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Research Article

Some Sharp Results on Coefficient Estimate Problems for Four-Leaf-Type Bounded Turning Functions

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In this study, we focused on a subclass of bounded turning functions that are linked with a four-leaf-type domain. The primary goal of this study is to explore the limits of the first four initial coefficients, the Fekete-Szegö type inequality, the Zalcman inequality, the Kruskal inequality, and the estimation of the second-order Hankel determinant for functions in this class. All of the obtained findings have been sharp.

1. Introduction and Definitions

Before getting into the key findings, some prior information on function theory fundamentals is required. In this case, the symbols $\mathscr{A}$ and $\mathscr{S}$ indicate the families of normalised holomorphic and univalent functions, respectively. These families are specified in the set-builder form:

$$\mathscr{A}=\left\{g\in\mathscr{Q}(\mathscr{U}_d):\ g(0)=g'(0)-1=0(z\in\mathscr{U}_d)\right\}, \tag{1}$$

$$\mathscr{S}=\{g\in\mathscr{A}:\ g\ \text{is univalent in}\ \mathscr{U}_d\}, \tag{2}$$

where $\mathscr{Q}(\mathscr{U}_d)$ stands for the set of analytic (holomorphic) functions in the disc $\mathscr{U}_d=\{z\in\mathbb{C}\text{and}|z|<1\}$. Thus, if $g\in\mathscr{A}$, then, it can be stated in the series expansion form by

$$g(z)=z+\sum_{k=2}^{\infty}a_k z^k(z\in\mathscr{U}_d). \tag{3}$$

For the given functions $G_1, G_2\in\mathscr{Q}(\mathscr{U}_d)$, the function G_1 is subordinated by G_2 (stated mathematically by $G_1\prec G_2$) if there exists a holomorphic function v in $\mathscr{U}_d$ with the restrictions $v(0)=0$ and $|v(z)|<1$ such that $G_1(z)=G_2(v(z))$. Moreover, if G_2 is univalent in $\mathscr{U}_d$, then

$$G_1(z)\prec G_2(z), (z\in\mathscr{U}_d)\Leftrightarrow G_1(0)=G_2(0)\ \text{and}\ G_1(\mathscr{U}_d)\subset G_2(\mathscr{U}_d). \tag{4}$$

Although the function theory was created in 1851, Bieberbach [1] presented the coefficient hypothesis in 1916, and it made the topic a hit as a promising new research field. De-Brages [2] proved this conjecture in 1985. From 1916 to 1985, many of the world's most distinguished scholars sought to prove or disprove this claim. As a result, they investigated a number of subfamilies of the class $\mathscr{S}$ of univalent functions that are associated with various image domains [3–5]. The most fundamental and significant subclasses of the set $\mathcal{S}$ are the families of starlike and convex functions, represented by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Ma and Minda [6] defined the unified form of the family in 1992 as

$$\mathcal{S}^*(\phi) := \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \phi(z)(z \in \mathcal{U}_d) \right\}, \quad (5)$$

where ϕ indicates the analytic function with $\phi'(0) > 0$ and $\mathfrak{Re}\phi > 0$. Also, the region $\phi(\mathcal{U}_d)$ is star-shaped about $\phi(0) = 1$ and is symmetric along the real axis. They examined some interesting aspects of this class. Some significant subfamilies of the collection $\mathcal{A}$ have recently been investigated as unique instances of the class $\mathcal{S}^*(\phi)$. In particular;

(i) The class $\mathcal{S}^*[L, M] \equiv \mathcal{S}^*(1 + Lz/1 + Mz)$, $-1 \le M < L \le 1$, is obtained by selecting $\phi(z) = 1 + Lz/1 + Mz$ and was established in [7]. Moreover, $\mathcal{S}^*(\xi) := \mathcal{S}^*[1 - 2\xi, -1]$ displays the well-known order ξ $(0 \le \xi < 1)$ starlike function class

(ii) The class $\mathcal{S}^*_{\mathcal{L}} := \mathcal{S}^*(\phi(z))$ with $\phi(z) = \sqrt{1 + z}$ was designed by the researchers Sokól and Stankiewicz in [8]. Also, they showed that the image of the function $\phi(z) = \sqrt{1 + z}$ is bounded by $|w^2 - 1| < 1$.

(iii) The set $\mathcal{S}^*_{car} := \mathcal{S}^*(\phi(z))$ with $\phi(z) = 1 + 4/3z + 2/3 z^2$ has been deduced by Sharma and his coauthors [9] in which they located the image domain of $\phi(z) = 1 + 4/3z + 2/3z^2$, which is bounded by the below cardioid

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0. \quad (6)$$

(iv) By selecting $\phi(z) = 1 + \sin z$, we get the class $\mathcal{S}^*(\phi(z)) = \mathcal{S}^*_{\sin}$, which was defined in [10] while $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ was contributed by the authors [11] and, subsequently, explored some more properties of it in [12]. This class was recently generalized by Srivastava et al. [13] in which the authors determined upper bound of Hankel determinant of order three

(v) The family $\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}^*_{\cosh} := \mathcal{S}^*(\cosh(z))$ were offered, respectively, by Raza and Bano [14] and Alotaibi et al. [15]. In both the papers, the authors studied some good properties of these families

(vi) By choosing $\phi(z) = 1 + \sinh^{-1} z$, we obtain the recently studied class $\mathcal{S}^*_\rho := \mathcal{S}^*(1 + \sinh^{-1} z)$ created by Al-Sawalha [16]. Barukab and his coauthors [17] studied the sharp Hankel determinant of third-order for the following class in 2021

$$\mathcal{R}_s = \left\{ g \in \mathcal{A} : g'(z) \prec 1 + \sinh^{-1} z, z \in \mathcal{U}_d \right\}. \quad (7)$$

In [18, 19], Pommerenke provided the following Hankel determinant $\mathcal{D}_{q,n}(g)$ containing coefficients of a function $g \in \mathcal{S}$

$$\mathcal{D}_{q,n}(g) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (8)$$

with $q, n \in \mathbb{N} = \{1, 2, \cdots\}$. By varying the parameters q and n, we get the determinants listed below:

$$\mathcal{D}_{2,1}(g) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad (9)$$

$$\mathcal{D}_{2,2}(g) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2, \quad (10)$$

$$\mathcal{D}_{3,1}(g) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2 a_4 - a_3^2\right) - a_4(a_4 - a_2 a_3) + a_5\left(a_3 - a_2^2\right), \quad (11)$$

that referred as first-, second-, and third-order Hankel determinants, respectively. The Hankel determinant for functions belonging to the general family $\mathcal{S}$ has just a few references in the literature. The best established sharp inequality for the function $g \in \mathcal{S}$ is $|\mathcal{D}_{2,n}(g)| \le \lambda\sqrt{n}$, where λ is a constant, and it is because of Hayman [20]. Additionally, it was determined in [21] for the class $\mathcal{S}$ that

$$\left|\mathcal{D}_{2,2}(g)\right| \le \lambda, \text{ for } 1 \le \lambda \le \frac{11}{3}, \quad (12)$$

$$\left|\mathcal{D}_{3,1}(g)\right| \le \mu, \text{ for } \frac{4}{9} \le \mu \le \frac{32 + \sqrt{285}}{15}. \quad (13)$$

Several mathematicians were drawn to the problem of finding the sharp bounds of Hankel determinants in a given family of functions. In this context, Janteng et al. [22, 23] estimated the sharp bounds of $|\mathcal{D}_{2,2}(g)|$, for three basic subfamilies of the set $\mathcal{S}$. These families are $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$ (functions of a bounded turning class), and these bounds are stated as

$$\left|\mathcal{D}_{2,2}(g)\right| \le \begin{cases} \frac{1}{8}, & \text{for } g \in \mathcal{K}, \\ 1, & \text{for } g \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for } g \in \mathcal{R}. \end{cases} \quad (14)$$

This determinant's exact bound for the unified collection $\mathcal{S}^*(\phi)$ was determined in [24] and subsequently investigated in [25]. In [26–28], this problem was also solved for various families of biunivalent functions.

The formulae provided in (11) make it abundantly evident that the computation of $|\mathcal{D}_{3,1}(g)|$ is much more difficult than determining the bound of $|\mathcal{D}_{2,2}(g)|$. Babalola [29] was the first mathematician who studied third-order Hankel determinant for the $\mathcal{K}$,$\mathcal{S}^*$, and $\mathcal{R}$ families in 2010. Following that, several academics [30–34] used the same method to publish papers regarding $|\mathcal{D}_{3,1}(g)|$ for specific subclasses of univalent functions. However, Zaprawa's work [35] caught the researcher's attention, in which he improved Babalola's results by utilising a revolutionary method to show that

$$|\mathcal{D}_{3,1}(g)| \le \begin{cases} \frac{49}{540}, & \text{for } g \in \mathcal{K}, \\ 1, & \text{for } g \in \mathcal{S}^*, \\ \frac{41}{60}, & \text{for } g \in \mathcal{R}. \end{cases} \quad (15)$$

He also pointed out that these bounds are not sharp. In 2018, Kwon et al. [36] achieved a more acceptable finding for $g \in \mathcal{S}^*$ and demonstrated that $|\mathcal{D}_{3,1}(g)| \le 8/8$, and this limit was further enhanced by Zaprawa and his coauthors [37] in 2021. They got $|\mathcal{D}_{3,1}(g)| \le 5/9$ for $g \in \mathcal{S}^*$. In recent years, Kowalczyk et al. [38] and Lecko et al. [39] got a sharp bound of third Hankel determinant given by

$$|\mathcal{D}_{3,1}(g)| \le \begin{cases} \frac{4}{135}, & \text{for } g \in \mathcal{K}, \\ \frac{1}{9}, & \text{for } g \in \mathcal{S}^*\left(\frac{1}{2}\right), \end{cases} \quad (16)$$

where $\mathcal{S}^*(1/2)$ is the starlike functions family of order 1/2. In [40], the authors obtained the sharp bounds of third Hankel determinant for the subclass of $\mathcal{S}^*_{\sin}$, and Mahmood et al. [41] calculated the third Hankel determinant for starlike functions in q-analogue. For some new literature on sharp third-order Hankel determinant, see [42–45].

In [46], Gandhi introduced a family of bounded turning function connected with a four-leaf function defined by

$$\mathcal{S}^*_{4\mathcal{L}} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, (z \in \mathcal{U}_d) \right\}, \quad (17)$$

and characterized it with some important properties.

Similar to the definition of $\mathcal{S}^*_{4\mathcal{L}}$, we now define a new subfamily of bounded turning functions by the following set builder notation:

$$\mathcal{BT}_{4\mathcal{L}} := \left\{ g \in \mathcal{S} : g'(z) \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, (z \in \mathcal{U}_d) \right\}. \quad (18)$$

The aim of the current manuscript is to determine the exact bounds of the coefficient inequalities, Fekete-Szegö type problem, Kruskal inequality, and Hankel determinant of order two for functions of bounded turning class linked with four-leaf domain.

2. A Set of Lemmas

We say a function $p \in \mathcal{P}$ if and only if it has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n (z \in \mathcal{U}_d), \quad (19)$$

along with the $\Re p(z) \ge 0 (z \in \mathcal{U}_d)$.

Lemma 1. *Let $p \in \mathcal{P}$ be represented by (19). Then*

$$|c_n| \le 2\, n \ge 1. \quad (20)$$

$$|c_{n+k} - \mu c_n c_k| \le 2 \max\{1, |2\mu - 1|\} = \begin{cases} 2 & \text{for } 0 \le \mu \le 1; \\ 2|2\mu - 1| & \text{otherwise.} \end{cases} \quad (21)$$

Also, If $B \in [0, 1]$ with $B(2B - 1) \le D \le B$, we have

$$|c_3 - 2Bc_1c_2 + Dc_1^3| \le 2. \quad (22)$$

These inequalities (20), (21), and (22) are taken from [47, 48].

Lemma 2. *Let $p \in \mathcal{P}$ and be given by (19). Then, for $x, \delta, \rho \in \bar{\mathcal{U}}_d$, we have*

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad (23)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\delta, \quad (24)$$

For the formula c_2, see [48]. The formula c_3 was due to Zlotkiewicz and Libera [49] while the formula for c_4 was proved in [50].

Lemma 3 [51]. *Let α, β, γ, and a satisfy that $a, \alpha \in (0, 1)$ and*

$$8a(1 - a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a + \alpha) - \beta)^2\right) + \alpha(1 - \alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1 - \alpha)^2(1 - a). \quad (25)$$

If $p \in \mathcal{P}$ and be given by (19), then

$$\left| \gamma c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right| \le 2. \quad (26)$$

3. Coefficient Inequalities for the Class $\mathcal{BT}_{4\mathcal{L}}$

We begin this section by finding the absolute values of the first four initial coefficients for the function $\mathcal{BT}_{4\mathcal{L}}$.

Theorem 4. *If $g \in \mathscr{BT}_{4\mathscr{L}}$ and has the series representation (3), then*

$$|a_2| \leq \frac{5}{12}, \tag{27}$$

$$|a_3| \leq \frac{5}{18}, \tag{28}$$

$$|a_4| \leq \frac{5}{24}, \tag{29}$$

$$|a_5| \leq \frac{1}{6}. \tag{30}$$

These bounds are best possible.

Proof. Let $g \in \mathscr{BT}_{4\mathscr{L}}$. Then, (18) can be written in the form of Schwarz function as

$$g'(z) = 1 + \frac{5}{6}\omega(z) + \frac{1}{6}(\omega(z))^5, (z \in \mathscr{U}_d). \tag{31}$$

If $p \in \mathscr{P}$, and it may be written in terms of Schwarz function $w(z)$ as

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{32}$$

Equivalently, we have

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{33}$$

where

$$\omega(z) = \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)z^4 + \cdots. \tag{34}$$

From (3), we get

$$g'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{35}$$

By simplication and using the series expansion of (34), we get

$$1 + \frac{5}{6}\omega(z) + \frac{1}{6}(\omega(z))^5 = 1 + \left(\frac{5}{12}c_1\right)z + \left(-\frac{5}{24}c_1^2 + \frac{5}{12}c_2\right)z^2 + \left(-\frac{5}{12}c_1c_2 + \frac{5}{12}c_3 + \frac{5}{48}c_1^3\right)z^3 + \left(\frac{5}{12}c_4 + \frac{5}{16}c_1^2c_2 - \frac{5}{96}c_1^4 - \frac{5}{24}c_2^2 - \frac{5}{12}c_1c_3\right)z^4 + \cdots. \tag{36}$$

By comparing (35) and (36), we obtain

$$a_2 = \frac{5}{24}c_1, \tag{37}$$

$$a_3 = \frac{1}{3}\left(-\frac{5}{24}c_1^2 + \frac{5}{12}c_2\right), \tag{38}$$

$$a_4 = \frac{1}{4}\left(-\frac{5}{12}c_1c_2 + \frac{5}{12}c_3 + \frac{5}{48}c_1^3\right), \tag{39}$$

$$a_5 = \frac{1}{5}\left(\frac{5}{12}c_4 + \frac{5}{16}c_1^2c_2 - \frac{5}{96}c_1^4 - \frac{5}{24}c_2^2 - \frac{5}{12}c_1c_3\right). \tag{40}$$

For a_2, implementing (20), in (37), we get

$$|a_2| \leq \frac{5}{12}. \tag{41}$$

For a_3, (38) can be written as

$$a_3 = \frac{5}{36}\left(c_2 - \frac{1}{2}c_1^2\right). \tag{42}$$

Using (21), we get

$$|a_3| \leq \frac{5}{18}. \tag{43}$$

For a_4, we can write (39) as

$$|a_4| = \frac{5}{48}\left|\left(c_3 - 2\left(\frac{1}{2}\right)c_1c_2 + \frac{1}{4}c_1^3\right)\right|. \tag{44}$$

From (22), we have

$$0 \leq B = \frac{1}{2} \leq 1, B = \frac{1}{2} \geq D = \frac{1}{4}, \tag{45}$$

$$B(2B-1) = 0 \leq D = \frac{1}{4}. \tag{46}$$

Application of triangle inequality plus (22) leads us to

$$|a_4| \leq \frac{5}{24}. \tag{47}$$

For a_5, we may write (40) as

$$|a_5| = \left| -\frac{1}{96}c_1^4 - \frac{1}{24}c_2^2 - \frac{1}{12}c_1c_3 + \frac{1}{16}c_1^2c_2 + \frac{1}{12}c_4 \right|. \quad (48)$$

After simplifying, we have

$$|a_5| = \frac{1}{12}\left| \frac{1}{8}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{1}{2}\right)c_1^2c_2 - c_4 \right|. \quad (49)$$

Comparing the right side of (49) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \quad (50)$$

we get

$$\gamma = \frac{1}{8}, a = \frac{1}{2}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}. \quad (51)$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0, \end{aligned} \quad (52)$$

$$4a\alpha^2(1-\alpha)^2(1-a) = \frac{1}{16}. \quad (53)$$

From (26), we deduce that

$$|a_5| \le \frac{1}{6}. \quad (54)$$

These bounds are best possible and can be determined by the following extremal functions:

$$g_0(z) = \int_0^z \left(1 + \frac{5}{6}(t) + \frac{1}{6}(t^5)\right) dt = z + \frac{5}{12}z^2 + \frac{1}{36}z^6 + \cdots, \quad (55)$$

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}(t^2) + \frac{1}{6}(t^{10})\right) dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots, \quad (56)$$

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}(t^3) + \frac{1}{6}(t^{15})\right) dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots, \quad (57)$$

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}(t^4) + \frac{1}{6}(t^{20})\right) dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \quad (58)$$

□

Theorem 5. *If g is of the form (3) belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{5}{18}, \frac{25|\gamma|}{144}\right\}, \quad \text{for} \quad \gamma \in \mathbb{C}. \quad (59)$$

This inequality is sharp.

Proof. By using (37) and (38), we may have

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{5}{36}c_2 - \frac{5}{72}c_1^2 - \frac{25}{576}\gamma c_1^2\right|. \quad (60)$$

By rearranging, it yields

$$\left|a_3 - \gamma a_2^2\right| = \frac{5}{36}\left|\left(c_2 - \left(\frac{5\gamma + 8}{16}\right)c_1^2\right)\right|. \quad (61)$$

Application of (21) leads us to

$$\left|a_3 - \gamma a_2^2\right| \le \frac{10}{36}\max\left\{1, \left|\frac{5\gamma + 8}{8} - 1\right|\right\}. \quad (62)$$

After the simplification, we get

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{5}{18}, \frac{25|\gamma|}{144}\right\}. \quad (63)$$

This required result is sharp and is determined by

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}(t^2) + \frac{1}{6}(t^{10})\right) dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots. \quad (64)$$

□

Theorem 6. *If g has the form (3) belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|a_2a_3 - a_4\right| \le \frac{5}{24}. \quad (65)$$

This inequality is best possible.

Proof. By employing (37), (38), and (39), we have

$$\left|a_2a_3 - a_4\right| = \frac{5}{48}\left|c_3 - 2\left(\frac{23}{36}\right)c_1c_2 + \frac{7}{18}c_1^3\right|. \quad (66)$$

From (22), we have

$$0 \le B = \frac{23}{36} \le 1, B = \frac{23}{36} \ge D = \frac{7}{18}, \quad (67)$$

$$B(2B - 1) = \frac{115}{648} \le D = \frac{7}{18}. \quad (68)$$

Using (22), we obtain

$$|a_2a_3 - a_4| \leq \frac{5}{24}. \tag{69}$$

This inequality is best possible and can be obtained by

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^3\right) + \frac{1}{6}\left(t^{15}\right)\right) dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots. \tag{70}$$

□

Theorem 7. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$|a_5 - a_2a_4| \leq \frac{1}{6}. \tag{71}$$

This result is sharp.

Proof. From (37), (39), and (40), we obtain

$$|a_5 - a_2a_4| = \left| -\frac{73}{4608}c_1^4 - \frac{1}{24}c_2^2 - \frac{121}{1152}c_1c_3 + \frac{97}{1152}c_1^2c_2 + \frac{1}{12}c_4 \right|. \tag{72}$$

After simplifying, we have

$$|a_5 - a_2a_4| = \frac{1}{12}\left| \frac{73}{384}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{121}{192}\right)c_1c_3 - \frac{3}{2}\left(\frac{97}{144}\right)c_1^2c_2 - c_4 \right|. \tag{73}$$

Comparing the right side of (73) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \tag{74}$$

we get

$$\gamma = \frac{73}{384}, a = \frac{1}{2}, \alpha = \frac{121}{192}, \beta = \frac{97}{144}. \tag{75}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00735, \end{aligned} \tag{76}$$

and

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.05431. \tag{77}$$

From (26), we deduce that

$$|a_5 - a_2a_4| \leq \frac{1}{6}. \tag{78}$$

The required result is sharp and can be determined by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right) dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{79}$$

□

Theorem 8. *If $g \in \mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$\left|a_5 - a_3^2\right| \leq \frac{1}{6}. \tag{80}$$

This inequality is best possible.

Proof. By using (38) and (40), we have

$$\left|a_5 - a_3^2\right| = \left| -\frac{79}{5184}c_1^4 - \frac{79}{1296}c_2^2 - \frac{1}{12}c_1c_3 + \frac{53}{648}c_1^2c_2 + \frac{1}{12}c_4 \right|. \tag{81}$$

After simplifying, we have

$$\left|a_5 - a_3^2\right| = \frac{1}{12}\left| \frac{79}{432}c_1^4 + \frac{79}{108}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{53}{81}\right)c_1^2c_2 - c_4 \right|. \tag{82}$$

Comparing the right side of (82) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \tag{83}$$

we get

$$\gamma = \frac{79}{432}, a = \frac{79}{108}, \alpha = \frac{1}{2}, \beta = \frac{53}{81}. \tag{84}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00616, \end{aligned} \tag{85}$$

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.04910. \tag{86}$$

From (26), we deduce that

$$\left|a_5 - a_3^2\right| \leq \frac{1}{6}. \tag{87}$$

This inequality is best possible and can be achieved by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right) dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{88}$$

□

4. Kruskal Inequality for the Class $\mathscr{BT}_{4\mathscr{L}}$

In this section, we will give a direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \le 2^{p(n-1)} - n^p, \tag{89}$$

over the class $\mathscr{BT}_{4\mathscr{L}}$ for the choice of $n = 4, p = 1$, and for $n = 5, p = 1$. Krushkal introduced and proved this inequality for the whole class of univalent functions in [52].

Theorem 9. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$\left|a_4 - a_2^3\right| \le \frac{5}{24}. \tag{90}$$

This result is sharp.

Proof. From (37) and (39), we obtain

$$\left|a_4 - a_2^3\right| = \frac{5}{48}\left|c_3 - 2\left(\frac{1}{2}\right)c_1c_2 + \frac{47}{288}c_1^3\right|. \tag{91}$$

From (22), we have

$$0 \le B = \frac{1}{2} \le 1, B = \frac{1}{2} \ge D = \frac{47}{288}, \tag{92}$$

$$B(2B-1) = 0 \le D = \frac{47}{288}. \tag{93}$$

Using (22), we obtain

$$\left|a_4 - a_2^3\right| \le \frac{5}{24}. \tag{94}$$

This result is sharp and can be obtained by

$$g_2(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^3\right) + \frac{1}{6}\left(t^{15}\right)\right)dt = z + \frac{5}{24}z^4 + \frac{1}{96}z^{16} + \cdots. \tag{95}$$

□

Theorem 10. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, and be of the form (3). Then*

$$\left|a_5 - a_2^4\right| \le \frac{1}{6}. \tag{96}$$

This inequality is best possible.

Proof. From (37) and (40), we obtain

$$\left|a_5 - a_2^4\right| = \left|-\frac{4081}{331776}c_1^4 - \frac{1}{24}c_2^2 - \frac{1}{12}c_1c_3 + \frac{1}{16}c_1^2c_2 + \frac{1}{12}c_4\right|. \tag{97}$$

After simplifying, we have

$$\left|a_5 - a_2^4\right| = \frac{1}{12}\left|\frac{4081}{27648}c_1^4 + \frac{1}{2}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{1}{2}\right)c_1^2c_2 - c_4\right|. \tag{98}$$

Comparing the right side of (98) with

$$\left|\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right|, \tag{99}$$

we get

$$\gamma = \frac{4081}{27648}, a = \frac{1}{2}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}. \tag{100}$$

It follows that

$$\begin{aligned} &8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) \\ &\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 = 0.00408, \end{aligned} \tag{101}$$

$$4a\alpha^2(1-\alpha)^2(1-a) = \frac{1}{16}. \tag{102}$$

From (26), we deduce that

$$\left|a_5 - a_2^4\right| \le \frac{1}{6}. \tag{103}$$

This inequality is best possible and can be achieved by

$$g_3(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right)dt = z + \frac{1}{6}z^5 + \frac{1}{126}z^{21} + \cdots. \tag{104}$$

Next, we will calculate the Hankel determinant of order two $|\mathscr{D}_{2,2}(g)|$ for the class $g \in \mathscr{BT}_{4\mathscr{L}}$. □

Theorem 11. *If g belongs to $\mathscr{BT}_{4\mathscr{L}}$, then*

$$\left|\mathscr{D}_{2,2}(g)\right| \le \frac{25}{324}. \tag{105}$$

This inequality is sharp.

Proof. The $\mathscr{D}_{2,2}(g)$ can be written as follows:

$$\mathscr{D}_{2,2}(g) = a_2a_4 - a_3^2. \tag{106}$$

From (37), (38), and (39), we have

$$\mathscr{D}_{2,2}(g) = \frac{25}{1152}c_1c_3 - \frac{25}{10368}c_1^2c_2 + \frac{25}{41472}c_1^4 - \frac{25}{1296}c_2^2. \tag{107}$$

Using (23) and (24) to express c_2 and c_3 in terms of c_1 and, noting that without loss in generality we can write $c_1 = c$, with $0 \le c \le 2$, we obtain

$$\left|\mathscr{D}_{2,2}(g)\right| = \left| -\frac{25}{4608}c^2\left(4-c^2\right)x^2 + \frac{25}{2304}c\left(4-c^2\right)\left(1-|x|^2\right)\delta - \frac{25}{5184}\left(4-c^2\right)^2 x^2 \right|, \tag{108}$$

with the aid of the triangle inequality and replacing $|\delta| \le 1, |x| = k$, where $k \le 1$ and taking $c \in [0, 2]$. So,

$$\left|\mathscr{D}_{2,2}(g)\right| \le \frac{25}{4608}c^2\left(4-c^2\right)k^2 + \frac{25}{2304}c\left(4-c^2\right)\left(1-k^2\right) + \frac{25}{5184}\left(4-c^2\right)^2 k^2 \coloneqq \Xi(c,k). \tag{109}$$

It is not hard to observe that $\Xi'(c,k) \ge 0$ for $[0, 1]$, so we have $\Xi(c,k) \le \Xi(c,1)$. Putting $k = 1$ gives

$$\left|\mathscr{D}_{2,2}(g)\right| \le \frac{25}{4608}c^2\left(4-c^2\right) + \frac{25}{5184}\left(4-c^2\right)^2 \coloneqq \Xi(c,1). \tag{110}$$

It is clear that $\Xi'(c,1) < 0$, so $\Xi(c,1)$ is a decreasing function and attains its maximum value at $c = 0$. Thus, we have

$$\left|\mathscr{D}_{2,2}(g)\right| \le \frac{25}{324}. \tag{111}$$

The required second Hankel determinant is sharp and is obtained by

$$g_1(z) = \int_0^z \left(1 + \frac{5}{6}\left(t^2\right) + \frac{1}{6}\left(t^{10}\right)\right)dt = z + \frac{5}{18}z^3 + \frac{1}{66}z^{11} + \cdots. \tag{112}$$

□

5. Conclusion

In our present investigation, we considered a subclass of bounded turning functions associated with a four-leaf-type domain. We obtained some useful results for such a class, such as the limits of the first four initial coefficients, as well as the Fekete-Szego type inequality, the Zalcman inequality, the Kruskal inequality, and the estimation of the second-order Hankel determinant. All of the obtained results have been proven to be sharp. This work has been used to obtain higher-order Hankel determinants, such as in the investigation of the bounds of fourth-order and fifth-order Hankel determinants. These two determinants have been studied in [45, 53–56], respectively. Also, one can easily use this new methodology to obtain sharp bounds of the third-order Hankel determinant for other subclasses of univalent functions.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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