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Research Article

Stability Results of Some Fractional Neutral Integrodifferential Equations with Delay

Abdellatif Ben Makhlouf[1] **El-sayed El-hady**[1] **Salah Boulaaras**[2] **and Lassaad Mchiri**[3]

[1]*Mathematics Department, College of Science, Jouf University, P.O. Box 2014, Sakaka, Saudi Arabia*
[2]*Department of Mathematics, College of Sciences and Arts, Qassim University, Arras, Saudi Arabia*
[3]*Department of Statistics and Operations Research, College of Sciences, King Saud University, P.O. Box 2455 Riyadh 11451, Saudi Arabia*

Correspondence should be addressed to Salah Boulaaras; s.boularas@qu.edu.sa

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Differential equations with fractional derivative are being extensively used in the modelling of the transmission of many infective diseases like HIV, Ebola, and COVID-19. Analytical solutions are unreachable for a wide range of such kind of equations. Stability theory in the sense of Ulam is essential as it provides approximate analytical solutions. In this article, we utilize some fixed point theorem (FPT) to investigate the stability of fractional neutral integrodifferential equations with delay in the sense of Ulam-Hyers-Rassias (UHR). This work is a generalized version of recent interesting works. Finally, two examples are given to prove the applicability of our results.

1. Introduction

Fractional calculus (FC) has proved to be an efficient tool in many domains like biology, mechanics, electricity, signal processing, chemistry, economics, polymer rheology, aerodynamics, and other areas of interest problems (see, e.g., [1–8] and the references therein). This is because of the powerful tools (see, e.g., [5]) that are not available in the classical calculus. In particular, FC enable researches to model in an efficient way many complicated real-world problems like COVID-19 (see [9]), HIV (see [10]), Rubella disease (see [11]), Ebola virus (see [12]), and HBV infection (see [13]).

Neutral FDEs (NFDEs) play an essential role in many applications. For instance, NFDEs with delay model have electrical networks containing lossless transmission lines (see, e.g., [14]).

As a consequence of the importance and applications of this class of equations, numerous numerical and approximate tools have been proposed to solve such kind of equations. One of such tools that provide close exact analytical solutions is the theory of stability. Stability theory popped up as a result of Ulam's famous question (see, e.g., [15]). Various answers have been introduced for Ulam's problem by many mathematicians. For instance, in 1941, D. H. Hyers (see [15]) presented a positive answer to the Ulam question and the stability problem is called Ulam-Hyers or Hyers-Ulam stability problem. The most important result after Hyers, Aoki, and Bourgin answer (see [16, 17]) was that of Rassias in 1978 (see [18]). The idea of Rassias is a generalization of the result of Hyers. The result introduced by Rassias in [18] is now known as the UHR stability.

During the last seventy years, the stability subject for many kinds of equations has been a common issue of investigations in many directions and there are a lot of articles as well as books published in this subject (see, e.g., [19, 20] for further references). Obloza in 1993 (see [21]) is the first who investigated the Ulam stability of differential equations (see also [22]). Alsina and Ger in 1998 (see [23]) studied the Ulam-Hyers stability (UHS) of the ordinary differential equation $y'(s) = y(s)$ and end up with the estimation $|h(x) - y_0(x)| \leq 3\varepsilon$, where $y_0(x)$ is a solution of the equation, and some $h : \Omega \longrightarrow \mathbb{R}$ is a differentiable mapping satisfying the corresponding differential inequality with some interval Ω. Takahasi et al. in 2002 (see [24]) extended the result of Alsina and Ger. In particular, Takahasi et al. studied the stability of the differential equation $g'(s) = \lambda g(s)$ in Banach spaces. Miura et al. in 2003 (see [25, 26]) generalized the work of Alsina and Ger to higher order differential equations.

As a consequence of the interesting results presented in this direction, many articles devoted to this subject have been introduced (see, e.g., [27, 28]). For instance, Jung in 2010 (see [29]) used some FPT to study the stability of the equation $\chi' = k(s, \chi)$. It should be remarked that Jung in [29] generalized the work of Alsina and Ger (in [23]) to the nonlinear case. Bojor in 2012 (see [30]) used different assumptions to study the stability of the equation

$$h'(x_1) + m(x_1)h(x_1) = r(x_1), \tag{1}$$

and improved the result of Jung in [29]. Tunç and Biçer in 2015 (see [31]) improved the approach of Jung in [29] for the equation

$$l'(x_1) = F(x_1, l(x_1), l(x_1 - \tau)). \tag{2}$$

Huang et al. in 2015 (see [32]) studied the stability of some general form of a nonlinear differential equation. Popa and Pugna in 2016 (see [33]) studied the Hyers-Ulam stability of Euler's equation. Shen in 2017 (see [34]) introduced Ulam stability results for differential equations on time scales.

Rahim and Akbar in 2018 (see [27]) used a FPT-based approach to study the stability of a delay Volterra integrodifferential equation. Shikhare and Kucche in 2019 (see [35]) employed weakly Picard operator to investigate the UHS of some kind of equations. Furthermore, they obtained stability in the sense of UHR for such kind of equations via Pachpatte's integral inequalities. Also, Shah and Zada in 2019 (see [36]) used some FPT to investigate the stability of impulsive Volterra integral equation. In 2020, the authors in [37] investigated the stability of some general equation using FPT. In [38], the authors studied the stability of some Caputo fractional differential equations using FPT (see also [39, 40]).

A great number of research articles have been introduced to study the stability of fractional differential equations. For instance, in [41], the authors studied the Ulam stability for some fractional differential equations in complex domain. In [42], Ulam-Hyers stability for Cauchy fractional differential equation in the unit disk is investigated. In [43], the authors investigated the existence of Ulam stability for iterative fractional differential equations based on fractional entropy. In [44], Mittag-Leffler-Ulam stabilities of fractional evolution equations have been introduced. The author in [45] investigated the generalized Ulam-Hyers stability for the following fractional differential equation

$$D_z^\alpha f(z) = G\left(f(z), zf'(z), z^2 f''; z\right), 2 < \alpha \leq 3, \tag{3}$$

in a complex Banach space. In [46], the authors investigated Ulam stability for nonlinear Hilfer fractional stochastic differential systems in finite dimensional stochastic setting. In [47], the authors studied the existence of a mild solution and exponential stability for a class of second-order impulsive fractional neutral stochastic differential equations. In [48], the authors employed some fixed point theory to study the existence of mild solution for the analysis of the moment stability of fractional stochastic differential inclusions driven by the Rosenblatt process and Poisson jumps with impulses in a Hilbert space. As far as we know, there is no existing work using the fixed point approach to investigate the stability of fractional neutral integrodifferential equations with delay in the sense of Ulam-Hyers-Rassias (UHR).

The main contributions of our paper are as follows:

(1) Investigating the stability of fractional neutral integrodifferential equations with delay in the sense of Ulam-Hyers-Rassias (UHR)

(2) Extending some interesting work by adding the neutral term and the fixed point theorem (see [27, 38])

The article is divided into three sections. In the next section, we recall some preliminaries; in Section 3, we present the stability results in UHR sense; in Section 4, we illustrate our results with two examples; and in Section 5, we present the conclusion.

2. Preliminaries

Definition 1 (see [8]). The Hadamard fractional integral of order λ for a function h is defined as

$$^{H}I^{\lambda}h(\omega) = \frac{1}{\Gamma(\lambda)}\int_1^\omega \left(\log\frac{\omega}{\nu}\right)^{\lambda-1}\frac{h(\nu)}{\nu}d\nu, \quad \lambda > 0, \tag{4}$$

provided the integral exists.

Definition 2 (see [8]). The Hadamard derivative of fractional order $\lambda \in (0, 1)$ for a function $h : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$D^{\lambda}h(\omega) = \frac{1}{\Gamma(1-\lambda)}\left(\omega\frac{d}{d\omega}\right)\int_1^\omega \left(\log\frac{\omega}{\nu}\right)^{-\lambda}\frac{h(\nu)}{\nu}d\nu. \tag{5}$$

Definition 3 (see [8]). The Riemann-Liouville fractional integral of order λ for a function $h : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$I^{\lambda}h(\omega) = \frac{1}{\Gamma(\lambda)}\int_1^\omega (\omega-\nu)^{\lambda-1}h(\nu)d\nu, \quad \lambda > 0, \tag{6}$$

provided the integral exists.

Definition 4. Let $\alpha > 0$, $l \in \mathbb{C}$. The Mittag-Leffler function (see, e.g., [36]) $\mathbb{E}_\alpha$ is defined as

$$\mathbb{E}_\alpha(l) \coloneqq \sum_{n=0}^{\infty} \frac{l^n}{\Gamma(\alpha n + 1)}. \tag{7}$$

Remark 1. The function $\kappa(\vartheta) = \mathbb{E}_\varsigma(\gamma(\vartheta - 1)^\varsigma)$ satisfies $I^\varsigma \kappa(\vartheta) = (1/\gamma)(\kappa(\vartheta) - 1)$, where $\gamma \in \mathbb{R}^*$.

Theorem 1 (see [49]). *Suppose (M, d) is a complete metric space and $L : M \longrightarrow M$ satisfies $d(L(z), L(e)) \leq \delta d(z, e)$ (with $0 < \delta < 1$) for all $z, e \in M$. Assume that $v \in M$, $\lambda > 0$ and $d(v, L(v)) \leq \lambda$. Then, there is a unique $k \in M$ with $k = L(k)$. Moreover,*

$$d(v, k) \leq \frac{\lambda}{1 - \delta}. \tag{8}$$

The goal of the article is to investigate the stability of the solution of the following fractional order differential equations:

$$D^\delta \left[y(\rho) - \sum_{i=1}^{m} I^{\lambda_i} \xi_i \left(\rho, y_\rho \right) \right] = \Psi\left(\rho, y_\rho \right), \rho \in I \coloneqq [1, b], \tag{9}$$

with initial conditions $y(\rho) = \phi(\rho)$, $\rho \in [1 - \upsilon, 1]$, where $\phi \in C([1 - \upsilon, 1], \mathbb{R})$, with $\phi(1) = 0{,}0 < \delta < 1$, $\lambda_i > 0$, Ψ, ξ_i are given functions.

For any function y defined on $[1 - \upsilon, b]$ and any $\rho \in I$, we denote by y_ρ the element of $C_\upsilon \coloneqq C([-\upsilon, 0], \mathbb{R})$ defined by $y_\rho(s) = y(\rho + s)$, $s \in [-\upsilon, 0]$, with norm $\|y_\rho\| = \sup\{y(\rho + s); -\upsilon \leq s \leq 0\}$.

3. Stability Results

In this section, we present our main results.

Theorem 2. *Suppose that $\Psi : I \times \mathbb{R}$, $\xi_i : I \times \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$ satisfy*

$$\begin{aligned} |\Psi(\tau, \phi_1) - \Psi(\tau, \phi_2)| &\leq \sigma_0 \|\phi_1 - \phi_2\|, \\ |\xi_i(\tau, \phi_1) - \xi_i(\tau, \phi_2)| &\leq \sigma_i \|\phi_1 - \phi_2\|, \end{aligned} \tag{10}$$

for all $\tau \in I$, $\phi_j \in C_\upsilon$, $j = 1, 2$ and for some $\sigma_i > 0$.

If $l \in C^1([1 - \upsilon, b], \mathbb{R})$ satisfies $l(1) = 0$ and

$$\left| D^\delta \left[l(\tau) - \sum_{i=1}^{m} I^{\lambda_i} \xi_i(\tau, l_\tau) \right] - \Psi(\tau, l_\tau) \right| \leq \varepsilon \gamma(\tau), \tag{11}$$

for all $\tau \in [1, b]$, where $\varepsilon > 0$ and $\gamma(\tau)$ is a nondecreasing, continuous, positive function, then there is a solution l^* of (9) with $l^*(\tau) = l(\tau)$, $\tau \in [1 - \upsilon, 1]$, such that

$$|l(\tau) - l^*(\tau)| \leq \frac{M b^\mu \prod_{i=1}^{m} E_{\lambda_i}\left(\mu_i (b - 1)^{\lambda_i} \right)}{\Gamma(\delta + 1)\left(1 - \left((\sigma_0/\mu^\delta) + \sum_{i=1}^{m} \sigma_i/\mu_i \right) \right)} \varepsilon \gamma(\tau), \quad \forall \tau \in [1, b], \tag{12}$$

where $M = \sup_{s \in [1,b]}((\log s)^\delta / (s^\mu \prod_{i=1}^{m} E_{\lambda_i}(\mu_i (s - 1)^{\lambda_i})))$ and μ, μ_i are some positive constants such that

$$\left(\frac{\sigma_0}{\mu^\delta} + \sum_{i=1}^{m} \frac{\sigma_i}{\mu_i} \right) < 1. \tag{13}$$

Proof. Consider the metric d on $E = C([1 - \upsilon, b], \mathbb{R})$ by

$$d(y_1, y_2) = \inf \left\{ k \in [0,\infty) : \frac{|y_1(\tau) - y_2(\tau)|}{\beta(\tau)} \leq k \tilde{\gamma}(\tau), \forall \tau \in [1 - \upsilon, b] \right\}, \tag{14}$$

with $\beta(\tau) = \tau^\mu \prod_{i=1}^{m} E_{\lambda_i}(\mu_i (\tau - 1)^{\lambda_i})$ for $\tau \in [1, b]$ and $\beta(\tau) = 1$ for $\tau \in [1 - \upsilon, 1]$ and $\tilde{\gamma}(\tau) = \gamma(\tau)$ for $\tau \in [1, b]$ and $\tilde{\gamma}(\tau) = \gamma(1)$ for $\tau \in [1 - \upsilon, 1]$.

We consider the operator $\mathscr{B} : E \longrightarrow E$ such that $(\mathscr{B} y)(\tau) = l(\tau)$, for $\tau \in [1 - \upsilon, 1]$, and

$$(\mathscr{B} y)(\tau) = \sum_{i=1}^{m} I^{\lambda_i} \xi_i(\tau, y_\tau) + \frac{1}{\Gamma(\delta)} \int_1^\tau \left(\log \frac{\tau}{s} \right)^{\delta - 1} \frac{\Psi(s, y_s)}{s} ds, \tag{15}$$

for $\tau \in [1, b]$.

Let $y_1, y_2 \in E$, we have $(\mathscr{B} y_1)(\tau) - (\mathscr{B} y_2)(\tau) = 0$, for all $\tau \in [1 - \upsilon, 1]$.

For $\tau \in [1, b]$, we get

$$\begin{aligned} |(\mathscr{B} y_1)(\tau) - (\mathscr{B} y_2)(\tau)| &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\lambda_i)} \int_1^\tau (\tau - s)^{\lambda_i - 1} |\xi_i(s, y_{1s}) \\ &\quad - \xi_i(s, y_{2s})| ds + \frac{1}{\Gamma(\delta)} \int_1^\tau \left(\log \frac{\tau}{s} \right)^{\delta - 1} \left| \frac{\Psi(s, y_{1s}) - \Psi(s, y_{2s})}{s} \right| ds \\ &\leq \sum_{i=1}^{m} \frac{\sigma_i}{\Gamma(\lambda_i)} \int_1^\tau (\tau - s)^{\lambda_i - 1} \|y_{1s} - y_{2s}\| ds \\ &\quad + \frac{\sigma_0}{\Gamma(\delta)} \int_1^\tau \left(\log \frac{\tau}{s} \right)^{\delta - 1} \frac{\|y_{1s} - y_{2s}\|}{s} ds. \end{aligned} \tag{16}$$

For $s \in [1, \tau]$, there is $q \in [-\upsilon, 0]$ such that $\|y_{1s} - y_{2s}\| = |y_1(s + q) - y_2(s + q)|$. Then,

$$\begin{aligned} \|y_{1s} - y_{2s}\| &= \frac{|y_1(s + q) - y_2(s + q)|}{\beta(s + q) \tilde{\gamma}(s + q)} \beta(s + q) \tilde{\gamma}(s + q) \\ &\leq d(y_1, y_2) \beta(s + q) \tilde{\gamma}(s + q) \leq d(y_1, y_2) \beta(s) \gamma(s). \end{aligned} \tag{17}$$

Therefore,

$$|(\mathscr{B}y_1)(\tau)-(\mathscr{B}y_2)(\tau)|\le\sum_{i=1}^{m}\frac{\sigma_i}{\Gamma(\lambda_i)}d(y_1,y_2)\gamma(\tau)\int_1^\tau(\tau-s)^{\lambda_i-1}\beta(s)ds+\frac{\sigma_0}{\Gamma(\delta)}d(y_1,y_2)\gamma(\tau)\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\delta-1}\frac{\beta(s)}{s}ds\le\sum_{i=1}^{m}\frac{\sigma_i}{\Gamma(\lambda_i)}d(y_1,y_2)\gamma(\tau)\tau^\mu\prod_{j=1;j\neq i}^{m}E_{\lambda_j}\left(\mu_j(\tau-1)^{\lambda_j}\right)\int_1^\tau(\tau-s)^{\lambda_i-1}E_{\lambda_i}\left(\mu_i(s-1)^{\lambda_i}\right)ds+\frac{\sigma_0}{\Gamma(\delta)}d(y_1,y_2)\gamma(\tau)\prod_{j=1}^{m}E_{\lambda_j}\left(\mu_j(\tau-1)^{\lambda_j}\right)\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\delta-1}\frac{s^\mu}{s}ds. \tag{18}$$

By using the change of variable $u=\mu\log\tau-\mu\log s$, we get

$$\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\delta-1}\frac{s^\mu}{s}ds=\int_0^{\mu\log\tau}\left(\frac{u}{\mu}\right)^{\delta-1}\tau^\mu\frac{e^{-u}}{\mu}du\le\frac{\tau^\mu}{\mu^\delta}\Gamma(\delta). \tag{19}$$

Using the inequality (17) and Remark 1, we get

$$|(\mathscr{B}y_1)(\tau)-(\mathscr{B}y_2)(\tau)|\le\left(\frac{\sigma_0}{\mu^\delta}+\sum_{i=1}^{m}\frac{\sigma_i}{\mu_i}\right)d(y_1,y_2)\gamma(\tau)\beta(\tau). \tag{20}$$

Thus, $\mathscr{B}$ is contractive.

For $\tau\in[1-\upsilon,1]$, we have $(\mathscr{B}l)(\tau)-l(\tau)=0$.

We have

$$\left|D^\delta\left[l(\tau)-\sum_{i=1}^{m}I^{\lambda_i}\xi_i(\tau,l_\tau)\right]-\Psi(\tau,l_\tau)\right|\le\varepsilon\gamma(\tau),\quad\forall\tau\in[1,b]. \tag{21}$$

By using Lemma 2.1 in [50], we get

$$|l(\tau)-\mathscr{B}l(\tau)|\le\frac{\varepsilon}{\Gamma(\delta)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\delta-1}\frac{\gamma(s)}{s}ds\le\frac{\varepsilon\gamma(\tau)}{\Gamma(\delta)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\delta-1}\frac{1}{s}ds\le\frac{\varepsilon\gamma(\tau)}{\Gamma(\delta+1)}(\log\tau)^\delta,\quad\forall\tau\in[1,b]. \tag{22}$$

Hence,

$$\frac{|l(\tau)-\mathscr{B}l(\tau)|}{\beta(\tau)}\le\frac{\varepsilon}{\Gamma(\delta+1)}\gamma(\tau)\frac{(\log\tau)^\delta}{\beta(\tau)}\le\frac{\varepsilon M}{\Gamma(\delta+1)}\gamma(\tau),\quad\forall\tau\in[1,b], \tag{23}$$

then

$$d(l,\mathscr{B}l)\le\varepsilon\frac{M}{\Gamma(\delta+1)}. \tag{24}$$

Using Theorem 1, there is a solution l^* of (9) such that

$$d(l,l^*)\le\varepsilon\frac{1}{1-\left(\left(\sigma_0/\mu^\delta\right)+\sum_{i=1}^{m}\sigma_i/\mu_i\right)}\frac{M}{\Gamma(\delta+1)}, \tag{25}$$

so that

$$|l(\tau)-l^*(\tau)|\le\frac{Mb^\mu\prod_{i=1}^{m}E_{\lambda_i}\left(\mu_i(b-1)^{\lambda_i}\right)}{\Gamma(\delta+1)\left(1-\left(\left(\sigma_0/\mu^\delta\right)+\sum_{i=1}^{m}\sigma_i/\mu_i\right)\right)}\varepsilon\gamma(\tau), \tag{26}$$

for all $\tau\in[1,b]$. □

The following theorem represents the UHS of (9).

Theorem 3. *Suppose that $\Psi:I\times\mathbb{R}$, $\xi_i:I\times\mathbb{R}$ for $i\in\{1,2,\cdots,m\}$ satisfy*

$$|\Psi(\tau,\phi_1)-\Psi(\tau,\phi_2)|\le\sigma_0\|\phi_1-\phi_2\|,\quad |\xi_i(\tau,\phi_1)-\xi_i(\tau,\phi_2)|\le\sigma_i\|\phi_1-\phi_2\|, \tag{27}$$

for all $\tau\in I$, $\phi_j\in C_\upsilon$, $j=1,2$ and for some $\sigma_i>0$.

If $l\in C^1([1-\upsilon,b],\mathbb{R})$ satisfies $l(1)=0$ and

$$\left|D^\delta\left[l(\tau)-\sum_{i=1}^{m}I^{\lambda_i}\xi_i(\tau,l_\tau)\right]-\Psi(\tau,l_\tau)\right|\le\varepsilon, \tag{28}$$

for all $\tau\in[1,b]$, where $\varepsilon>0$, then there is a solution l^* of (9) with $l^*(\tau)=l(\tau)$, $\tau\in[1-\upsilon,1]$, such that

$$|l(\tau)-l^*(\tau)|\le\frac{Mb^\mu\prod_{i=1}^{m}E_{\lambda_i}\left(\mu_i(b-1)^{\lambda_i}\right)}{\Gamma(\delta+1)\left(1-\left(\left(\sigma_0/\mu^\delta\right)+\sum_{i=1}^{m}\sigma_i/\mu_i\right)\right)}\varepsilon,\quad\forall\tau\in[1,b], \tag{29}$$

where $M=\sup_{s\in[1,b]}\left((\log s)^\delta/\left(s^\mu\prod_{i=1}^{m}E_{\lambda_i}\left(\mu_i(s-1)^{\lambda_i}\right)\right)\right)$ and μ, μ_i are some positive constants such that

$$\left(\frac{\sigma_0}{\mu^\delta}+\sum_{i=1}^{m}\frac{\sigma_i}{\mu_i}\right)<1. \tag{30}$$

Proof. The proof is similar to Theorem 2. □

Remark 2. Note that the author in [51] has studied the existence and uniqueness of (9), where he assumes some conditions on σ_i.

4. Examples

Example 1. Consider equation (9) for $m = 1$, $\upsilon = 0.3, \delta = 0.4$, $\lambda_1 = 0.5$, $b = 2$, $\xi_1(\tau, \phi) = (1 + \tau)^2 \sin(\phi(-\upsilon))$, and $\Psi(\tau, \phi) = (2 + \tau^2) \cos(\phi(-\upsilon))$.

We have

$$|\xi_1(\tau, \phi_1) - \xi_1(\tau, \phi_2)| \le 9\|\phi_1 - \phi_2\|, \quad \forall \tau \in [1, 2], \phi_1, \phi_2 \in C_{0.3},$$
$$|\Psi(\tau, \phi_1) - \Psi(\tau, \phi_2)| \le 6\|\phi_1 - \phi_2\|, \quad \forall \tau \in [1, 2], \phi_1, \phi_2 \in C_{0.3}. \tag{31}$$

Then, $\sigma_0 = 6$ and $\sigma_1 = 9$.

Suppose that $l \in C^1([0.7,2], \mathbb{R})$ satisfies $l(1) = 0$ and

$$\left|D^{0.4}\left[l(\tau) - I^{0.5}\xi_1(\tau, l_\tau)\right] - \Psi(\tau, l_\tau)\right| \le 0.01\tau, \tag{32}$$

for all $\tau \in [1, 2]$.

Here, $\gamma(\tau) = \tau$ and $\varepsilon = 0.01$. By Theorem 2 there is a solution l^* of equation (9) and $K > 0$ such that

$$|l(\tau) - l^*(\tau)| \le 0.01K\tau, \quad \forall \tau \in [1, 2]. \tag{33}$$

Example 2. Consider equation (9) for $m = 2$, $\upsilon = 0.5, \delta = 0.6$, $\lambda_1 = 0.7$, $\lambda_2 = 0.4$, $b = 3$, $\xi_1(\tau, \phi) = \cos(\phi(-\upsilon))$, $\xi_2(\tau, \phi) = 2\tau^2 \sin(\phi(-\upsilon))$, and $\Psi(\tau, \phi) = 5\tau \sin(\phi(-\upsilon))$.

We have

$$|\xi_1(\tau, \phi_1) - \xi_1(\tau, \phi_2)| \le \|\phi_1 - \phi_2\|, \quad \forall \tau \in [1, 3], \phi_1, \phi_2 \in C_{0.5},$$
$$|\xi_2(\tau, \phi_1) - \xi_2(\tau, \phi_2)| \le 18\|\phi_1 - \phi_2\|, \quad \forall \tau \in [1, 3], \phi_1, \phi_2 \in C_{0.5},$$
$$|\Psi(\tau, \phi_1) - \Psi(\tau, \phi_2)| \le 15\|\phi_1 - \phi_2\|, \quad \forall \tau \in [1, 3], \phi_1, \phi_2 \in C_{0.5}. \tag{34}$$

Then, $\sigma_0 = 15$, $\sigma_1 = 1$, and $\sigma_2 = 18$.

Suppose that $l \in C^1([0.5,3], \mathbb{R})$ satisfies $l(1) = 0$ and

$$\left|D^{0.6}\left[l(\tau) - I^{0.7}\xi_1(\tau, l_\tau) - I^{0.4}\xi_2(\tau, l_\tau)\right] - \Psi(\tau, l_\tau)\right| \le 0.01, \tag{35}$$

for all $\tau \in [1, 3]$.

Here, $\varepsilon = 0.01$. By Theorem 3, there is a solution l^* of equation (9) and $K > 0$ such that

$$|l(\tau) - l^*(\tau)| \le 0.01K, \forall \tau \in [1, 3]. \tag{36}$$

5. Conclusion

We managed to utilize some version of Banach FPT to show that according to certain conditions, functions that fulfill some neutral fractional integrodifferential delay equations (NFIDDE) approximately are close in some sense to the exact solutions of such problems. In fact, we present UHR stability results for some NFIDDE. To illustrate our results, we presented two examples. We think that this work can be extended for various types of differential equations. Potential future work could be to invent a new method to obtain such stability results or to investigate the stability of much more complicated differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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