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Research Article

On New Banach Sequence Spaces Involving Leonardo Numbers and the Associated Mapping Ideal

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In the present study, we have constructed new Banach sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$, where $\mathfrak{L} = (\mathfrak{l}_{v,k})$ is a regular matrix defined by $\mathfrak{l}_{v,k} = \begin{cases} (\mathfrak{l}_k/\mathfrak{l}_{v+2} - (v+2)), & 0 \leq k \leq v, \\ 0, & k > v, \end{cases}$ for all $v, k = 0, 1, 2, \cdots$, where $\mathfrak{l} = (\mathfrak{l}_k)$ is a sequence of Leonardo numbers. We study their topological and inclusion relations and construct Schauder bases of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$. Besides, α-, β- and γ-duals of the aforementioned spaces are computed. We state and prove results of the characterization of the matrix classes between the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$ to any one of the spaces ℓ_1,c_0,c, and ℓ_∞. Finally, under a definite functional ρ and a weighted sequence of positive reals r, we introduce new sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$. We present some geometric and topological properties of these spaces, as well as the eigenvalue distribution of ideal mappings generated by these spaces and s-numbers.

1. Introduction and Preliminaries

Let ω denote the set of all real- or complex-valued sequences. A linear subspace of ω is called a sequence space. Some of the well-known examples of sequence spaces are the space of absolutely p-summable sequences, the space of null sequences, the space of convergent sequences, and the space of bounded sequences, denoted by ℓ_p,c_0,c, and ℓ_∞, respectively. Here and afterwards, $1 \leq p < \infty$, unless stated otherwise. Let bs and cs denote the spaces of all bounded and convergent series, respectively. A Banach sequence space with continuous coordinates is called a BK-space. The spaces $\mathfrak{Z}$ and ℓ_p are BK-spaces equipped with the supremum norm $\|\mathfrak{z}\|_{\ell_\infty} = \sup_{k \in \mathbb{N}_0} |\mathfrak{z}_k|$ and the ℓ_p norm $\|\mathfrak{z}\|_{\ell_p} = (\sum_{k=0}^{\infty} |\mathfrak{z}_k|^p)^{1/p}$, respectively, where $\mathbb{N}_0$ is the set of nonnegative integers and $\mathfrak{Z}$ is any one of the spaces c_0,c, or ℓ_∞.

Let $\mathfrak{A} = (\mathfrak{a}_{v,k})$ be an infinite matrix over the complex field $\mathbb{C}$. The $\mathfrak{A}$-transform of a sequence $\mathfrak{z} = (\mathfrak{z}_k)$ is a sequence $\mathfrak{A}\mathfrak{z} = \{(\mathfrak{A}\mathfrak{z})_v\} = \{\sum_{k=0}^{\infty} \mathfrak{a}_{v,k}\mathfrak{z}_k\}$, provided that the series $\sum_{k=0}^{\infty} \mathfrak{a}_{v,k}\mathfrak{z}_k$ exists, for each $v \in \mathbb{N}_0$. In addition, if $\mathfrak{Z}$ and $\mathfrak{U}$ are two sequence spaces and $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$, for every sequence $\mathfrak{z} \in \mathfrak{Z}$, then the matrix $\mathfrak{A}$ is said to define a matrix mapping from $\mathfrak{Z}$ to $\mathfrak{U}$. The notation $(\mathfrak{Z}, \mathfrak{U})$ represents the family of all matrices that map from $\mathfrak{Z}$ to $\mathfrak{U}$. Furthermore, the matrix $\mathfrak{A} = (\mathfrak{a}_{v,k})$ is called a triangle if $\mathfrak{a}_{v,v} \neq 0$ and $\mathfrak{a}_{v,k} = 0$, for $v < k$. For any $\mathfrak{Z} \subset \omega$, define $\mathfrak{Z}_{\mathfrak{A}} = \{\mathfrak{z} \in \omega : \mathfrak{A}\mathfrak{z} \in \omega\}$. Then, $\mathfrak{Z}_{\mathfrak{A}}$ is a sequence space and is called the matrix domain of $\mathfrak{A}$ in the space $\mathfrak{Z}$. It is well known that if $\mathfrak{Z}$ is a BK-space and $\mathfrak{A}$ is a triangle, then, the matrix domain $\mathfrak{Z}_{\mathfrak{A}}$ is also a BK-space under the norm $\|\mathfrak{z}\|_{\mathfrak{Z}_{\mathfrak{A}}} = \|\mathfrak{A}\mathfrak{z}\|_{\mathfrak{Z}}$. We refer to [1–13] for papers related to theory of sequence spaces and summability.

1.1. Some Special Integer Sequences and the Associated Sequence Spaces. We shall briefly highlight the literature concerning special integer sequences and the construction of the associated sequence spaces.

Let $(\mathfrak{f}_k)_{k=0}^{\infty}$ be the sequence of Fibonacci numbers defined by the recurrence relation $\mathfrak{f}_v = \mathfrak{f}_{v-1} + \mathfrak{f}_{v-2}, v \geq 2$, with $\mathfrak{f}_0 = 1$ and $\mathfrak{f}_1 = 1$. Several authors constructed different types of sequence spaces involving Fibonacci numbers. For instance, Kara [3] studied the Fibonacci sequence spaces $\ell_p(\mathfrak{F}) := (\ell_p)_{\mathfrak{F}}$ and $\ell_\infty(\mathfrak{F}) := (\ell_\infty)_{\mathfrak{F}}$ and examined certain topological and geometrical structures of these Banach sequence spaces, where $\mathfrak{F} = (\mathfrak{f}_{v,k})$ is a double band matrix of Fibonacci numbers defined by

$$\mathfrak{f}_{v,k} = \begin{cases} -\dfrac{\mathfrak{f}_{v+1}}{\mathfrak{f}_v}, & k = v-1, \\ \dfrac{\mathfrak{f}_v}{\mathfrak{f}_{v+1}}, & k = v, \\ 0, & \text{otherwise}. \end{cases} \quad (1)$$

$v, k \in \mathbb{N}_0$. Besides, Basarir et al. [7] studied the sequence spaces $\ell_p(\mathfrak{F}) := (\ell_p)_{\mathfrak{F}}, c_0(\mathfrak{F}) := (c_0)_{\mathfrak{F}}$ and $c(\mathfrak{F}) := (c)_{\mathfrak{F}}$, where $0 < p < 1$. The studies on Fibonacci sequence spaces are further strengthened by Kara and Basarir [4] by introducing the matrix domain $\mathfrak{Z}(\tilde{\mathfrak{F}}) := (\mathfrak{Z})_{\tilde{\mathfrak{F}}}$, where $\mathfrak{Z}$ represents any one of the sequence spaces ℓ_p, c_0, c, or ℓ_∞ and $\tilde{\mathfrak{F}} = (\tilde{\mathfrak{f}}_{v,k})$ is a regular matrix of Fibonacci numbers defined by

$$\tilde{\mathfrak{f}}_{v,k} = \begin{cases} \dfrac{\mathfrak{f}_k^2}{\mathfrak{f}_v \mathfrak{f}_{v+1}}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \quad (2)$$

for all $v, k \in \mathbb{N}_0$. Furthermore, another regular matrix $\bar{\mathfrak{F}} = (\bar{\mathfrak{f}}_{v,k})$ of Fibonacci numbers is defined by Debnath and Saha [1] as follows:

$$\bar{\mathfrak{f}}_{v,k} = \begin{cases} \dfrac{\mathfrak{f}_k}{\mathfrak{f}_{v+2} - 1}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \quad (3)$$

for all $v, k \in \mathbb{N}_0$. By using this matrix, Debnath and Saha [1] and Ercan and Bektas [2] defined and studied the matrix domains $\ell_p(\bar{\mathfrak{F}}) := (\ell_p)_{\bar{\mathfrak{F}}}, c_0(\bar{\mathfrak{F}}) := (c_0)_{\bar{\mathfrak{F}}}, c(\bar{\mathfrak{F}}) := (c)_{\bar{\mathfrak{F}}}$, and $\ell_\infty(\bar{\mathfrak{F}}) := (\ell_\infty)_{\bar{\mathfrak{F}}}$. More studies concerning construction of Banach sequence spaces involving Fibonacci numbers can be tracked in the literature that are generalization or extension of any one of the above discussed Fibonacci sequence spaces. We refer to [5, 6, 8, 9], for such studies.

The number sequence $(\mathfrak{t}_v)_{v=0}^{\infty} := (1, 1, 2, 4, 7, 13, 24, \cdots)$ defined by the recurrence relation $\mathfrak{t}_v = \mathfrak{t}_{v-1} + \mathfrak{t}_{v-2} + \mathfrak{t}_{v-3}$, $v \geq 3$, with $\mathfrak{t}_0 = \mathfrak{t}_1 = 1$ and $\mathfrak{t}_2 = 2$, is called tribonacci sequence. Recently, Yaying and Hazarika [10] introduced tribonacci sequence spaces $\ell_p(\mathfrak{T}) := (\ell_p)_{\mathfrak{T}}$ and $\ell_\infty(\mathfrak{T}) := (\ell_\infty)_{\mathfrak{T}}$, where $\mathfrak{T} = (\bar{\mathfrak{t}}_{v,k})$ is an infinite matrix of tribonacci numbers defined by

$$\bar{\mathfrak{t}}_{v,k} = \begin{cases} \dfrac{2\mathfrak{t}_k}{\mathfrak{t}_{v+2} + \mathfrak{t}_v - 1}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \quad (4)$$

for all $v, k \in \mathbb{N}_0$. Quite recently, Yaying and Kara [11] studied the matrix domains $c_0(\mathfrak{T}) := (c_0)_{\mathfrak{T}}$ and $c(\mathfrak{T}) := (c)_{\mathfrak{T}}$. Moreover, Yaying et al. [12] studied Banach sequence spaces defined by the sequence of Padovan numbers $(\mathfrak{p}_v)_{v=0}^{\infty} = (1, 1, 1, 2, 2, 3, 4, 5, 7, \cdots)$. Besides, A. M. Karakas and M. Karakas [13] also constructed BK-sequence spaces defined by using Lucas numbers $(\mathfrak{l}'_v)_{v=0}^{\infty} = (2, 1, 3, 4, 7, 11, 18, 29, 47, \cdots)$.

1.2. Leonardo Numbers. The number sequence 1, 1, 3, 5, 9, 15, 25, 41, 67, $\cdots$ is termed as Leonardo sequence. Let $\mathfrak{l}_v$, $v = 0, 1, 2, \cdots$, denote the v^{th} Leonardo number. Then, the Leonardo numbers are defined by the following recurrence relation:

$$\mathfrak{l}_v = \mathfrak{l}_{v-1} + \mathfrak{l}_{v-2} + 1, v \geq 2, \quad \text{with } \mathfrak{l}_0 = \mathfrak{l}_1 = 1. \quad (5)$$

It is believed that Leonardo sequence is invented by Leonardo de Pisa, also known as Leonardo Fibonacci. But not much studies related to Leonardo numbers can be traced in the literature due to scarcity of research related to this integer sequence. Leonardo sequence has a very close relationship with the well-known Fibonacci sequence $(\mathfrak{f}_v)_{v=0}^{\infty}$ and the Lucus sequence $(\mathfrak{l}'_v)_{v=0}^{\infty}$:

$$\mathfrak{l}_v = 2\mathfrak{f}_{v+1} - 1, \mathfrak{l}_v = \frac{2}{5}\left(\mathfrak{l}'_v + \mathfrak{l}'_{v+2}\right) - 1, \quad v \geq 0. \quad (6)$$

Quite recently, Catarino and Borges [14] studied basic properties of Leonardo numbers and established several interesting identities, some of which are listed below:

$$
\begin{aligned}
&\sum_{k=0}^{\nu} \mathfrak{l}_{\nu} = \mathfrak{l}_{\nu+2} - (\nu+2), \quad \nu \in \mathbb{N}_0, \\
&\mathfrak{l}_{\nu}^2 - \mathfrak{l}_{\nu-r}\mathfrak{l}_{\nu+r} = \mathfrak{l}_{\nu-r} + \mathfrak{l}_{\nu+r} - 2\mathfrak{l}_{\nu} - (-1)^{\nu-r}(\mathfrak{l}_{\nu-1}+1)^2, \quad \nu > r, r \geq 1 \left(\text{Catalan}'\text{s identity}\right), \\
&\mathfrak{l}_{\nu}^2 - \mathfrak{l}_{\nu-1}\mathfrak{l}_{\nu+1} = \mathfrak{l}_{\nu-1} + \mathfrak{l}_{\nu-2} + 4(-1)^{\nu}, \quad \nu \geq 2 \left(\text{Cassini}'\text{s identity}\right), \\
&\mathfrak{l}_{\nu} = 2\left(\frac{\mathfrak{x}^{\nu+1} - \mathfrak{y}^{\nu+1}}{\mathfrak{x} - \mathfrak{y}}\right) - 1, \nu \geq 0, \quad \text{where } \mathfrak{x} = \frac{1+\sqrt{5}}{2}, \mathfrak{y} = \frac{1-\sqrt{5}}{2}.
\end{aligned} \tag{7}
$$

Besides, Alp and Koçer [15] also established interesting relationships between Fibonacci, Lucus, and Leonardo numbers. Vieira et al. [16] worked in the matrix form of the Leonardo numbers and established several interesting relations. Moreover, Shannon [17] also worked on the extension and generalization of the Leonardo numbers.

Inspired by the above studies, we define an infinite matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ involving Leonardo numbers and construct sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$. We study their topological and inclusion properties and obtain Schauder bases of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$. In Section 3, α-, β-, and γ-duals of these new spaces are determined. In Section 4, matrix classes from the space $\mathfrak{Z} \in \{\ell_p(\mathfrak{L}), c_0(\mathfrak{L}), c(\mathfrak{L}), \ell_\infty(\mathfrak{L})\}$ to any one of the spaces ℓ_1, c_0,c, and ℓ_∞ are characterized. In Sections 5 and 6, we introduce new sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ under a definite functional ρ and weighted sequence of positive reals r and discuss certain geometric and topological properties of $(\ell_p(\mathfrak{L}, r))_\rho$ and the eigenvalue distribution of mappings ideally generated by these spaces and s-numbers are presented.

2. Leonardo Sequence Spaces

Define an infinite matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ by

$$
\mathfrak{l}_{\nu,k} = \begin{cases} \dfrac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)}, & 0 \leq k \leq \nu, \\ 0, & k > \nu, \end{cases} \tag{8}
$$

for all $\nu, k \in \mathbb{N}_0$. Equivalently,

$$
\mathfrak{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & \cdots \\ \frac{1}{10} & \frac{1}{10} & \frac{3}{10} & \frac{5}{10} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{9}
$$

The inverse of the matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ is given by the matrix $\mathfrak{L}^{-1} = (\mathfrak{l}_{\nu,k}^{-1})$ defined by

$$
\mathfrak{l}_{\nu,k}^{-1} = \begin{cases} (-1)^{\nu-k}\dfrac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_{\nu}}, & \nu \leq k \leq \nu+1, \\ 0, & k > \nu, \end{cases} \tag{10}
$$

for all $\nu, k \in \mathbb{N}_0$.

Now, we define the following sequence spaces:

$$
\begin{aligned}
\ell_p(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \sum_{\nu=0}^{\infty} \left| \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right|^p < \infty \right\}, \\
c_0(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k = 0 \right\}, \\
c(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \text{ exists} \right\}, \\
\ell_\infty(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right| < \infty \right\},
\end{aligned} \tag{11}
$$

where the sequence $\mathfrak{w} = (\mathfrak{w}_k)$ defined by

$$
\mathfrak{w}_\nu = (\mathfrak{L}\mathfrak{z})_\nu = \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k, \tag{12}
$$

for each $\nu \in \mathbb{N}_0$, which is known as the $\mathfrak{L}$-transform of the sequence $\mathfrak{z} = (\mathfrak{z}_k)$. In what follows, the sequences $\mathfrak{z}$ and $\mathfrak{w}$ are related by (12). It is trivial that the above defined sequence spaces can be expressed in the form $\mathfrak{Z}(\mathfrak{L}) = (\mathfrak{Z})_{\mathfrak{L}}$, where $\mathfrak{Z}$ represents any one of the spaces ℓ_p,c_0,c, and ℓ_∞. That is, $\mathfrak{Z}(\mathfrak{L})$ is the domain of the matrix $\mathfrak{L}$ in the sequence space $\mathfrak{Z}$.

We observe by the definition of the matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ that $\sum_{k=0}^{\nu}\mathfrak{l}_{\nu,k} = 1$. That is, $\sup_{\nu \in \mathbb{N}_0} \sum_{k=0}^{\nu}\mathfrak{l}_{\nu,k} < \infty$ and $\lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\infty}\mathfrak{l}_{\nu,k} = 1$.

Additionally, $\lim_{\nu\longrightarrow\infty}\mathfrak{l}_{\nu,k}=0$, for each $k\in\mathbb{N}_0$. Thus, we conclude that the matrix $\mathfrak{L}$ is regular.

Theorem 1. *The following inclusion relations hold:*

(i) $\mathfrak{Z}\subset\mathfrak{Z}(\mathfrak{L})$, *where* $\mathfrak{Z}$ *is any one of the spaces* ℓ_p,c_0,c *or* ℓ_∞

(ii) $\ell_p(\mathfrak{L})\subset c_0(\mathfrak{L})\subset c(\mathfrak{L})\subset\ell_\infty(\mathfrak{L})$

(iii) $\ell_p(\mathfrak{L})\subset\ell_q(\mathfrak{L})$ *for* $1\le p<q$

Proof.

(i) The inclusion part is trivial. Assume that $\mathfrak{Z}\coloneqq c$ and consider the sequence $\mathfrak{g}=(1,0,1,0,\cdots)$. We observe that $\mathfrak{g}\notin c$. However

$$(\mathfrak{L}\mathfrak{g})_\nu=\sum_{k=0}^{\nu}\frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2}-(\nu+2)}\mathfrak{g}_k=\frac{\mathfrak{l}_0+\mathfrak{l}_2+\cdots+\mathfrak{l}_\nu}{\mathfrak{l}_{\nu+2}-(\nu+2)}\quad(\nu\in\mathbb{N}_0),\tag{13}$$

which converges. Thus, $\mathfrak{g}\in c(\mathfrak{L})\setminus c$. In the similar manner strictness can be established for the other inclusions.

(ii) It is known that the matrix $\mathfrak{L}$ is regular and the inclusion $\ell_p\subset c_0\subset c\subset\ell_\infty$ holds. These imply that the inclusion part holds. Now, consider the sequence $\mathfrak{h}=(1,1,1,1,\cdots)$. Then, $(\mathfrak{L}\mathfrak{h})_\nu=\sum_{k=0}^{\nu}(\mathfrak{l}_k/\mathfrak{l}_{\nu+2}-(\nu+2))\mathfrak{h}_k=1$, for all $\nu\in\mathbb{N}_0$. Thus, $\mathfrak{L}\mathfrak{h}\in c\setminus c_0$. That is, $\mathfrak{h}\in c(\mathfrak{L})\setminus c_0(\mathfrak{L})$. This verifies the strictness of the inclusion $c_0(\mathfrak{L})\subset c(\mathfrak{L})$. In the similar fashion, strictness of other inclusions can be established.

(iii) Assume that $1\le p<q$. Since $\mathfrak{L}$ is regular and the inclusion $\ell_p\subset\ell_q$ holds, therefore the desired inclusion holds. To prove the strictness part, we consider a sequence $\mathfrak{g}=(\mathfrak{g}_k)\in\ell_q\setminus\ell_p$. Define a sequence $\mathfrak{h}=(\mathfrak{h}_k)$ by $\mathfrak{h}_k=((\mathfrak{g}_k(\mathfrak{l}_{k+2}-(k+2))-\mathfrak{g}_{k-1}(\mathfrak{l}_{k+1}-(k+1)))/\mathfrak{l}_k),k\in\mathbb{N}_0$. Then, we get

$$\begin{aligned}(\mathfrak{L}\mathfrak{h})_\nu&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}\mathfrak{l}_k\mathfrak{h}_k\\&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}\{\mathfrak{g}_k(\mathfrak{l}_{k+2}-(k+2))\\&\quad-\mathfrak{g}_{k-1}(\mathfrak{l}_{k+1}-(k+1))\}\\&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\mathfrak{g}_\nu(\mathfrak{l}_{\nu+2}-(\nu+2))\\&=\mathfrak{g}_\nu,\end{aligned}\tag{14}$$

for each $\nu\in\mathbb{N}_0$, where the terms with negative subscripts are considered to be zero. Thus, we deduce that $\mathfrak{L}\mathfrak{h}=\mathfrak{g}\in\ell_q\setminus\ell_p$ which implies $\mathfrak{h}\in\ell_q(\mathfrak{L})\setminus\ell_p(\mathfrak{L})$. Thus there exists at least one sequence that is contained in $\ell_q(\mathfrak{L})$ but not in $\ell_p(\mathfrak{L})$. Hence, the desired inclusion is strict. This completes the proof. □

Theorem 2. *We have the following results:*

(i) *The sequence spaces* $\ell_\infty(\mathfrak{L}),c_0(\mathfrak{L})$, *and* $c(\mathfrak{L})$ *are BK -spaces equipped with the bounded norm* $\|\mathfrak{z}\|_{\ell_\infty}=\sup_{k\in\mathbb{N}_0}|(\mathfrak{L}\mathfrak{z})_k|$

(ii) *The sequence space* $\ell_p(\mathfrak{L})$ *is a BK-space equipped with the norm* $\|\mathfrak{z}\|_{\ell_p}=(\sum_{\nu=0}^{\infty}|(\mathfrak{L}\mathfrak{z})_\nu|^p)^{1/p}$

Proof. The proof is a routine exercise and so omitted. □

Theorem 3. $\mathfrak{Z}(\mathfrak{L})\cong\mathfrak{Z}$, *where* $\mathfrak{Z}$ *is any one of the spaces* ℓ_p,c_0, *c, or* ℓ_∞.

Proof. We present the proof for the space ℓ_p. Define the mapping $\varphi:\ell_p(\mathfrak{L})\longrightarrow\ell_p$ by $\mathfrak{w}=\varphi\mathfrak{z}=\mathfrak{L}\mathfrak{z}$, for all $\mathfrak{z}\in\ell_p(\mathfrak{L})$. We observe that the mapping φ is linear and injective.

In view of the relation (12), we write

$$\mathfrak{z}_k=\sum_{j=k-1}^{k}(-1)^{k-j}\frac{\mathfrak{l}_{j+2}-(j+2)}{\mathfrak{l}_k}\mathfrak{w}_j⊠,\tag{15}$$

for each $k\in\mathbb{N}_0$ and $\mathfrak{w}=(\mathfrak{w}_k)\in\ell_p$. Then,

$$\begin{aligned}\|\mathfrak{z}\|_{\ell_p(\mathfrak{L})}^p&=\sum_{\nu=0}^{\infty}\left|\sum_{j=0}^{k}\frac{\mathfrak{l}_j}{\mathfrak{l}_{k+2}-(k+2)}\mathfrak{z}_j\right|^p\\&=\sum_{\nu=0}^{\infty}\left|\sum_{j=0}^{k}\frac{\mathfrak{l}_j}{\mathfrak{l}_{k+2}-(k+2)}\right.\\&\quad\left.\cdot\left(\sum_{u=j-1}^{j}(-1)^{j-u}\frac{\mathfrak{l}_{u+2}-(u+2)}{\mathfrak{l}_j}\mathfrak{w}_u\right)\right|^p\\&=\sum_{\nu=0}^{\infty}|\mathfrak{w}|^p=\|\mathfrak{w}\|_{\ell_p}^p.\end{aligned}\tag{16}$$

Thus, $\|\mathfrak{z}\|_{\ell_p(\mathfrak{L})}=\|\mathfrak{w}\|_{\ell_p}<\infty$. Thus, $\mathfrak{z}\in\ell_p(\mathfrak{L})$, and this implies that φ is surjective and norm preserving. Thus, $\ell_p(\mathfrak{L})\cong\ell_p$. In the similar manner, we can prove the existence of isomorphism between other given spaces. This completes the proof. □

Let us consider the following sequences:

$$\mathfrak{g}=\left(1,1,-\frac{2}{3},0,0,\cdots\right),\mathfrak{h}=\left(1,-3,\frac{2}{3},0,0,\cdots\right).\tag{17}$$

Observe that $\mathfrak{L}\mathfrak{g}=(1,1,0,0,0,\cdots)$ and $\mathfrak{L}\mathfrak{h}=(1,-1,0,0,0,\cdots)$. Since $\mathfrak{L}$ is linear, so $\mathfrak{L}(\mathfrak{g}+\mathfrak{h})=(2,0,0,0,\cdots)$ and

$\mathfrak{L}(\mathfrak{g}-\mathfrak{h})=(0,2,0,0,\cdots)$. With some elementary calculation, we deduce that

$$\begin{aligned}\|\mathfrak{g}+\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}+\|\mathfrak{g}-\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}&=8\neq 2^{2(1+(1/p))}\\&=2\left(\|\mathfrak{g}\|^2_{\ell_p(\mathfrak{L})}+\|\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}\right).\end{aligned} \tag{18}$$

Thus, we realize that the norm $\|\cdot\|_{\ell_p(\mathfrak{L})}$ violates the parallelogram identity for $p\neq 2$. This immediately allows us to write the following result.

Theorem 4. *The sequence space $\ell_p(\mathfrak{L})$ is not a Hilbert space for $p\neq 2$.*

Proof. The proof is immediate from the above discussion. □

We are well awarded that a matrix domain $\mathfrak{Z}_{\mathfrak{A}}$, where $\mathfrak{A}$ is a triangle, has a basis, if and only if, $\mathfrak{Z}$ has a basis (cf. [18]). Thus, in the light of Theorem 3, we have the following result:

Theorem 5. *Define the sequence $\mathfrak{b}^{(k)}=(\mathfrak{b}_\nu^{(k)})$ by*

$$\mathfrak{b}_\nu^{(k)}=\begin{cases}(-1)^{\nu-k}\dfrac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_\nu}, & \nu-1\leq k\leq \nu+1,\\ 0, \quad k>\nu\end{cases} \tag{19}$$

for each fixed $k\in\mathbb{N}_0$. Then,

(i) *The sequence $(\mathfrak{b}^{(k)})_{k\in\mathbb{N}_0}$ is the Schauder basis of the sequence spaces $\ell_p(\mathfrak{L})$ and $c_0(\mathfrak{L})$, and every $\mathfrak{z}$ in $\ell_p(\mathfrak{L})$ or $c_0(\mathfrak{L})$ is expressed uniquely in the form $\mathfrak{z}=\sum_{k=0}^{\infty}\mathfrak{b}^{(k)}\mathfrak{w}_k$, where $\mathfrak{w}=(\mathfrak{w}_k)$ is the $\mathfrak{L}$-transform of the sequence $\mathfrak{z}=(\mathfrak{z}_k)$*

(ii) *The sequence $(\mathfrak{e},\mathfrak{b}^{(0)},\mathfrak{b}^{(1)},\mathfrak{b}^{(2)},\cdots)$ is the Schauder basis of the sequence space $c(\mathfrak{L})$, and every $\mathfrak{z}$ in $c(\mathfrak{L})$ is expressed uniquely in the form $\mathfrak{z}=\tau\mathfrak{e}+\sum_{k=0}^{\infty}(\mathfrak{w}_k-\tau)\mathfrak{b}^{(k)}$, where $\tau=\lim_{\nu\longrightarrow\infty}\mathfrak{w}_\nu$ and $\mathfrak{e}$ is the unit sequence*

(iii) *The sequence space $\ell_\infty(\mathfrak{L})$ has no Schauder basis*

Corollary 6. *The sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$ are separable spaces.*

3. α-, β-, and γ-Duals

In this section, we obtain the α-, β-, and γ-duals of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$. Before proceeding, we recall the definitions of α-, β-, and γ-duals. Define the multiplier sequence space $\mathscr{M}(\mathfrak{Z},\mathfrak{U})$ by

$$\mathscr{M}(\mathfrak{Z},\mathfrak{U}):=\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\mathfrak{d}\mathfrak{z}=(\mathfrak{d}_k\mathfrak{z}_k)\in\mathfrak{U},\forall\mathfrak{z}=(\mathfrak{z}_k)\in\mathfrak{U}\}. \tag{20}$$

In particular, if $\mathfrak{U}$ is ℓ_1,cs or bs, then, the sets

$$\mathfrak{Z}^\alpha=\mathscr{M}(\mathfrak{Z},\ell_1),\ \mathfrak{Z}^\beta=\mathscr{M}(\mathfrak{Z},cs),\ \mathfrak{Z}^\gamma=\mathscr{M}(\mathfrak{Z},bs) \tag{21}$$

are, respectively, termed as α-, β-, and γ-dual of the sequence space $\mathfrak{Z}$.

We present Lemma 7 which is essential to compute the dual spaces. In what follows, we denote the collection of all finite subsets of $\mathbb{N}_0$ by $\mathscr{N}$ and $q=p/p-1$.

Lemma 7 (see [19]). *The following statements hold:*

(i) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(\ell_1,\ell_1)$, if and only if, $\sup_{k\in\mathbb{N}_0}\sum_{\nu=0}^{\infty}|\mathfrak{a}_{\nu,k}|<\infty$*

(ii) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(\ell_p,\ell_1)$, if and only if*

$$\sup_{K\in\mathscr{N}}\sum_{\nu=0}^{\infty}\left|\sum_{k\in K}\mathfrak{a}_{\nu,k}\right|^q<\infty \tag{22}$$

(iii) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(c_0,\ell_1)=(c,\ell_1)=(\ell_\infty,\ell_1)$, if and only if, (22) holds with $q=1$*

(iv) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(\ell_1,\ell_\infty)$, if and only if, $\sup_{\nu,k\in\mathbb{N}_0}|\mathfrak{a}_{\nu,k}|<\infty$*

(v) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(\ell_p,\ell_\infty)$, if and only if,*

$$\sup_{\nu\in\mathbb{N}_0}\sum_{k=0}^{\infty}|\mathfrak{a}_{\nu,k}|^q<\infty \tag{23}$$

(vi) *$\mathfrak{A}=(\mathfrak{a}_{\nu,k})\in(c_0,\ell_\infty)=(c,\ell_\infty)=(\ell_\infty,\ell_\infty)$, if and only if (23) holds with $q=1$*

Theorem 8. *Consider the following sets:*

$$\begin{aligned}\mathfrak{D}_1^{(q)}&:=\left\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\sum_{k=0}^{\infty}\left|\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_k}\mathfrak{d}_k\right|^q<\infty\right\},\\ \mathfrak{D}_1&:=\left\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\sup_{k\in\mathbb{N}_0}\left|\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_k}\mathfrak{d}_k\right|<\infty\right\}.\end{aligned} \tag{24}$$

Then, $[\ell_1(\mathfrak{L})]^\alpha:=\mathfrak{D}_1$, $[\ell_p(\mathfrak{L})]^\alpha:=\mathfrak{D}_1^{(q)}$, and $[c_0(\mathfrak{L})]^\alpha=[c(\mathfrak{L})]^\alpha=[\ell_\infty(\mathfrak{L})]^\alpha:=\mathfrak{D}_1^{(1)}$, where $1<p<\infty$.

Proof. We observe that

$$\mathfrak{d}_\nu\mathfrak{z}_\nu=\sum_{k=\nu-1}^{\nu}(-1)^{\nu-k}\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_\nu}\mathfrak{d}_\nu\mathfrak{w}_k=(\mathfrak{B}\mathfrak{w})_\nu, \tag{25}$$

for each $\nu \in \mathbb{N}_0$ and $\mathfrak{d} = (\mathfrak{d}_k) \in \omega$, where the matrix $\mathfrak{B} = (\mathfrak{b}_{\nu,k})$ is defined by

$$\mathfrak{b}_{\nu,k} = \begin{cases} (-1)^{\nu-k} \dfrac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu, & \nu - 1 \leq k \leq \nu, \\ 0, & k > \nu, \end{cases} \tag{26}$$

for all $\nu, k \in \mathbb{N}_0$. We notice that the sequence $\mathfrak{d}\mathfrak{z} = (\mathfrak{d}_\nu \mathfrak{z}_\nu) \in \ell_1$ whenever $\mathfrak{z} = (\mathfrak{z}_\nu) \in c_0(\mathfrak{L})$, if and only if, the sequence $\mathfrak{B}\mathfrak{w} \in \ell_1$ whenever the sequence $\mathfrak{w} = (\mathfrak{w}_\nu) \in c_0$. We realize that the sequence $\mathfrak{d} = (\mathfrak{d}_\nu) \in [c_0(\mathfrak{L})]^\alpha$, if and only if, $\mathfrak{B} \in (c_0, \ell_1)$. Thus, by employing Part (iii) of Lemma 7, we get that

$$\sup_{\mathfrak{B} \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{\nu \in \mathfrak{B}} \mathfrak{b}_{\nu,k} \right| < \infty. \tag{27}$$

Moreover, for any $\mathfrak{B} \in \mathcal{N}$, (27) holds, if and only if,

$$\sum_{k=0}^{\infty} \left| \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k \right| < \infty. \tag{28}$$

Consequently, $[c_0(\mathfrak{L})]^\alpha \coloneqq \mathfrak{D}_1^{(1)}$.

In the similar manner, α-dual of the other sequence spaces can be obtained by employing Part (i), Part (ii), and Part (iii) of Lemma 7. □

Theorem 9. *Consider the following sets:*

$$\begin{aligned} \mathfrak{D}_2^{(q)} &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \sum_{k=0}^{\infty} \left| \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right|^q < \infty \right\}, \\ \mathfrak{D}_2 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \sup_{k \in \mathbb{N}_0} \left| \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right| < \infty \right\}, \\ \mathfrak{D}_3 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \lim_{k \longrightarrow \infty} \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k = 0 \right\}, \\ \mathfrak{D}_4 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \lim_{k \longrightarrow \infty} \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k \text{ exists} \right\}, \end{aligned} \tag{29}$$

where $\Delta(\mathfrak{d}_k/\mathfrak{l}_k) = (\mathfrak{d}_k/\mathfrak{l}_k) - (\mathfrak{d}_{k+1}/\mathfrak{l}_{k+1})$. *Then,* $[\ell_1(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2$, $[\ell_p(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(q)}$, $[c_0(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(1)}$, $[c(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_2^{(1)} \cap \mathfrak{D}_4$, *and* $[\ell_\infty(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_2^{(1)} \cap \mathfrak{D}_3$, *where* $1 < p < \infty$.

Proof. Let $\mathfrak{d} = (\mathfrak{d}_k) \in \omega$. Then, we have

$$\begin{aligned} \sum_{k=0}^{\nu} \mathfrak{d}_k \mathfrak{z}_k &= \sum_{k=0}^{\nu} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\mathfrak{l}_{j+2} - (j+2)}{\mathfrak{l}_k} \mathfrak{w}_j \mathfrak{d}_k \\ &= \sum_{k=0}^{\nu-1} \left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k} - \frac{\mathfrak{d}_{k+1}}{\mathfrak{l}_{k+1}} \right)(\mathfrak{l}_{k+2} - (k+2))\mathfrak{w}_k + \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu \mathfrak{w}_\nu \\ &= \sum_{k=0}^{\nu-1} \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2))\mathfrak{w}_k + \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu \mathfrak{w}_\nu, \end{aligned} \tag{30}$$

for each $\nu \in \mathbb{N}_0$. By employing Theorem 2 and Corollary 1 of Malkowsky and Savas [20], we get

$$\begin{aligned} [\ell_1(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_\infty \text{ and} \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [\ell_p(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_q, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [c_0(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [c(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in c \right\}, \\ [\ell_\infty(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in c_0 \right\}. \end{aligned} \tag{31}$$

This completes the proof. □

Theorem 10. *We have the following results:*

(i) $[\ell_1(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2$.

(ii) $[\ell_p(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(q)}$, *where* $1 < p < \infty$

(iii) $[c_0(\mathfrak{L})]^\gamma = [c(\mathfrak{L})]^\gamma = [\ell_\infty(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(1)}$

Proof. It can be obtained by using relation (30) and Parts (iv), (v), and (vi) of Lemma 7, respectively. □

4. Characterization of Matrix Classes

Let $\mathfrak{A} = (\mathfrak{a}_{\nu,k})$ be an infinite matrix over the field of complex numbers. Denote

$$\tilde{\mathfrak{A}}_\nu \coloneqq \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{A}_\nu = \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{a}_{\nu,k} \right)_{k=0}^{\infty}, \quad \text{for each } \nu \in \mathbb{N}_0, \tag{32}$$

$$\mathfrak{m}_{v,k} := (\mathfrak{l}_{k+2} - (k+2))\Delta\left(\frac{\mathfrak{a}_{v,k}}{\mathfrak{l}_k}\right), \quad \text{for all } v, k \in \mathbb{N}_0. \tag{33}$$

Now, we state the following result:

Lemma 11. *Let $\mathfrak{Z}$ denote either of the spaces ℓ_p or c_0, $1 \le p \le \infty$. Then, $\mathfrak{A} \in (\mathfrak{Z}_{\mathfrak{L}}, \mathfrak{U})$, if and only if, $\tilde{\mathfrak{A}} \in \mathscr{M}(\mathfrak{Z}, c_0)$, for each $v \in \mathbb{N}_0$, and $\mathfrak{M} \in (\mathfrak{Z}, \mathfrak{U})$, where $\tilde{\mathfrak{A}}$ and $\mathfrak{M} = (\mathfrak{m}_{v,k})$ are defined in (32) and (33), respectively.*

Proof. It follows straightly from ([20], Theorem 3]). □

Lemma 12. $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$, *if and only if,*

$$\tilde{\mathfrak{A}}_v \in c, \text{ that is } \lim_{k\longrightarrow\infty} \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} = \alpha_v (v \in \mathfrak{N}_0), \tag{34}$$

$$\mathfrak{M} \in (c_0, \mathfrak{U}) = (\ell_\infty, \mathfrak{U}), \tag{35}$$

$$(\alpha_v)_{v=0}^{\infty} \in \mathfrak{U}. \tag{36}$$

Proof. Assume that $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$. Then, $\mathfrak{A}_v \in [c(\mathfrak{L})]^\beta$ which immediately shows the necessity of condition (35). Also, it is known that $((c)_{\mathfrak{L}}, \mathfrak{U}) \subset ((c_0)_{\mathfrak{L}}, \mathfrak{U})$. Hence, by employing Lemma 11, we get that $\mathfrak{M} \in (c_0, \mathfrak{U})$. Thus, in the light of (30), we get that

$$\sum_{k=0}^{\infty} \mathfrak{a}_{v,k}\mathfrak{z}_k = \sum_{k=0}^{\infty} \mathfrak{m}_{v,k}\mathfrak{w}_k + \mathfrak{z}\alpha_v. \tag{37}$$

It is clear from the assumptions that $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$ and $\mathfrak{M}\mathfrak{w} \in \mathfrak{U}$. These together yield $(\alpha_v)_{v=0}^{\infty} \in \mathfrak{U}$.

Conversely, we assume that conditions (34), (35), and (36) hold. We realize that conditions (34) and (35) together imply that $\mathfrak{A}_v \in [c(\mathfrak{L})]^\beta$. Again condition (34) implies (37). By condition (35), $\mathfrak{M}\mathfrak{w} \in \mathfrak{U}$, for all $\mathfrak{w} \in c$. This together with condition (36) implies that $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$, for all $\mathfrak{z} \in c(\mathfrak{L})$. This proves that $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$. □

Now, using Lemmas 11 and 12 together with the properties $\mathscr{M}(\ell_p, c_0) = \mathscr{M}(c_0, c_0) = \ell_\infty (1 \le p < \infty)$,$\mathscr{M}(c, c) = c$, and $\mathscr{M}(\ell_\infty, c_0) = c_0$, we deduce the following results:

Corollary 13. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), \ell_\infty)$, *if and only if*

$$\sup_{k\in\mathbb{N}_0} \left| \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} \right| < \infty, \tag{38}$$

$$\sup_{v,k\in\mathbb{N}_0} \left|\mathfrak{m}_{v,k}\right| < \infty \tag{39}$$

(ii) Let $1 < p < \infty$. Then $\mathfrak{A} \in (\ell_p(\mathfrak{L}), \ell_\infty)$, if and only if, (38) holds, and

$$\sup_{v\in\mathbb{N}_0} \sum_{k=0}^{\infty} \left|\mathfrak{m}_{v,k}\right|^q < \infty \tag{40}$$

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), \ell_\infty)$, if and only if, (38) holds, and (40) holds with $q = 1$

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), \ell_\infty)$, if and only if, (34) and (40) hold with $q = 1$, and

$$\sup_{v\in\mathbb{N}_0} |\alpha_v| < \infty, \tag{41}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), \ell_\infty)$, if and only if,

$$\lim_{k\longrightarrow\infty} \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} = 0, \quad \text{for all } v \in \mathbb{N}_0, \tag{42}$$

and (40) holds with $q = 1$

Corollary 14. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), c_0)$, if and only if, (38) and (39) hold, and

$$\lim_{v\longrightarrow\infty} \mathfrak{m}_{v,k} = 0, \quad \text{for all } k \in \mathbb{N}_0, \tag{43}$$

also holds

(ii) Let $1 < p < \infty$. Then, $\mathfrak{A} \in (\ell_p(\mathfrak{L}), c_0)$, if and only if, (38), (40), and (43) hold

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), c_0)$, if and only if, (38) and (40) hold with $q = 1$, and (43) also holds

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), c_0)$, if and only if, (34), (40) with $q = 1$ and (43) hold, and

$$\lim_{v\longrightarrow\infty} \alpha_v = 0, \tag{44}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), c_0)$, if and only if, (42) holds, and

$$\lim_{v\longrightarrow\infty} \sum_{k=0}^{\infty} \left|\mathfrak{m}_{v,k}\right| = 0, \tag{45}$$

also holds

Corollary 15. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), c)$, if and only if, (38), (39), and

$$\lim_{v\longrightarrow\infty} \mathfrak{m}_{v,k} \text{ exists}, \quad \text{for all } k \in \mathbb{N}_0, \tag{46}$$

also holds

(ii) Let $1 < p < \infty$. Then $\mathfrak{A} \in (\ell_p(\mathfrak{L}), c)$, if and only if, (38), (40) and (46) holds

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), c)$, *if and only if, (38) holds, (40) with* $q = 1$ *and (46) hold*

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), c)$, *if and only if, (34), (40) with* $q = 1$ *and (46) hold, and*

$$\lim_{v \longrightarrow \infty} \alpha_v \text{ exists} \tag{47}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), c)$, *if and only if, (42) and (46) hold, and*

$$\sum_{k=0}^{\infty} |\mathfrak{m}_{v,k}| \text{ converges uniformly in } v \tag{48}$$

also holds

Corollary 16. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds and*

$$\sup_{v \in \mathbb{N}_0} \sum_{v=0}^{\infty} |\mathfrak{m}_{v,k}| < \infty, \tag{49}$$

also holds

(ii) Let $1 < p < \infty$. *Then* $\mathfrak{A} \in (\ell_p(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds, and*

$$\sup_{\mathfrak{B} \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{v \in \mathfrak{B}} \mathfrak{m}_{v,k} \right|^q < \infty, \tag{50}$$

also holds

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds, and (50) holds with* $q = 1$

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), \ell_1)$, *if and only if, (34) and (50) hold with* $q = 1$, *and*

$$\sum_{v=0}^{\infty} |\alpha_v| < \infty, \tag{51}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), \ell_1)$, *if and only if, (42) and (50) hold with* $q = 1$

5. Mapping Ideal

In this section, we construct s-type mapping ideals on Leonardo sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$. By $\mathcal{B}$, we denote the class of all bounded linear mappings between any two Banach spaces. In particular, $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ denote the class of all bounded linear mappings acting from Banach space $\mathfrak{X}$ to Banach space $\mathfrak{Y}$. We note down certain notations and definitions before moving to our results:

Definition 17 (see [21, 22]). Let ω^+ represent the set of non-negative real sequences. Then, s-number is a mapping $s : \mathcal{B}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow \omega^+$ that satisfies the following settings:

(i) $\|\phi\| = s_0(\phi) \geq s_1(\phi) \geq s_2(\phi) \geq \cdots \geq 0$, for each $\phi \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$

(ii) $s_{a+b-1}(\phi + \psi) \leq s_a(\phi) + s_b(\psi)$, for each $\phi, \psi \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and $a, b \in \mathbb{N}_0$

(iii) $s_a(\phi\theta\psi) \leq \|\phi\| s_a(\theta) \|\psi\|$, for all $\phi \in \mathcal{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and $\psi \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are any two Banach sequence spaces

(iv) Let $\phi \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and $\nu \in \mathbb{C}$. Then, $s_a(\nu\phi) = |\nu| s_a(\phi)$

(v) If rank $(\phi) \leq a$, then $s_a(\phi) = 0$ for all $\phi \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$

(vi) $s_v(\mathfrak{I}_a) = 0$ for $v \geq a$ or $s_v(\mathfrak{I}_a) = 1$ for $v < a$, where $\mathfrak{I}_a$ denotes the identity mapping on the a-dimensional Hilbert space ℓ_2^a

In an assorted illustration of s-numbers, we intimate the next settings:

(1) The a-th Kolmogorov number, denoted by $d_a(X)$, is defined as

$$d_a(X) = \inf_{\dim J \leq a} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\| \tag{52}$$

(2) The a-th approximation number, denoted by $\alpha_a(X)$, is defined as

$$\alpha_a(X) = \inf \{\|X - Y\| : Y \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}), \text{rank } (Y) \leq a\} \tag{53}$$

Definition 18 (see [23]). Let $\mathcal{W} \subset \mathcal{B}$ and denote $\mathcal{W}(\mathfrak{X}, \mathfrak{Y}) = \mathcal{W} \cap \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then, $\mathcal{W}$ is known as a mapping ideal if it satisfies the following settings:

(i) $\mathfrak{I}_{\mathfrak{D}} \in \mathcal{W}$, where $\mathfrak{D}$ is a Banach sequence space of one dimension

(ii) $\mathcal{W}(\mathfrak{X}, \mathfrak{Y})$ is a linear space over $\mathbb{C}$

(iii) If $\psi \in \mathcal{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and $\phi \in \mathcal{B}(\mathfrak{Y}_0, \mathfrak{Y})$, then $\phi\theta\psi \in \mathcal{B}(\mathfrak{X}_0, \mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are any two normed spaces

Definition 19 (see [24]). A prequasi norm on the ideal $\mathcal{W}$ is a mapping $\mu : \mathcal{W} \longrightarrow \omega^+$ satisfying the following settings:

(i) $\mu(\phi) \geq 0$ and $\mu(\phi) = 0$ if and only if $\phi = 0$, for all $\phi \in \mathcal{W}(\mathfrak{X}, \mathfrak{Y})$

(ii) There exists $m_0 \geq 1$ such that $\mu(\varsigma\phi) \leq m_0|\varsigma|\mu(\phi)$, for all $\phi \in \mathscr{W}(\mathfrak{X}, \mathfrak{Y})$

(iii) There exists $n_0 \geq 1$ such that $\mu(\phi + \psi) \leq n_0(\mu(\phi) + \mu(\psi))$, for all $\phi, \psi \in \mathscr{W}(\mathfrak{X}, \mathfrak{Y})$

(iv) There exists $p_0 \geq 1$ such that $\mu(\phi\theta\psi) \leq p_0\|\phi\|\mu(\theta)\|\psi\|$ whenever $\psi \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\phi \in \mathscr{B}(\mathfrak{Y}_0, \mathfrak{Y})$

Definition 20 (see [24]). The subspace $\mathfrak{Z} \subset \omega$ is said to be a private sequence space (or in short pss) if it satisfies the following settings:

(i) $\mathfrak{e}_\nu \in \mathfrak{Z}$, for each $\nu \in \mathbb{N}_0$, where $\mathfrak{e}_\nu$ denotes the sequence with 1 in the ν^{th} position and 0 elsewhere

(ii) If $\mathfrak{g} = (\mathfrak{g}_\nu) \in \omega, |\mathfrak{h}| = (|\mathfrak{h}_\nu|) \in \mathfrak{Z}$ and $|\mathfrak{g}_\nu| \leq |\mathfrak{h}_\nu|$, for $\nu \in \mathbb{N}_0$, then $|\mathfrak{g}| \in \mathfrak{Z}$

(iii) $(|g_{[\nu/2]}|) \in \mathfrak{Z}$ whenever $(|g_\nu|) \in \mathfrak{Z}$, where $[\nu/2]$ denotes the integral part of $\nu/2$

Definition 21 (see [24]). A subspace of the pss is said to be a premodular pss, if there is a function $\upsilon : \mathfrak{Z} \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) For every $j \in \mathfrak{Z}$, $j = 0 \Leftrightarrow \upsilon(|j|) = 0$, and $\upsilon(j) \geq 0$, with 0 is the zero vector of $\mathfrak{Z}$

(ii) If $j \in \mathfrak{Z}$ and $\rho \in \mathbb{C}$, then there are $E_0 \geq 1$ with $\upsilon(\rho j) \leq |\rho| E_0 \upsilon(j)$

(iii) $\upsilon(h + j) \leq G_0(\upsilon(h) + \upsilon(j))$ holds for some $G_0 \geq 1$, with $f, g \in \mathfrak{Z}$

(iv) Assume $x \in \mathbb{N}_0$, $|h_x| \leq |j_x|$, we have $\upsilon((|h_x|)) \leq \upsilon((|j_x|))$

(v) The inequality, $\upsilon((|j_x|)) \leq \upsilon((|j_{[x/2]}|)) \leq D_0 \upsilon((|j_x|))$ verifies, for $D_0 \geq 1$

(vi) $\bar{\mathfrak{C}} = \mathfrak{Z}_\upsilon$, where $\bar{\mathfrak{C}}$ denotes the closure of the space of all sequences with infinite zero coordinates

(vii) We have $\eta > 0$ such that $\upsilon(\rho, 0, 0, 0, \cdots) \geq \eta |\rho| \upsilon(1, 0, 0, 0, \cdots)$, with $\rho \in \mathbb{C}$

Definition 22 (see [24]). The *pss* $\mathfrak{Z}_\upsilon$ is said to be a prequasi normed *pss*, if υ confirms the setups (i)-(iii) of Definition 21. If $\mathfrak{Z}$ is complete equipped with υ, then $\mathfrak{Z}_\upsilon$ is called a prequasi Banach *pss*.

Lemma 23 (see [24]). *Every premodular pss is a prequasi normed pss.*

In what follows, we will use the following inequality:

$$|\mathfrak{g} + \mathfrak{h}|^p \leq 2^{p-1}\left(|\mathfrak{g}|^p + |\mathfrak{h}|^p\right), \tag{54}$$

where $1 \leq p < \infty$ and $\mathfrak{g}, \mathfrak{h} \in \mathbb{C}$. For detailed studies concerning s-numbers and mapping ideals, we refer to [23–28].

Definition 24. We define the following sequence spaces:

$$\begin{aligned} &\left(\ell_p(\mathfrak{L}, r)\right)_{\rho_1} \\ &\quad := \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \rho_1(\mathfrak{z}) = \sum_{\nu=0}^{\infty} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right|^p < \infty \right\}, \\ &\left(c_0(\mathfrak{L}, r)\right)_{\rho_2} \\ &\quad := \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k = 0 \right\}, \end{aligned} \tag{55}$$

where $r = (r_k) \in \omega^+$ and $\rho_2(\mathfrak{z}) = \sup_{\nu \in \mathbb{N}_0} \left|\sum_{k=0}^{\nu} (r_k \mathfrak{l}_k / (\mathfrak{l}_{\nu+2} - (\nu+2))) \mathfrak{z}_k\right|$.

By $\mathfrak{I}_\nearrow$ and $\mathfrak{I}_\searrow$, we will denote the space of all monotonic increasing and decreasing sequences of positive reals, respectively.

Theorem 25. *$c_0(\mathfrak{L}, r)$ is a pss, whenever $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\nearrow \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$.*

Proof.

(i) Let $\mathfrak{g}, \mathfrak{h} \in c_0(\mathfrak{L}, r)$, we obtain

$$\begin{aligned} &\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\mathfrak{g}_k + \mathfrak{h}_k) \right| \\ &\quad \leq \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{g}_k \right| \\ &\quad + \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{h}_k \right| < \infty \end{aligned} \tag{56}$$

Thus, $\mathfrak{g} + \mathfrak{h} \in c_0(\mathfrak{L}, r)$.

Assume that $\varsigma \in \mathbb{C}$ and $\mathfrak{g} \in c_0(\mathfrak{L}, r)$. Then, we have

$$\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\varsigma \mathfrak{g}_k) \right| = |\varsigma| \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{g}_k \right| < \infty. \tag{57}$$

Thus $c_0(\mathfrak{L}, r)$ is a linear space. Moreover

$$\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\mathfrak{e}_a)_k \right| = r_a \mathfrak{l}_a \sup_{\nu \in \mathbb{N}_0} \left(\frac{1}{\mathfrak{l}_{\nu+2} - (\nu+2)} \right) < \infty. \tag{58}$$

This implies $\mathbf{e}_a \in c_0(\mathfrak{L}, r)$, for each $a \in \mathbb{N}_0$.

(ii) Assume that $|\mathfrak{g}_k| \le |\mathfrak{h}_k|$, for all $k \in \mathbb{N}_0$ and $|\mathfrak{h}| \in c_0(\mathfrak{L}, r)$. Then, we have

$$\sup_{v \in \mathbb{N}_0} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} |\mathfrak{g}_k| \right| \le \sup_{v \in \mathbb{N}_0} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} |\mathfrak{h}_k| \right| < \infty \tag{59}$$

This concludes that $|\mathfrak{g}| \in c_0(\mathfrak{L}, r)$.

(iii) Let $(|\mathfrak{g}_k|) \in c_0(\mathfrak{L}, r)$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \ge 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \le C r_k \mathfrak{l}_k$. Then, we have

$$\begin{aligned}
&\sup_{v \in \mathbb{N}_0} \left(\sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} \left| \mathfrak{g}_{[\frac{k}{2}]} \right| \right) \\
&\le \sup_{v \in \mathbb{N}_0} \left(\sum_{k=0}^{2v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{2v+2} - (2v+2)} \left| \mathfrak{g}_{[\frac{k}{2}]} \right| \right) + \sup_{v \in \mathbb{N}_0} \left(\sum_{k=0}^{2v+1} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{2v+3} - (2v+3)} \left| \mathfrak{g}_{[\frac{k}{2}]} \right| \right) \\
&\le \sup_{v \in \mathbb{N}_0} \left[\frac{1}{\mathfrak{l}_{2v+2} - (2v+2)} \cdot \left\{ r_{2v}\mathfrak{l}_{2v}|\mathfrak{g}_v| + \sum_{k=0}^{v} \left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k| \right) \right\} \right] \\
&\quad + \sup_{v \in \mathbb{N}_0} \left[\frac{1}{\mathfrak{l}_{2v+3} - (2v+3)} \cdot \sum_{k=0}^{v} \left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k| \right) \right] \\
&\le \sup_{v \in \mathbb{N}_0} \left(\frac{C}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} r_k \mathfrak{l}_k |\mathfrak{g}_k| \right) + \sup_{v \in \mathbb{N}_0} \left(\frac{2C}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} r_k \mathfrak{l}_k |\mathfrak{g}_k| \right) \\
&\quad + \sup_{v \in \mathbb{N}_0} \left(\frac{2C}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} r_k \mathfrak{l}_k |\mathfrak{g}_k| \right) \\
&\le 5C \sup_{v \in \mathbb{N}_0} \left(\frac{1}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} \mathfrak{l}_k |\mathfrak{g}_k| \right) < \infty
\end{aligned} \tag{60}$$

Thus, $(\mathfrak{g}_{[k/2]}) \in c_0(\mathfrak{L}, r)$.
This completes the proof. □

Theorem 26. *$\ell_p(\mathfrak{L}, r)$ is a pss, whenever $1 < p < \infty$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\nearrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \ge 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \le C r_k \mathfrak{l}_k$.*

Proof.

(i) Let $\mathfrak{g}, \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$. By using (54), we obtain

$$\begin{aligned}
&\sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} (\mathfrak{g}_k + \mathfrak{h}_k) \right|^p \\
&\le 2^{p-1} \left\{ \sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} \mathfrak{g}_k \right|^p + \sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} \mathfrak{h}_k \right|^p \right\} < \infty
\end{aligned} \tag{61}$$

Thus, $\mathfrak{g} + \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$.
Assume that $\varsigma \in \mathbb{C}$ and $\mathfrak{g} \in \ell_p(\mathfrak{L}, r)$. Then, we have

$$\sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} (\varsigma \mathfrak{g}_k) \right|^p = |\varsigma|^p \sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} \mathfrak{g}_k \right|^p < \infty. \tag{62}$$

Thus, $\ell_p(\mathfrak{L}, r)$ is a linear space. Moreover,

$$\sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} (\mathfrak{e}_a)_k \right|^p = r_a^p \mathfrak{l}_a^p \sum_{v=r}^{\infty} \left(\frac{1}{\mathfrak{l}_{v+2} - (v+2)} \right)^p < \infty. \tag{63}$$

This implies $\mathfrak{e}_a \in \ell_p(\mathfrak{L}, r)$ for each $a \in \mathbb{N}_0$.

(ii) Assume that $|\mathfrak{g}_k| \le |\mathfrak{h}_k|$, for all $k \in \mathbb{N}_0$ and $|\mathfrak{h}| \in \ell_p(\mathfrak{L}, r)$. Then, we have

$$\sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} |\mathfrak{g}_k| \right|^p \le \sum_{v=0}^{\infty} \left| \sum_{k=0}^{v} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{v+2} - (v+2)} |\mathfrak{h}_k| \right|^p < \infty \tag{64}$$

This concludes that $|\mathfrak{g}| \in \ell_p(\mathfrak{L}, r)$.

(iii) Let $(|\mathfrak{g}_k|) \in \ell_p(\mathfrak{L}, r)$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, we have

$$\begin{aligned}
&\sum_{v=0}^{\infty}\left(\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p \\
&= \sum_{v=0}^{\infty}\left(\sum_{k=0}^{2v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2v+2}-(2v+2)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p \\
&\quad + \sum_{v=0}^{\infty}\left(\sum_{k=0}^{2v+1}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2v+3}-(2v+3)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p \\
&\leq \sum_{v=0}^{\infty}\left[\frac{1}{\mathfrak{l}_{2v+2}-(2v+2)}\right. \\
&\quad \left.\cdot\left\{r_{2v}\mathfrak{l}_{2v}|\mathfrak{g}_v| + \sum_{k=0}^{v}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right\}\right]^p \\
&\quad + \sum_{v=0}^{\infty}\left[\frac{1}{\mathfrak{l}_{2v+3}-(2v+3)}\sum_{k=0}^{v}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right]^p \\
&\leq 2^{p-1}\left[\sum_{v=0}^{\infty}\left(\frac{C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p\right. \\
&\quad \left.+ \sum_{v=0}^{\infty}\left(\frac{2C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p\right] \\
&\quad + \sum_{v=0}^{\infty}\left(\frac{2C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p \\
&\leq \left(2^{2p-1}+2^p+2^{p-1}\right)C^p \\
&\quad \cdot \sum_{v=0}^{\infty}\left(\frac{1}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}\mathfrak{l}_k|\mathfrak{g}_k|\right)^p < \infty
\end{aligned} \tag{65}$$

Thus, $(\mathfrak{g}_{[k/2]}) \in \ell_p(\mathfrak{L}, r)$.
This completes the proof. □

Define the sets $\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$, $\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$, and $\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$ by

$$\begin{aligned}
\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (s_a(\phi)) \in \mathfrak{Z}\}, \\
\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (\alpha_a(\phi)) \in \mathfrak{Z}\}, \\
\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (d_a(\phi)) \in \mathfrak{Z}\},
\end{aligned} \tag{66}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are any two Banach sequence spaces. We denote $\mathscr{B}^s_{\mathfrak{Z}} := \{\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$., $\mathscr{B}^{\alpha}_{\mathfrak{Z}} := \{\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$, and $\mathscr{B}^d_{\mathfrak{Z}} := \{\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$, respectively.

Lemma 27 (see [24]). *Let the linear sequence space $\mathfrak{Z}$ be a pss. Then, $\mathscr{B}^s_{\mathfrak{Z}}$ is a mapping ideal.*

Theorem 28. *Let $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $\mathscr{B}^s_{c_0(\mathfrak{L},r)}$ is a mapping ideal.*

Proof. It follows straightly from Lemma 27. □

Theorem 29. *Let $1 < p < \infty$,$(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $\mathscr{B}^s_{\ell_p(\mathfrak{L},r)}$ is a mapping ideal.*

Proof. It follows straightly from Lemma 27. □

Theorem 30. *Let $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $(c_0(\mathfrak{L}, r))_{\rho}$ is a premodular pss.*

Proof.

(i) Clearly, for all $\mathfrak{g} \in (c_0(\mathfrak{L}, r))_{\rho}$ that $\rho(\mathfrak{g}) \geq 0$ and $\rho(|\mathfrak{g}|) = 0$, if and only if, $\mathfrak{g} = 0$

(ii) For any $\varepsilon \geq 1$. Then $\rho(\alpha\mathfrak{g}) \leq \varepsilon|\alpha|\rho(\mathfrak{g})$, for all $\mathfrak{g} \in c_0(\mathfrak{L}, r)$ and $\alpha \in \mathbb{C}$

(iii) Observe that $\rho(\mathfrak{g} + \mathfrak{h}) \leq \rho(\mathfrak{g}) + \rho(\mathfrak{h})$, for all $\mathfrak{g}, \mathfrak{h} \in c_0(\mathfrak{L}, r)$

(iv) We have $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{h}_k|))$, whenever $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$ (see Proof Part (ii), Theorem 25).

(v) It is immediate from Proof Part (iii) of Theorem 25 that $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{g}_{[k/2]}|)) \leq \delta\rho((|\mathfrak{g}_k|))$ with $\delta = 5C$.

$$\bar{\mathfrak{C}} = c_0(\mathfrak{L}, r) \tag{67}$$

(vi) We have, when $\alpha \neq 0$ then $0 < \gamma \leq 1$, for $\rho(\alpha, 0, 0, \cdots) \geq \gamma|\alpha|\rho(1, 0, 0, \cdots)$ and when $\alpha = 0$ then $\gamma > 0$

This completes the proof. □

Theorem 31. *Let $1 < p < \infty$,$(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow} \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $(\ell_p(\mathfrak{L}, r))_{\rho}$ is a premodular pss.*

Proof.

(i) Clearly, for all $\mathfrak{g} \in (\ell_p(\mathfrak{L}, r))_{\rho}$ that $\rho(\mathfrak{g}) \geq 0$ and $\rho(|\mathfrak{g}|) = 0$, if and only if, $\mathfrak{g} = 0$

(ii) Let $\varepsilon = \max\{1, |\alpha|^{p-1}\} \geq 1$. Then, $\rho(\alpha\mathfrak{g}) \leq \varepsilon|\alpha|\rho(\mathfrak{g})$, for all $\mathfrak{g} \in \ell_p(\mathfrak{L}, r)$ and $\alpha \in \mathbb{C}$

(iii) Observe that $\rho(\mathfrak{g} + \mathfrak{h}) \leq 2^{p-1}(\rho(\mathfrak{g}) + \rho(\mathfrak{h}))$, for all $\mathfrak{g}, \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$

(iv) We have $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{h}_k|))$, whenever $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$ (see Proof Part (ii), Theorem 26).

(v) It is immediate from Proof Part (iii) of Theorem 26 that $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{g}_{[k/2]}|)) \leq \delta\rho((|\mathfrak{g}_k|))$ with $\delta = (2^{2p-1} + 2^p + 2^{p-1})C^p$

$$\bar{\mathfrak{C}} = \ell_p(\mathfrak{L}, r) \tag{68}$$

(vi) We have, when $\alpha \neq 0$ then $0 < \gamma < |\alpha|^{p-1}$, for $\rho(\alpha, 0, 0, \cdots) \geq \gamma|\alpha|\rho(1, 0, 0, \cdots)$, and when $\alpha = 0$ then $\gamma > 0$

This completes the proof. □

Theorem 32. *Assume that* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow} \cap \ell_{\infty}$, *and there exists* $C \geq 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. *Then,* $(c_0(\mathfrak{L}, r))_{\rho}$ *is a prequasi Banach pss.*

Proof. In view of Theorem 30 and Lemma 23, it is enough to prove that every Cauchy sequence in $(c_0(\mathfrak{L}, r))_{\rho}$ is convergent in $(c_0(\mathfrak{L}, r))_{\rho}$. We assume that $\mathfrak{g}^{(m)} = (\mathfrak{g}_k^{(m)})$ is a Cauchy sequence in $(c_0(\mathfrak{L}, r))_{\rho}$. Then, for all $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}_0$ such that

$$\rho\left(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}\right) = \sup_{\nu=0}^{\infty} \left| \frac{1}{\mathfrak{l}_{\nu+2} - (\nu+2)} \sum_{k=0}^{\nu} \mathfrak{l}_k \left(\mathfrak{g}_k^{(m)} - \mathfrak{g}^{(n)}\right)_k \right| < \varepsilon, \tag{69}$$

for all $m, n \geq n_0$. This implies that $(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}) < \varepsilon$, for all $m, n \geq n_0$. Thus, $(\mathfrak{g}^{(m)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\lim_{m \longrightarrow \infty} \mathfrak{g}_k^{(m)} = \mathfrak{g}_k$, for a fixed $k \in \mathbb{N}_0$. This yields, by using (69), that $\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) < \varepsilon$, for all $m \geq n_0$. Besides, we have $\rho(\mathfrak{g}) \leq \rho(\mathfrak{g}^{(m)} - \mathfrak{g}) + \rho(\mathfrak{g}^{(m)}) < \infty$. This concludes that $\mathfrak{g} \in (c_0(\mathfrak{L}, r))_{\rho}$. Thus, $(c_0(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach *pss*. □

Theorem 33. *Assume that* $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_{\infty}$, *and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. *Then,* $(\ell_p(\mathfrak{L}, r))_{\rho}$ *is a prequasi Banach pss.*

Proof. In view of Theorem 31 and Lemma 23, it is enough to prove that every Cauchy sequence in $(\ell_p(\mathfrak{L}, r))_{\rho}$ is convergent in $(\ell_p(\mathfrak{L}, r))_{\rho}$. We assume that $\mathfrak{g}^{(m)} = (\mathfrak{g}_k^{(m)})$ is a Cauchy sequence in $(\ell_p(\mathfrak{L}, r))_{\rho}$. Then, for all $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}_0$ such that

$$\rho\left(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}\right) = \sum_{\nu=0}^{\infty} \left| \frac{1}{\mathfrak{l}_{\nu+2} - (\nu+2)} \sum_{k=0}^{\nu} \mathfrak{l}_k \left(\mathfrak{g}_k^{(m)} - \mathfrak{g}^{(n)}\right)_k \right| < \varepsilon^p, \tag{70}$$

for all $m, n \geq n_0$. This implies that $(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}) < \varepsilon$, for all $m, n \geq n_0$. Thus, $(\mathfrak{g}^{(m)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\lim_{m \longrightarrow \infty} \mathfrak{g}_k^{(m)} = \mathfrak{g}_k$, for a fixed $k \in \mathbb{N}_0$. This yields, by using (70), that $\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) < \varepsilon^p$, for all $m \geq n_0$. Besides, we have $\rho(\mathfrak{g}) \leq 2^{p-1}(\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) + \rho(\mathfrak{g}^{(m)})) < \infty$. This concludes that $\mathfrak{g} \in (\ell_p(\mathfrak{L}, r))_{\rho}$. Thus $(\ell_p(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach *pss*. □

Theorem 34 (see [27]). *Suppose* $s - type\, \mathscr{E}_{\rho} \coloneqq \{h = (s_x(H)) \in \omega^+ : H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) and\, \rho(h) < \infty\}$. *If* $\mathbb{B}^s_{\mathscr{E}_{\rho}}$ *is a mapping ideal, then the following conditions are verified:*

(1) $\mathfrak{C} \subset s - type\, \mathscr{E}_{\rho}$

(2) Suppose $(s_x(H_1))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$ *and* $(s_x(H_2))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$; *then* $(s_x(H_1 + H_2))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ *and* $(s_x(H))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$; *then* $|\lambda|(s_x(H))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$

(4) The sequence space $\mathscr{E}_{\rho}$ *is solid; i.e., if* $(s_x(J))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$ *and* $s_x(H) \leq s_x(J)$, *for all* $x \in \mathbb{N}_0$ *and* $H, J \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$; *then* $(s_x(H))_{x=0}^{\infty} \in s - type\, \mathscr{E}_{\rho}$

In view of Theorem 34, we construct the next properties of the $s - type\, (c_0(\mathfrak{L}, r))_{\rho}$ *and the* $s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$.

Theorem 35. *Let* $s - type\, (c_0(\mathfrak{L}, r))_{\rho} \coloneqq \{f = (s_n(X)) \in \omega^+ : X \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})\, and\, \rho(f) < \infty\}$. *The next conditions are established:*

(1) One has $s - type\, (c_0(\mathfrak{L}, r))_{\rho} \supset \mathfrak{C}$

(2) Suppose $(s_r(X_1))_{r=0}^{\infty} \in s - type\, (c_0(\mathfrak{L}, r))_{\rho}$ *and* $(s_r(X_2))_{r=0}^{\infty} \in s - type\, (c_0(\mathfrak{L}, r))_{\rho}$; *then* $(s_r(X_1 + X_2))_{r=0}^{\infty} \in s - type\, (c_0(\mathfrak{L}, r))_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ *and* $(s_r(X))_{r=0}^{\infty} \in s - type\, (c_0(\mathfrak{L}, r))_{\rho}$; *hence* $|\lambda|(s_r(X))_{r=0}^{\infty} \in s - type\, (c_0(\mathfrak{L}, r))_{\rho}$

(4) The $s - type\, (c_0(\mathfrak{L}, r))_{\rho}$ *is solid*

Theorem 36. *Let* $s - type\, (\ell_p(\mathfrak{L}, r))_{\rho} \coloneqq \{f = (s_n(X)) \in \omega^+ : X \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})\, and\, \rho(f) < \infty\}$. *The next conditions are established:*

(1) One has $s - type\, (\ell_p(\mathfrak{L}, r))_{\rho} \supset \mathfrak{C}$

(2) Suppose $(s_r(X_1))_{r=0}^{\infty} \in s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$ *and* $(s_r(X_2))_{r=0}^{\infty} \in s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$; *then* $(s_r(X_1 + X_2))_{r=0}^{\infty} \in s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ *and* $(s_r(X))_{r=0}^{\infty} \in s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$; *hence* $|\lambda|(s_r(X))_{r=0}^{\infty} \in s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$

(4) The $s - type\, (\ell_p(\mathfrak{L}, r))_{\rho}$ *is solid*

6. Characteristics of the Prequasi Ideal

The conventions listed below will be followed throughout the article; if the species is preowned, we will give it to you.

Conventions 1. Please see the following conventions:

$\mathbb{F}$: the ideal of finite rank mappings between any arbitrary Banach spaces

$\mathscr{A}$: the ideal of approximable mappings between any arbitrary Banach spaces

$\mathscr{K}$: the ideal of compact mappings between any arbitrary Banach spaces

$\mathbb{F}(\mathfrak{X}, \mathfrak{Y})$: the space of finite rank mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathbb{F}(\mathfrak{X})$: the space of finite rank mappings from a Banach space $\mathfrak{X}$ into itself

$\mathscr{A}(\mathfrak{X}, \mathfrak{Y})$: the space of approximable mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathscr{A}(\mathfrak{X})$: the space of approximable mappings from a Banach space $\mathfrak{X}$ into itself

$\mathscr{K}(\mathfrak{X}, \mathfrak{Y})$: the space of compact mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathscr{K}(\mathfrak{X})$: the space of compact mappings from a Banach space $\mathfrak{X}$ into itself

Lemma 37 (see [28]). *If $M \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $M \notin \mathscr{A}(\mathfrak{X}, \mathfrak{Y})$, then there are operators $Q \in \mathscr{B}(\mathfrak{X})$ and $L \in \mathscr{B}(\mathfrak{Y})$ so that $LMQe_x = e_x$, for all $x \in \mathbb{N}_0$.*

Definition 38 (see [28]). A Banach space $\mathscr{E}$ is called simple if the algebra $\mathscr{B}(\mathscr{E})$ includes one and only one nontrivial closed ideal.

Theorem 39 (see [28]). *Suppose $\mathscr{E}$ is a Banach space with* dim $(\mathscr{E}) = \infty$*; then*

$$\mathbb{F}(\mathscr{E}) \subset \mathscr{A}(\mathscr{E}) \subset \mathscr{K}(\mathscr{E}) \subset \mathscr{B}(\mathscr{E}). \quad (71)$$

In this section, firstly, we introduce the enough setups (not necessary) on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that $\bar{\mathbb{F}} = \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\bar{\mathbb{F}} = \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. This investigates a negative answer of Rhoades [29] open problem about the linearity of $s-$type $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ spaces. Secondly, for which conditions on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$, are $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ closed and complete? Thirdly, we explain the enough setups on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ are strictly contained for different weights and powers. We offer the setups so that $\mathscr{B}^\alpha_{(\ell_p(\mathfrak{L},r))_\rho}$ is minimum. Fourthly, we introduce the conditions so that the Banach prequasi ideal $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ are simple Banach spaces. Fifthly, we investigate the enough conditions on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that the space of all bounded linear operators which sequence of eigenvalues in $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ equal $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$, respectively.

6.1. Finite Rank Prequasi Ideal

Theorem 40. $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) = \mathbb{F}(\bar{\mathfrak{X}, \mathfrak{Y}})$*; suppose the setups $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are satisfied. But the converse is not necessarily true.*

Proof. To investigate that $\mathbb{F}(\bar{\mathfrak{X}, \mathfrak{Y}}) \subseteq \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, as $e_l \in (c_0(\mathfrak{L}, r))_\rho$, for every $l \in \mathbb{N}_0$,$(c_0(\mathfrak{L}, r))_\rho$ is a linear space. Let $Z \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})$, one gets $(s_l(Z))_{l=0}^\infty \in \mathfrak{C}$. To explain that $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \subseteq \mathbb{F}(\bar{\mathfrak{X}, \mathfrak{Y}})$, assume $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we obtain $(s_l(Z))_{l=0}^\infty \in (c_0(\mathfrak{L}, r))_\rho$. Since $\rho(s_l(Z))_{l=0}^\infty < \infty$, let $\rho \in (0, 1)$; hence, there is $l_0 \in \mathbb{N}_0 - \{0\}$ with $\rho((s_l(Z))_{l=l_0}^\infty) < \rho/16d$, for some $d \geq 1$. Since $s_l(Z) \in \mathfrak{I}_\rceil$, we get

$$\begin{aligned} \sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_{2l_0}(Z)}{\mathfrak{l}_{l+2} - (l+2)} &\leq \sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}. \end{aligned} \quad (72)$$

Hence, there is $Y \in \mathbb{F}_{2l_0}(\mathfrak{X}, \mathfrak{Y})$ so that $\text{rank}(Y) \leq 2l_0$ and

$$\sup_{l=2l_0+1}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} \leq \sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}, \quad (73)$$

we have

$$\sum_{j=0}^{l_0} r_j \mathfrak{l}_j \|Z - Y\| < \frac{\rho}{16}. \quad (74)$$

Therefore, one has

$$\sup_{l=0}^{l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}. \quad (75)$$

In view of inequalities (72)–(75), and $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\rceil$, one gets

$$\begin{aligned} d(Z, Y) &= \rho(s_l(Z - Y))_{l=0}^\infty \\ &\leq \sup_{l=0}^{3l_0-1} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=3l_0}^{\infty} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l=0}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2l_0+2} - (l+2l_0+2)} \\ &\leq \sup_{l=0}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} \end{aligned}$$

$$\begin{aligned}
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)+\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right) \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{l} r_{j+2l_0} \mathfrak{l}_{j+2l_0} s_{j+2l_0}(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+2\sum_{j=0}^{l_0}\mathfrak{l}_j r_j\|Z-Y\| \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}<\rho.
\end{aligned} \tag{76}$$

On the opposite side, one has a negative example as $I_3 \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, where $r=(0,0,0,1,0,1,0,)$. This shows the proof. □

Theorem 41. $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})=\mathbb{F}(\bar{\mathfrak{X}},\mathfrak{Y})$*; suppose the setups* $1<p<\infty, (r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_{\searrow}$ *or* $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_Z\cap\ell_\infty$*, and there exists* $C\ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1}\le C r_k\mathfrak{l}_k$ *are confirmed. But the converse is not necessarily true.*

Proof. To investigate that $\mathbb{F}(\bar{\mathfrak{X}},\mathfrak{Y})\subseteq\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, as $e_l\in(\ell_p(\mathfrak{L},r))_\rho$, for every $l\in\mathbb{N}_0$, $(\ell_p(\mathfrak{L},r))_\rho$ is a linear space. Let $Z\in\mathbb{F}(\mathfrak{X},\mathfrak{Y})$; one gets $(s_l(Z))_{l=0}^{\infty}\in\mathfrak{C}$. To explain that $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})\subseteq\mathbb{F}(\bar{\mathfrak{X}},\mathfrak{Y})$, assume $Z\in\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$; we obtain $(s_l(Z))_{l=0}^{\infty}\in(\ell_p(\mathfrak{L},r))_\rho$. Since $\rho(s_l(Z))_{l=0}^{\infty}<\infty$, let $\rho\in(0,1)$, hence, there is $l_0\in\mathbb{N}_0-\{0\}$ with $\rho((s_l(Z))_{l=l_0}^{\infty})<(\rho/2^{p+3}\eta d)$, for some $d\ge 1$, where $\eta=\max\{1,\sum_{l=l_0}^{\infty}(1/(\mathfrak{l}_{l+2}-(l+2)))^p\}$. Since $s_l(Z)\in\mathfrak{I}_{]}$, we get

$$\begin{aligned}
\sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{2l_0}(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\le \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&< \frac{\rho}{2^{p+3}\eta d}.
\end{aligned} \tag{77}$$

Hence, there is $Y\in\mathbb{F}_{2l_0}(\mathfrak{X},\mathfrak{Y})$ so that $\operatorname{rank}(Y)\le 2l_0$ and

$$\begin{aligned}
\sum_{l=2l_0+1}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\le \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&< \frac{\rho}{2^{p+3}\eta d}.
\end{aligned} \tag{78}$$

Since $1<p<\infty$, we have

$$\left(\sum_{j=0}^{l_0} r_j \mathfrak{l}_j\|Z-Y\|\right)^p<\frac{\rho}{2^{2p+2}\eta}. \tag{79}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p<\frac{\rho}{2^{p+3}\eta d}. \tag{80}$$

In view of inequalities (54), (77)–(80), and $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_{]}$, one gets

$$\begin{aligned}
d(Z,Y)&=\rho(s_l(Z-Y))_{l=0}^{\infty} \\
&=\sum_{l=0}^{3l_0-1}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=3l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2l_0+2}-(l+2l_0+2)}\right)^p \\
&\le \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)+\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p
\end{aligned}$$

$$+2^{p-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right]$$
$$\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + 2^{p-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_{j+2l_0} \mathfrak{l}_{j+2l_0} s_{j+2l_0}(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right]$$
$$\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + 2^p\left(\sum_{j=0}^{l_0} \mathfrak{l}_j r_j \|Z-Y\|\right)^p \cdot \sum_{l=l_0}^{\infty}\left(\frac{1}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + 2^{p-1}\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p < \rho. \tag{81}$$

On the opposite side, one has a negative example as $I_4 \in \mathscr{B}^s_{(\ell_{0.5}(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, where $r = (0,0,0,0,1,1,)$. This shows the proof. □

6.2. Banach and Closed Prequasi Ideal

Theorem 42 (see [24]). *The function Ψ is a prequasi norm on $\mathscr{B}^s_{(\mathscr{E})_\rho}$, where $\Psi(Y) = \rho(s_b(Y))_{b=0}^{\infty}$, for every $Y \in \mathscr{B}^s_{(\mathscr{E})_\rho}(\mathfrak{X},\mathfrak{Y})$, if $(\mathscr{E})_\rho$ is a premodular pss.*

Theorem 43. *If the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are satisfied, then $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi Banach ideal, where $\psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(c_0(\mathfrak{L},r))_\rho$ is a premodular *pss*, hence from Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$. Suppose $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. As $\mathscr{B}(\mathfrak{X},\mathfrak{Y}) \supseteq \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, one obtains

$$\Psi(X_a - X_b) = \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_a - X_b)}{\mathfrak{l}_{l+2}-(l+2)} \geq r_0\,\|X_a - X_b\|. \tag{82}$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Since $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ is a Banach space, then there is $X \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{b\longrightarrow\infty}\|X_b - X\| = 0$. Since $(s_l(X_b))_{l=0}^{\infty} \in (c_0(\mathfrak{L},r))_\rho$, every $b \in \mathbb{N}_0$. According to Definition 21 setups (ii), (iii), and (v), one gets

$$\begin{aligned}\Psi(X) &= \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2}-(l+2)} + \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2}-(l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2}-(l+2)} + D_0 \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2}-(l+2)} < \infty.\end{aligned} \tag{83}$$

Therefore, $(s_l(X))_{l=0}^{\infty} \in (c_0(\mathfrak{L},r))_\rho$; then $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. □

Theorem 44. *If the setups $1 < p < \infty$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are confirmed; then $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi Banach ideal, where $\psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(\ell_p(\mathfrak{L},r))_\rho$ is a premodular *pss*, hence from Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. Suppose $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. As $\mathscr{B}(\mathfrak{X},\mathfrak{Y}) \supseteq \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, one obtains

$$\Psi(X_a - X_b) = \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_a - X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \geq (r_0\|X_a - X_b\|)^p. \tag{84}$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Since $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ is a Banach space, then there is $X \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{b\longrightarrow\infty}\|X_b - X\| = 0$.Since $(s_l(X_b))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L},r))_\rho$, every $b \in \mathbb{N}_0$. According to Definition 21 setups (ii), (iii), and (v), one gets

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\ &\leq 2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + 2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\ &\leq 2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p + 2^{p-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p < \infty.\end{aligned} \tag{85}$$

Therefore, $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$; then $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

Theorem 45. *Assume $\mathfrak{X}$, $\mathfrak{Y}$ are normed spaces; the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$; and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are satisfied; then $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi closed ideal, where $\Psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(c_0(\mathfrak{L}, r))_\rho$ is a premodular *pss*, by using Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$. Assume $X_b \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, every $b \in \mathbb{N}_0$ and $\lim_{b\longrightarrow\infty}\Psi(X_b - X) = 0$. As $\mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\Psi(X - X_b) = \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)} \geq r_0 \|X - X_b\|. \quad (86)$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a convergent sequence in $\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (c_0(\mathfrak{L}, r))_\rho$, for every $b \in \mathbb{N}_0$. In view of Definition 21 setups (ii), (iii), and (v), one has

$$\begin{aligned}\Psi(X) &= \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2} - (l+2)} + D_0 \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2} - (l+2)} < \infty.\end{aligned} \quad (87)$$

We get $(s_l(X))_{l=0}^{\infty} \in (c_0(\mathfrak{L}, r))_\rho$, so $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

Theorem 46. *Assume $\mathfrak{X}$, $\mathfrak{Y}$ are normed spaces; the setups $1 < p < \infty$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$; and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are satisfied; hence, $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi closed ideal, where $\Psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(\ell_p(\mathfrak{L}, r))_\rho$ is a premodular *pss*, by using Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. Assume $X_b \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, for every $b \in \mathbb{N}_0$ and $\lim_{b\longrightarrow\infty}\Psi(X_b - X) = 0$. As $\mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\Psi(X - X_b) = \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \geq (r_0 \|X - X_b\|)^p. \quad (88)$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a convergent sequence in $\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$, every $b \in \mathbb{N}_0$. In view of Definition 21 setups (ii), (iii), and (v), one has

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\leq 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\quad + 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\leq 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\quad + 2^{p-1} D_0 \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p < \infty.\end{aligned} \quad (89)$$

We get $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$, so $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

6.3. Minimum Prequasi Ideal

Theorem 47. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_]$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are confirmed with $0 < r_l^{(2)} \leq r_l^{(1)}$, for all $l \in \mathbb{N}_0$, hence*

$$\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X}, \mathfrak{Y}). \quad (90)$$

Proof. Let $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$; then $(s_l(Z)) \in (c_0(\mathfrak{L}, (r_l^{(1)})))_\rho$. One obtains

$$\sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} \leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} < \infty. \quad (91)$$

Then, $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$. Next, if we choose $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = \mathfrak{l}_{l+2} - (l+2)$ and $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2} - (l+2))/(l+1)(\mathfrak{l}_{l+2} - (l+2)/l+1)$, one gets $Z \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$ and $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$.

Clearly, $\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Next, if we put $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z) = \mathfrak{l}_{l+2} - (l+2)$. We have $Z \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$. This explains the proof. □

Theorem 48. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are confirmed with $1 < h^{(1)} < h^{(2)}$ and $0 < r_l^{(2)} \leq r_l^{(1)}$ for all $l \in \mathbb{N}_0$; hence,*

$$\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X},\mathfrak{Y}). \quad (92)$$

Proof. Let $Z \in \mathscr{B}^s_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y})$; then $(s_l(Z)) \in (\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho$. One obtains

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}} < \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(1)}} < \infty. \quad (93)$$

Then $Z \in \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$. Next, if we choose $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2}-(l+2))/\sqrt[h1]{l+1}(\mathfrak{l}_{l+2} - (l+2)/\sqrt[h1]{l+1})$, one gets $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ such that

$$\begin{aligned}
\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(1)}} &= \sum_{l=0}^{\infty}\frac{1}{l+1} = \infty, \\
\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}} &\leq \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}} \\
&= \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{\frac{h^{(2)}}{h^{(1)}}} < \infty.
\end{aligned} \quad (94)$$

Therefore, $Z \notin \mathscr{B}^s_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y})$ and $Z \in \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Clearly, $\mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Next, if we put $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2}-(l+2))/\sqrt[h2]{l+1}$. We have $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$. This explains the proof. □

Theorem 49. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are established with $((\sum_{j=0}^{l} r_j \mathfrak{l}_j)/(\mathfrak{l}_{l+2}-(l+2)))_{l\in\mathbb{N}_0} \notin \ell_p$; then, $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}$ is minimum.*

Proof. Suppose the enough setups are confirmed; then $(\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$, where $\Psi(Z) = \sum_{l=0}^{\infty}((\sum_{j=0}^{l} r_j \mathfrak{l}_j \alpha_j(Z))/(\mathfrak{l}_{l+2}-(l+2)))^p$, is a prequasi Banach ideal. Suppose $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}(\mathfrak{X},\mathfrak{Y})$; hence, there is $\eta > 0$ with $\Psi(Z) \leq \eta\|Z\|$, for every $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. According to Dvoretzky's Theorem [23], for every $b \in \mathbb{N}_0$, one obtains quotient spaces $\mathfrak{X}/Y_b$ and subspaces M_b of $\mathfrak{Y}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leq 2$ and $\|X_b\|\|X_b^{-1}\| \leq 2$. Let I_b be the identity operator on ℓ_2^b and T_b be the quotient operator from $\mathfrak{X}$ onto $\mathfrak{X}/Y_b$, and J_b is the natural embedding operator from M_b into $\mathfrak{Y}$. Suppose m_z is the Bernstein numbers [26]; then

$$\begin{aligned}
1 = m_z(I_b) &= m_z\left(X_b X_b^{-1} I_b V_b V_b^{-1}\right) \\
&\leq \|X_b\| m_z\left(X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\| \\
&= \|X_b\| m_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\| \\
&\leq \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\| \\
&= \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\| \\
&\leq \|X_b\| \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|,
\end{aligned} \quad (95)$$

for $0 \leq l \leq b$. We have

$$\begin{aligned}
\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)} &\leq \frac{\sum_{z=0}^{l}\|X_b\| r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|}{\mathfrak{l}_{l+2}-(l+2)} \\
&\Longrightarrow \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\leq \left(\|X_b\|\left\|V_b^{-1}\right\|\right)^p \left(\frac{\sum_{z=0}^{l} r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p.
\end{aligned} \quad (96)$$

Hence, for some $\rho \geq 1$, one gets

$$\begin{aligned}
\sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\leq \rho\|X_b\|\left\|V_b^{-1}\right\| \\
&\cdot \sum_{l=0}^{b}\left(\frac{\sum_{z=0}^{l} r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\leq \rho\|X_b\|\left\|V_b^{-1}\right\|\Psi\left(J_b X_b^{-1} I_b V_b T_b\right) \\
&\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\leq \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1} I_b V_b T_b\right\| \\
&\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\leq \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1}\right\|\|I_b\|\|V_b T_b\| \\
&= \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|X_b^{-1}\right\|\|I_b\|\|V_b\| \leq 4\rho\eta.
\end{aligned} \quad (97)$$

Therefore, we have a contradiction, if $b \longrightarrow \infty$. Then, $\mathfrak{X}$ and $\mathfrak{Y}$ both cannot be infinite dimensional if $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. This shows the proof. □

By the same manner, we can easily conclude the next theorem.

Theorem 50. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are established with $((\sum_{j=0}^{l} r_j \mathfrak{l}_j)/(\mathfrak{l}_{l+2} - (l+2)))_{l \in \mathbb{N}_0} \notin \ell_p$; then $\mathscr{B}^{d}_{(\ell_p(\mathfrak{L},r))_\rho}$ is minimum.*

6.4. Simple Banach Prequasi Ideal

Theorem 51. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are confirmed with $0 < r_l^{(2)} \leq r_l^{(1)}$, for all $l \in \mathbb{N}_0$; then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \quad (98)$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned}\|I_b\|_{\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sup_{l \in \mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \|ZXY\| \|I_b\|_{\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\leq \sup_{l \in \mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}.\end{aligned} \quad (99)$$

This contradicts Theorem 47. Then $X \in \mathscr{A}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Corollary 52. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are established with $0 < r_l^{(2)} \leq r_l^{(1)}$, for all $l \in \mathbb{N}_0$; then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{K}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \quad (100)$$

Proof. Clearly, $\mathscr{A} \subset \mathscr{K}$. □

Theorem 53. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are confirmed with $1 < h^{(1)} < h^{(2)}$ and $0 < r_l^{(2)} \leq r_l^{(1)}$, for all $l \in \mathbb{N}_0$; then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \quad (101)$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned}\|I_b\|_{\mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^{h^{(1)}} \\ &\leq \|ZXY\| \|I_b\|_{\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\leq \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^{h^{(2)}}.\end{aligned} \quad (102)$$

This contradicts Theorem 48. Then $X \in \mathscr{A}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Corollary 54. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that r_{2k+1}*

$\mathfrak{l}_{2k+1} \le Cr_k\mathfrak{l}_k$ *are established with* $1 < h^{(1)} < h^{(2)}$ *and* $0 < r_l^{(2)} \le r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$; *then*

$$\mathscr{B}\left(\mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{K}\left(\mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{103}$$

Proof. Clearly, $\mathscr{A} \subset \mathscr{K}$. □

Theorem 55. *Suppose* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \le Cr_k\mathfrak{l}_k$ *are confirmed with* $1 < h < \infty$ *and* $0 < r_l^{(2)} \le r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$, *then*

$$\mathscr{B}\left(\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{104}$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned}\|I_b\|_{\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sum_{l=0}^\infty\left(\frac{\sum_{j=0}^l r_j^{(1)}\mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^h \\ &\le \|ZXY\|\|I_b\|_{\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\le \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^l r_j^{(2)}\mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2}-(l+2)}.\end{aligned} \tag{105}$$

This contradicts $\ell_h(\mathfrak{L},(r_l^{(1)})) \subset c_0(\mathfrak{L},(r_l^{(2)}))$. Then $X \in \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Theorem 56. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \le Cr_k\mathfrak{l}_k$ *are satisfied; hence,* $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ *is simple.*

Proof. Assume the closed ideal $\mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ includes an operator $X \notin \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, we have $Y, Z \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})} \in \mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Then, $\mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})) = \mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Hence, $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ is simple Banach space. □

Theorem 57. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \le Cr_k\mathfrak{l}_k$ *are satisfied; hence,* $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ *is simple.*

Proof. Assume the closed ideal $\mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ includes an operator $X \notin \mathscr{A}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, we have $Y, Z \in \mathscr{B}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})} \in \mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Then, $\mathscr{B}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})) = \mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Hence, $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ is simple Banach space. □

6.5. Eigenvalues of S-Type Operators

Conventions 2. Please see the following conventions:

$$\begin{aligned}(\mathscr{B}^s_{\mathscr{E}})^\rho &:= \{(\mathscr{B}^s_{\mathscr{E}})^\rho(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}\text{ and }\mathfrak{Y}\text{ are Banach Spaces}\},\text{ where} \\ (\mathscr{B}^s_{\mathscr{E}})^\rho(\mathfrak{X},\mathfrak{Y}) &:= \{X \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})\colon\ ((\rho_l(X))_{n=0}^\infty \\ &\quad \in \mathscr{E}\text{ and }\|X-\rho_l(X)I\|\text{ is not invertible, for all }l\in\mathbb{N}_0\}\end{aligned} \tag{106}$$

Theorem 58. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \le Cr_k\mathfrak{l}_k$ *are verified with* $\inf_{l\in\mathbb{N}_0}((\sum_{j=0}^l r_j\mathfrak{l}_j)/(\mathfrak{l}_{l+2}-(l+2))) > 0$; *then* $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Proof. Let $X \in (\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X},\mathfrak{Y})$; hence, $(\rho_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$ and $\|X-\rho_l(X)I\| = 0$, for all $l \in \mathbb{N}_0$. We have $X = \rho_l(X)I$, for all $l \in \mathbb{N}_0$; hence, $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for every $l \in \mathbb{N}_0$. Therefore, $(s_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$; then $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Secondly, suppose $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. Then $(s_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$. Hence, we have

$$\lim_{l\longrightarrow\infty}\frac{\sum_{j=0}^l r_j\mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)} \ge \inf_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^l r_j\mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\lim_{l\longrightarrow\infty}s_l(X). \tag{107}$$

Therefore, $\lim_{l\longrightarrow\infty}s_l(X) = 0$. Assume $\|X-s_l(X)I\|^{-1}$ exists, for every $l \in \mathbb{N}_0$. Hence, $\|X-s_l(X)I\|^{-1}$ exists and bounded, for every $l \in \mathbb{N}_0$. Then, $\lim_{l\longrightarrow\infty}\|X-s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi operator ideal, we get

$$\begin{aligned} &I = XX^{-1} \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \\ &\quad \Longrightarrow (s_l(I))_{l=0}^\infty \in (c_0(\mathfrak{L}, r))_\rho \Longrightarrow \lim_{l\longrightarrow\infty} s_l(I) = 0. \end{aligned} \tag{108}$$

So we have a contradiction, since $\lim_{l\longrightarrow\infty} s_l(I) = 1$. Hence, $\|X - s_l(X)I\| = 0$, for every $l \in \mathbb{N}_0$. This gives $X \in (\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$. This shows the proof. □

Theorem 59. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* dim $(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ or $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are satisfied with $\inf_l((\sum_{j=0}^l r_j\mathfrak{l}_j)/(\mathfrak{l}_{l+2} - (l + 2)))^p > 0$; then $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y}) = \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$.*

Proof. Assume $X \in (\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$; hence, $(\rho_l(X))_{l=0}^\infty \in (\ell_p(\mathfrak{L}, r))_\rho$ and $\|X - \rho_l(X)I\| = 0$, for all $l \in \mathbb{N}_0$. We have $X = \rho_l(X)I$, for all $l \in \mathbb{N}_0$; hence, $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for every $l \in \mathbb{N}_0$. Therefore, $(s_l(X))_{l=0}^\infty \in (\ell_p(\mathfrak{L}, r))_\rho$; then $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$.

Secondly, suppose $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. Then $(s_l(X))_{l=0}^\infty \in (\ell_p(\mathfrak{L}, r))_\rho$. Hence, we have

$$\sum_{l=0}^\infty \left(\frac{\sum_{j=0}^l r_j\mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \geq \inf_l \left(\frac{\sum_{j=0}^l r_j\mathfrak{l}_j}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \sum_{l=0}^\infty [s_l(X)]^p. \tag{109}$$

Therefore, $\lim_{l\longrightarrow\infty} s_l(X) = 0$. Assume $\|X - s_l(X)I\|^{-1}$ exists, for every $l \in \mathbb{N}_0$. Hence, $\|X - s_l(X)I\|^{-1}$ exists and bounded, for every $l \in \mathbb{N}_0$. Then, $\lim_{l\longrightarrow\infty} \|X - s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi operator ideal, we get

$$\begin{aligned} &I = XX^{-1} \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \\ &\quad \Longrightarrow (s_l(I))_{l=0}^\infty \in (\ell_p(\mathfrak{L}, r))_\rho \Longrightarrow \lim_{l\longrightarrow\infty} s_l(I) = 0. \end{aligned} \tag{110}$$

So we have a contradiction, since $\lim_{l\longrightarrow\infty} s_l(I) = 1$. Hence, $\|X - s_l(X)I\| = 0$, for every $l \in \mathbb{N}_0$. This gives $X \in (\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$. This shows the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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