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Research Article

On q-Rung Orthopair Fuzzy Subgroups

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The q-rung orthopair fuzzy environment is an innovative tool to handle uncertain situations in various decision-making problems. In this work, we characterize the idea of a q-rung orthopair fuzzy subgroup and examine various algebraic attributes of this newly defined notion. We also present q-rung orthopair fuzzy coset and q-rung orthopair fuzzy normal subgroup along with relevant fundamental theorems. Moreover, we introduce the concept of q-rung orthopair fuzzy level subgroup and proved related results. At the end, we explore the consequence of group homomorphism on the q-rung orthopair fuzzy subgroup.

1. Introduction

In classical fuzzy set theory, a fuzzy subset of a crisp set S is represented by a function from S to $[0, 1] \subseteq \mathbb{R}$. The inequalities and equations are used to define operations and characteristic. The original notion of the fuzzy set was proposed in 1965 by Zadeh [1]. Since then, it has been used in almost every field of science especially where mathematical logic and set theory are significantly involved. A fuzzy subset $\mathscr{R}$ of a crisp set S is an object $\{s, \mu_{\mathscr{R}}(s)\colon s \in S\}$ such that $\mu_{\mathscr{R}} : S \longrightarrow [0, 1]$ is called membership mapping of $\mathscr{R}$ and $\mu_{\mathscr{R}}(s)$ is known as a degree of membership of s in $\mathscr{R}$. One can see that fuzzy sets are the extensions of characteristic functions of classical sets, by expanding the range of the function from $\{0, 1\}$ to $[0, 1]$. After the proposal of fuzzy sets, a lot of theories have been put forward to handle uncertain and imprecision circumstances. Some of these theories are expansions of fuzzy sets, whereas others strive to cope with uncertainties in another appropriate manner. Atanassov [2] introduced an intuitionistic fuzzy set (IFS) which is the generalization of fuzzy set. An intuitionistic fuzzy subset $\mathscr{R}$ of a crisp set S is an object $\{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\colon s \in S\}$, where $\mu_{\mathscr{R}} : S \longrightarrow [0, 1]$ and $\nu_{\mathscr{R}} : S \longrightarrow [0, 1]$ are membership and nonmembership functions, respectively, such that $\mu_{\mathscr{R}}(s) + \nu_{\mathscr{R}}(s) \leq 1$ for all $s \in S$. Compared with classical fuzzy sets, the positive and negative membership functions of intuitionistic fuzzy sets ensure its effective handling of uncertain and vague situations in physical problem, especially in the field of decision-making [3–6]. In 2013, Yager [7] generalized intuitionistic fuzzy sets by presenting the idea of Pythagorean fuzzy set (PFS). The Pythagorean fuzzy subset $\mathscr{R}$ of a crisp set S is an object $\{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\colon s \in S\}$, where $\mu_{\mathscr{R}} : S \longrightarrow [0, 1]$ and $\nu_{\mathscr{R}} : S \longrightarrow [0, 1]$ are membership and nonmembership functions, respectively, such that $(\mu_{\mathscr{R}}(s))^2 + (\nu_{\mathscr{R}}(s))^2 \leq 1$ for all $s \in S$. This concept is designed to convert uncertain and vague environment in the form of mathematics and to find more effective solutions of such real-world problems [8–11]. Although Pythagorean fuzzy subsets solve different types of real-life problems in an efficient way but even then, there is a room for improvement because there exists so many cases where Pythagorean fuzzy subsets fail to work. For example, if positive and negative membership values proposed by a decision-maker are 0.75 and 0.85, respectively, then $(0.75)^2 + (0.85)^2 > 1$; therefore, Pythagorean fuzzy subsets fail to deal with such problems. In order to find a reasonable solution of such kinds of situations, Yager defines the notion of q-rung orthopair fuzzy set (q-ROFS), where q is a natural number [12]. The q-rung orthopair fuzzy subset $\mathscr{R}$ of a crisp set S is an object $\{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\colon s \in S\}$, where $\mu_{\mathscr{R}} : S \longrightarrow [0, 1]$ and

$\nu_{\mathscr{R}} : S \longrightarrow [0, 1]$ are positive and negative membership functions, respectively, such that $(\mu_{\mathscr{R}}(s))^q + (\nu_{\mathscr{R}}(s))^q \leq 1$ for all $s \in S$. In order to find real-world applications of q-ROFSs, we suggest reading [13, 14].

Theory of groups is one of the prominent branches of mathematics with numerous applications in physics [15], chemistry [16], cryptography [17–19], differential equations [20], and graph theory [21, 22]. Rosenfeld [23] initiated the study of fuzzy subgroups. Since then, many mathematicians studied classical group theoretic results in different fuzzy environments. In [24], Das presented a comprehensive study of level subgroups of a fuzzy subgroup. Sherwood defined products of fuzzy subgroups and proved important results related to this notion [25]. Bhattacharya and Mukherjee [26] presented the idea of fuzzy relations on fuzzy subgroups and presented some new results in this direction. Choudhury et al. presented a study on fuzzy subgroups and fuzzy homomorphism in [27]. Some new results on normal fuzzy subgroups have been proven in [28]. Kumar [29] discussed some properties of fuzzy cosets and fuzzy ideals. In [30], some problems related to equivalence relation on fuzzy subgroups have been studied. Biswas [31] initiated the work on intuitionistic fuzzy subgroups in 1989. Hur et al. [32] defined the notion of intuitionistic fuzzy coset and discussed some of its algebraic characteristics. In [33], Sharma defined direct product of intuitionistic fuzzy subgroups. To find more about intuitionistic fuzzy subgroups, we recommend reading [34–38]. Recently, Bhunia et al. [39] defined Pythagorean fuzzy subgroups and explored different attributes of this concept.

Considering the above literature and the significance of q-rung orthopair fuzzy sets and theory of groups, this article reveals the study of q-rung orthopair fuzzy subgroups (q-ROFSGs). The basic purpose and the principal contribution of this work are to

(1) discuss various important algebraic attributes of q-rung orthopair fuzzy subgroups

(2) define the concepts of q-rung orthopair fuzzy coset and q-rung orthopair fuzzy normal subgroup along with the study of relevant fundamental theorems

(3) introduce the idea of q-rung orthopair fuzzy level subgroup and prove some important results of this notion

2. The q-Rung Orthopair Fuzzy Subgroups

In this section, some important algebraic attributes of q-rung orthopair fuzzy subgroups will be discussed. We start this section with the definition of q-rung orthopair fuzzy subgroup (q-ROFSG).

Definition 1. Let G be a group; then, a q-ROFS $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ of G is called a q-rung orthopair fuzzy subgroup (q-ROFSG) of G if the following conditions hold:

(i) $(\mu_{\mathscr{R}}(s_1 s_2))^q \geq \min\{(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_1))^q\}$ and $(\nu_{\mathscr{R}}(s_1 s_2))^q \leq \max\{(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q\}$ for all $s_1, s_2 \in G$

(ii) $(\mu_{\mathscr{R}}(s^{-1}))^q \geq (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s^{-1}))^q \leq (\nu_{\mathscr{R}}(s))^q$ for all $s \in G$

Theorem 2. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s): s \in G, (\mu_{\mathscr{R}}(s))^q + (\nu_{\mathscr{R}}(s))^q \leq 1\}$ be a q-ROFSG of G. Then, the following conditions are true:*

(i) $(\mu_{\mathscr{R}}(e))^q \geq (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(e))^q \leq (\nu_{\mathscr{R}}(s))^q$ for all $s \in G$

(ii) $(\mu_{\mathscr{R}}(s^{-1}))^q = (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s^{-1}))^q = (\nu_{\mathscr{R}}(s))^q$ for all $s \in G$

Proof.

(i) Let $s \in G$; then, $(\mu_{\mathscr{R}}(s^{-1}s))^q \geq \min\{(\mu_{\mathscr{R}}(s^{-1}))^q, (\mu_{\mathscr{R}}(s))^q\} \Longrightarrow (\mu_{\mathscr{R}}(e))^q \geq \min\{(\mu_{\mathscr{R}}(s^{-1}))^q, (\mu_{\mathscr{R}}(s))^q\} = (\mu_{\mathscr{R}}(s))^q \Longrightarrow (\mu_{\mathscr{R}}(e))^q \geq (\mu_{\mathscr{R}}(s))^q$. Similarly, we can show that $(\nu_{\mathscr{R}}(e))^q \leq (\nu_{\mathscr{R}}(s))^q$

(ii) Since $(\mu_{\mathscr{R}}(s^{-1}))^q \geq (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s^{-1}))^q \leq (\nu_{\mathscr{R}}(s))^q$ for all $s \in G$, therefore $(\mu_{\mathscr{R}}((s^{-1})^{-1}))^q \geq (\mu_{\mathscr{R}}(s^{-1}))^q$ and $(\nu_{\mathscr{R}}((s^{-1})^{-1}))^q \leq (\nu_{\mathscr{R}}(s^{-1}))^q$ which means that $(\mu_{\mathscr{R}}(s))^q \geq (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s))^q \leq (\nu_{\mathscr{R}}(s^{-1}))^q$. Thus, $(\mu_{\mathscr{R}}(s^{-1}))^q = (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s^{-1}))^q = (\nu_{\mathscr{R}}(s))^q$ for all $s \in G$

□

The following theorem shows that every Pythagorean fuzzy subgroup (PFSG) of G is q-ROFSG of G.

Theorem 3. *Let G be a group and $P = \{s, \mu_P(s), \nu_P(s): s \in G, (\mu_P(s))^2 + (\nu_P(s))^2 \leq 1\}$ be a PFSG of G. Then, P is a q-ROFSG of G.*

Proof. Let $s_1, s_2 \in G$; then

$$\begin{aligned} (\mu_P((s_1 s_2)))^2 &\geq \min\{(\mu_P(s_1))^2, (\mu_P(s_2))^2\}, \\ (\nu_P((s_1 s_2)))^2 &\leq \max\{(\nu_P(s_1))^2, (\nu_P(s_2))^2\}, \\ (\mu(s^{-1}))^2 &\geq (\mu(s))^2, \\ (\nu(s^{-1}))^2 &\leq (\nu(s))^2. \end{aligned} \tag{1}$$

This implies that

$$(\mu_P((s_1 s_2)))^q \geq \min\{(\mu_P(s_1))^q, (\mu_P(s_2))^q\}, \tag{2}$$

$$(\nu_P((s_1 s_2)))^q \leq \max\{(\nu_P(s_1))^q, (\nu_P(s_2))^q\}, \tag{3}$$

$$(\mu(s^{-1}))^q \geq (\mu(s))^q, \tag{4}$$

$$\left(\nu_P\left(s^{-1}\right)\right)^q \le \left(\nu_P(s)\right)^q. \tag{5}$$

Since $(\mu_P(s_1))^2, (\mu_P(s_2))^2, (\mu_P(s))^2, (\nu_P(s_1))^2, (\nu_P(s_2))^2, (\nu_P(s))^2 \in [0,1]$, therefore for all $q > 2$, we have $(\mu_P(s_1))^q \le (\mu_P(s_1))^2$, $(\mu_P(s_2))^q \le (\mu_P(s_2))^2$, $(\mu_P(s))^q \le (\mu_P(s))^2$, $(\nu_P(s_1))^q \le (\nu_P(s_1))^2$,$(\nu_P(s_2))^q \le (\nu_P(s_2))^2$, and $(\nu_P(s))^q \le (\nu_P(s))^2$. Thus,

$$(\mu_P(s_1))^q + (\nu_P(s_1))^q \le 1, \tag{6}$$

$$(\mu_P(s_2))^q + (\nu_P(s_2))^q \le 1, \tag{7}$$

$$(\mu_P(s))^q + (\nu_P(s))^q \le 1. \tag{8}$$

The inequalities (2)–(8) reveal that P is a q-ROFSG of G. □

The next example provides evidence of invalidity of the converse of the above theorem.

Example 1. Consider dihedral group D_4, that is,

$$\begin{aligned} D_4 &= \left\langle s_1, s_2 : (s_1)^2 = (s_2)^4 = (s_1 s_2)^2 = e \right\rangle \\ &= \left\{ e, s_1, s_2, (s_2)^2, (s_2)^3, s_2 s_1, (s_2)^2 s_1, (s_2)^3 s_1 \right\}. \end{aligned} \tag{9}$$

One can easily verify that

$$R = \left\{ \begin{array}{c} (e, 0.95, 0.10), ((s_2)^2, 0.80, 0.45), (s_2, 0.80, 0.75), ((s_2)^3, 0.80, 0.75), \\ (s_2 s_1, 0.80, 0.78), ((s_2)^2 s_1, 0.80, 0.78), ((s_2)^3 s_1, 0.80, 0.78), (s_1, 0.80, 0.78) \end{array} \right\} \tag{10}$$

is 3-ROFSG of D_4, but it is not a PFSG of D_4 as $(0.80)^2 + (0.78)^2 > 1$.

Theorem 4. *A q-ROFS $R = \{s, \mu_R(s), \nu_R(s)\}$ of group G is a q-ROFSG of G if and only if $(\mu_P(st^{-1}))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\}$ and $(\nu_{\mathscr{R}}(st^{-1}))^q \le \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q\}$ for all $s, t \in G$.*

Proof. Let $R = \{s, \mu_R(s), \nu_R(s): s \in G, (\mu_R(s))^q + (\nu_R(s))^q \le 1\}$ be a q-ROFSG of G. Then, for all $s, t \in G$, $(\mu_P(st^{-1}))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t^{-1}))^q\} = \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\}$ and $(\nu_{\mathscr{R}}(st^{-1}))^q \le \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t^{-1}))^q\} = \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q\}$.

Conversely, suppose that $(\mu_P(st^{-1}))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\}$ and $(\nu_{\mathscr{R}}(st^{-1}))^q \le \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q\}$ for all $s, t \in G$. Then, $(\mu_P(st))^q = (\mu_P(s(t^{-1})^{-1}))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t^{-1}))^q\} = \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\}$. Thus,

$$(\mu_P(st))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\}. \tag{11}$$

Similarly,

$$(\nu_{\mathscr{R}}(st))^q \le \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q\}. \tag{12}$$

Next, $(\mu_P(s^{-1}))^q = (\mu_P(es^{-1}))^q \ge \min\{(\mu_{\mathscr{R}}(e))^q, (\mu_{\mathscr{R}}(s))^q\} = (\mu_{\mathscr{R}}(s))^q$, that is,

$$\left(\mu_R\left(s^{-1}\right)\right)^q \ge (\mu_{\mathscr{R}}(s))^q. \tag{13}$$

Similarly,

$$\left(\nu_{\mathscr{R}}\left(s^{-1}\right)\right)^q \le (\nu_{\mathscr{R}}(s))^q. \tag{14}$$

The inequalities (11)–(14) show that R is a q-ROFSG of G. □

Theorem 5. *Let $R_1 = \{s, \mu_{R_1}(s), \nu_{R_1}(s)\}$ and $R_2 = \{s, \mu_{R_2}(s), \nu_{R_2}(s)\}$ be two q-ROFSGs of G; then, $R_1 \cap R_2$ is a q-ROFSG of G.*

Proof. Suppose that R_1 and R_2 are two q-ROFSGs of G. Then, for all $s_1, s_2 \in G$, we have

$$\begin{aligned} \left(\mu_{R_1\cap R_2}\left(s_1 s_2^{-1}\right)\right)^q &= \min\left[\left(\mu_{R_1}\left(s_1 s_2^{-1}\right)\right)^q, \left(\mu_{R_2}\left(s_1 s_2^{-1}\right)\right)^q\right] \\ &\ge \min\left[\min\left(\left(\mu_{R_1}(s_1)\right)^q, \left(\mu_{R_1}(s_2)\right)^q\right), \min\left(\left(\mu_{R_2}(s_1)\right)^q, \left(\mu_{R_2}(s_2)\right)^q\right)\right] \\ &= \min\left[\min\left(\left(\mu_{R_1}(s_1)\right)^q, \left(\mu_{R_2}(s_1)\right)^q\right), \min\left(\left(\mu_{R_1}(s_2)\right)^q, \left(\mu_{R_2}(s_2)\right)^q\right)\right] \\ &= \min\left[\left(\mu_{R_1\cap R_2}(s_1)\right)^q, \left(\mu_{R_1\cap R_2}(s_2)\right)^q\right]. \end{aligned} \tag{15}$$

That is,

$$\left(\mu_{R_1\cap R_2}\left(s_1 s_2^{-1}\right)\right)^q \ge \min\left[\left(\mu_{R_1\cap R_2}(s_1)\right)^q, \left(\mu_{R_1\cap R_2}(s_2)\right)^q\right]. \tag{16}$$

Similarly,

$$\left(\mu_{R_1\cap R_2}\left(s_1 s_2^{-1}\right)\right)^q \le \max\left[\left(\mu_{R_1\cap R_2}(s_1)\right)^q, \left(\mu_{R_1\cap R_2}(s_2)\right)^q\right]. \tag{17}$$

The application of (16) and (17) together with the theorem give $R_1 \cap R_2$ which is a q-ROFSG of G. □

Theorem 6. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFSG of G. Then, $(\mu_R(s^m))^q \ge (\mu_R(s))^q$ and $(\nu_R(s^m))^q \le (\nu_R(s))^q$ for all $s \in G$ and $m \in \mathbb{N}$.*

Proof. We will use mathematical induction to prove this theorem. Suppose $s \in G$; then, $(\mu_R(s^2))^q = (\mu_R(ss))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(s))^q\} = (\mu_{\mathscr{R}}(s))^q$.

Therefore, the inequality is valid for $m = 2$. Assume that the inequality holds for $m = n - 1$, that is, $(\mu_R(s^{n-1}))^q \ge (\mu_R(s))^q$. Then, $(\mu_R(s^n))^q = (\mu_R(ss^{n-1}))^q \ge \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(s^{n-1}))^q\} = (\mu_R(s))^q$. Thus, by mathematical induction, we have $(\mu_R(s^m))^q \ge (\mu_R(s))^q$ for all $m \in \mathbb{N}$.

Similarly, we can show $(\nu_R(s^m))^q \le (\nu_R(s))^q$ for all $m \in \mathbb{N}$. □

Theorem 7. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFSG of G. If $\mu_{\mathscr{R}}(s_1) \ne \mu_{\mathscr{R}}(s_2)$ and $\nu_{\mathscr{R}}(s_1) \ne \nu_{\mathscr{R}}(s_2)$ for some $s_1, s_2 \in G$,* *then* $(\mu_{\mathscr{R}}(s_1 s_2))^q = \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q]$ *and* $(\nu_{\mathscr{R}}(s_1 s_2))^q = \max[(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q]$.

Proof. Suppose that for some $s_1, s_2 \in G$, we have $\mu_{\mathscr{R}}(s_1) > \mu_{\mathscr{R}}(s_2)$; then, obviously $(\mu_{\mathscr{R}}(s_1))^q > (\mu_{\mathscr{R}}(s_2))^q$.

Consider

$$\begin{aligned}(\mu_{\mathscr{R}}(s_2))^q &= \left(\mu_{\mathscr{R}}\left(s_1^{-1}s_1 s_2\right)\right)^q \geq \min\left[\left(\mu_{\mathscr{R}}\left(s_1^{-1}\right)\right)^q, (\mu_{\mathscr{R}}(s_1 s_2))^q\right] \\ &= \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_1 s_2))^q].\end{aligned} \tag{18}$$

Since $(\mu_{\mathscr{R}}(s_1))^q > (\mu_{\mathscr{R}}(s_2))^q$, therefore from relation (18), we obtain

$$(\mu_{\mathscr{R}}(s_2))^q \geq (\mu_{\mathscr{R}}(s_1 s_2))^q. \tag{19}$$

Also, $(\mu_{\mathscr{R}}(s_1 s_2))^q \geq \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q] = (\mu_{\mathscr{R}}(s_2))^q$, that is,

$$(\mu_{\mathscr{R}}(s_1 s_2))^q \geq (\mu_{\mathscr{R}}(s_2))^q. \tag{20}$$

From (19) and (20), we have

$$(\mu_{\mathscr{R}}(s_1 s_2))^q = (\mu_{\mathscr{R}}(s_2))^q = \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q]. \tag{21}$$

Similarly, the result can be proven if $\mu_{\mathscr{R}}(s_2) > \mu_{\mathscr{R}}(s_1)$.

Next, assume that $\nu_{\mathscr{R}}(s_1) < \nu_{\mathscr{R}}(s_2)$; therefore, $(\nu_{\mathscr{R}}(s_1))^q < (\nu_{\mathscr{R}}(s_2))^q$. Then,

$$\begin{aligned}(\nu_{\mathscr{R}}(s_2))^q &= \left(\nu_{\mathscr{R}}\left(s_1^{-1}s_1 s_2\right)\right)^q \leq \max\left[\left(\nu_{\mathscr{R}}\left(s_1^{-1}\right)\right)^q, (\nu_{\mathscr{R}}(s_1 s_2))^q\right] \\ &= \max[(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_1 s_2))^q].\end{aligned} \tag{22}$$

Since $(\nu_{\mathscr{R}}(s_1))^q < (\nu_{\mathscr{R}}(s_2))^q$, therefore from relation (22), we obtain

$$(\nu_{\mathscr{R}}(s_2))^q \leq (\nu_{\mathscr{R}}(s_1 s_2))^q. \tag{23}$$

Also, $(\nu_{\mathscr{R}}(s_1 s_2))^q \leq \max[(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q] = (\nu_{\mathscr{R}}(s_2))^q$, that is,

$$(\nu_{\mathscr{R}}(s_1 s_2))^q \leq (\nu_{\mathscr{R}}(s_2))^q. \tag{24}$$

From (23) and (24), we have

$$(\nu_{\mathscr{R}}(s_1 s_2))^q \leq (\nu_{\mathscr{R}}(s_2))^q = \max[(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q]. \tag{25}$$

Similarly, $(\nu_{\mathscr{R}}(s_1 s_2))^q = \max[(\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q]$, if $\nu_{\mathscr{R}}(s_1) < \nu_{\mathscr{R}}(s_2)$. □

Theorem 8. *Let e denote the identity element of G and* $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ *be a q-ROFSG of G. Then,*

(i) if $(\mu_{\mathscr{R}}(s_1))^q = (\mu_{\mathscr{R}}(e))^q$ *for some* $s_1 \in G$, *then* $(\mu_{\mathscr{R}}(s_1 s_2))^q = (\mu_{\mathscr{R}}(s_2))^q$ *for all* $s_2 \in G$

(ii) if $(\nu_{\mathscr{R}}(s_1))^q = (\nu_{\mathscr{R}}(e))^q$ *for some* $s_1 \in G$, *then* $(\nu_{\mathscr{R}}(s_1 s_2))^q = (\nu_{\mathscr{R}}(s_2))^q$ *for all* $s_2 \in G$

Proof. Suppose that $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ is a q-ROFSG of G.

(i) Let $(\mu_{\mathscr{R}}(s_1))^q = (\mu_{\mathscr{R}}(e))^q$for some $s_1 \in G$. Then,

$$\begin{aligned}(\mu_{\mathscr{R}}(s_2))^q &= \left(\mu_{\mathscr{R}}\left(s_1^{-1}s_1 s_2\right)\right)^q \geq \min\left[\left(\mu_{\mathscr{R}}\left(s_1^{-1}\right)\right)^q, (\mu_{\mathscr{R}}(s_1 s_2))^q\right] \\ &= \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_1 s_2))^q] \\ &= \min[(\mu_{\mathscr{R}}(e))^q, (\mu_{\mathscr{R}}(s_1 s_2))^q].\end{aligned} \tag{26}$$

Since $(\mu_{\mathscr{R}}(e))^q \geq (\mu_{\mathscr{R}}(s_2))^q$, therefore from relation (26), we obtain

$$(\mu_{\mathscr{R}}(s_2))^q \geq (\mu_{\mathscr{R}}(s_1 s_2))^q. \tag{27}$$

Also, $(\mu_{\mathscr{R}}(s_1 s_2))^q \geq \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q] = (\mu_{\mathscr{R}}(s_2))^q$, that is,

$$(\mu_{\mathscr{R}}(s_1 s_2))^q \geq (\mu_{\mathscr{R}}(s_2))^q. \tag{28}$$

From (27) and (28), we have

$$(\mu_{\mathscr{R}}(s_1 s_2))^q = (\mu_{\mathscr{R}}(s_2))^q. \tag{29}$$

(ii) The proof is similar to that of (i).

□

Theorem 9. *Let e denote the identity element of G and* $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ *be a q-ROFSG of G. Then,* $H = \{s \in G : (\mu_{\mathscr{R}}(s))^q = (\mu_{\mathscr{R}}(e))^q$ *and* $(\nu_{\mathscr{R}}(s))^q = (\nu_{\mathscr{R}}(e))^q\}$ *is a subgroup of G.*

Proof. By definition of H, we have $e \in H$. Therefore, H is nonempty subset of G.

Let $s_1, s_2 \in H$; then, $(\mu_{\mathscr{R}}(s_1))^q = (\mu_{\mathscr{R}}(e))^q = (\mu_{\mathscr{R}}(s_2))^q$ and $(\nu_{\mathscr{R}}(s_1))^q = (\nu_{\mathscr{R}}(e))^q = (\nu_{\mathscr{R}}(s_2))^q$.

Now,

$$\begin{aligned}\left(\mu_{\mathscr{R}}\left(s_1 s_2^{-1}\right)\right)^q &\geq \min\left[(\mu_{\mathscr{R}}(s_1))^q, \left(\mu_{\mathscr{R}}\left(s_2^{-1}\right)\right)^q\right] \\ &= \min[(\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q] \\ &= \min[(\mu_{\mathscr{R}}(e))^q, (\mu_{\mathscr{R}}(e))^q] = (\mu_{\mathscr{R}}(e))^q.\end{aligned} \tag{30}$$

Also, by Theorem 2, we have $(\mu_{\mathscr{R}}(e))^q \geq (\mu_{\mathscr{R}}(s_1 s_2^{-1}))^q$. Therefore, $(\mu_{\mathscr{R}}(s_1 s_2^{-1}))^q = (\mu_{\mathscr{R}}(e))^q$. Similarly, we can show that $(\nu_{\mathscr{R}}(s_1 s_2^{-1}))^q = (\nu_{\mathscr{R}}(e))^q$. Thus, $s_1 s_2^{-1} \in H$, which completes the proof. □

3. q-Rung Orthopair Fuzzy Coset and q-Rung Orthopair Fuzzy Normal Subgroup

In this section, we define q-rung orthopair fuzzy coset and q-rung orthopair fuzzy normal subgroup. Moreover, we prove some important results regarding q-rung orthopair fuzzy normal subgroup (q-ROFNS).

Definition 10. Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFSG of G. Then, for $x \in G$,

(a) a q-ROFS $x\mathscr{R} = \{s, \mu_{x\mathscr{R}}(s), \nu_{x\mathscr{R}}(s)\}$ of G, where $(\mu_{x\mathscr{R}}(s))^q = (\mu_{\mathscr{R}}(x^{-1}s))^q$ and $(\nu_{x\mathscr{R}}(s))^q = (\nu_{\mathscr{R}}(x^{-1}s))^q$, is called q-rung orthopair fuzzy left coset of $\mathscr{R}$ in G determined by x

(b) a q-ROFS $\mathscr{R}x = \{s, \mu_{\mathscr{R}x}(s), \nu_{\mathscr{R}x}(s)\}$ of G, where $(\mu_{\mathscr{R}x}(s))^q = (\mu_{\mathscr{R}}(sx^{-1}))^q$ and $(\nu_{\mathscr{R}x}(s))^q = (\nu_{\mathscr{R}}(sx^{-1}))^q$, is called q-rung orthopair fuzzy right coset of $\mathscr{R}$ in G determined by x

Definition 11. A q-ROFSG $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ of G is called q-rung orthopair fuzzy normal subgroup (q-ROFNS) of G if $x\mathscr{R} = \mathscr{R}x$ for all $x \in G$.

Theorem 12. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFSG of G. Then, $\mathscr{R}$ is q-ROFNS of G if and only if $(\mu_{\mathscr{R}}(s_1 s_2))^q = (\mu_{\mathscr{R}}(s_2 s_1))^q$ and $(\nu_{\mathscr{R}}(s_1 s_2))^q = (\nu_{\mathscr{R}}(s_2 s_1))^q$ for all $s_1, s_2 \in G$.*

Proof. Assume that $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ is a q-ROFNSG of G. Then, $x\mathscr{R} = \mathscr{R}x$ for all $x \in G$; it means that $(\mu_{x\mathscr{R}}(s))^q = (\mu_{\mathscr{R}x}(s))^q$ and $(\nu_{x\mathscr{R}}(s))^q = (\nu_{\mathscr{R}x}(s))^q$ for all $s \in G$. Therefore, $(\mu_{\mathscr{R}}(x^{-1}s))^q = (\mu_{\mathscr{R}}(sx^{-1}))^q$ and $(\nu_{\mathscr{R}}(x^{-1}s))^q = (\nu_{\mathscr{R}}(sx^{-1}))^q$ for all $x, s \in G$.

Now,

$$(\mu_{\mathscr{R}}(s_1 s_2))^q = \left(\mu_{\mathscr{R}}\left(s_1\left(s_2^{-1}\right)^{-1}\right)\right)^q = \left(\mu_{\mathscr{R}}\left(\left(s_2^{-1}\right)^{-1} s_1\right)\right)^q = (\mu_{\mathscr{R}}(s_2 s_1))^q,$$
$$(\nu_{\mathscr{R}}(s_1 s_2))^q = \left(\nu_{\mathscr{R}}\left(s_1\left(s_2^{-1}\right)^{-1}\right)\right)^q = \left(\nu_{\mathscr{R}}\left(\left(s_2^{-1}\right)^{-1} s_1\right)\right)^q (\nu_{\mathscr{R}}(s_2 s_1))^q. \quad (31)$$

Conversely, suppose that $(\mu_{\mathscr{R}}(s_1 s_2))^q = (\mu_{\mathscr{R}}(s_2 s_1))^q$ and $(\nu_{\mathscr{R}}(s_1 s_2))^q = (\nu_{\mathscr{R}}(s_2 s_1))^q$ for all $s_1, s_2 \in G$. Then, $(\mu_{\mathscr{R}}(s_1 (s_2^{-1})^{-1}))^q = (\mu_{\mathscr{R}}((s_2^{-1})^{-1} s_1))^q$ and $(\nu_{\mathscr{R}}(s_1 (s_2^{-1})^{-1}))^q = (\nu_{\mathscr{R}}((s_2^{-1})^{-1} s_1))^q$. Using $s_1 = s$ and $s_2^{-1} = x$ gives $(\mu_{\mathscr{R}}(sx^{-1}))^q = (\mu_{\mathscr{R}}(x^{-1}s))^q$ and $(\nu_{\mathscr{R}}(sx^{-1}))^q = (\nu_{\mathscr{R}}(x^{-1}s))^q$. This means that $(\mu_{\mathscr{R}x}(s))^q = (\mu_{x\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}x}(s))^q = (\nu_{x\mathscr{R}}(s))^q$; therefore, $x\mathscr{R} = \mathscr{R}x$. Hence, $\mathscr{R}$ is q-ROFNS of G. □

Theorem 13. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFSG of G. Then, $\mathscr{R}$ is q-ROFNS of G if and only if $(\mu_{\mathscr{R}}(sxs^{-1}))^q = (\mu_{\mathscr{R}}(x))^q$ and $(\nu_{\mathscr{R}}(sxs^{-1}))^q = (\nu_{\mathscr{R}}(x))^q$ for all $s, x \in G$.*

Proof. Let $\mathscr{R}$ be a q-ROFNS of G and $s, x \in G$. Then,

$$\begin{aligned}\left(\mu_{\mathscr{R}}\left(sxs^{-1}\right)\right)^q &= \left(\mu_{\mathscr{R}}\left((sx)s^{-1}\right)\right)^q = \left(\mu_{\mathscr{R}}\left(s^{-1}(sx)\right)\right)^q \\ &= \left(\mu_{\mathscr{R}}\left(\left(ss^{-1}\right)x\right)\right)^q = (\mu_{\mathscr{R}}(ex))^q = (\mu_{\mathscr{R}}(x))^q\end{aligned} \quad (32)$$

(by Theorem 12).

Similarly, we can prove $(\nu_{\mathscr{R}}(sxs^{-1}))^q = (\nu_{\mathscr{R}}(x))^q$.

Conversely, suppose that $(\mu_{\mathscr{R}}(sxs^{-1}))^q = (\mu_{\mathscr{R}}(x))^q$ and $(\nu_{\mathscr{R}}(sxs^{-1}))^q = (\nu_{\mathscr{R}}(x))^q$ for all $s, x \in G$. Let $s_1, s_2 \in G$; then,

$$\begin{aligned}(\mu_{\mathscr{R}}(s_1 s_2))^q &= \left(\mu_{\mathscr{R}}\left(\left(s_2^{-1} s_2\right) s_1 s_2\right)\right)^q = \left(\mu_{\mathscr{R}}\left(\left(s_2^{-1} s_2\right) s_1 \left(s_2^{-1}\right)^{-1}\right)\right)^q \\ &= \left(\mu_{\mathscr{R}}\left(s_2^{-1}(s_2 s_1)\left(s_2^{-1}\right)^{-1}\right)\right)^q = (\mu_{\mathscr{R}}(s_2 s_1))^q.\end{aligned} \quad (33)$$

Similarly,

$$(\nu_{\mathscr{R}}(s_1 s_2))^q = (\nu_{\mathscr{R}}(s_2 s_1))^q. \quad (34)$$

Then, the application of (33) and (34) together with Theorem 12 gives $\mathscr{R}$ which is q-ROFNS of G. □

Theorem 14. *Let $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s)\}$ be a q-ROFNSG of G . Then, $H = \{s \in G : (\mu_{\mathscr{R}}(s))^q = (\mu_{\mathscr{R}}(e))^q$ and $(\nu_{\mathscr{R}}(s))^q = (\nu_{\mathscr{R}}(e))^q\}$ is a normal subgroup of G.*

Proof. The application of Theorem 9 gives H which is a subgroup of G. Let $h \in H$ and $s \in G$; then,

$$\left(\mu_{\mathscr{R}}\left(shs^{-1}\right)\right)^q = (\mu_{\mathscr{R}}(h))^q \text{ (by Theorem 3.2)} = (\mu_{\mathscr{R}}(e))^q \text{ (since } h \in H). \quad (35)$$

Similarly,

$$\left(\nu_{\mathscr{R}}\left(shs^{-1}\right)\right)^q = (\nu_{\mathscr{R}}(e))^q. \quad (36)$$

Thus, $shs^{-1} \in H$, which implies that $\mathscr{H}$ is a normal subgroup of G. □

4. q-Rung Orthopair Fuzzy Level Subgroup

This section reveals the idea of q-rung orthopair fuzzy level subgroup. We also prove some relevant results.

Definition 15. Let $\mathscr{R} = \{\jmath, \mu_{\mathscr{R}}(\jmath), \nu_{\mathscr{R}}(\jmath)\}$ be a q-ROFS of crisp set G and $\gamma, \delta \in [0, 1]$ such that $0 \leq (\gamma)^q + (\delta)^q \leq 1$. Then $\mathscr{R}_{(\gamma,\delta)} = \{\jmath \in G : (\mu_{\mathscr{R}}(\jmath))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(\jmath))^q \leq \delta\}$ is called q-rung orthopair fuzzy level subset (q-ROFLS) of q-ROFS $\mathscr{R}$ of G.

Theorem 16. *Let $\mathscr{R} = \{\jmath, \mu_{\mathscr{R}}(\jmath), \nu_{\mathscr{R}}(\jmath)\}$ be a q-ROFS of G and $\gamma, \delta, \gamma', \delta' \in [0, 1]$. Then,*

(i) $\mathscr{R}_{(\gamma,\delta)} \subseteq \mathscr{R}_{(\gamma',\delta')}$ if $\gamma' \leq \gamma$ and $\delta \leq \delta'$

(ii) $\mathscr{R}_{(\gamma,\delta)} \subseteq \mathscr{R}'_{(\gamma,\delta)}$ *if* $\mathscr{R} \subseteq \mathscr{R}'$

Proof.

(i) Let $s \in \mathscr{R}_{(\gamma,\delta)}$; then, $(\mu_{\mathscr{R}}(s))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(s))^q \leq \delta$. Since $\gamma' \leq \gamma$ and $\delta \leq \delta'$, therefore $(\mu_{\mathscr{R}}(s))^q \geq \gamma \geq \gamma'$ and $(\nu_{\mathscr{R}}(s))^q \leq \delta \leq \delta'$. It means that $s \in \mathscr{R}_{(\gamma',\delta')}$; hence, $\mathscr{R}_{(\gamma,\delta)} \subseteq \mathscr{R}_{(\gamma',\delta')}$

(ii) Let $s \in \mathscr{R}_{(\gamma,\delta)}$; then, $(\mu_{\mathscr{R}}(s))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(s))^q \leq \delta$. Since $\mathscr{R} \subseteq \mathscr{R}'$, therefore $(\mu_{\mathscr{R}}(s))^q \leq (\mu_{\mathscr{R}'}(s))^q$ and $(\nu_{\mathscr{R}}(s))^q \geq (\nu_{\mathscr{R}'}(s))^q$. So, $(\mu_{\mathscr{R}'}(s))^q \geq (\mu_{\mathscr{R}}(s))^q \geq \gamma$ and $(\nu_{\mathscr{R}'}(s))^q \leq (\nu_{\mathscr{R}}(s))^q \leq \delta$, which implies that $s \in \mathscr{R}'_{(\gamma,\delta)}$. Thus, $\mathscr{R}_{(\gamma,\delta)} \subseteq \mathscr{R}'_{(\gamma,\delta)}$

□

Theorem 17. *A q-ROFS $\mathscr{R}$ of a group G is q-ROFSG of G if and only if q-ROFLS $\mathscr{R}_{(\gamma,\delta)}$ of G is a subgroup of G.*

Proof. We know $\mathscr{R}_{(\gamma,\delta)} = \{s \in G : (\mu_{\mathscr{R}}(s))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(s))^q \leq \delta\}$. Since for all $\gamma, \delta \in [0, 1]$, we have $(\mu_{\mathscr{R}}(e))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(e))^q \leq \delta$. Therefore, at least $e \in \mathscr{R}_{(\gamma,\delta)}$, which implies that $\mathscr{R}_{(\gamma,\delta)}$ is nonempty.

Suppose $s, t \in \mathscr{R}_{(\gamma,\delta)}$, which means that $(\mu_{\mathscr{R}}(s))^q$, $(\mu_{\mathscr{R}}(t))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q \leq \delta$. Since $\mathscr{R}$ is q-ROFSG of G, therefore

$$\begin{aligned}
\left(\mu_{\mathscr{R}}\left(st^{-1}\right)\right)^q &\geq \min\left\{(\mu_{\mathscr{R}}(s))^q, \left(\mu_{\mathscr{R}}\left(t^{-1}\right)\right)^q\right\} \\
&= \min\{(\mu_{\mathscr{R}}(s))^q, (\mu_{\mathscr{R}}(t))^q\} \geq \min\{\gamma, \gamma\} = \gamma, \\
\left(\nu_{\mathscr{R}}\left(st^{-1}\right)\right)^q &\leq \max\left\{(\nu_{\mathscr{R}}(s))^q, \left(\nu_{\mathscr{R}}\left(t^{-1}\right)\right)^q\right\} \\
&= \max\{(\nu_{\mathscr{R}}(s))^q, (\nu_{\mathscr{R}}(t))^q\} \leq \max\{\delta, \delta\} = \delta.
\end{aligned} \tag{37}$$

Thus, $st^{-1} \in \mathscr{R}_{(\gamma,\delta)}$; therefore, $\mathscr{R}_{(\gamma,\delta)}$ is a subgroup of G.

Conversely, let $\mathscr{R}$ be a q-ROFS of G, and for all $\gamma, \delta \in [0, 1]$, $\mathscr{R}_{(\gamma,\delta)}$ is a subgroup of G. Suppose $s_1, s_2 \in G$ such that $(\mu_{\mathscr{R}}(s_1))^q = \gamma_1$, $(\mu_{\mathscr{R}}(s_2))^q = \gamma_2$, $(\nu_{\mathscr{R}}(s_1))^q = \delta_1$, and $(\nu_{\mathscr{R}}(s_2))^q = \delta_2$. Then, $s_1, s_2 \in \mathscr{R}_{(\min(\gamma_1,\gamma_2),\min(\delta_1,\delta_2))}$; since $\mathscr{R}_{(\min(\gamma_1,\gamma_2),\max(\delta_1,\delta_2))}$ is a subgroup of G, therefore $s_1s_2 \in \mathscr{R}_{(\min(\gamma_1,\gamma_2),\max(\delta_1,\delta_2))}$. This implies that $(\mu_{\mathscr{R}}(s_1s_2))^q \geq \min(\gamma_1, \gamma_2) = \min((\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q)$ and $(\nu_{\mathscr{R}}(s_1s_2))^q \leq \max(\delta_1, \delta_2) = \max((\nu_{\mathscr{R}}(s_1))^q, (\nu_{\mathscr{R}}(s_2))^q)$.

Next, let $s \in G$ such that $(\mu_{\mathscr{R}}(s))^q = \gamma$ and $(\nu_{\mathscr{R}}(s))^q = \delta$. Then, $s \in \mathscr{R}_{(\gamma,\delta)}$; since $\mathscr{R}_{(\gamma,\delta)}$ is a subgroup of G, therefore $s_1^{-1} \in \mathscr{R}_{(\gamma,\delta)}$. It means that $(\mu_{\mathscr{R}}(s_1^{-1}))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(s_1^{-1}))^q \leq \delta$, which implies that $(\mu_{\mathscr{R}}(s_1^{-1}))^q \geq (\mu_{\mathscr{R}}(s))^q$ and $(\nu_{\mathscr{R}}(s_1^{-1}))^q \leq (\nu_{\mathscr{R}}(s))^q$.

Hence, $\mathscr{R}$ is a q-ROFSG of G. □

Theorem 18. *If $\mathscr{R}$ is a q-ROFNSG of G, then q-ROFLS $\mathscr{R}_{(\gamma,\delta)}$ of G is a normal subgroup of G.*

Proof. By Theorem 17, $\mathscr{R}_{(\gamma,\delta)}$ is a subgroup of G. Let $x \in \mathscr{R}_{(\gamma,\delta)}$ and $s \in G$. Then, $(\mu_{\mathscr{R}}(x))^q \geq \gamma$ and $(\nu_{\mathscr{R}}(x))^q \leq \delta$. Since $x \in \mathscr{R}_{(\gamma,\delta)} \subseteq G$, therefore by using Theorem 13, we have $(\mu_{\mathscr{R}}(sxs^{-1}))^q = (\mu_{\mathscr{R}}(x))^q$ and $(\nu_{\mathscr{R}}(sxs^{-1}))^q = (\nu_{\mathscr{R}}(x))^q$. Ultimately, it gives $(\mu_{\mathscr{R}}(sxs^{-1}))^q \geq \gamma$ and $(\mu_{\mathscr{R}}(sxs^{-1}))^q \leq \delta$, which means that $sxs^{-1} \in \mathscr{R}_{(\gamma,\delta)}$. Thus, $\mathscr{R}_{(\gamma,\delta)}$ is a normal subgroup of G. □

5. Homomorphism on q-Rung Orthopair Fuzzy Subgroups

This section is devoted to explore the impact of group homomorphism on q-rung orthopair fuzzy subgroups.

Theorem 19. *Suppose $\theta : G \longrightarrow G'$ is an onto group homomorphism and $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s) : s \in G\}$ is a q-ROFAG of G. Then, $\theta(\mathscr{R}) = \{s', \mu_{\theta(\mathscr{R})}(s'), \nu_{\theta(\mathscr{R})}(s') : s' \in G'\}$ is a q-ROFSG of G'.*

Proof. Since $\theta : G \longrightarrow G'$ is an onto homomorphism, therefore $\theta(G) = G'$.

Let $s_1', s_2' \in G'$; then, there exists $s_1, s_2 \in G$ such that $\theta(s_1) = s_1'$, $\theta(s_2) = s_2'$, and $\theta(s_1s_2) = \theta(s_1)\theta(s_2) = s_1's_2'$.

$$\begin{aligned}
\left(\mu_{\theta(\mathscr{R})}\left(s_1's_2'\right)\right)^q &= \left(\max\left\{\mu_{\mathscr{R}}(z) : z \in G, \theta(z) = s_1's_2'\right\}\right)^q \\
&= \max\left\{(\mu_{\mathscr{R}}(z))^q : z \in G, \theta(z) = s_1's_2'\right\} \\
&= \max\left\{\begin{array}{c}(\mu_{\mathscr{R}}(s_1s_2))^q : s_1s_2 = z \in G, \theta(s_1) = s_1', \theta(s_2) = s_2' \\ \text{and}\ \theta(s_1s_2) = \theta(s_1)\theta(s_2) = s_1's_2'\end{array}\right\} \\
&\quad \cdot (\theta \text{ is homomorphism}) \\
&\geq \max\left\{\begin{array}{c}\min\left((\mu_{\mathscr{R}}(s_1))^q, (\mu_{\mathscr{R}}(s_2))^q\right): \\ s_1, s_2 \in G, \theta(s_1) = s_1', \theta(s_2) = s_2'\end{array}\right\} \\
&\quad \cdot (\mathscr{R} \text{ is a } q\text{-ROFSG of } G) \\
&= \min\left(\begin{array}{c}\max\left\{(\mu_{\mathscr{R}}(s_1))^q : s_1 \in G, \theta(s_1) = s_1'\right\}, \\ \max\left\{(\mu_{\mathscr{R}}(s_2))^q : s_2 \in G, \theta(s_2) = s_2'\right\}\end{array}\right) \\
&= \min\left(\left(\mu_{\theta(\mathscr{R})}\left(s_1'\right)\right)^q, \left(\mu_{\theta(\mathscr{R})}\left(s_2'\right)\right)^q\right).
\end{aligned} \tag{38}$$

So, $(\mu_{\theta(\mathscr{R})}(s_1's_2'))^q \geq \min\left((\mu_{\theta(\mathscr{R})}(s_1'))^q, (\mu_{\theta(\mathscr{R})}(s_2'))^q\right)$ for all $s_1', s_2' \in G'$. In a similar way, it can be shown that $(\nu_{\theta(\mathscr{R})}(s_1's_2'))^q \geq \min\left((\nu_{\theta(\mathscr{R})}(s_1'))^q, (\nu_{\theta(\mathscr{R})}(s_2'))^q\right)$.

Again, suppose that $s' \in G'$; then,

$$\begin{aligned}\left(\mu_{\theta(\mathscr{R})}\left(s'^{-1}\right)\right)^q &= \left(\max\left\{\mu_{\mathscr{R}}(z): z \in G, \theta(z) = s'^{-1}\right\}\right)^q \\ &= \left(\max\left\{\mu_{\mathscr{R}}\left(z^{-1}\right): z^{-1} \in G, \theta\left(z^{-1}\right) = s'\right\}\right)^q \\ &= \left(\mu_{\theta(\mathscr{R})}\left(s'\right)\right)^q.\end{aligned} \tag{39}$$

Similarly, we have $(\nu_{\theta(\mathscr{R})}(s'^{-1}))^q = (\nu_{\theta(\mathscr{R})}(s'))^q$ for all $s' \in G'$.

Thus, $\theta(\mathscr{R}) = \{s', \mu_{\theta(\mathscr{R})}(s'), \nu_{\theta(\mathscr{R})}(s'): s' \in G'\}$ is a q-ROFSG of G'. □

Theorem 20. *Suppose $\theta : G \longrightarrow G'$ is a bijective homomorphism and $\mathscr{T} = \{s', \mu_{\mathscr{T}}(s'), \nu_{\mathscr{T}}(s'): s' \in G'\}$ is a q-ROFSG of G'. Then, $\theta^{-1}(\mathscr{T}) = \{s, \mu_{\theta^{-1}(\mathscr{T})}(s), \nu_{\theta^{-1}(\mathscr{T})}(s): s \in G\}$ is a q-ROFSG of G.*

Proof. Let $s_1, s_2 \in G$; then, $s_1 s_2 \in G$.

Next,

$$\begin{aligned}\left(\mu_{\theta^{-1}(\mathscr{T})}(s_1 s_2)\right)^q &= (\mu_{\mathscr{T}}(\theta(s_1 s_2)))^q \\ &= (\mu_{\mathscr{T}}(\theta(s_1)\theta(s_2)))^q (\theta \text{ is homomorphism}) \\ &\geq \min\{(\mu_{\mathscr{T}}(\theta(s_1)))^q, (\mu_{\mathscr{T}}(\theta(s_2)))^q\} \\ &\quad \cdot \left(\mathscr{T} \text{ is a } q\text{-ROFSG of } G'\right) \\ &= \min\left\{\left(\mu_{\theta^{-1}(\mathscr{T})}(s_1)\right)^q, \left(\mu_{\theta^{-1}(\mathscr{T})}(s_2)\right)^q\right\}.\end{aligned} \tag{40}$$

In a similar fashion, we can show $(\nu_{\theta^{-1}(\mathscr{T})}(s_1 s_2))^q \leq \max\{(\nu_{\theta^{-1}(\mathscr{T})}(s_1))^q, (\nu_{\theta^{-1}(\mathscr{T})}(s_2))^q\}$, for all $s_1, s_2 \in G$.

Again, suppose that $s \in G$; then,

$$\left(\mu_{\theta^{-1}(\mathscr{T})}\left(s^{-1}\right)\right)^q = \left(\mu_{\mathscr{T}}\left(\theta\left(s^{-1}\right)\right)\right)^q = (\mu_{\mathscr{T}}(\theta(s)))^q = \left(\mu_{\theta^{-1}(\mathscr{T})}(s)\right)^q. \tag{41}$$

Similarly, we can prove $(\nu_{\theta^{-1}(\mathscr{T})}(s^{-1}))^q = (\nu_{\theta^{-1}(\mathscr{T})}(s))^q$.

Thus, $\theta^{-1}(\mathscr{T})$ is a q-ROFSG of G. □

Theorem 21. *Suppose $\theta : G \longrightarrow G'$ is an onto group homomorphism and $\mathscr{R} = \{s, \mu_{\mathscr{R}}(s), \nu_{\mathscr{R}}(s): s \in G\}$ is a q-ROFNSG of G. Then, $\theta(\mathscr{R}) = \{s', \mu_{\theta(\mathscr{R})}(s'), \nu_{\theta(\mathscr{R})}(s'): s' \in G'\}$ is a q-ROFNSG of G'.*

Proof. The application of Theorem 19 yields that $\theta(\mathscr{R})$ is a q-ROFSG of G'. Let $s_1', s_2' \in G'$; then, there exists $s_1, s_2 \in G$ such that $\theta(s_1) = s_1'$, $\theta(s_2) = s_2'$, and $\theta(s_1 s_2) = \theta(s_1)\theta(s_2) = s_1' s_2'$.

$$\begin{aligned}\left(\mu_{\theta(\mathscr{R})}\left(s_1' s_2'\right)\right)^q &= \left(\max\left\{\mu_{\mathscr{R}}(z): z \in G, \theta(z) = s_1' s_2'\right\}\right)^q \\ &= \max\left\{(\mu_{\mathscr{R}}(z))^q : z \in G, \theta(z) = s_1' s_2'\right\} \\ &= \max\left\{\begin{array}{c}(\mu_{\mathscr{R}}(s_1 s_2))^q : s_1 s_2 = z \in G, \theta(s_1) = s_1', \theta(s_2) = s_2' \\ \text{and } \theta(s_1 s_2) = \theta(s_1)\theta(s_2) = s_1' s_2'\end{array}\right\} \\ &\quad \cdot (\theta \text{ is homomorphism}) \\ &= \max\left\{\begin{array}{c}(\mu_{\mathscr{R}}(s_2 s_1))^q : s_2 s_1 = z' \in G, \theta(s_1) = s_1', \\ \theta(s_2) = s_2' \text{ and } \theta(s_2 s_1) = \theta(s_2)\theta(s_1) = s_2' s_1'\end{array}\right\} \\ &\quad \cdot (\mathscr{R} \text{ is a } q\text{-ROFNSG of } G) \\ &= \max\left\{\left(\mu_{\mathscr{R}}\left(z'\right)\right)^q : z' \in G, \theta\left(z'\right) = s_2' s_1'\right\} \\ &= \left(\max\left\{\mu_{\mathscr{R}}\left(z'\right): z' \in G, \theta\left(z'\right) = s_2' s_1'\right\}\right)^q \\ &= \left(\mu_{\theta(\mathscr{R})}\left(s_2' s_1'\right)\right)^q.\end{aligned} \tag{42}$$

So, $(\mu_{\theta(\mathscr{R})}(s_1' s_2'))^q = (\mu_{\theta(\mathscr{R})}(s_2' s_1'))^q$ for all $s_1', s_2' \in G'$. In a similar way, it can be shown that $(\nu_{\theta(\mathscr{R})}(s_1' s_2'))^q = (\nu_{\theta(\mathscr{R})}(s_2' s_1'))^q$ for all $s_1', s_2' \in G'$.

Hence, by Theorem 12, $\theta(\mathscr{R}) = \{s', \mu_{\theta(\mathscr{R})}(s'), \nu_{\theta(\mathscr{R})}(s'): s' \in G'\}$ is a q-ROFNSG of G'. □

Theorem 22. *Suppose $\theta : G \longrightarrow G'$ is a bijective homomorphism and $\mathscr{T} = \{s', \mu_{\mathscr{T}}(s'), \nu_{\mathscr{T}}(s'): s' \in G'\}$ is a q-ROFNSG of G'. Then, $\theta^{-1}(\mathscr{T}) = \{s, \mu_{\theta^{-1}(\mathscr{T})}(s), \nu_{\theta^{-1}(\mathscr{T})}(s): s \in G\}$ is a q-ROFNSG of G.*

Proof. By using Theorem 13, we have $\theta^{-1}(\mathscr{T})$ is a q-ROFSG of G. Let $s_1, s_2 \in G$; then, $s_1 s_2 \in G$.

Now,

$$\begin{aligned}\left(\mu_{\theta^{-1}(\mathscr{T})}(s_1 s_2)\right)^q &= (\mu_{\mathscr{T}}(\theta(s_1 s_2)))^q \\ &= (\mu_{\mathscr{T}}(\theta(s_1)\theta(s_2)))^q (\theta \text{ is homomorphism}) \\ &= (\mu_{\mathscr{T}}(\theta(s_2)\theta(s_1)))^q \left(\mathscr{T} \text{ is a } q\text{-ROFNSG of } G'\right) \\ &= (\mu_{\mathscr{T}}(\theta(s_2 s_1)))^q = \left(\mu_{\theta^{-1}(\mathscr{T})}(s_2 s_1)\right)^q.\end{aligned} \tag{43}$$

Therefore, $(\mu_{\theta^{-1}(\mathscr{T})}(s_1 s_2))^q = (\mu_{\theta^{-1}(\mathscr{T})}(s_2 s_1))^q$ for all $s_1, s_2 \in G$. In a similar way, it can be shown that $(\nu_{\theta^{-1}(\mathscr{T})}(s_1 s_2))^q = (\nu_{\theta^{-1}(\mathscr{T})}(s_2 s_1))^q$ for all $s_1, s_2 \in G$.

Hence, by Theorem 12, $\theta^{-1}(\mathscr{T}) = \{s, \mu_{\theta^{-1}(\mathscr{T})}(s), \nu_{\theta^{-1}(\mathscr{T})}(s): s \in G\}$ is a q-ROFNSG of G. □

6. Conclusion

This article aims to initiate the study of q-rung orthopair fuzzy group theory.

We have introduced the notion of q-ROFSG and many algebraic characteristics of this newly defined concept. We have proven that every PFSG is q-ROFSG, but the converse is not true. We presented the idea of q-rung orthopair fuzzy coset and q-ROFNSG and found a necessary and sufficient condition for ROFSG to be a ROFNSG. In addition, we found that q-ROFLS q-rung orthopair fuzzy level subset of a group G is a normal subgroup of G. Lastly, we have explored the impact of group homomorphism on q-ROFSG. We are working on some other classical group theoretic topics like quotient groups, Lagrange's theorem, isomorphism theorems, conjugate subgroups, Caley's theorem, subgroups of nilpotent, solvable, Hamiltonian, and P-Hall groups under q-rung orthopair fuzzy environment. We will share this work in our upcoming papers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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