{ \frac{m_X(t)}{m_Y(t)} - 1 \right\} \bar{F}(t) f_T(t), \text{ for all } t \geq 0. \quad (53)$$

We can take $\psi(x) = -\log(\bar{F}_T(x))$ in Lemma 2 from which we realize that the sequence $\{(-\log(\bar{F}_T(t)))^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ . It thus follows from (51) that $\phi_2(t) = 0$, for all $t \geq 0$, which provides that $m_X(t)/m_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are identical in distribution. To give credit to the assertion (ii), we assume that $E(X(T_{U_n})/v_Y(T_{U_n})) < +\infty$. Therefore, similarly as in the proof of

Theorem 3 (ii), we obtain

$$E\left(\frac{X(T_{U_n})}{v_Y(T_{U_n})}\right) = E\left(\frac{v_X(T_{n,n}^{**})}{v_Y(T_{n,n}^{**})}\right). \quad (54)$$

The proof of the “only if” part is very simple. To prove the “if” part for all $n = 1, 2, \dots$, we can write

$$\begin{aligned} E\left(\frac{X(T_{U_n})}{v_Y(T_{U_n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} f_{T_{n,n}^{**}}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \left(\frac{\bar{F}_T(t) (-\log(\bar{F}_T(t)))^n f_T(t)}{\int_0^{+\infty} \bar{F}_T(t) (-\log(\bar{F}_T(t)))^n f_T(t) dt} \right) dt. \end{aligned} \quad (55)$$

By assumption $E(X(T_{U_n})/v_Y(T_{U_n})) - 1 = 0$, for all $n \in \mathbb{N}$ if

$$\int_0^{+\infty} \phi_2^*(t) (-\log(\bar{F}_T(t)))^n dt = 0, \text{ for all } n \in \mathbb{N}, \quad (56)$$

where

$$\phi_1^*(t) = \left\{ \frac{v_X(t)}{v_Y(t)} - 1 \right\} \bar{F}_T(t) f_T(t), \text{ for all } t \geq 0. \quad (57)$$

It is found by Lemma 2 that the sequence $\{(-\log(\bar{F}_T(t)))^n : n \in \mathbb{N}\}$ is complete on \mathbb{R}^+ , from which Equation (55) gives $\phi_2^*(t) = 0$, for all $t \geq 0$, that is $v_X(t)/v_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are identical in distribution. To prove the assertion (iii), we write

$$E\left(\frac{I[X > T_{U_n}]}{\bar{F}_Y(T_{U_n})}\right) = E\left(\frac{\bar{F}_X(T_{U_n})}{\bar{F}_Y(T_{U_n})}\right). \quad (58)$$

The proof of the “only if” part is plain to follow from (57). To prove the “if” part, note that for all $n = 1, 2, \dots$, we have

$$\begin{aligned} E\left(\frac{I[X > T_{U_n}]}{\bar{F}_Y(T_{U_n})}\right) - 1 &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} f_{T_{U_n}}(t) dt \\ &= \int_0^{+\infty} \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} \frac{(-\log(\bar{F}_T(t)))^n}{n!} f_T(t) dt. \end{aligned} \quad (59)$$

It holds by assumption that $E((I[X > T_{U_n}])/\bar{F}_Y(T_{U_n})) - 1 = 0$, for all $n \in \mathbb{N}$ which holds true iff

$$\int_0^{+\infty} \phi_2^{**}(t) (-\log(\bar{F}_T(t)))^n dt = 0, \text{ for all } n \in \mathbb{N}, \quad (60)$$

where

$$\phi_2^{**}(t) = \left\{ \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} - 1 \right\} f_T(t), \text{ for all } t \geq 0. \quad (61)$$

Equation (59) together with Lemma 2 provides that $\phi_2^{**}(t) = 0$, for all $t \geq 0$. This concludes that $\bar{F}_X(t)/\bar{F}_Y(t) - 1 = 0$, for all $t \geq 0$, i.e., X and Y are equally distributed, and hence, the proof is completed. \square

The following result will be similarly as Theorem 5 can be proved.

Theorem 6. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. nonnegative absolutely continues random variables with s.f. \bar{F}_T . Let X be a random lifetime which is independent of T_n , for all $n = 1, 2, \dots$. Let Y be another random lifetime. Then,

- (i) $X =^{st} Y$ iff $E(X_{T_{L_n}}/m_Y(T_{L_n})) = 1$, for all $n \in \mathbb{N}$ where m_Y is the m.r.l. function of Y
- (ii) $X =^{st} Y$ iff $E(X(T_{L_n})/v_Y(T_{L_n})) = 1$, for all $n \in \mathbb{N}$ where m_Y is the m.r.l. function of Y where v_Y is the vitality function of Y
- (iii) $X =^{st} Y$ iff $E((I[X > T_{L_n}])/\bar{F}_Y(T_{L_n})) = 1$, for all $n \in \mathbb{N}$ where \bar{F}_Y is the s.f. of Y

It must be mentioned here that the results obtained in this work differ from the characterization properties in Kayid and Izadkhah [12]. In fact, the characterization properties in this work represent a further development of the results obtained by Kayid and Izadkhah [12], since in our approach the distribution of Y can be any lifetime distribution. However, if Y is chosen to be an exponential r.v., then our characterization results in Theorems 3(i), 4(i), and 5(i) reduce to Theorems 3.1 and 3.2 in Kayid and Izadkhah [12], respectively.

5. Conclusion

In this paper, we have studied several characterization problems using the residual lifetime of an original item relative to the mean residual lifetime, the vitality function, or the survival function of another item. The ages at which the residual lifetimes applied have been considered to be independent random times as they are the order statistics of a partial set of a sequence of random lifetimes or the record statistics arising from the infinite version of the sequence of random lifetimes. The main idea for proving the results has been a technical Lemma (Lemma 2) which is a basic result in functional analysis. The random residual lives are applicable to model the remaining useful life of high-quality products with respect to frailer products. Further strategies and descriptions using the introduced concept in reliability theory and survival analysis could be given.

Recently, many researchers have focused on the relating aging properties of two lifespan by the monotonicity of ratio of their associated reliability quantities (see, e.g., Finkelstein [40], Kayid et al. [41], and He and Xie [42]). The authors believes that using the measures $R_1(X, Y; t) = (X - t/m_X(t) | X > t)$ and $R_2(X, Y; t) = (X/v_X(t) | X > t)$, further perspectives on the relative behaviour of a system with lifetime X

in comparison with another system in the same environment with lifetime Y may be produced. The relative measure $R_1(X, Y; t)$ is the scaled residual live of X in which the reciprocal m.r.l. function of Y adjusts the scale. In contrast, the relative measure $R_2(X, Y; t)$ is the scaled right tail of the live of X where the reciprocal vitality function of Y regulates the scale. The random quantities R_1 and R_2 are indeed scale-free measures. For example, the moments of R_1 and R_2 may be of some interest when X and Y are equally distributed, as in this case $E(R_1^2(X, Y; t)) = E(X_t)^2/E^2(X_t)$ (resp. $E(R_2^2(X, Y; t)) = E(X(t))^2/E^2(X(t))$) is closely related with coefficient of variation of X_t (resp. $X(t)$) which is a basic tool in the study of aging phenomenon or the vitality of the systems. However, extensions to the case where the age is random makes the study more dynamic and further the conclusions become quite broader. In situations where R_1 and R_2 are regarded as two stochastic processes, observation of the process will be sequentially in some random time not continually with time. In fact, it has been widely acknowledged that a stochastic process is partially observable. For example, in degradation models in which the lifetime is considered to be the first-passage time of some stochastic process (see, for example, Bordes et al. [43]). Therefore, the random relative residual lives and their averages measured in some sequences of random time such as order statistics or record values may be practically beneficial. There are various situations where the times of occurrence of events are record values of a sequence or order statistics of a sample. For example, in the context of reliability engineering, a coherent system fails with the the failure of the consecutive order statistics of the lifetime of its components. The residual lifetime aftershocks arrived to a system may be an important quantity where the shocks which make further degradation in a highly reliable system are the ones which are record values.

In the future of this study, the problem of characterizations of distributions in the cases when the lifetime depends on random ages will be studied. The question may be whether the current characterization properties remain valid and hold at the disposal of dependencies between lifetime and random ages. Constructing similar relative measures as $R_i, i = 1, 2$ to contribute in the area of relative aging of systems and also further characterization properties will be sought.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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