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Research Article

Analysis of Fractional Differential Inclusion Models for COVID-19 via Fixed Point Results in Metric Space

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We examine in this paper some new problems on coincidence point and fixed point theorems for multivalued mappings in metric space. By applying the characterizations of a modified $\widetilde{\mathscr{MT}}$-function, under the name $\mathscr{D}$-function, a few novel fixed point results different from the existing fixed point theorems are launched. It is well-known that differential equation of either integer or fractional order is not sufficient to capture ambiguity, since the derivative of a solution to any differential equation inherits all the regularity properties of the mapping involved and of the solution itself. This does not hold in the case of differential inclusions. In particular, fractional-order differential inclusion models are more suitable for describing epidemics. Thus, as a generalization of a newly launched existence result for fractional-order model for COVID-19, using Banach and Shauder fixed point theorems, we investigate solvability criteria of a novel Caputo-type fractional-order differential inclusion model for COVID-19 by applying a standard fixed point theorem of multivalued contraction. Stability analysis of the proposed model in the framework of Ulam-Hyers is also discussed. Nontrivial comparative illustrations are constructed to show that our ideas herein complement, unify and, extend a significant number of existing results in the corresponding literature.

1. Introduction and Preliminaries

Numerous challenges in practical world defined by nonlinear functional equations can be simplified by reconfiguring them to their equivalent fixed point problems. Fixed point theory yields relevant tools for solving problems emanating in various arms of sciences. The fixed point theorem, commonly named as the Banach fixed point theorem (see [1]), came up in clear form in Banach thesis in 1922, where it was availed to study the existence of a solution to an integral equation. Since then, because of its importance, it has gained a number of refinements by many authors. In some modifications of the principle, the inequality is weakened, see, for example [2, 3], and in others, the topology of the ambient space is relaxed, see [4–7] and the references therein. Along the lane, three prominent improvements of the Banach fixed point theorem was presented by Ciric [2], Reich [8], and Rus [9].

Nadler [10] launched a multivalued improvement of the Banach contraction mapping principle. Nadler's contraction mapping principle opened up the concept of metric fixed point theory of multivalued contraction in nonlinear analysis. In line with [10], a number of refinements of fixed point theorems of multivalued contractions have been presented, famously, by Berinde-Berinde [11], Du [12, 13], Mizoguchi and Takahashi [14], Pathak [15], and Reich [16, 17], to cite a few. Fixed point theorems for multivalued mappings are highly advantageous in optimal control theory and have been commonly used to solve several problems in economics, game theory, biomathematics, qualitative physics, viability theory, and many more.

Differential inclusions are found to be of great usefulness in studying dynamical systems and stochastic processes. A few examples include sweeping process, granular systems, nonlinear dynamics of wheeled vehicles, and control problems. In particular, fractional differential inclusions arise in several problems in mathematical physics, biomathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic, traffic theory, etc. Usually, the first most concerned problem in the study of differential inclusion is the conditions for existence of its solutions. In this direction, several authors have applied different fixed point approaches and topological methods to obtain existence results of differential inclusions in abstract spaces. In the current literature, we can find many works on fractional-order models proposing different measures for curbing the novel corona virus (COVID-19) (see, for example, Ali et al. [18], Yu et al. [19], Xu et al. [20], Shaikh et al. [21], and the references therein). Recently, Ahmed et al. [22] constructed a Caputo-type fractional-order model and studied the significance and effect of the lockdown in curbing COVID-19. They ([22]) investigated the existence and uniqueness of solutions of the fractional-order corona virus model by applying the Banach and Schauder fixed point theorems. One of the pioneer results of fixed point theory using fractional-order model was presented by Boccaletti et al. [23]. For some recent results and applications of fraction calculus, we refer [24–26].

Following the above developments, we consider in this paper some problems on coincidence point and fixed point theorems for multivalued mappings. By applying the characterizations of $\mathscr{D}$-function, a few new fixed point results different from the fixed point theorems due to Berinde-Berinde [11], Du [13], Mizoguchi-Takahashi [14], Nadler [10], Reich [17], and Rus [27] are launched. It is a common knowledge that differential equation of either integer or fractional order is not sufficient to capture ambiguity, since the derivative $j'(.)$ of a solution $j(.)$ to the differential equation $j'(t) = g(t, j(t))$ inherits the regularity properties of the mapping g and of the function $j(.)$. This is no longer the case with differential inclusions. In particular, fractional-order differential inclusions models are more suitable for describing epidemics (see, e.g., [28]). Differential inclusions are not only models for handling dynamic processes but also provide powerful analytic tools to prove existence theorems such as in control theory, to derive sufficient conditions of optimality, play a significant role in the theory of control conditions under uncertainty. Thus, as a generalization of the existence theorem presented by Ahmed et al. [22], in the sequel, we investigate solvability conditions of a new Caputo-type fractional differential inclusions model for COVID-19 by applying a fixed point theorem of multivalued contraction. Stability analysis of the proposed model in the context of Ulam-Hyers is also obtained. Our results herein complement, unify, and extend the above-mentioned articles and a few others in the comparable literature. A few nontrivial comparative illustrations are constructed to indicate that our obtained ideas properly advanced corresponding results in the literature.

In what follows, we recall some preliminary concepts that are useful to our main results. Throughout this paper, the set $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ represent the set of real numbers, non-negative real numbers, and the set of natural numbers, respectively. Let $(\mho, \mu)$ be a metric space. Denote by $\mathscr{N}(\mho)$, $CB(\mho)$, and $\mathscr{K}(\mho)$, the family of nonempty subsets of $\mho$, the collection of all nonempty closed and bounded subsets of $\mho$, and the class of all nonempty compact subsets of $\mho$, respectively. For $A, B \in CB(\mho)$, the mapping $\tilde{H} : CB(\mho) \times CB(\mho) \longrightarrow \mathbb{R}$ is given by

$$\tilde{H}(A, B) = \max \left\{ \sup_{j\in B} \mu(j, A), \sup_{\ell\in A} \mu(\ell, B) \right\}, \tag{1}$$

where $\mu(j, A) = \inf_{\ell\in A} \mu(j, \ell)$ is named the Hausdorff-Pompeiu metric induced by the metric μ. For example, if we consider the set of real numbers endowed with the standard metric, then for any two closed intervals $[a, b]$ and $[c, d]$, we have $\tilde{H}([a, b], [c, d]) = \max \{|a - c|, |b - d|\}$.

Let $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be point-valued mappings and $Y : \mho \longrightarrow \mathscr{N}(\mho)$ be a multivalued mapping. A point u in $\mho$ is a coincidence point of Δ, Θ, Λ and Y if $\Delta u = \Theta u = \Lambda u \in Yu$. If $\Delta = \Theta = \Lambda = I_\mho$ is the identity mapping on $\mho$, then $u = \Delta u = \Theta u = \Lambda u \in Yu$ is named a fixed point of Y. We denote the set of fixed points of Y and the set of coincidence point of Δ, Θ, Λ and Y by $\mathscr{F}_{ix}(Y)$ and $\mathscr{COP}(\Delta, \Theta, \Lambda, Y)$, respectively.

Let g be a real-valued function. For $t \in \mathbb{R}$, we recall that

$$\limsup_{r\longrightarrow t} g(r) = \inf_{\varepsilon>0} \sup_{0<|r-t|<\varepsilon} g(r) \text{and} \limsup_{r\longrightarrow t^+} g(r) = \inf_{\varepsilon>0} \sup_{0<r-t<\varepsilon} g(r). \tag{2}$$

Definition 1. (see [12]). $\psi_{\widetilde{\mathscr{MT}}} : (0,\infty) \longrightarrow [0, 1)$ is named an $\widetilde{\mathscr{MT}}$-function if it obeys the Mizoguchi-Takahashi's condition, that is, $\limsup_{r\longrightarrow t^+} \psi_{\widetilde{\mathscr{MT}}}(r) < 1$, for each $t \in \mathbb{R}_+ = [0,\infty)$.

Remark 2. (see [12]).

(i) If $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ is given as $\psi_{\widetilde{\mathscr{MT}}}(t) = \alpha \in [0, 1)$, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function

(ii) If the function $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ is either increasing or decreasing, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function

Definition 3. $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is named a $\mathscr{D}$-function if it obeys the condition: For each $t \in \mathbb{R}_+$, we can find $k \in (1,\infty)$ such that $\limsup_{r\longrightarrow t^+} \psi(r) < 1/k$.

Definition 4. (see [12]). A function $\psi : \mathbb{R}_+ \longrightarrow [0, 1)$ is named a function of contractive factor, if for any strictly decreasing sequence $\{j_n\}_{n\geq1}$ in $\mathbb{R}_+$, we have $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) < 1$.

Definition 5. A function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is named a function of $1/k$-contractive factor, if for any sequence $\{j_n\}_{n\geq1}$ in $\mathbb{R}_+$ from and after some fixed terms, it is strictly nonincreasing and $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) < 1/k$, for some $k \in (1,\infty)$.

The following example recognizes the existence of $\mathscr{D}$-function and function of $1/k$-contractive factor.

Example 6.
Let $\{j_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}_+$ given by

$$j_n = \begin{cases} 3^{2n} - 1, & \text{if } n \leq 7 \\ 3 + \dfrac{1}{2n}, & \text{if } n > 7. \end{cases} \tag{3}$$

Define $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by

$$\psi(\widehat{t}) = \begin{cases} \dfrac{1}{17 + \widehat{t}^2}, & \text{if } 0 \leq \widehat{t} < 2 \\ \dfrac{1}{3} - \dfrac{\widehat{t}}{3^7}, & \text{if } 2 \leq \widehat{t} < 50 \\ 0 & \textit{otherwise}. \end{cases} \tag{4}$$

Then, it is clear that ψ is a $\mathscr{D}$-function, $\{j_n\}_{n\geq 1}$ is a strictly decreasing sequence from and after the eight term and $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) = 727/2187 < 1/k$ for some $k \in (1,\infty)$. Whence, ψ is also a function of $1/k$-contractive factor. An example which is not a $\mathscr{D}$-function is provided hereunder.

Example 7.
Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be given by

$$\psi(\widehat{t}) = \begin{cases} \dfrac{\sin \widehat{t}}{\widehat{t}}, & \text{if } \widehat{t} \in \left(0, \dfrac{\pi}{2}\right] \\ \dfrac{1}{\widehat{t} + k^2}, & \textit{elsewhere}. \end{cases} \tag{5}$$

Since $\limsup_{r\longrightarrow 0^+} \psi(r) = 1$, then ψ is not a $\mathscr{D}$-function.

Remark 8.

(i) Note that if $\psi_{\widetilde{\mathscr{MT}}} = k\psi(\widehat{t})$ for all $\widehat{t} \in \mathbb{R}_+$ and for some $k \in (1,\infty)$, then $\psi_{\widetilde{\mathscr{MT}}}$ becomes an $\widetilde{\mathscr{MT}}$-function, provided ψ is a $\mathscr{D}$-function

(ii) If we define $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ as $\psi(\widehat{t}) = 1/k^n$ for all $n \geq 2$ and $k \in (1,\infty)$, then ψ is a $\mathscr{D}$-function

The following Lemma is in consistent with [16, Lemma 18].

Lemma 9.
Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Then $\rho : \mathbb{R}_+ \longrightarrow [0, (1/k))$ given by $\rho(\widehat{\widehat{t}}) = (\psi(\widehat{\widehat{t}}) + (1/k))/2$ is also a $\mathscr{D}$-function for each $\widehat{\widehat{t}} \in \mathbb{R}_+$ and some $k \in (1,\infty)$.

Proof. Obviously, $\psi(\widehat{t}) < \rho(\widehat{t})$ and $0 < \rho(\widehat{t}) < (1/k)$. Let $\widehat{t} \in \mathbb{R}_+$ be fixed. Since $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is a $\mathscr{D}$-function, we can find $\sigma_{\widehat{t}} \in [0, (1/k))$ and $\delta_{\widehat{t}} > 0$ such that $\psi(s) \leq \sigma_{\widehat{t}}$ for all $s \in [\widehat{t}, \widehat{t} + \delta_{\widehat{t}})$. Assume that $\eta_{\widehat{t}} \coloneqq (\sigma_{\widehat{t}} + (1/k))/2 \in [0, (1/k))$. Then, $\rho(s) \leq \eta_{\widehat{t}}$ for all $s \in [\widehat{t}, \widehat{t} + \delta_{\widehat{t}})$. Thus, ρ is a $\mathscr{D}$-function. □

The following result due to Nadler [26] is the first metric fixed point theorem for multivalued contractions.

Theorem 10. (see [10]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued λ-contraction, that is, we can find $\lambda \in (0, 1)$ such that*

$$\tilde{H}(Yj, Y\ell) \leq \lambda\mu(j, \ell), \tag{6}$$

for all $j, \ell \in \mho$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

In 2007, Berinde-Berinde [11] presented the following notable fixed point Theorem.

Theorem 11. (see [11]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Assume that we can find $L \geq 0$ such that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell) + L\mu(\ell, Yj), \tag{7}$$

for all $j, \ell \in \mho$ with $j \neq \ell$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Observe that if we take $L = 0$ in Theorem 11, we realize the Mizoguchi-Takahashi fixed point theorem [14] which partially answered the problem posed in Reich [8].

Theorem 12. (see [8]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow \mathscr{K}(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Suppose that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell), \tag{8}$$

for all $j, \ell \in \mho$ with $j \neq \ell$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

In [8], Reich raised the question whether Theorem 12 is also valid when $\mathscr{K}(\mho)$ is replaced with $CB(\mho)$. In 1989, Mizoguch-Takahashi [14] responded to this puzzle in affirmative via the following result.

Theorem 13. (see [14]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Suppose that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell), \tag{9}$$

for all $j, \ell \in \mho$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Let A be a nonempty subset of $\mho$ and $Y : \mho \longrightarrow \mho$ be a mapping. We recall that the set A is Y-invariant if $Y(A) \subseteq A$. Not long ago, Du [13] obtained the following important fixed point and coincidence point result.

Theorem 14. (see [13]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $g : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Assume that the following conditions hold:*

(Du_1) Yj is g-invariant for each $j \in \mho$;

(Du_2) we can find a function $h : \mho \longrightarrow \mathbb{R}_+$ such that

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell) + h(g\ell)\mu(g\ell, Yj), \quad (10)$$

for all $j, \ell \in \mho$. Then, $\mathscr{COP}(g, Y) \cap \mathscr{F}_{ix}(Y) \neq \emptyset$.

Notice that Mizoguchi-Takahashi fixed point theorem (13) is an extension of Nadler's fixed point theorem (10), but its original proof is not friendly. Alternative proof presented in [29] is also difficult.

Definition 15. (see [9]). Let $(\mho, \mu)$ be a metric space. A single-valued mapping $Y : \mho \longrightarrow \mho$ is named:

Rus contraction if we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $j, \ell \in \mho$,

$$\mu(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(\ell, Y\ell). \quad (11)$$

Ciric-Reich-Rus contraction if we can find $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $j, \ell \in \mho$,

$$\mu(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell). \quad (12)$$

In [9], it was proved that every Rus and Ciric-Reich-Rus contraction has a unique fixed point. These results have been extended to multivalued mappings in the following manner.

Theorem 16. (see [27]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $j, \ell \in \mho$:*

$$\tilde{H}(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(\ell, Y\ell). \quad (13)$$

Then, $\mathscr{F}_{ix}(Y) \neq \emptyset$.

Theorem 17. (see [17]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $j, \ell \in \mho$:*

$$\tilde{H}(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell). \quad (14)$$

Then, $\mathscr{F}_{ix}(Y) \neq \emptyset$.

For more variants of fixed point results of multivalued contractions, the interested reader may consult [30–33] and the references therein.

2. Main Results

In line with the characterizations of $\widetilde{\mathscr{MT}}$-function, we begin this section by launching a few characterizations of $\mathscr{D}$-function in Lemma 18. Its proof is a slight adaption of [17, Theorem 2.1].

Lemma 18.

Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k)), k \in (1, \infty)$. Then, the following statements are equivalent:

(i) ψ is a $\mathscr{D}$-function

(ii) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(1)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(1)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(1)}$ for all $s \in (\hat{t}, \hat{t} + \delta_{\hat{t}}^{(1)})$

(iii) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(2)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(2)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(2)}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}}^{(2)}]$

(iv) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(3)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(3)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(3)}$ for all $s \in (\hat{t}, \hat{t} + \delta_{\hat{t}}^{(3)}]$

(v) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(4)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(4)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(4)}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}}^{(4)}]$

(vi) For any sequence $\{j_n\}_{n \geq 1}$ in $\mathbb{R}_+$, from and after some fixed term, it is nonincreasing and $0 \leq \sup_{n \in \mathbb{N}} \psi(j_n) < (1/k)$

(vii) ψ is a function of $1/k$-contractive factor, that is, for any sequence $\{j_n\}_{n \geq 1}$ in $\mathbb{R}_+$, from and after some fixed term, it is strictly decreasing and $0 \leq \sup_{n \in \mathbb{N}} \psi(j_n) < (1/k)$

The following existence theorem for coincidence point and fixed point is one of the main results of this paper.

Theorem 19.

Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be continuous point-valued mappings, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that the following conditions are obeyed:

(ax_1) for each $j \in \mho$, $\{\Delta\ell = \Theta\ell = \Lambda\ell : \ell \in Yj\} \subseteq Yj$;

(ax_2) we can find three mappings $f, g, h : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}\tilde{H}(Yj, Y\ell) \leq{} & \psi(\mu(j, \ell))[a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\ & + f(\Delta\ell)\mu(\Delta\ell, Yj) + g(\Theta\ell)\mu(\Theta\ell, Yj) \\ & + h(\Lambda\ell)\mu(\Lambda\ell, Yj),\end{aligned} \quad (15)$$

for all $j, \ell \in \mho$, where $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) \neq \emptyset$.

Proof. By (ax_1), we note that for each $j \in \mho$, $\mu(\Delta\ell, Yj) = \mu(\Theta\ell, Yj) = \mu(\Lambda\ell, Yj) = 0$ for all $\ell \in Yj$. So for each $j \in \mho$, it follows from (ax_2) that for all $\ell \in Yj$,

$$\tilde{H}(Yj, Y\ell) \le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]. \tag{16}$$

□

Further, for each $\ell \in Yj$, $\mu(\ell, Y\ell) \le \tilde{H}(Yj, Y\ell)$. Whence, for each $j \in \mho$, (16) gives

$$\begin{aligned}\mu(\ell, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]\\ &\le \frac{\psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj)]}{1 - c\psi(\mu(j,\ell))}\\ &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj)].\end{aligned} \tag{17}$$

Let $j_0 \in \mho$ and choose $j_1 \in Yj_0$. If $\mu(j_0, j_1) = 0$, then $j_0 = j_1 \in Yj_0$, that is, $j_0 \in \mathscr{F}_{ix}(Y)$, and the proof is finished. Otherwise, if $\mu(j_0, j_1) > 0$, then consider a function $\rho : \mathbb{R}_+ \longrightarrow [0, (1/k))$ given by $\rho(t) = ((1/k) + \psi(t))/2$. By Lemma 9, ρ is a $\mathscr{D}$-function and $0 \le \psi(t) < \rho(t) < (1/k)$ for all $t \in \mathbb{R}_+$. From (2.2), it follows that

$$\begin{aligned}\mu(j_1, Yj_1) &\le \psi(\mu(j_0, j_1))[a\mu(j_0, j_1) + b\mu(j_0, Yj_0)]\\ &< \rho(\mu(j_0, j_1))[a\mu(j_0, j_1) + b\mu(j_0, j_1)]\\ &= \rho(\mu(j_0, j_1))[(a+b)\mu(j_0, j_1)].\end{aligned} \tag{18}$$

Since $a + b + c < 1$, then we can find $\eta \in (0,1)$ such that $a + b < \eta = 1 - c < 1$. Thus, (18) can be written as

$$\mu(j_1, Yj_1) < \eta\rho(\mu(j_0, j_1))\mu(j_0, j_1) < \rho(\mu(j_0, j_1))\mu(j_0, j_1). \tag{19}$$

From (19), we claim that we can find $j_2 \in Yj_1$ such that

$$\mu(j_1, j_2) < \rho(\mu(j_0, j_1))\mu(j_0, j_1). \tag{20}$$

Assume that this claim is not true, that is, $\mu(j_1, j_2) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1)$. Then, we get

$$\mu(j_1, j_2) \ge \inf_{\gamma \in Yj_1} \mu(j_1, \gamma) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1), \tag{21}$$

that is, $\mu(j_1, Yj_1) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1)$, contradicting (19). Now, if $\mu(j_1, j_2) = 0$, then $j_1 = j_2 \in Yj_1$ and so $j_1 \in \mathscr{F}_{ix}(Y)$. Otherwise, we can find $j_3 \in Yj_2$ such that

$$\mu(j_2, j_3) < \rho(\mu(j_1, j_2))\mu(j_1, j_2). \tag{22}$$

Let $\tau_n = \mu(j_{n-1}, j_n)$ for each $n \in \mathbb{N}$. Proceeding on similar steps as above, we can construct a sequence $\{j_n\}_{n\in\mathbb{N}}$ in $\mho$ with $j_n \in Yj_{n-1}$ for each $n \in \mathbb{N}$ and

$$\tau_{n+1} < \rho(\tau_n)\tau_n. \tag{23}$$

Given that ψ is a $\mathscr{D}$-function, then by Lemma 18:

$$0 \le \sup_{n\in\mathbb{N}} \psi(\tau_n) < \sup_{n\in\mathbb{N}} \rho(\tau_n) < \frac{1}{k}. \tag{24}$$

Whence,

$$0 < \sup_{n\in\mathbb{N}} \rho(\tau_n) = \left\{\frac{(1/k) + \psi(\tau_n)}{2} : n \in \mathbb{N}, k \in (1,\infty)\right\} < \frac{1}{k} < 1. \tag{25}$$

Take $\xi \coloneqq \sup_{n\in\mathbb{N}} \rho(\tau_n)$, then $0 < \xi < 1$. Since $\rho(t) < (1/k) < 1$ for all $t \in \mathbb{R}_+$, then by (23), $\{\tau_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive real numbers. Therefore, for each $n \in \mathbb{N}$, we have

$$\tau_{n+1} < \rho(\tau_n) \le \xi\tau_n. \tag{26}$$

Whence, it follows from (26) that

$$\mu(j_n, j_{n+1}) = \tau_{n+1} \le \xi\tau_n \le \cdots \le \xi^n\tau_1 = \xi^n d(j_0, j_1). \tag{27}$$

For any $m, n, n_0 \in \mathbb{N}$ with $m > n > n_0$, by (27), we get

$$\begin{aligned}\mu(j_m, j_n) &\le \sum_{j=n}^{m-1} \mu\left(j_j, j_{j+1}\right) \le \sum_{j=n}^{m-1} \xi^j \mu(j_0, j_1) \le \sum_{j=n}^{\infty} \xi^j \mu(j_0, j_1)\\ &\le \frac{\xi^n}{1-\xi}\mu(j_0, j_1) \longrightarrow 0 (\text{as} n \longrightarrow \infty).\end{aligned} \tag{28}$$

Thus, $\limsup_{n\longrightarrow\infty}\{\mu(j_m, j_n): m > n\} = 0$. This proves that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mho$. The completeness of $\mho$ implies that we can find $u \in \mho$ such that $j_n \longrightarrow u$ as $n \longrightarrow \infty$. Since $j_n \in Yj_{n-1}$ for each $n \in \mathbb{N}$, it follows from condition (ax_1) that for each $n \in \mathbb{N}$,

$$\Delta j_n = \Theta j_n = \Lambda j_n \in Yj_{n-1}. \tag{29}$$

Using the continuity of the functions Δ, Θ and Λ, we have

$$\begin{aligned}u &= \lim_{n\longrightarrow\infty} \Delta j_n = \lim_{n\longrightarrow\infty} \Theta j_n = \lim_{n\longrightarrow\infty} \Lambda j_n = \lim_{n\longrightarrow\infty} \Delta u\\ &= \lim_{n\longrightarrow\infty} \Theta u = \lim_{n\longrightarrow\infty} \Lambda u.\end{aligned} \tag{30}$$

We claim that $u \in Yu$. Assume contrary so that $\mu(u, Yu) > 0$. Since the function $j \mapsto \mu(j, Yu)$ is continuous, then from condition (ax_2), we realize

$$\begin{aligned}\mu(u, Yu) &= \lim_{n\longrightarrow\infty} \mu(j_n, Yu) \le \lim_{n\longrightarrow\infty} \tilde{H}(Yj_{n-1}, Yu)\\ &\le \lim_{n\longrightarrow\infty} \{\psi(\mu(j_{n-1}, u))[a\mu(j_{n-1}, u) + b\mu(j_{n-1}, Yj_{n-1})\\ &\quad + c\mu(u, Yu)] + f(\Delta u)\mu(\Delta u, Yj_{n-1})\\ &\quad + g(\Theta u)\mu(\Theta u, Yj_{n-1}) + h(\Lambda u)\mu(\Lambda u, Yj_{n-1})\}\\ &< \lim_{n\longrightarrow\infty} \{\rho(\mu(j_{n-1}, u))[a\mu(j_{n-1}, u) + b\mu(j_{n-1}, j_n)\\ &\quad + c\mu(u, Yu)] + f(\Delta u)\mu(\Delta u, j_n)\\ &\quad + g(\Theta u)\mu(\Theta u, j_n) + h(\Lambda u)\mu(\Lambda u, j_n)\}\\ &< \frac{c}{k}(\mu(u, Yu)) < \mu(u, Yu),\end{aligned} \tag{31}$$

a contradiction. Whence, $\mu(u, Yu) = 0$. Since Yu is closed, we have $u \in Yu$. By condition (ax_1), $\Delta u = \Theta u = \Lambda u \in Yu$. Consequently, $u \in \mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y)$.

The following example shows the generality of our Theorem 19 over Theorems 10, 11, 17, and 16 due to Nadler, Berinde-Berinde, Reich, and Rus, respectively.

Example 20.

Let $\mho = \{0, (1/5), 2\}$ and $\mu(j, \ell) = |j - \ell|$ for all $j, \ell \in \mho$. Let $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping and $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be mappings given by

$$Yj = \begin{cases} \{0\}, & \text{if } j = 0 \\ \left\{0, \dfrac{1}{5}\right\}, & \text{if } j = \dfrac{1}{5} \\ \{0, 2\}, & \text{if } j = 2, \end{cases} \tag{32}$$

and $\Delta = \Theta = \Lambda = I_{\mho}$, the identity mapping on $\mho$. Define the function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by $\psi(t) = 1/k^2$ for all $t \in \mathbb{R}_+$ and some $k \in (1, \infty)$. Also, define the mappings $f, g, h : \mho \longrightarrow \mathbb{R}_+$ by $f(j) = g(j) = h(j) = 1/3$ for all $j \in \mho$. Then, we realize the following:

(i) for each $j \in \mho$, $\{\Delta\ell = \Theta\ell = \Lambda\ell : \ell \in Yj\} \subseteq Yj$;

(ii) $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) = \{0, (1/5), 2\}$;

(iii) Δ, Θ and Λ are continuous

Clearly, $\limsup_{s \longrightarrow t^+} \psi(s) = (1/k^2) < (1/k)$ for all $t \in \mathbb{R}_+$ and some $k \in (1, \infty)$. Whence, ψ is a $\mathscr{D}$-function. Furthermore, it is a routine to verify that condition (ax_2) holds for all $j, \ell \in \mho$.

Now, notice that the mapping Y does not obey the hypotheses of Theorem 10 due to Nadler. To see this, let $j = 0$ and $\ell = 2$, then

$$\tilde{H}(Y0, Y2) = \tilde{H}(\{0\}, \{0, 2\}) = 2 > \lambda\mu(0, 2), \tag{33}$$

for all $\lambda \in (0, 1)$. Moreover, to see that Theorem 11 due to Berinde-Berinde fails in this instance, let $L = 1/9$ and $\psi_{\widetilde{\mathscr{MT}}}(t) = k\psi(t)$ for all $t \in \mathbb{R}_+, k \in (1, \infty)$. Then, for all $\lambda \in (0, 1)$,

$$\tilde{H}(Y0, Y2) = 2 > \lambda\mu(0, 2) + \frac{1}{9}\mu(2, Y0). \tag{34}$$

Moreover, to see that Theorems 17 and 16 of Reich and Rus are also not applicable to this example, again take $j = 0$ and $\ell = 2$. Then, by setting $b = c = 0$ and $a = 0$ in Theorems 1.17 and 1.16, respectively, we have

$$\begin{aligned} &\tilde{H}(Y0, Y2) = 2 > a\mu(0, 2) \text{ for all } a \in (0, 1), \\ &\tilde{H}(Y0, Y2) = 2 > b\mu(2, Y2) \text{ for all } b \in (0, 1). \end{aligned} \tag{35}$$

A slight modification of Example A of Du [13] provided below shows the generality of our Theorem 19 over Mizoguch-Takahash's [14] and Du's [13] fixed point theorems.

Example 21.

Let l^∞ be the Banach space of all bounded real sequences endowed with the uniform norm $\|.\|_\infty$, and let $\{e_n\}$ be the canonical basis of l^∞. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers obeying $\tau_1 = \tau_2$ and $\tau_{2n-1} < \tau_n$ for all $n \geq 2$ (for example, take $\tau_1 = 1/9$ and $\tau_n = 1/3^n, n \geq 2$). It follows that $\{\tau_n\}_{n\in\mathbb{N}}$ is convergent. Set $v_n = \tau_n e_n$ for all $n \in \mathbb{N}$, and let $\mho = \{v_n\}_{n\in\mathbb{N}}$ be a bounded and complete subset of l^∞. Then, $(\mho, \|.\|_\infty)$ is a complete metric space and $\|v_n - v_m\|_\infty = \tau_n$ if $m > n$.

Let $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping and Δ, $\Theta, \Lambda : \mho \longrightarrow \mho$ be three mappings, respectively, given by

$$\begin{aligned} &Yv_n = \begin{cases} \{v_1, v_2, v_3\}, & \text{if } n \in \{1, 2, 3\} \\ \{v_{n+1}\}, & \text{if } n > 3, \end{cases} \\ &\Delta v_n = \Theta v_n = \Lambda v_n = \begin{cases} v_2, & \text{if } n \in \{1, 2, 3\} \\ v_{n+1}, & \text{if } n > 3. \end{cases} \end{aligned} \tag{36}$$

Then, we notice that the following results hold:

$$\begin{aligned} &(ax_1)\text{for each } j \in \mho, \{\Delta\ell = \Theta\ell = \Lambda\ell \in Yj\} \subseteq Yj, \\ &(ax_1)\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) = \{v_1, v_2, v_3\}. \end{aligned} \tag{37}$$

To show that Δ, Θ and Λ are continuous, it is suffices to prove that Δ, Θ and Λ are nonexpansive. So we consider the following six possibilities:

(i) $\|\Delta v_1 - \Delta v_2\|_\infty = 0 < \tau_1 = \|v_1 - v_2\|_\infty$

(ii) $\|\Delta v_1 - \Delta v_3\|_\infty = 0 < \tau_1 = \|v_1 - v_3\|_\infty$

(iii) $\|\Delta v_1 - \Delta v_m\|_\infty = \tau_2 = \tau_1 = \|v_1 - v_m\|_\infty$ for any $m > 3$

(iv) $\|\Delta v_2 - \Delta v_m\|_\infty = \tau_2 = \|v_2 - v_m\|_\infty$ for any $m > 3$

(v) $\|\Delta v_3 - \Delta v_m\|_\infty = \tau_2 = \|v_3 - v_m\|_\infty$ for any $m > 3$

(vi) $\|\Delta v_n - \Delta v_m\|_\infty = \tau_{n+1} < \tau_n = \|v_n - v_m\|_\infty$ for any $m > 3$ and $m > n$

Consequently, Δ is nonexpansive, and, since $\Delta = \Theta = \Lambda$, then Δ, Θ and Λ are continuous.

Next, define the function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by

$$\psi(t) = \begin{cases} \dfrac{\tau_{n+2}}{\tau_n}, & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N} \\ 0, & \textit{elsewhere}. \end{cases} \tag{38}$$

Also, define the mappings $f, g, h : \mho \longrightarrow \mho$ by

$$f(\nu_n) = g(\nu_n) = h(\nu_n) = \begin{cases} 0, & \text{if } n \in \{1, 2, 3\} \\ \tau_1 n, & \text{if } n > 3. \end{cases} \quad (39)$$

Then, we observe that $\limsup_{s \longrightarrow t^+} \psi(s) = 0 < (1/k)$ for all $t \in \mathbb{R}_+$ and some $k \in (1, \infty)$. It follows that ψ is a $\mathscr{D}$-function. Moreover, we claim that

$$\begin{aligned} \tilde{H}_\infty(Yj, Y\ell) \le \psi(\|j - \ell\|_\infty) &[a\|j - \ell\|_\infty + b\|j - Yj\|_\infty \\ &+ c\|\ell - Y\ell\|_\infty] + f(\Delta\ell)\|\Delta\ell - Yj\|_\infty \\ &+ g(\Theta\ell)\|\Theta\ell - Yj\|_\infty + h(\Lambda\ell)\|\Lambda\ell - Yj\|_\infty, \end{aligned} \quad (40)$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$, where $\tilde{H}_\infty$ is the Hausdorff metric induced by the norm $\|.\|_\infty$.

To see (40), we consider the following cases:

Case 1. For $n = 1, m = 2$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_1 - \nu_2\|_\infty)(a\|\nu_1 - \nu_2\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty \\ &\quad + c\|\nu_2 - Y\nu_2\|_\infty) + f(\Delta\nu_2)\|\Delta\nu_2 - Y\nu_1\|_\infty \\ &\quad + g(\Theta\nu_2)\|\Theta\nu_2 - Y\nu_1\|_\infty + h(\Lambda\nu_2)\|\Lambda\nu_2 - Y\nu_1\|_\infty \\ &= \frac{\tau_3}{2} > 0 = \tilde{H}_\infty(Y\nu_1, Y\nu_2). \end{aligned} \quad (41)$$

Case 2. For $n = 1, m = 3$ and $a = 1/4, b = c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_1 - \nu_3\|_\infty)(a\|\nu_1 - \nu_3\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty + c\|\nu_3 - Y\nu_3\|_\infty) \\ &\quad + f(\Delta\nu_3)\|\Delta\nu_3 - Y\nu_1\|_\infty + g(\Theta\nu_3)\|\Theta\nu_3 - Y\nu_1\|_\infty \\ &\quad + h(\Lambda\nu_3)\|\Lambda\nu_3 - Y\nu_1\|_\infty \\ &= \frac{\tau_3}{4} > 0 = \tilde{H}_\infty(Y\nu_1, Y\nu_3). \end{aligned} \quad (42)$$

Case 3. For $n = 1, m > 3$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_1 - \nu_m\|_\infty)(a\|\nu_1 - \nu_m\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty + c\|\nu_m - Y\nu_m\|_\infty) \\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_1\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_1\|_\infty \\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_1\|_\infty \\ &= \frac{\tau_3}{2}(1 + 6\tau_1(m + 1)) > \tau_1 = \tilde{H}_\infty(Y\nu_1, Y\nu_m). \end{aligned} \quad (43)$$

Case 4. For $n = 2, m > 3$ and $a = 1/4, b = c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_2 - \nu_m\|_\infty)(a\|\nu_2 - \nu_m\|_\infty + b\|\nu_2 - Y\nu_2\|_\infty + c\|\nu_m - Y\nu_m\|_\infty) \\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_2\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_2\|_\infty \\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_2\|_\infty \\ &= \frac{\tau_4}{4}\left(1 + \frac{12\tau_1}{\tau_4}(m + 1)\tau_3\right) > \tau_1 = \tilde{H}_\infty(Y\nu_2, Y\nu_m). \end{aligned} \quad (44)$$

Case 5. For $n = 3, m > 3$ and $a = 1/3 = b, c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_3 - \nu_m\|_\infty)(a\|\nu_3 - \nu_m\|_\infty + b\|\nu_3 - Y\nu_3\|_\infty + c\|\nu_m - Y\nu_m\|_\infty) \\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_3\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_3\|_\infty \\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_3\|_\infty \\ &= \frac{\tau_5}{3}(1 + 9\tau_1(m + 1)\tau_3) > \tau_1 = \tilde{H}_\infty(Y\nu_3, Y\nu_m). \end{aligned} \quad (45)$$

Case 6. For $n > 3, m > n$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned} &\psi(\|\nu_n - \nu_m\|_\infty)(a\|\nu_n - \nu_m\|_\infty + b\|\nu_n - Y\nu_n\|_\infty + c\|\nu_m - Y\nu_m\|_\infty) \\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_n\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_n\|_\infty \\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_n\|_\infty \\ &= \frac{\tau_{n+2}}{2} + 3(m + 1)\tau_{n+1} > \tau_{n+1} = \tilde{H}_\infty(Y\nu_n, Y\nu_m). \end{aligned} \quad (46)$$

Therefore, from Cases (1)–(6), we have shown that Condition (40) is obeyed. Consequently, all the assertions of Theorem 19 are obeyed. It follows that $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) \ne \emptyset$.

Now, observe that if we take the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ as earlier given, that is, $\tau_1 = \tau_2, \tau_{2n-1} < \tau_n$, where $\tau_n = 1/3^n$ for all $n \ge 2$ and let $\psi_{\widetilde{\mathscr{MT}}}(t) = 2\psi(t) (i.e. k = 2 \in (1, \infty))$ for all $t \in \mathbb{R}_+$, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function, provided ψ is a $\mathscr{D}$-function. Thus,

(a) for $n = 1$ and any $m > 3$, we have

$$\begin{aligned} \tilde{H}_\infty(Y\nu_1, Y\nu_m) = \tau_1 &> 2\tau_3 \\ &= \psi_{\widetilde{\mathscr{MT}}}(\|\nu_1 - \nu_m\|_\infty)\|\nu_1 - \nu_m\|_\infty. \end{aligned} \quad (47)$$

Whence, Mizoguch-Takahashi's Theorem 13 does not hold in this case.

(b) Let the function $f : \mho \longrightarrow \mho$ be given by

$$f(\nu_n) = \begin{cases} 0, & \text{if } n \in \{1, 2, 3\} \\ \dfrac{\tau_1}{k\tau_n}, & \text{if } n > 3, k \in (1, \infty), \end{cases} \quad (48)$$

and g and h be as given in the above Example. Then, for $n = 1$ and $m > 3$ with $a = 1/2$, $b = c = 0$, the above Case 3 becomes

Case 3$'$:

$$\begin{aligned}
&\psi_{\widetilde{\mathscr{MT}}}\left(\|\nu_1 - \nu_m\|_\infty\right)\left(a\|\nu_1 - \nu_m\|_\infty\right) + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_1\|_\infty \\
&\quad + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_1\|_\infty + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_1\|_\infty \\
&\quad = \tau_3 + \frac{\tau_1}{k\tau_{m+1}} + 2\tau_1(m+1)\tau_3 > \tau_1 = \tilde{H}_\infty(Y\nu_1, Y\nu_m),
\end{aligned} \tag{49}$$

that is, Case 3 also hold. On the other hand, notice that

$$\begin{aligned}
\tilde{H}_\infty(Y\nu_1, Y\nu_m) &= \tau_1 > \tau_3 + \frac{\tau_1}{k\tau_{m+1}} \\
&= \psi_{\widetilde{\mathscr{MT}}}\left(\|\nu_1 - \nu_m\|_\infty\right)\|\nu_1 - \nu_m\|_\infty \\
&\quad + f(\Delta\nu_m)\|\nu_1 - \nu_m\|_\infty,
\end{aligned} \tag{50}$$

that is, the main result of Du [17, Theorem 19] is not applicable here.

3. Consequences

In this section, we deduce some significant consequences of Theorem 19.

Corollary 2.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e. $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find a mapping $f : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + f(\Delta\ell)\mu(\Delta\ell, Yj),
\end{aligned} \tag{51}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. Take $g, h : \mho \longrightarrow \mathbb{R}_+$ as $g(j) = h(j) = 0$ for all $j \in \mho$ in Theorem 19. □

The following result is a direct consequence of Corollary 2.

Corollary 23.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e., $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find $\xi \ge 0$ and a mapping $\widehat{f} : \mho \longrightarrow [0, \xi]$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + \widehat{f}(\Delta\ell)\mu(\Delta\ell, Yj),
\end{aligned} \tag{52}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Corollary 24.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e. $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find $\xi \ge 0$ such that

$$\begin{aligned}
\tilde{H}(Yj, Yy) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + \xi\mu(\Delta\ell, Yj),
\end{aligned} \tag{53}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. Define $\widehat{f} : \mho \longrightarrow [0, \xi]$ as $\widehat{f}(j) = \xi$ for all $j \in \mho$ in Corollary 23. □

By applying Corollary 2, we deduce a generalized version of the primitive Ciric-Reich-Rus fixed point theorem for multivalued mapping as follows.

Corollary 25.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that we can find a mapping $f : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + f(\ell)\mu(\ell, Yj),
\end{aligned} \tag{54}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. Take $\Delta := I_{\mho}$, the identity mapping on $\mho$ in Corollary 2. □

Remark 26.

(i) If we take $\psi_{\widetilde{\mathscr{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1, \infty)$, ψ is a $\mathscr{D}$-function, and set $b = c = 0$, then Corollary 25 reduces to Theorem 13 due to Mizoguchi-Takahashi [14].

(ii) If ψ is a monotonic increasing function such that $0 \leq \psi(t) < (1/k)$ for each $t \in \mathbb{R}_+$ and $k \in (1,\infty)$, then by setting $\psi_{\widetilde{\mathscr{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1,\infty)$ and $b = c = 0$, Corollary 24 generalizes [14, Corollary 2.2]. Also, Corollary 24 includes Theorem 1.2 in [29] as a special case, by extending the range of Y from the family of bounded proximal subsets of $\mho$ to $CB(\mho)$.

(iii) If we take $f(j) = 0$ and $\psi(t) = a\mu(j, \ell)/k^2[a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]$ for all $j, \ell \in \mho$ and $k \in (1,\infty)$, where not all of a, b and c are identically zeros, then Corollary 25 reduces to Theorem 1.10

(iv) If we put $\psi_{\widetilde{\mathscr{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1,\infty)$, ψ is a $\mathscr{D}$-function, take $\Delta := I_{\mho}$, the identity mapping on $\mho$, and set $b = c = 0$, then Corollary 24 reduces to Theorem 11 due to Berinde-Berinde [11].

(v) If we define the multivalued mapping $Y : \mho \longrightarrow CB(\mho)$ as $Yj = \{\phi j\}$ for all $j \in \mho$, where ϕ is a single-valued mapping on $\mho$, then all the results presented herein can be reduced to their single-valued counterparts

(vi) It is clear that more consequences of our main result can be deduced, but we skip them due to the length of the paper

4. Applications to Caputo-Type Fractional Differential Inclusions Model for COVID-19

Very recently, Ahmed et al. [22] investigated the significance of lockdown in curbing the spread of COVID-19 via the following fractional-order epidemic model:

$$\begin{cases} {}^{C}D_{0^+}^{\nu}\tilde{G}(\hat{t}) = \Lambda^{\nu} - \beta^{\nu}\tilde{G}I - \lambda_1\tilde{G}L - \bar{\mu}^{\nu}\tilde{G} + \gamma_1^{\nu}I + \gamma_2^{\nu}I_L + \theta_1^{\nu}\tilde{G}_L, \\ {}^{C}D_{0^+}^{\nu}\tilde{G}_L(\hat{t}) = \lambda_1^{\nu}\tilde{G}L - \bar{\mu}^{\nu}\tilde{G}_L - \theta_1^{\nu}\tilde{G}_L, \\ {}^{C}D_{0^+}^{\nu}I(\hat{t}) = \beta^{\nu}\tilde{G}I - \gamma_1^{\nu} - \alpha_1^{\nu} - \bar{\mu}^{\nu}I + \lambda_2^{\nu}IL + \theta_2^{\nu}I_L, \\ {}^{C}D_{0^+}^{\nu}I_L(\hat{t}) = \lambda_2^{\nu}IL - \bar{\mu}^{\nu}I_L - \theta_2^{\nu} - \gamma_2^{\nu} - \alpha_2^{\nu}I_L, \\ {}^{C}D_{0^+}^{\nu}L(\hat{t}) = \mu^{\nu}I - \phi^{\nu}L, \end{cases} \tag{55}$$

where the total population under study, $N(\hat{t})$ is divided into four components, namely susceptible population that are not under lockdown $\tilde{G}(\hat{t})$, susceptible population that are under lock-down $\tilde{G}_L(\hat{t})$, infective population that are not under lockdown $I(\hat{t})$, infective population that are under lock-down $I_L(\hat{t})$, and cumulative density of the lockdown program $L(\hat{t})$. For the meaning of the rest parameters and numerical simulations of (55), we refer the reader to [22]. The above model (55) is simplified as follows:

$$\begin{cases} {}^{C}D_{0^+}^{\nu}\tilde{G}(\hat{t}) = \Theta_1\left(\hat{t}, \tilde{G}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D_{0^+}^{\nu}\tilde{G}_L(\hat{t}) = \Theta_2\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D_{0^+}^{\nu}I(\hat{t}) = \Theta_3\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D_{0^+}^{\nu}I_L(\hat{t}) = \Theta_4\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D_{0^+}^{\nu}L(\hat{t}) = \Theta_5\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \end{cases} \tag{56}$$

where

$$\begin{cases} \Theta_1\left(\hat{t}, \tilde{G}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \Lambda^{\nu} - \beta^{\nu}\tilde{G}I - \lambda_1\tilde{G}L - \bar{\mu}^{\nu}\tilde{G} + \gamma_1^{\nu}I + \gamma_2^{\nu}I_L + \theta_1^{\nu}\tilde{G}_L, \\ \Theta_2\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \lambda_1^{\nu}\tilde{G}L - \bar{\mu}^{\nu}\tilde{G}_L - \theta_1^{\nu}\tilde{G}_L, \\ \Theta_3\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \beta^{\nu}\tilde{G}I - \gamma_1^{\nu} - \alpha_1^{\nu} - \bar{\mu}^{\nu}I + \lambda_2^{\nu}IL + \theta_2^{\nu}I_L, \\ \Theta_4\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \lambda_2^{\nu}IL - \bar{\mu}^{\nu}I_L - \theta_2^{\nu} - \gamma_2^{\nu} - \alpha_2^{\nu}I_L, \\ \Theta_5\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \mu^{\nu}I - \phi^{\nu}L. \end{cases} \tag{57}$$

Consequently, the model (55) takes the form:

$$\begin{cases} {}^{C}D_{0}^{\nu} j(\hat{t}) = g(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = [0.b], 0 < \nu < 1 \\ j(0) = j_0 \geq 0, \end{cases} \tag{58}$$

with the condition:

$$\begin{cases} j(\hat{t}) = \left(\tilde{G}, \tilde{G}_L, I, I_L, L\right)^{tr}, \\ j(0) = \left(\tilde{G}_0, \tilde{G}_{L_0}, I_0, I_{L_0}, L_0\right)^{tr}, \\ g(\hat{t}, j(\hat{t})) = \left(\Theta_i\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right)\right)^{tr}, i = 1, \cdots, 5, \end{cases} \tag{59}$$

where $(.)^{tr}$ denotes the transpose operation.

In this section, we extend problem (55) to its multivalued analogue given by

$$\begin{cases} {}^{C}D_{0}^{\nu} j(\hat{t}) \in M(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = (0, \delta) \\ j(0) = j_0 \geq 0, \end{cases} \tag{60}$$

where $M : \Omega \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a multivalued mapping ($P(\mathbb{R})$ is the power set of $\mathbb{R}$). We launch existence criteria for solutions of the inclusion problem (60) for which the right hand side is nonconvex with the aid of standard fixed point theorem for multivalued contraction mapping. First, we outline some preliminary concepts of fractional calculus and multivalued analysis as follows.

Definition 27. (see [34]). Let $\mathrm{v} > 0$ and $\mathrm{f} \in \mathrm{L}'([0, \delta], \mathbb{R})$. Then, the Riemann-Liouville fractional integral order v for a function f is given as

$$I_{0^+}^{\nu} f(\hat{t}) = \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \mu\tau, \hat{t} > 0, \tag{61}$$

where $\Gamma(.)$ is the gamma function given by $\Gamma(\nu) = \int_0^{\infty} \tau^{\nu-1} e^{-\tau} \mu\tau$.

Definition 28. (see [34]). Let $\mathrm{n} - 1 < \mathrm{v} < \mathrm{n}, \mathrm{n} \in \mathbb{N}$, and $\mathrm{f} \in \mathrm{C}^{\mathrm{n}}(0, \delta)$. Then, the Caputo fractional derivative of order v for a function f is given as

$${}^{C}D_{0^+}^{\nu} f(\hat{t}) = \frac{1}{\Gamma(n-\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{n-\nu-1} f^n(\tau) \mu\tau, \hat{t} > 0. \tag{62}$$

Lemma 29. (see [34]). *Let $\mathfrak{R}(\nu) > 0$, $n = [\mathfrak{R}(\nu)] + 1$, and $f \in AC^n(0, \delta)$. Then,*

$$\left(I_{0^+}^{\nu} {}^{C}D_{0^+}^{\nu} f\right)(\hat{t}) = f(\hat{t}) - \frac{\sum_{k=1}^{m} \left(D_{0^+}^{k} f\right)(0^+)}{k!}. \tag{63}$$

In particular, if $0 < \nu \leq 1$, then $(I_{0^+}^{\nu} {}^{C}D_{0^+}^{\nu} f)(\hat{t}) = f(\hat{t}) - f(0)$.

In view of Lemma 29, the integral reformulation of problem 16 which is equivalent to the model 13 is given by

$$j(\hat{t}) = j_0 + I_{0^+}^{\nu} g(\hat{t}, j(\hat{t})) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} g(\tau, j(\tau)) \mu\tau. \tag{64}$$

Let $\mho = C(\Omega, \mathbb{R})$ denotes the Banach space of all continuous functions j from Ω to $\mathbb{R}$ equipped with the norm given by

$$\|j\| = \sup \left\{ |j(\hat{t})| : \hat{t} \in \Omega = [0, \delta] \right\}, \tag{65}$$

where

$$|j(\hat{t})| = |\tilde{G}(\hat{t})| + |\tilde{G}_L(\hat{t})| + |I(\hat{t})| + |I_L(\hat{t})| + |L(\hat{t})|, \tag{66}$$

and $\tilde{G}, \tilde{G}_L, I, I_L, L \in \mho$.

Definition 30.

Let $\mho$ be a nonempty set. A single-valued mapping $\mathrm{f} : \mho \longrightarrow \mho$ is named a selection of a multivalued mapping $\mathrm{M} : \mho \longrightarrow \mathrm{P}(\mho)$, if $\mathrm{f}(\mathrm{j}) \in \mathrm{M}(\mathrm{j})$ for each $\mathrm{j} \in \mho$.

For each $j \in \mho$, we define the set of all selections of a multi-valued mapping M by

$$\tilde{G}_{M,j} = \left\{ f \in L'(\Omega, \mathbb{R}) : f(\hat{t}) \in M(\hat{t}, j(\hat{t})) \text{ for } a.e. \hat{t} \in \Omega \right\}. \tag{67}$$

Definition 31. A function $\mathrm{j} \in \mathrm{C}'(\Omega, \mathbb{R})$ is a solution of problem (60) if there is a function $\varphi \in \mathrm{L}'(\Omega, \mathbb{R})$ with $\varphi(\hat{\mathrm{t}}) \in \mathrm{M}(\hat{\mathrm{t}}, \mathrm{j}(\hat{\mathrm{t}}))$a.e. on Ω such that

$$j(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi(\tau) \mu\tau \tag{68}$$

and $j(0) = j_0 \geq 0$.

Definition 32. A multivalued mapping $\mathrm{M} : \Omega \longrightarrow \mathrm{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable, if for every $\varpi \in \mathbb{R}$, the function $\hat{\mathrm{t}} \mapsto \mu(\varpi, \mathrm{M}(\hat{\mathrm{t}})) = \inf \{|\varpi - \zeta| : \zeta \in \mathrm{M}(\hat{\mathrm{t}})\}$ is measurable.

The following is the main result of this section.

Theorem 33. *Assume that the following conditions are obeyed:*

(N_1) $M : \Omega \times \mathbb{R} \longrightarrow \mathscr{K}(\mathbb{R})$ is such that $M(., j) : \Omega \longrightarrow \mathscr{K}(\mathbb{R})$ is measurable for each $j \in \mathbb{R}$

(N_2) We can find a continuous function $h : \Omega \longrightarrow \mathbb{R}_+$ such that for all $j, \ell \in \mathbb{R}$,

$$\tilde{H}(M(\hat{t}, j), M(\hat{t}, \ell)) \leq h(\hat{t}) |j - \ell|, \tag{69}$$

for almost all $\hat{t} \in \Omega$ and $\mu(0, M(\hat{t}, 0)) \leq h(\hat{t})$ for almost all $\hat{t} \in \Omega$.

Then, the differential inclusion problem (60) has at least one solution on Ω, provided that $\Phi\|h\| < 1$, where $\Phi = b^{\nu}/(\Gamma(\nu+1))$.

Proof. First, we convert the differential inclusions (60) into a fixed point problem. For this, let $\mho = C(\Omega, \mathbb{R})$ and consider the multivalued mapping $Y : \mho \longrightarrow P(\mho)$ given by

$$Y(j) = \left\{ \begin{array}{c} \nabla \in \mho : \\ \nabla(\hat{t}) = j_0 + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi(\tau) \mu\tau, \varphi \in \tilde{G}_{M,j} \end{array} \right\}. \tag{70}$$

□

Clearly, the fixed points of Y are solutions of problem (60). Now, we prove that Y obeys all the conditions of Theorem 10 under the following cases.

Case 1. $Y(j)$ is nonempty and closed for every $\varphi \in \tilde{G}_{M,j}$. Since the multi-valued mapping $M(.,j(.))$ is measurable, by the measurable selection theorem (see, e.g. [35], Theorem III. 6), it admits a measurable selection $\varphi : \Omega \longrightarrow \mathbb{R}$. Furthermore, by condition (N_2), we get $|\varphi(\hat{t})| \leq h(\hat{t}) + h(\hat{t})|j(\hat{t})|$, that is, $\varphi \in L'(\Omega, \mathbb{R})$, and hence M is integrably bounded. Thus, $\tilde{G}_{M,j}$ is nonempty. Now, we show that $Y(j)$ is closed for each $j \in \mho$. Let $\{\varsigma_n\}_{n\in\mathbb{N}} \in Y(j)$ be such that $\varsigma_n \longrightarrow u$ $(n \longrightarrow \infty)$ in $\mho$. Then, $u \in \mho$, and we can find $\varphi_n \in \tilde{G}_{M,j_n}$ such that for each $\hat{t} \in \Omega$,

$$\varsigma_n(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi_n(\tau) \mu\tau. \tag{71}$$

Since M has compact values, we pass onto a subsequence to obtain that φ_n converges to $u \in L'(\Omega, \mathbb{R})$. Therefore, $u \in \tilde{G}_{M,j}$ and for each $\hat{t} \in \Omega$, we have

$$\varsigma_n(\hat{t}) \longrightarrow u(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi(\tau) \mu\tau. \tag{72}$$

Thus, $u \in Y(j)$.

Case 2. Next, we prove that we can find $a \in (0,1)(a = \Phi\|h\|)$ such that $\tilde{H}(Y(j), Y(\ell)) \leq a\|j - \ell\|$ for each $j, \ell \in \mho$. Let $j, \ell \in \mho$ and $\nabla_1 \in Y(j)$. Then, we can find $\varphi_1(\hat{t}) \in M(\hat{t}, j(\hat{t}))$ such that for each $\hat{t} \in \Omega$,

$$\nabla_1(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} + \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi_1(\tau) \mu\tau. \tag{73}$$

By (N_2), $\tilde{H}(M(\hat{t}, j), M(\hat{t}, \ell)) \leq h(\hat{t})\|j - \ell\|$. Whence, we can find $\rho \in M(\hat{t}, \ell(\hat{t}))$ such that

$$|\nabla_1(\hat{t}) - \rho(\hat{t})| \leq h(\hat{t}) |j(\hat{t}) - \ell(\hat{t})|, \hat{t} \in \Omega. \tag{74}$$

Define $\Xi : \Omega \longrightarrow P(\mathbb{R})$ by

$$\Xi(\hat{t}) = \left\{ \hat{t} \in \mathbb{R} : |\nabla_1(\hat{t}) - \rho(\hat{t})| \leq h(\hat{t}) |j(\hat{t}) - \ell(\hat{t})| \right\}. \tag{75}$$

Since the multivalued mapping $\Xi(\hat{t}) \cap M(\hat{t}, \ell(\hat{t}))$ is measurable (see ([35], Proposition III.4)), we can find a function φ_2 which is a measurable selection of Ξ. Thus, $\varphi_2(\hat{t}) \in M(\hat{t}, \ell(\hat{t}))$, and for each $\hat{t} \in \Omega$, we have $|\varphi_1(\hat{t}) - \varphi_2(\hat{t})| \leq h(\hat{t})|j(\hat{t}) - \ell(\hat{t})|$. For each $\hat{t} \in \Omega$, take

$$\nabla_2(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi_2(\tau) \mu\tau. \tag{76}$$

Then, from (73) and (76), we realize

$$\begin{aligned} |\nabla_1(\hat{t}) - \nabla_2(\hat{t})| &\leq \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} [|\varphi_1(\tau) - \varphi_2(\tau)|] \mu\tau \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \left[h(\hat{t}) |j(\hat{t}) - \ell(\hat{t})||| \right] \mu\tau \\ &\leq \frac{b^{\nu}}{\Gamma(\nu+1)} \|h\| \|j - \ell\| = \Phi\|h\|\|j - \ell\|. \end{aligned} \tag{77}$$

Therefore, $\|\nabla_1 - \nabla_2\| \leq \Phi\|h\|\|j - \ell\|$. On similar steps, interchanging the roles of j and ℓ, we have

$$\tilde{H}(Y(j), Y(\ell)) \leq \Phi\|h\|\|j - \ell\| = a\|j - \ell\|. \tag{78}$$

Note that if we take $f(j) = 0$ and $\psi(\hat{t}) = (\Phi\|h\|\|j - \ell\|)/(k^2[\Phi\|h\|\|j - \ell\| + b\|j - Yj\| + c\|\ell - Y\ell\|])$ for all $j, \ell \in \mho$ and $k \in (1, \infty)$, then (54) coincides with (78). Whence, Corollary 25 can be applied to conclude that the mapping Y has at least one fixed point in $\mho$ which corresponds to the solutions of Problem 4.6.

Example 34. Consider the Caputo-type fractional differential inclusion problem given by

$$\begin{cases} {}^{C}D_0^{3/5} j(\hat{t}) \in M(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = [0, 1], \\ j(0) = 0, \end{cases} \tag{79}$$

where the multivalued mapping $M : [0,1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is given as

$$M(\hat{t}, j(\hat{t})) = \left[\frac{1}{50}, \frac{1}{9 + 10\hat{t}} \left(\frac{\sin^2 j(\hat{t})}{2 - \sin |j(\hat{t})|} \right) + \frac{1}{30} \right]. \tag{80}$$

Obviously, the mapping $j \mapsto [1/50, (1/9 + 10\hat{t})(\sin^2 j(\hat{t})/2 - \sin |j(\hat{t})|) + 1/30]$ is measurable for each $j \in \mathbb{R}$. In this case, we can take $h(\hat{t}) = 1/(9 + 10\hat{t})$ for all $\hat{t} \in [0, 1]$, and thus, $\mu(0, M(\hat{t}, 0)) = 1/30 \leq h(\hat{t})$ for almost all $\hat{t} \in [0, 1]$. Note that for each $j, \ell \in \mathbb{R}$, we have

$$\begin{aligned}
&\tilde{H}\left(M\left(\hat{t}, j\left(\hat{t}\right)\right), M\left(\hat{t}, \ell\left(\hat{t}\right)\right)\right) \\
&\quad= \left(\left[\frac{1}{50}, \frac{1}{9+10\hat{t}}\left(\frac{\sin^2 j\left(\hat{t}\right)}{2 - \sin\left|j\left(\hat{t}\right)\right|}\right) + \frac{1}{30}\right],\right. \\
&\quad\quad \left.\cdot\left[\frac{1}{50}, \frac{1}{9+10\hat{t}}\left(\frac{\sin^2 \ell\left(\hat{t}\right)}{2 - \sin\left|\ell\left(\hat{t}\right)\right|}\right) + \frac{1}{30}\right]\right) \\
&\quad\leq \frac{1}{9+10\hat{t}}\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right| = h\left(\hat{t}\right)\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right|.
\end{aligned} \tag{81}$$

Moreover, $\|h\| = 1/9$. Whence, $\Phi\|h\| \approx 0.124355 < 1$. Consequently, by Theorem 38, Problem (68) has at least one solution on $[0, 1]$.

5. Stability Results

Investigated as a type of data dependence, the concept of Ulam stability was initiated by Ulam [36] and developed by Hyers [37], Rassias [38], and later on by many authors. In this section, we study an Ulam-Hyers type stability of the proposed fractional-order model 4.6. In [22], the stability result of the model 4.4 has been obtained in the framework of single-valued mappings. But, it is a known fact that multivalued mappings often have more fixed points than their corresponding single-valued mappings. Whence, the set of fixed points of set-valued mappings becomes more interesting for the study of stability. First, we give some needed definitions as follows.

Let $\varepsilon > 0$ and consider the following inequality:

$$\left|{}^{C}D_{0^+}^{\nu} j^*\left(\hat{t}\right) - j^*\left(\hat{t}\right)\right| \leq \varepsilon, \hat{t} \in \Omega a.e. \tag{82}$$

Definition 35. The proposed problem (60) is Ulam-Hyers stable if we can find a real number $\varsigma^* > 0$ such that for every $\varepsilon > 0$ and for each solution $j^* \in C(\Omega, \mathbb{R})$ of the inequality (82), we can find a solution $j \in C(\Omega, \mathbb{R})$ of problem (60) and two functions $\varphi^*, \varphi \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ and $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$ a.e. on Ω such that

$$\left\|j^*\left(\hat{t}\right) - j\left(\hat{t}\right)\right\| \leq \varsigma^* \varepsilon, \tag{83}$$

for almost all $\hat{t} \in \Omega$, where $\|j\| = \sup\{|j(\hat{t})|: \hat{t} \in \Omega a.e.\}$.

Remark 36. A function $j^* \in C(\Omega, \mathbb{R})$ is a solution of the inequality (82) if and only if we can find a continuous function $m : \Omega \longrightarrow \mathbb{R}$ and $\varphi^* \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ a.e. on Ω such that the following properties hold:

(i) $|m(\hat{t})| \leq \varepsilon, m = \max\,(m_j)^{tr}, \hat{t} \in \Omega a.e.$

(ii) ${}^{C}D_{0^+}^{\nu} j^*\left(\hat{t}\right) = j^*\left(\hat{t}\right) + m(\hat{t}), \hat{t} \in \Omega$ *a.e.*

Lemma 37. *Suppose that $j^* \in C(\Omega, \mathbb{R})$ obeys the inequality (82), then we can find a function $\varphi^* \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$a.e. on Ω such that*

$$\left|j^*\left(\hat{t}\right) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t} - \tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau\right| \leq \Phi\varepsilon. \tag{84}$$

Proof. From (ii) of Remark 36, we have ${}^{C}D_{0^+}^{\nu} j^*(\hat{t}) = j^*(\hat{t}) + m(\hat{t})$, and by Lemma 29, we get

$$\begin{aligned}
j^*\left(\hat{t}\right) = j_0^* &+ \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t} - \tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau \\
&+ \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t} - \tau\right)^{\nu-1}m(\tau)\mu\tau.
\end{aligned} \tag{85}$$

Therefore, from (i) of Remark 36, we realize

$$\begin{aligned}
&\left|j^*\left(\hat{t}\right) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t} - \tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau\right| \\
&\quad\leq \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t} - \tau\right)^{\nu-1}|m(\tau)|\mu\tau \leq \Phi\varepsilon.
\end{aligned} \tag{86}$$

□

Now, we present the main result of this section as follows.

Theorem 38. *Assume that the following conditions are obeyed:*

(i) the multivalued mapping $M(.,j)$: $\Omega \longrightarrow \mathcal{K}(\mho)$ is measurable for each $j \in \mathbb{R}$

(ii) for all $j, \ell \in \mathbb{R}$, we can find $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$, $\varphi^*(\hat{t}) \in M(\hat{t}, \ell(\hat{t}))$ a.e. on Ω and a continuous function $h : \Omega \longrightarrow \mathbb{R}_+$ such that for almost all $\hat{t} \in \Omega$,

$$\left|\varphi\left(\hat{t}\right) - \varphi^*\left(\hat{t}\right)\right| \leq h\left(\hat{t}\right)\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right|. \tag{87}$$

(iii) $\|h\| < 1/\Phi$, where $\Phi = b^\nu/(\Gamma(\nu + 1))$.

Then the fractional-order inclusion model (60) is Ulam-Hyers stable.

Proof.

Let $j, j^* \in C(\Omega, \mathbb{R})$, where j obeys (82) and j is a solution of problem (60). Then, we can find two functions $\varphi^*, \varphi \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ and $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$*a.e.*

on Ω such that for every $\varepsilon > 0$, Lemma 37 can be applied to have

$$\begin{aligned}
\left|j^*(\widehat{t}) - j(\widehat{t})\right| &= \left|j^*(\widehat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\widehat{t}} (\widehat{t}-\tau)^{\nu-1}\varphi(\tau)\mu\tau\right| \\
&= \left|j^*(\widehat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\widehat{t}} (\widehat{t}-\tau)^{\nu-1}\right. \\
&\quad \left. \cdot [\varphi(\tau) - \varphi^*(\tau) + \varphi^*(\tau)]\mu\tau\right| \\
&\le \left|j^*(\widehat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\widehat{t}} (\widehat{t}-\tau)^{\nu-1}\varphi^*(\tau)\mu\tau\right| \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^{\widehat{t}} (\nu-\tau)^{\nu-1}|\varphi(\tau) - \varphi^*(\tau)|\mu\tau \\
&\le \Phi\varepsilon + \frac{b^\nu}{\Gamma(\nu+1)}\|h\|\|j^* - j\| \\
&= \Phi\varepsilon + \Phi\|h\|\|j^* - j\|,
\end{aligned} \tag{88}$$

that is, $\|j^* - j\| \le \varsigma^*\varepsilon$, where $\varsigma^* = \Phi/(1-\Phi\|h\|)$. Consequently, the proposed problem (60) is Ulam-Hyers stable. □

6. Conclusions

A new coincidence and fixed point theorem of multivalued mapping defined on a complete metric space has been presented in this work by using the characterizations of a modified $\widetilde{\mathscr{MT}}$-function, named $\mathscr{D}$-function. It has been noted herein that our result is a generalization of the fixed point theorems due Berinde-Berinde [11], Du [13], Mizoguchi-Takahashi [14], Nadler [10], Reich [17], Rus [27], and a few others in the corresponding literature. Though the conjecture raised by Reich [17] has now been proven valid in an almost complete form in [11, 13, 14], however, our main result (Theorem 19) provided a more general affirmative response to this problem. Moreover, from application perspective, we launched an existence theorem for nonlinear fractional-order differential inclusions model for COVID-19 via a standard fixed point theorem of multivalued mapping. Ulam-Hyers stability analysis of the considered model was also discussed. It is interesting to note that more useful analysis and results may be obtained if the metric on the ground set in this context is either quasi or pseudo metric. For better management of uncertainty, and since every fixed point theorem of contractive multivalued mapping has its fuzzy set-valued analogue, the result of this paper can as well be discussed in the framework of fuzzy fixed point theory and related hybrid models of fuzzy mathematics. Furthermore, in order to obtain effective measures for curbing Covid-19, other than observing the significance of lockdown, numerical simulations and better analytic tools of the proposed fractional-order differential inclusions model are another future directions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Conceptualization was made by M. Alansari. Methodology was made by M. S. Shagari. Formal analysis was made by M. S. Shagari. Review and editing was made by M. Alansari. Funding acquisition was made by M. Alansari. Writing, review, and editing was made by M. S. Shagari. In addition, all authors have read and approved the final manuscript for submission and possible publication.

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