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Research Article

A Refinement of the Integral Jensen Inequality Pertaining Certain Functions with Applications

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In this paper, we present a new refinement of the integral Jensen inequality by utilizing certain functions and give its applications to various means. We utilize the refinement to obtain some new refinements of the Hermite-Hadamard and Hölder's inequalities as well. Also, we present its applications in information theory. At the end of this paper, we give a more general form of the proposed refinement of the Jensen inequality, associated to several functions.

1. Introduction

Being an important part of modern applied analysis, the field of mathematical inequalities has recorded an exponential growth with significant impact on various parts of science and technology [1–5]. These inequalities are also extended and generalized in various aspects; one can see such results in [6–15]. The Jensen weighted integral inequality is a central tool among them; its basic form is follows as [16].

Theorem 1. *Assume a convex function $f : I \longrightarrow \mathbb{R}$ and $g, h : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ are measurable functions such that $g(\theta) \in I$ and $h(\theta) \geq 0 \forall \theta \in [\theta_1, \theta_2]$. Also, suppose that $h, gh, (f \circ g).h$ are all integrable functions on $[\theta_1, \theta_2]$ and $\int_{\theta_1}^{\theta_2} h(\theta) d\theta > 0$, then*

$$f\left(\frac{\int_{\theta_1}^{\theta_2} g(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right) \leq \frac{\int_{\theta_1}^{\theta_2} (f \circ g)(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}. \qquad (1)$$

The Jensen inequality is one of the fundamental inequalities in modern applied analysis. This inequality is of pivotal importance because various other classical inequalities, for example, the Beckenbach-Dresher, Minkowski's, the Hermite-Hadamard, Ky-Fan's, Hölder's, the arithmetic-geometric, and Levinson's and Young's inequalities, can be deduced from this inequality. Also, this inequality can be treated as a problem solving oriented tool in different areas of science and technology, and an extensive literature is dedicated to this inequality regarding its counterparts, generalizations, improvements, and converse results (see, for instance, [17–21]) and the references therein.

The Hermite-Hadamard inequality is presented as follows ([22], page 10 in [23]).

Theorem 2. *Assume a convex function $f : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$, then the following double inequalities hold:*

$$f\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} f(\theta) d\theta \leq \frac{f(\theta_1) + f(\theta_2)}{2}. \qquad (2)$$

The Hölder inequality in its integral form is presented as follows [23].

Theorem 3. *Let $1 < p, q$ be such that $1/p + 1/q = 1$ and assume two measurable functions say $h, g : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ such that the functions $|h(\theta)|^p$ and $|g(\theta)|^q$ are integrable on $[\theta_1, \theta_2]$. Then*

$$\int_{\theta_1}^{\theta_2} |h(\theta)g(\theta)| d\theta \leq \left(\int_{\theta_1}^{\theta_2} |h(\theta)|^p d\theta \right)^{1/p} \left(\int_{\theta_1}^{\theta_2} |g(\theta)|^q d\theta \right)^{1/q}. \tag{3}$$

The remaining paper is organized in the following manner: Section 2 proposes a new refinement of the integral Jensen inequality, associated to four functions whose sum is equal to unity in pairs. Utilizing this refinement, we derive some new refinements of the Hölder and Hermite-Hadamard inequalities and some new inequalities for power and quasi-arithmetic means. In Section 3, we focus to deduce inequalities for the Csiszâr divergence, variational distance, Shannon entropy, and Kullback-Leibler divergence. In the last section, we present a more general form of the proposed refinement concerning several certain functions.

2. Main Results

Assuming a real valued convex function f defined on the interval I and suppose that $h, u, v, w, z, g : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ are some integrable functions with the following conditions $g(\theta) \in I, h(\theta), u(\theta), v(\theta), w(\theta), z(\theta) \in \mathbb{R}^+$ with $u(\theta) + w(\theta) = 1$, $v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$ and $\mathscr{H} := \int_{\theta_1}^{\theta_2} h(\theta) d\theta$. Also, let for a nonempty subinterval $\mathscr{J}$ of $[\theta_1, \theta_2]$, we put $\bar{\mathscr{J}} = [\theta_1, \theta_2] \setminus \mathscr{J}$; then, for the above facts, we can define the following functional

$$\begin{aligned} \mathbb{Z}^{\int}(f, u, v, h, w, z, g ; \mathscr{J}) &= \frac{1}{\mathscr{H}} \int_{\mathscr{J}} u(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathscr{J}} u(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} u(\theta)h(\theta)d\theta} \right) \\ &+ \frac{1}{\mathscr{H}} \int_{\mathscr{J}} w(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathscr{J}} w(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} w(\theta)h(\theta)d\theta} \right) \\ &+ \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta} \right) \\ &+ \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta} \right). \end{aligned} \tag{4}$$

We give the following refinement of the integral Jensen inequality associated to four functions.

Theorem 4. *Let f be a convex function defined on the interval I. Also, let $h, u, v, w, z, g : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be some integrable functions with the following conditions such as $g(\theta) \in I, h(\theta), u(\theta), v(\theta), w(\theta), z(\theta) \in \mathbb{R}^+$ with $u(\theta) + w(\theta) = 1$, $v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$ and $\mathscr{H} := \int_{\theta_1}^{\theta_2} h(\theta) d\theta$. Then, for any nonempty subinterval $\mathscr{J}$ of $[\theta_1, \theta_2]$ with $\bar{\mathscr{J}} = [\theta_1, \theta_2] \setminus \mathscr{J}$, the following inequalities hold*

$$\begin{aligned} \frac{1}{\mathscr{H}} \int_{\theta_1}^{\theta_2} h(\theta) f(g(\theta)) d\theta &\geq \mathbb{Z}^{\int}(f, u, v, h, w, z, g ; \mathscr{J}) \\ &\geq f\left(\frac{1}{\mathscr{H}} \int_{\theta_1}^{\theta_2} h(\theta)g(\theta)d\theta \right). \end{aligned} \tag{5}$$

For a concave function f, the reverse inequalities hold in (5).

Proof. Since $u(\theta) + w(\theta) = 1, v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$, therefore for the subinterval $\mathscr{J}$ of $[\theta_1, \theta_2]$ with $\bar{\mathscr{J}} = [\theta_1, \theta_2] \setminus \mathscr{J}$, we have

$$\begin{aligned} \int_{\theta_1}^{\theta_2} h(\theta)g(\theta)d\theta &= \int_{\mathscr{J}} h(\theta)g(\theta)d\theta + \int_{\bar{\mathscr{J}}} h(\theta)g(\theta)d\theta \\ &= \int_{\mathscr{J}} (u(\theta) + w(\theta))h(\theta)g(\theta)d\theta \\ &+ \int_{\bar{\mathscr{J}}} (v(\theta) + z(\theta))h(\theta)g(\theta)d\theta. \end{aligned} \tag{6}$$

Multiplying equation (6) by $1/\mathscr{H}$ and assigning it to the function f, then by convexity of f, we have

$$\begin{aligned} & f\left(\frac{1}{\mathscr{H}} \int_{\theta_1}^{\theta_2} h(\theta)g(\theta)d\theta \right) \\ &= f\left(\frac{1}{\mathscr{H}} \int_{\mathscr{J}} u(\theta)h(\theta)g(\theta)d\theta + \frac{1}{\mathscr{H}} \int_{\mathscr{J}} w(\theta)h(\theta)g(\theta)d\theta \right. \\ &\quad \left. + \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} v(\theta)h(\theta)g(\theta)d\theta + \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} z(\theta)h(\theta)g(\theta)d\theta \right) \\ &= f\left(\frac{\int_{\mathscr{J}} u(\theta)h(\theta)d\theta}{\mathscr{H}} \cdot \frac{\int_{\mathscr{J}} u(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} u(\theta)h(\theta)d\theta} + \frac{\int_{\mathscr{J}} w(\theta)h(\theta)d\theta}{\mathscr{H}} \right. \\ &\quad \cdot \frac{\int_{\mathscr{J}} w(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} w(\theta)h(\theta)d\theta} + \frac{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta}{\mathscr{H}} \cdot \frac{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta} \\ &\quad \left. + \frac{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta}{\mathscr{H}} \cdot \frac{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta} \right) \\ &\leq \frac{1}{\mathscr{H}} \int_{\mathscr{J}} u(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathscr{J}} u(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} u(\theta)h(\theta)d\theta} \right) \\ &\quad + \frac{1}{\mathscr{H}} \int_{\mathscr{J}} w(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathscr{J}} w(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathscr{J}} w(\theta)h(\theta)d\theta} \right) \\ &\quad + \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} v(\theta)h(\theta)d\theta} \right) \\ &\quad + \frac{1}{\mathscr{H}} \int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathscr{J}}} z(\theta)h(\theta)d\theta} \right) \\ &= \mathbb{Z}^{\int}(f, u, v, h, w, z, g ; \mathscr{J}). \end{aligned} \tag{7}$$

Also, by making use of the integral Jensen inequality, one has

$$\begin{aligned}
\mathbb{Z}^{\int}(f,u,v,h,w,z,g\,;\mathcal{J}) &= \frac{1}{\mathcal{H}}\int_{\mathcal{J}} u(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}}u(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}}u(\theta)h(\theta)d\theta}\right)\\
&\quad+\frac{1}{\mathcal{H}}\int_{\mathcal{J}} w(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}}w(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}}w(\theta)h(\theta)d\theta}\right)\\
&\quad+\frac{1}{\mathcal{H}}\int_{\bar{\mathcal{J}}} v(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)d\theta}\right)\\
&\quad+\frac{1}{\mathcal{H}}\int_{\bar{\mathcal{J}}} z(\theta)h(\theta)d\theta f\left(\frac{\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)g(\theta)d\theta}{\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)d\theta}\right)\\
&\le \frac{\int_{\mathcal{J}}u(\theta)h(\theta)d\theta}{\mathcal{H}}\cdot\frac{\int_{\mathcal{J}}u(\theta)h(\theta)f(g(\theta))d\theta}{\int_{\mathcal{J}}u(\theta)h(\theta)d\theta}\\
&\quad+\frac{\int_{\mathcal{J}}w(\theta)h(\theta)d\theta}{\mathcal{H}}\cdot\frac{\int_{\mathcal{J}}w(\theta)h(\theta)f(g(\theta))d\theta}{\int_{\mathcal{J}}w(\theta)h(\theta)d\theta}\\
&\quad+\frac{\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)d\theta}{\mathcal{H}}\cdot\frac{\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)f(g(\theta))d\theta}{\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)d\theta}\\
&\quad+\frac{\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)d\theta}{\mathcal{H}}\cdot\frac{\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)f(g(\theta))d\theta}{\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)d\theta}\\
&= \frac{1}{\mathcal{H}}\int_{\mathcal{J}}u(\theta)h(\theta)f(g(\theta))d\theta+\frac{1}{\mathcal{H}}\int_{\mathcal{J}}w(\theta)h(\theta)f(g(\theta))d\theta\\
&\quad+\frac{1}{\mathcal{H}}\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)f(g(\theta))d\theta+\frac{1}{\mathcal{H}}\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)f(g(\theta))d\theta\\
&= \frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2}h(\theta)f(g(\theta))d\theta. \qquad (8)
\end{aligned}$$

From (7) and (8), we get (5). □

Remark 5. The following is an equivalent form of the inequality (5)

$$\inf_{\phi\neq\mathcal{J}\subset[\theta_1,\theta_2]}\mathbb{Z}^{\int}(f,u,v,h,w,z,g\,;\mathcal{J})\ge f\left(\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2}h(\theta)g(\theta)d\theta\right), \qquad (9)$$

$$\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2}h(\theta)f(g(\theta))d\theta\ge\sup_{\phi\neq\mathcal{J}\subset[\theta_1,\theta_2]}\mathbb{Z}^{\int}(f,u,v,h,w,z,g\,;\mathcal{J}). \qquad (10)$$

From Theorem 4, we obtain a new refinement of the H-H inequality as follows.

Corollary 6. *Let f be a convex function defined on $[\theta_1,\theta_2]$. Also, let $u,v,w,z:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be integrable functions such that $u(\theta),v(\theta),w(\theta),z(\theta)\in\mathbb{R}^+$ with $u(\theta)+w(\theta)=1$ and $v(\theta)+z(\theta)=1$ for all $\theta\in[\theta_1,\theta_2]$. Then, for a subinterval $\mathcal{J}$ of $[\theta_1,\theta_2]$ with $\bar{\mathcal{J}}=[\theta_1,\theta_2]\setminus\mathcal{J}$, the following inequalities hold*

$$\begin{aligned}
\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}f(\theta)d\theta &\ge \frac{1}{\theta_2-\theta_1}\int_{\mathcal{J}}u(\theta)d\theta f\left(\frac{\int_{\mathcal{J}}\theta u(\theta)d\theta}{\int_{\mathcal{J}}u(\theta)d\theta}\right)\\
&\quad+\frac{1}{\theta_2-\theta_1}\int_{\mathcal{J}}w(\theta)d\theta f\left(\frac{\int_{\mathcal{J}}\theta w(\theta)d\theta}{\int_{\mathcal{J}}w(\theta)d\theta}\right)\\
&\quad+\frac{1}{\theta_2-\theta_1}\int_{\bar{\mathcal{J}}}v(\theta)d\theta f\left(\frac{\int_{\bar{\mathcal{J}}}\theta v(\theta)d\theta}{\int_{\bar{\mathcal{J}}}v(\theta)d\theta}\right)\\
&\quad+\frac{1}{\theta_2-\theta_1}\int_{\bar{\mathcal{J}}}z(\theta)d\theta f\left(\frac{\int_{\bar{\mathcal{J}}}\theta z(\theta)d\theta}{\int_{\bar{\mathcal{J}}}z(\theta)d\theta}\right)\ge f\left(\frac{\theta_1+\theta_2}{2}\right).
\end{aligned} \qquad (11)$$

The direction of the inequalities reverses in (11), when the function f becomes concave.

Proof. Utilizing Theorem 4 for $h(\theta)=1, g(\theta)=\theta$ for all $\theta\in[\theta_1,\theta_2]$, we get (11). □

From Theorem 4, we deduce the following refinement of Hölder's inequality.

Corollary 7. *If $p,q\in\mathbb{R}$ and the functions u,v,w,z,h_1,g_1, and g_2 defined on $[\theta_1,\theta_2]$ are nonnegative such that the functions $h_1g_1^p, h_1g_2^q, uh_1g_2^q, wh_1g_2^q, vh_1g_2^q, zh_1g_2^q, uh_1g_1g_2$ and $wh_1g_1g_2, vh_1g_1g_2, zh_1g_1g_2, h_1g_1g_2$ are integrable with $u(\theta)+w(\theta)=1$ and $v(\theta)+z(\theta)=1$ for all $\theta\in[\theta_1,\theta_2]$. Also, if $\mathcal{J}$ is a subinterval of $[\theta_1,\theta_2]$ with $\bar{\mathcal{J}}=[\theta_1,\theta_2]\setminus\mathcal{J}$, then*

(A) *For $p,q>1$ such that $(1/p)+(1/q)=1$, the following inequalities hold*

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}\left(\int_{\theta_1}^{\theta_2}h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\\
&\ge\left(\int_{\theta_1}^{\theta_2}h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\Bigg\{\left(\left(\int_{\mathcal{J}}u(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}\right.\\
&\quad\times\left(\int_{\mathcal{J}}u(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^p+\left(\int_{\mathcal{J}}w(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}\\
&\quad\times\left(\int_{\mathcal{J}}w(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^p+\left(\int_{\bar{\mathcal{J}}}v(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}\\
&\quad\times\left(\int_{\bar{\mathcal{J}}}v(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^p+\left(\int_{\bar{\mathcal{J}}}z(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}\\
&\quad\left.\times\left(\int_{\bar{\mathcal{J}}}z(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^p\right)^{1/p}\Bigg\}\\
&\ge\int_{\theta_1}^{\theta_2}h_1(\theta)g_1(\theta)g_2(\theta)d\theta
\end{aligned} \qquad (12)$$

(B) *For $0<p<1$ and $q=p/(p-1)$ with $\int_{\theta_1}^{\theta_2}h_1(\theta)g_2^q(\theta)d\theta>0$, the following inequalities hold*

$$\begin{aligned}
\int_{\theta_1}^{\theta_2}h_1(\theta)g_1(\theta)g_2(\theta)d\theta &\ge\left(\int_{\mathcal{J}}u(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\\
&\quad\times\left(\int_{\mathcal{J}}u(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}+\left(\int_{\mathcal{J}}w(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\\
&\quad\times\left(\int_{\mathcal{J}}w(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}+\left(\int_{\bar{\mathcal{J}}}v(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\\
&\quad\times\left(\int_{\bar{\mathcal{J}}}v(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}+\left(\int_{\bar{\mathcal{J}}}z(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}
\end{aligned}$$

$$\times\left(\int_{\bar{\mathcal{J}}} z(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p} \geq \left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}$$
$$\times\left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q} \tag{13}$$

(C) *For $p < 0$ and $q = p/(p-1)$ with $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta > 0$, the inequalities in (13) hold*

Proof.

(A) In the case when $p, q > 1$, let $\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta > 0$. Then, by using Theorem 4 for $f(\theta) = \theta^p$, $\theta > 0$, $h(\theta) = h_1(\theta)g_2^q(\theta)$, and $g(\theta) = g_1(\theta)g_2^{-q/p}(\theta)$, we obtain (12). Also, let $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta > 0$; then, applying the same procedure as above and replacing p, q, g_1, g_2 by q, p, g_2, g_1, respectively, we obtain (12)

For $\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta = 0$ and $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta = 0$, the inequalities in (12) also hold. This can be proved as follows, since we know that

$$0 \leq h_1(\theta)g_1(\theta)g_2(\theta) \leq \frac{1}{p}h_1(\theta)g_1^p(\theta) + \frac{1}{q}h_1(\theta)g_2^q(\theta). \tag{14}$$

Taking integral, then with the proposed conditions, we obtain $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1(\theta)g_2(\theta)d\theta = 0$, which concludes the result.

(B) In the case when $0 < p < 1$, we have $q < 0$. So, applying (12) for $p \longrightarrow 1/p > 1$, $q \longrightarrow (1-p)^{-1}$, $g_1(\theta) \longrightarrow (g_1(\theta)g_2(\theta))^p$, and $g_2(\theta) \longrightarrow (g_2(\theta))^{-p}$, we obtain (13)

(C) In the case when $p < 0$, we have $0 < q < 1$. So, applying the arguments of part B with replacing p, q, $g_1(\theta)$, $g_2(\theta)$ by $q, p, g_2(\theta), g_1(\theta)$, respectively, we get (13)

□

The Hölder inequality is refined by the following corollary.

Corollary 8. *Let $p, q \in \mathbb{R}$ and $u, v, w, z, h_1, g_1, g_2$ be nonnegative functions defined on $[\theta_1, \theta_2]$ such that $h_1g_1^p, h_1g_2^q, uh_1g_2^q, uh_1g_1^p, wh_1g_2^q, wh_1g_1^p, vh_1g_2^q, vh_1g_1^p, zh_1g_2^q$ and $zh_1g_1^p, h_1g_1g_2$ are integrable functions with $u(\theta) + w(\theta) = 1$ and $v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$. Also, assume that $\mathcal{J}$ is a subinterval of $[\theta_1, \theta_2]$ with $\bar{\mathcal{J}} = [\theta_1, \theta_2] \setminus \mathcal{J}$, then*

(A) *For $p > 1, q = p/(p-1)$ and $\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta > 0$, the following inequalities hold*

$$\left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}\left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}$$
$$\geq \left(\int_{\mathcal{J}} u(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\left(\int_{\mathcal{J}} u(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}$$
$$+ \left(\int_{\mathcal{J}} w(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\left(\int_{\mathcal{J}} w(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}$$
$$+ \left(\int_{\bar{\mathcal{J}}} v(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\left(\int_{\bar{\mathcal{J}}} v(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}$$
$$+ \left(\int_{\bar{\mathcal{J}}} z(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\left(\int_{\bar{\mathcal{J}}} z(\theta)h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}$$
$$\geq \int_{\theta_1}^{\theta_2} h_1(\theta)g_1(\theta)g_2(\theta)d\theta \tag{15}$$

(B) *For $0 < p < 1$ and $q = p/(p-1)$ with $\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta > 0$, the following inequalities hold*

$$\left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta\right)^{1/p}\left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}$$
$$\leq \left(\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta\right)^{1/q}\left\{\left(\int_{\mathcal{J}} u(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}\right.$$
$$\times\left(\int_{\mathcal{J}} u(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^{p} + \left(\int_{\mathcal{J}} w(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}$$
$$\times\left(\int_{\mathcal{J}} w(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^{p} + \left(\int_{\bar{\mathcal{J}}} v(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}$$
$$\times\left(\int_{\bar{\mathcal{J}}} v(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^{p} + \left(\int_{\bar{\mathcal{J}}} z(\theta)h_1(\theta)g_2^q(\theta)d\theta\right)^{1-p}$$
$$\left.\times\left(\int_{\bar{\mathcal{J}}} z(\theta)h_1(\theta)g_1(\theta)g_2(\theta)d\theta\right)^{p}\right\}^{1/p} \leq \int_{\theta_1}^{\theta_2} h_1(\theta)g_1(\theta)g_2(\theta)d\theta \tag{16}$$

(C) *For $p < 0$ and $q = p/(p-1)$ with $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta > 0$, the result in (16) holds*

Proof.

(A) Let $f(\theta) = \theta^{1/p}$, $\theta > 0$ which is clearly a concave function for $p > 1$. Thus, by using Theorem 4 for $f(\theta) = \theta^{1/p}$, $h(\theta) = h_1(\theta)g_2^q(\theta)$, and $g(\theta) = g_1^p(\theta)g_2^{-q}(\theta)$, we obtain (15). If $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta > 0$, then adopting the same procedure and replacing p, q, g_1, g_2 by q, p, g_2, g_1, respectively, we obtain (15)

For $\int_{\theta_1}^{\theta_2} h_1(\theta)g_2^q(\theta)d\theta = 0$ and $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1^p(\theta)d\theta = 0$, the inequalities in (12) also hold. This can be proved as follows, since we know that

$$0 \le h_1(\theta)g_1(\theta)g_2(\theta) \le \frac{1}{p}h_1(\theta)g_1^p(\theta) + \frac{1}{q}h_1(\theta)g_2^q(\theta). \quad (17)$$

Hence, taking integral and using the proposed conditions, we get $\int_{\theta_1}^{\theta_2} h_1(\theta)g_1(\theta)g_2(\theta)d\theta = 0$, which verifies the result.

(B) In the case when $0 < p < 1$, applying (15) for $p \longrightarrow 1/p > 1$, $q \longrightarrow (1-p)^{-1}$, $g_1(\theta) \longrightarrow (g_1(\theta)g_2(\theta))^p$, and $g_2(\theta) \longrightarrow (g_2(\theta))^{-p}$, we obtain (16)

(C) In the case when $0 > p$, we have $q \in (0,1)$, which shows that this case reflects case B; therefore, applying the arguments of case B with replacing $p, q, g_1(\theta)$, $g_2(\theta)$ by $q, p, g_2(\theta), g_1(\theta)$, respectively, we get (16)

□

Let h and g be positive integrable functions defined on $[\theta_1, \theta_2]$ and $\mathcal{J}$ be any nonempty subinterval of $[\theta_1, \theta_2]$, then the integral power means of order $r \in \mathbb{R}$ are defined as

$$\mathrm{M}_{[r;\mathcal{J}]}(h;g) = \begin{cases} \left(\dfrac{\int_{\mathcal{J}} h(\theta)g^r(\theta)d\theta}{\int_{\mathcal{J}} h(\theta)d\theta}\right)^{1/r} & \text{, if r} \ne 0, \\ \exp\left(\dfrac{\int_{\mathcal{J}} h(\theta)\log g(\theta)d\theta}{\int_{\mathcal{J}} h(\theta)d\theta}\right) & \text{, if } r = 0, \end{cases}$$

$$\mathrm{M}_{[r;\mathcal{J}]}(u.h;g) = \begin{cases} \left(\dfrac{\int_{\mathcal{J}} u(\theta)h(\theta)g^r(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)h(\theta)d\theta}\right)^{1/r} & \text{, if r} \ne 0, \\ \exp\left(\dfrac{\int_{\mathcal{J}} u(\theta)h(\theta)\log g(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)h(\theta)d\theta}\right) & \text{, if } r = 0, \end{cases} \quad (18)$$

$$\mathrm{M}_{[r;[\theta_1,\theta_2]]}(h;g) = \begin{cases} \left(\dfrac{\int_{\theta_1}^{\theta_2} h(\theta)g^r(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)^{1/r} & \text{, if r} \ne 0, \\ \exp\left(\dfrac{\int_{\theta_1}^{\theta_2} h(\theta)\log g(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right) & \text{, if } r = 0. \end{cases} \quad (19)$$

Corollary 9. *Assume some positive integrable functions h, u, v, w, z and g defined on $[\theta_1, \theta_2]$ with $u(\theta) + w(\theta) = 1$ and $v(\theta) + z(\theta) = 1$ for $\theta \in [\theta_1, \theta_2]$. Also, assume that $\alpha, \beta \in \mathbb{R}$ such that $\alpha \le \beta$, then*

(A) For $\alpha/\beta \in \mathbb{R} - \{(-1, 0)\}$, $\beta \ne 0$, the following inequalities hold

$$\begin{aligned} M_{[\alpha;[\theta_1,\theta_2]]}(h;g) \le \left[\frac{1}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right]^{1/\alpha} &\left[\left(\int_{\mathcal{J}} u(\theta)h(\theta)d\theta\right)M^{\alpha}_{[\beta;\mathcal{J}]}(u.h;g)\right. \\ &+ \left(\int_{\mathcal{J}} w(\theta)h(\theta)d\theta\right)M^{\alpha}_{[\beta;\mathcal{J}]}(w.h;g) \\ &+ \left(\int_{\bar{\mathcal{J}}} v(\theta)h(\theta)d\theta\right)M^{\alpha}_{[\beta;\bar{\mathcal{J}}]}(v.h;g) \\ &\left.+ \left(\int_{\bar{\mathcal{J}}} z(\theta)h(\theta)d\theta\right)M^{\alpha}_{[\beta;\bar{\mathcal{J}}]}(z.h;g)\right]^{1/\alpha} \\ &\le M_{[\beta;[\theta_1,\theta_2]]}(h;g), \alpha \ne 0, \end{aligned} \quad (20)$$

$$\begin{aligned} M_{[\alpha;[\theta_1,\theta_2]]}(h;g) \le \exp &\left[\left(\frac{\int_{\mathcal{J}} u(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\beta;\mathcal{J}]}(u.h;g)\right. \\ &+ \left(\frac{\int_{\mathcal{J}} w(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\beta;\mathcal{J}]}(w.h;g) \\ &+ \left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\beta;\bar{\mathcal{J}}]}(v.h;g) \\ &\left.+ \left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\beta;\bar{\mathcal{J}}]}(z.h;g)\right] \\ &\le M_{[\beta;[\theta_1,\theta_2]]}(h;g), \alpha = 0 \end{aligned} \quad (21)$$

(B) For $\beta/\alpha \in \mathbb{R} - \{(-1, 0)\}$, $\alpha \ne 0$, the following inequalities hold

$$\begin{aligned} M_{[\beta;[\theta_1,\theta_2]]}(h;g) \ge \left[\frac{1}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right]^{1/\beta} &\left[\left(\int_{\mathcal{J}} u(\theta)h(\theta)d\theta\right)M^{\beta}_{[\alpha;\mathcal{J}]}(u.h;g)\right. \\ &+ \left(\int_{\mathcal{J}} w(\theta)h(\theta)d\theta\right)M^{\beta}_{[\alpha;\mathcal{J}]}(w.h;g) \\ &+ \left(\int_{\bar{\mathcal{J}}} v(\theta)h(\theta)d\theta\right)M^{\beta}_{[\alpha;\bar{\mathcal{J}}]}(v.h;g) \\ &\left.+ \left(\int_{\bar{\mathcal{J}}} z(\theta)h(\theta)d\theta\right)M^{\beta}_{[\alpha;\bar{\mathcal{J}}]}(z.h;g)\right]^{1/\beta} \\ &\ge M_{[\alpha;[\theta_1,\theta_2]]}(h;g), \beta \ne 0, \end{aligned} \quad (22)$$

$$\begin{aligned} M_{[\beta;[\theta_1,\theta_2]]}(h;g) \ge \exp &\left[\left(\frac{\int_{\mathcal{J}} u(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\alpha;\mathcal{J}]}(u.h;g)\right. \\ &+ \left(\frac{\int_{\mathcal{J}} w(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\alpha;\mathcal{J}]}(w.h;g) \\ &+ \left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\alpha;\bar{\mathcal{J}}]}(v.h;g) \\ &\left.+ \left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)h(\theta)d\theta}{\int_{\theta_1}^{\theta_2} h(\theta)d\theta}\right)\log M_{[\alpha;\bar{\mathcal{J}}]}(z.h;g)\right] \end{aligned}$$

$$\geq M_{[\alpha;[\theta_1,\theta_2]]}(h\,;g),\ \beta=0 \tag{23}$$

Proof.

(A) Let $f(\theta)=\theta^{\alpha/\beta}$ for $\theta>0$, then the following possible cases can be discussed:

Case 1. If $\alpha,\beta\in\mathbb{R}^-$ with $\alpha\leq\beta$, then $\alpha/\beta\geq 1$, and the function $f(\theta)$ is convex. Therefore, utilizing (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\beta}(\theta)$, after that taking power $1/\alpha$, we obtain (20)

Case 2. If $\alpha,\beta\in\mathbb{R}^+$ with $\alpha<\beta$, then $0<\alpha/\beta<1$, and the function $f(\theta)$ is concave. Therefore, utilizing (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\beta}(\theta)$ and then taking power $1/\alpha$, we obtain (20)

Case 3. If $\beta\in\mathbb{R}^+$, $\alpha\in\mathbb{R}^-$ with $\alpha\leq\beta$, then $\alpha/\beta\leq 0$, and the function $f(\theta)$ is convex. Therefore, utilizing (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\beta}(\theta)$ and then taking power $1/\alpha$, we obtain (20)

For the case when $\alpha=0$, taking $\lim\limits_{\alpha\longrightarrow 0}$ of (20), we get (21).

(B) Let $f(\theta)=\theta^{\beta/\alpha}$ for $\theta>0$, then the following possible cases can be discussed:

Case 1. If $\alpha,\beta\in\mathbb{R}^+$ with $\alpha\leq\beta$, then $\beta/\alpha\geq 1$, and the function $f(\theta)$ is convex. Hence, using (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\alpha}(\theta)$ and then taking power $1/\beta$, we obtain (22)

Case 2. If $\alpha,\beta\in\mathbb{R}^-$ with $\alpha\leq\beta$, then $0<\beta/\alpha<1$, and the function $f(\theta)$ is concave. Hence, using (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\alpha}(\theta)$ and then taking power $1/\beta$, we obtain (22)

Case 3. Similarly, if $\alpha\in\mathbb{R}^-$, $\beta\in\mathbb{R}^+$ with $\alpha\leq\beta$, then $\beta/\alpha\leq -1$, and the function $f(\theta)$ is convex function. Hence, using (5) for $f(\theta)$ and $g(\theta)\longrightarrow g^{\alpha}(\theta)$ and taking power $1/\beta$, we obtain (22)

For the case when $\beta=0$, taking $\lim_{\beta\longrightarrow 0}$ in (22), we get (23). □

Let h and u be some positive integrable functions and g be an arbitrary integrable function defined on $[\theta_1,\theta_2]$. Further, if p is considered as a strictly monotone and continuous function whose domain is the image of g, then for any nonempty subinterval $\mathcal{J}$ of $[\theta_1,\theta_2]$, the quasi-arithmetic mean is defined by

$$\begin{aligned} \mathrm{M}_p^{[\mathcal{J}]}(h\,;g)&=p^{-1}\left(\frac{\int_{\mathcal{J}}h(\theta)p(g(\theta))d\theta}{\int_{\mathcal{J}}h(\theta)d\theta}\right),\\ \mathrm{M}_p^{[\mathcal{J}]}(u.h\,;g)&=p^{-1}\left(\frac{\int_{\mathcal{J}}u(\theta)h(\theta)p(g(\theta))d\theta}{\int_{\mathcal{J}}u(\theta)h(\theta)d\theta}\right), \end{aligned} \tag{24}$$

$$\mathrm{M}_p^{[\theta_1,\theta_2]}(h\,;g)=p^{-1}\left(\frac{\int_{\theta_1}^{\theta_2}h(\theta)p(g(\theta))d\theta}{\int_{\theta_1}^{\theta_2}h(\theta)d\theta}\right). \tag{25}$$

Some inequalities are given for the quasi-arithmetic mean as follows.

Corollary 10. *Assume some positive integrable functions h, u,v,w,z defined on $[\theta_1,\theta_2]$ such that $u(\theta)+w(\theta)=1$ and $v(\theta)+z(\theta)=1$ for $\theta\in[\theta_1,\theta_2]$ and further assume that g is an arbitrary integrable function defined on $[\theta_1,\theta_2]$. Also, suppose that p is a strictly monotone continuous function whose domain is the image of g. Then, for $(\Psi\circ p^{-1})(\theta)$ as a convex function, the following inequalities hold*

$$\begin{aligned} &\frac{1}{\int_{\theta_1}^{\theta_2}h(\theta)d\theta}\int_{\theta_1}^{\theta_2}h(\theta)\Psi(g(\theta))d\theta\\ &\geq\frac{1}{\int_{\theta_1}^{\theta_2}h(\theta)d\theta}\left\{\int_{\mathcal{J}}u(\theta)h(\theta)d\theta\Psi\left(M_p^{[\mathcal{J}]}(u.h\,;g)\right)\right.\\ &\quad+\int_{\mathcal{J}}w(\theta)h(\theta)d\theta\Psi\left(M_p^{[\mathcal{J}]}(w.h\,;g)\right)\\ &\quad+\int_{\bar{\mathcal{J}}}v(\theta)h(\theta)d\theta\Psi\left(M_p^{[\bar{\mathcal{J}}]}(v.h\,;g)\right)\\ &\quad\left.+\int_{\bar{\mathcal{J}}}z(\theta)h(\theta)d\theta\Psi\left(M_p^{[\bar{\mathcal{J}}]}(z.h\,;g)\right)\right\}\\ &\geq\Psi\left(M_p^{[\theta_1,\theta_2]}(h\,;g)\right). \end{aligned} \tag{26}$$

The direction of the inequalities reverses in (26), when the function $(\Psi\circ p^{-1})(\theta)$, becomes concave.

Proof. The desired inequalities can be calculated by utilizing (5) for $g\longrightarrow p\circ g$ and $f\longrightarrow\Psi\circ p^{-1}$. □

3. Applications in Information Theory

In this section, we use the main result to obtain some new and interesting estimates for various divergences and Shannon entropy in information theory. A literature about the inequalities related to these divergences can be found in [24].

Definition 11 (Csiszâr divergence [25]). Assume that $\Psi:I^+\longrightarrow\mathbb{R}$ is a function defined on a positive interval I^+. Also, assume that $p,q:[\theta_1,\theta_2]\longrightarrow(0,\infty)$ are two integrable functions such that $q(\theta)/p(\theta)\in I^+$ for all $\theta\in[\theta_1,\theta_2]$, then the integral form of Csiszâr-divergence is defined by

$$\mathrm{C}_{\Psi}(p,q)=\int_{\theta_1}^{\theta_2}p(\theta)\Psi\left(\frac{q(\theta)}{p(\theta)}\right)d\theta. \tag{27}$$

Theorem 12. *Let $\Psi:I^+\longrightarrow\mathbb{R}$ be a convex function defined on a positive interval I^+ and assume that $u,v,w,z,p,q:[\theta_1,\theta_2]\longrightarrow\mathbb{R}^+$ are integrable functions such that $q(\theta)/p(\theta)\in I^+$, $u(\theta)+w(\theta)=1$, and $v(\theta)+z(\theta)=1$ for all $\theta\in[\theta_1,\theta_2]$. Then, for a subinterval $\mathcal{J}$ of $[\theta_1,\theta_2]$ with $\bar{\mathcal{J}}=[\theta_1,\theta_2]\setminus\mathcal{J}$, the* *following inequalities hold*

$$\begin{aligned} C_{\Psi}(p,q) \geq &\int_{\mathcal{J}} u(\theta)p(\theta)d\theta\Psi\left(\frac{\int_{\mathcal{J}} u(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\mathcal{J}} w(\theta)p(\theta)d\theta\Psi\left(\frac{\int_{\mathcal{J}} w(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta\Psi\left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta\Psi\left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta}\right) \\ \geq &\Psi\left(\frac{\int_{\theta_1}^{\theta_2} q(\theta)d\theta}{\int_{\theta_1}^{\theta_2} p(\theta)d\theta}\right)\int_{\theta_1}^{\theta_2} p(\theta)d\theta. \end{aligned} \tag{28}$$

Proof. Using Theorem 4 for $f \longrightarrow \Psi$, $g(\theta) = q(\theta)/p(\theta)$, and $h(\theta) = p(\theta)$, we obtain (28). □

Definition 13 (Shannon entropy). The Shannon entropy for a positive probability density function $p(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$S(p) = -\int_{\theta_1}^{\theta_2} p(\theta)\log p(\theta)d\theta. \tag{29}$$

Corollary 14. *Let $u, v, w, z : [\theta_1, \theta_2] \longrightarrow \mathbb{R}^+$ be integrable functions such that $u(\theta) + w(\theta) = 1$ and $v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$. Also, let p, q be two positive probability density functions defined on $[\theta_1, \theta_2]$, then for a subinterval $\mathcal{J}$ of $[\theta_1, \theta_2]$ with $\bar{\mathcal{J}} = [\theta_1, \theta_2] \setminus \mathcal{J}$, the following inequalities hold*

$$\begin{aligned} &\int_{\theta_1}^{\theta_2} p(\theta)\log(q(\theta))d\theta + S(p) \\ &\leq \int_{\mathcal{J}} u(\theta)p(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} u(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right) \\ &\quad + \int_{\mathcal{J}} w(\theta)p(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} w(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right) \\ &\quad + \int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta}\right) \\ &\quad + \int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta}\right) \\ &\leq \log\left(\int_{\theta_1}^{\theta_2} q(\theta)d\theta\right). \end{aligned} \tag{30}$$

Proof. Using (28) for the function $\Psi(\zeta) = -\log\zeta$, $\zeta \in I^+$, we obtain (30). □

Remark 15. For $q(\theta) = 1$, the result (30) becomes

$$\begin{aligned} S(p) \leq &\int_{\mathcal{J}} u(\theta)p(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} u(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\mathcal{J}} w(\theta)p(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} w(\theta)d\theta}{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)d\theta}{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)d\theta}{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta}\right) \\ \leq &\log(\theta_2 - \theta_1). \end{aligned} \tag{31}$$

Definition 16 (Kullback-Leibler divergence). The Kullback-Leibler divergence for two positive probability densities $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$KL(q,p) = \int_{\theta_1}^{\theta_2} q(\theta)\log\frac{q(\theta)}{p(\theta)}d\theta. \tag{32}$$

Corollary 17. *Let $u, v, w, z, p, q : [\theta_1, \theta_2] \longrightarrow \mathbb{R}^+$ be integrable functions such that $p(\theta)$ and $q(\theta)$ are positive probability density functions and $u(\theta) + w(\theta) = 1$ and $v(\theta) + z(\theta) = 1$ for all $\theta \in [\theta_1, \theta_2]$. Then, for a subinterval $\mathcal{J}$ of $[\theta_1, \theta_2]$ with $\bar{\mathcal{J}} = [\theta_1, \theta_2] \setminus \mathcal{J}$, the following inequalities hold*

$$\begin{aligned} KL(q,p) \geq &\int_{\mathcal{J}} u(\theta)q(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} u(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\mathcal{J}} w(\theta)q(\theta)d\theta\log\left(\frac{\int_{\mathcal{J}} w(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} z(\theta)q(\theta)d\theta\log\left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta}\right) \\ \geq &0. \end{aligned} \tag{33}$$

Proof. Using (28) for the convex function $\Psi(\zeta) = \zeta\log\zeta$, $\zeta \in I^+$, we obtain (33). □

Definition 18 (variational distance). The variational distance for two positive probability densities $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$V(q,p) = \int_{\theta_1}^{\theta_2} |q(\theta) - p(\theta)|d\theta. \tag{34}$$

Corollary 19. *Let u, v, w, z, p, q and $\mathcal{J}, \bar{\mathcal{J}}$ be interpreted as in*

Corollary 17, then

$$V(q,p) \geq \left| \int_{\mathcal{J}} u(\theta)(q(\theta)-p(\theta))d\theta \right| + \left| \int_{\mathcal{J}} w(\theta)(q(\theta)-p(\theta))d\theta \right| + \left| \int_{\bar{\mathcal{J}}} v(\theta)(q(\theta)-p(\theta))d\theta \right| + \left| \int_{\bar{\mathcal{J}}} z(\theta)(q(\theta)-p(\theta))d\theta \right|. \tag{35}$$

Proof. Using the convex function $\Psi(\zeta) = |\zeta - 1|$ for $\zeta \in I^+$ in (28), we obtain (35). □

Definition 20 (Jeffrey's distance). Jeffrey's distance for two positive probability density functions $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$\mathrm{J}(q,p) = \int_{\theta_1}^{\theta_2} (q(\theta)-p(\theta)) \log\left(\frac{q(\theta)}{p(\theta)}\right) d\theta. \tag{36}$$

Corollary 21. *Let u, v, w, z, p, q and $\mathcal{J}, \bar{\mathcal{J}}$ be interpreted as in Corollary 17, then*

$$\begin{aligned} J(q,p) \geq &\int_{\mathcal{J}} u(\theta)(q(\theta)-p(\theta))d\theta \log\left(\frac{\int_{\mathcal{J}} u(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\mathcal{J}} w(\theta)(q(\theta)-p(\theta))d\theta \log\left(\frac{\int_{\mathcal{J}} w(\theta)q(\theta)d\theta}{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} v(\theta)(q(\theta)-p(\theta))d\theta \log\left(\frac{\int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta}\right) \\ &+ \int_{\bar{\mathcal{J}}} z(\theta)(q(\theta)-p(\theta))d\theta \log\left(\frac{\int_{\bar{\mathcal{J}}} z(\theta)q(\theta)d\theta}{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta}\right) \\ \geq 0. \end{aligned} \tag{37}$$

Proof. Using the convex function $\Psi(\zeta) = (\zeta - 1)\log\zeta, \zeta \in I^+$ in (28), we obtain (37). □

Definition 22 (Bhattacharyya coefficient). The Bhattacharyya coefficient for two positive probability density functions $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$\mathrm{B}(p,q) = \int_{\theta_1}^{\theta_2} \sqrt{p(\theta)q(\theta)}d\theta. \tag{38}$$

Corollary 23. *Let u, v, w, z, p, q and $\mathcal{J}, \bar{\mathcal{J}}$ be interpreted as in Corollary 17, then*

$$\begin{aligned} B(p,q) \leq &\sqrt{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta \int_{\mathcal{J}} u(\theta)q(\theta)d\theta} \\ &+ \sqrt{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta \int_{\mathcal{J}} w(\theta)q(\theta)d\theta} \\ &+ \sqrt{\int_{\bar{\mathcal{J}}} v(\theta)p(\theta)d\theta \int_{\bar{\mathcal{J}}} v(\theta)q(\theta)d\theta} \\ &+ \sqrt{\int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta \int_{\bar{\mathcal{J}}} z(\theta)p(\theta)d\theta} \leq 0. \end{aligned} \tag{39}$$

Proof. Using the convex function $\Psi(\zeta) = -\sqrt{\zeta}, \zeta \in I^+$ in (28), we obtain (39). □

Definition 24 (Hellinger distance). The Hellinger distance for two positive probability density functions $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$\mathrm{H}(q,p) = \int_{\theta_1}^{\theta_2} \left(\sqrt{q(\theta)} - \sqrt{p(\theta)}\right)^2 d\theta. \tag{40}$$

Corollary 25. *Let u, v, w, z, p, q and $\mathcal{J}, \bar{\mathcal{J}}$ be defined as in Corollary 17, then*

$$\begin{aligned} H(p,q) \geq &\left(\sqrt{\int_{\mathcal{J}} u(\theta)q(\theta)d\theta} - \sqrt{\int_{\mathcal{J}} u(\theta)p(\theta)d\theta}\right)^2 \\ &+ \left(\sqrt{\int_{\mathcal{J}} w(\theta)q(\theta)d\theta} - \sqrt{\int_{\mathcal{J}} w(\theta)p(\theta)d\theta}\right)^2 \\ &+ \left(\sqrt{\int_{\mathcal{J}} v(\theta)q(\theta)d\theta} - \sqrt{\int_{\mathcal{J}} v(\theta)p(\theta)d\theta}\right)^2 \\ &+ \left(\sqrt{\int_{\mathcal{J}} z(\theta)q(\theta)d\theta} - \sqrt{\int_{\mathcal{J}} z(\theta)p(\theta)d\theta}\right)^2 \\ \geq 0. \end{aligned} \tag{41}$$

Proof. Using (28) for the convex function $\Psi(\zeta) = (\sqrt{\zeta} - 1)^2$, $\zeta \in I^+$, we obtain (41). □

Definition 26 (triangular discrimination). The triangular discrimination for two positive probability density functions $p(\theta)$ and $q(\theta)$ defined on $[\theta_1, \theta_2]$ is given by

$$\Delta(q,p) = \int_{\theta_1}^{\theta_2} \frac{(q(\theta)-p(\theta))^2}{p(\theta)+q(\theta)} d\theta. \tag{42}$$

Corollary 27. *Let u, v, w, z, p, q and $\mathcal{J}, \bar{\mathcal{J}}$ be as stated in*

Corollary 17, then

$$\begin{aligned}\Delta(q,p) \geq & \frac{\left(\int_{\mathcal{J}} u(\theta)(q(\theta)-p(\theta))d\theta\right)^2}{\int_{\mathcal{J}} u(\theta)(p(\theta)+q(\theta))d\theta} \\ & + \frac{\left(\int_{\mathcal{J}} w(\theta)(q(\theta)-p(\theta))d\theta\right)^2}{\int_{\mathcal{J}} w(\theta)(p(\theta)+q(\theta))d\theta} \\ & + \frac{\left(\int_{\mathcal{J}} v(\theta)(q(\theta)-p(\theta))d\theta\right)^2}{\int_{\mathcal{J}} v(\theta)(p(\theta)+q(\theta))d\theta} \\ & + \frac{\left(\int_{\mathcal{J}} z(\theta)(q(\theta)-p(\theta))d\theta\right)^2}{\int_{\mathcal{J}} z(\theta)(p(\theta)+q(\theta))d\theta} \\ \geq & \ 0. \end{aligned} \tag{43}$$

Proof. Utilizing the convex function $\Psi(\zeta)=(\zeta-1)^2/(\zeta+1)$, $\zeta\in I^+$ in (28), we obtain (43). □

3.1. Further Generalization. In this section, we give more general form of the proposed refinement of the integral Jensen inequality concerning several certain functions.

Theorem 28. *Let $f: I \longrightarrow \mathbb{R}$ be a convex function defined on the interval I. Also, let $g, h, u_\ell^i : [\theta_1,\theta_2] \longrightarrow \mathbb{R}$ be integrable functions for each $i=1,2,\cdots,s$ where $g(\theta)\in I, h(\theta), u_\ell^i(\theta)\in \mathbb{R}^+$ for all $\theta\in[\theta_1,\theta_2] (\ell=1,2,\cdots,n, i=1,2,\cdots,s, s\in N)$ and $\mathcal{H} := \int_{\theta_1}^{\theta_2} h(\theta)d\theta, \sum_{\ell=1}^n u_\ell^i(\theta)=1$, for each i. Suppose that $L_1, L_2,\cdots,L_s$ be some nonempty subsets of $\{1,2,\cdots,n\}$ such that $L_k\cap L_t=\varnothing$ for $k\neq t$ and $\cup_{i=1}^s L_i=\{1,2,\cdots,n\}$. Furthermore, if $\mathcal{J}_1,\mathcal{J}_2,\cdots,\mathcal{J}_s$ are some nonempty subintervals of $[\theta_1,\theta_2]$ such that $\mathcal{J}_k\cap\mathcal{J}_t=\varnothing$ for $k\neq t$ and $\cup_{i=1}^s\mathcal{J}_i=[\theta_1,\theta_2]$, then the following inequalities hold:*

$$\begin{aligned} &\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2} h(\theta)f(g(\theta))d\theta \\ &\geq \frac{1}{\mathcal{H}}\int_{\mathcal{J}_1}\sum_{\ell\in L_1} u_\ell^1(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1(\theta)h(\theta)d\theta}\right) \\ &\quad +\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_1}\sum_{\ell\in L_s} u_\ell^1(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1(\theta)h(\theta)d\theta}\right) \\ &\quad +\frac{1}{\mathcal{H}}\int_{\mathcal{J}_2}\sum_{\ell\in L_1} u_\ell^2(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2(\theta)h(\theta)d\theta}\right) \\ &\quad +\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_2}\sum_{\ell\in L_s} u_\ell^2(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2(\theta)h(\theta)d\theta}\right) + \\ &\quad \vdots \\ &\quad +\frac{1}{\mathcal{H}}\int_{\mathcal{J}_s}\sum_{\ell\in L_1} u_\ell^s(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s(\theta)h(\theta)d\theta}\right) \\ &\quad +\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_s}\sum_{\ell\in L_s} u_\ell^s(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s(\theta)h(\theta)d\theta}\right) \\ &\geq f\left(\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2} h(\theta)g(\theta)d\theta\right). \end{aligned} \tag{44}$$

If the function f is concave, then the reverse inequalities hold in (44).

Proof. Since $\sum_{\ell=1}^n u_\ell^i(\theta)=\sum_{\ell\in\cup_{i=1}^s L_i}u_\ell^i(\theta)=1$ for all $\theta\in[\theta_1,\theta_2]$ and each $i=1,2,\cdots,s$, therefore for the subintervals $\mathcal{J}_i$ of $[\theta_1,\theta_2]$, we have

$$\begin{aligned} &\int_{\theta_1}^{\theta_2} h(\theta)f(g(\theta))d\theta \\ &= \int_{\mathcal{J}_1}\sum_{\ell\in\cup_{i=1}^s L_i} u_\ell^1(\theta)h(\theta)f(g(\theta))d\theta \\ &\quad + \int_{\mathcal{J}_2}\sum_{\ell\in\cup_{i=1}^s L_i} u_\ell^2(\theta)h(\theta)f(g(\theta))d\theta + \cdots + \int_{\mathcal{J}_s}\sum_{\ell\in\cup_{i=1}^s L_i} u_\ell^s(\theta)h(\theta)f(g(\theta))d\theta \\ &= \int_{\mathcal{J}_1}\sum_{\ell\in L_1} u_\ell^1(\theta)h(\theta)f(g(\theta))d\theta + \cdots + \int_{\mathcal{J}_1}\sum_{\ell\in L_s} u_\ell^1(\theta)h(\theta)f(g(\theta))d\theta \\ &\quad + \int_{\mathcal{J}_2}\sum_{\ell\in L_1} u_\ell^2(\theta)h(\theta)f(g(\theta))d\theta + \cdots + \int_{\mathcal{J}_2}\sum_{\ell\in L_s} u_\ell^2(\theta)h(\theta)f(g(\theta))d\theta + \\ &\quad \vdots \\ &\quad + \int_{\mathcal{J}_s}\sum_{\ell\in L_1} u_\ell^s(\theta)h(\theta)f(g(\theta))d\theta + \cdots + \int_{\mathcal{J}_s}\sum_{\ell\in L_s} u_\ell^s(\theta)h(\theta)f(g(\theta))d\theta. \end{aligned} \tag{45}$$

Applying the integral Jensen inequality to all terms on the right hand side of (45), we obtain

$$\begin{aligned} &\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2} h(\theta)f(g(\theta))d\theta \geq \frac{1}{\mathcal{H}}\left(\int_{\mathcal{J}_1}\sum_{\ell\in L_1} u_\ell^1(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1(\theta)h(\theta)d\theta}\right)\right. \\ &\quad +\cdots+\int_{\mathcal{J}_1}\sum_{\ell\in L_s} u_\ell^1(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1(\theta)h(\theta)d\theta}\right) \\ &\quad +\int_{\mathcal{J}_2}\sum_{\ell\in L_1} u_\ell^2(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2(\theta)h(\theta)d\theta}\right) \\ &\quad +\cdots+\int_{\mathcal{J}_2}\sum_{\ell\in L_s} u_\ell^2(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2(\theta)h(\theta)d\theta}\right) + \\ &\quad \vdots \\ &\quad +\int_{\mathcal{J}_s}\sum_{\ell\in L_1} u_\ell^s(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s(\theta)h(\theta)d\theta}\right) \\ &\quad \left.+\cdots+\int_{\mathcal{J}_s}\sum_{\ell\in L_s} u_\ell^s(\theta)h(\theta)d\theta f\left(\frac{\int_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s(\theta)h(\theta)g(\theta)d\theta}{\int_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s(\theta)h(\theta)d\theta}\right)\right) \\ &\geq f\left(\frac{1}{\mathcal{H}}\int_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1(\theta)h(\theta)g(\theta)d\theta+\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1(\theta)h(\theta)g(\theta)d\theta\right. \\ &\quad +\frac{1}{\mathcal{H}}\int_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2(\theta)h(\theta)g(\theta)d\theta+\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2(\theta)h(\theta)g(\theta)d\theta \\ &\quad +\cdots+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s(\theta)h(\theta)g(\theta)d\theta+\cdots \\ &\quad \left.+\frac{1}{\mathcal{H}}\int_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s(\theta)h(\theta)g(\theta)d\theta\right)=f\left(\frac{1}{\mathcal{H}}\int_{\theta_1}^{\theta_2}h(\theta)g(\theta)d\theta\right), \end{aligned} \tag{46}$$

which concludes the result (44). □

Remark 29. Using Theorem 28 for $n=s=2$, we obtain Theorem 4. Also, one can present similar applications of Theorem 28 as in the previous sections.

4. Conclusion

Jensen's inequality and its refinements can help in various aspects; for example, it helps to authenticate the positivity of Kullback-Leibler divergence, it can be used to obtain some useful estimates for Shannon and Zipf-Mandelbrot entropies and for various divergences in information theory, its gap can be utilized to obtain error bounds in the estimation of certain parameters, and it is useful in the stability analysis of discrete and continuous-time systems with time-varying delay. In this paper, a new refinement of the integral Jensen inequality is proposed with the help of four special type of functions. The refinement is utilized to obtain improved inequalities for various means. Also, new refinements of the Hölder and the Hermite-Hadamard inequalities are obtained. New estimates for several divergences and Shannon entropy are presented. At the end of this paper, more general form of the proposed refinement of the Jensen inequality associated to several special type of functions is established.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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