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Research Article

A New Optimization Approach Based on Bipolar Type-2 Fuzzy Soft Sets

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The main objective of this research study is to amplify the schematic representation of human reasoning by launching the most generalized fantastic theories of bipolar type-2 fuzzy set (BT2FS) and bipolar type-2 fuzzy soft set (BT2FS $_f$ S). These incredible models are exclusively developed for the simultaneous capturing of both polarity and abstruseness inherent in equivocal interpretations. The proposed BT2FS $_f$ S theory renders an outstanding parameterized framework that skilfully wipes out the high-order uncertainty of imprecise knowledge-based systems. First, we keenly provide the formal structure of both proposed models along with the deep exploration of their elementary properties. Moreover, we explore rudimentary set-theoretical operations of developed frameworks inclusive of equality, subset, complement, union, and intersection with their noticeable results. We brilliantly formulate a highly proficient algorithm using proposed theory to disentangle the real-world multiattribute decision-making conundrums with two-sided ambiguous information. Finally, we holistically scrutinize an empirical analysis for the selection best way to cultivate the dessert city to demonstrate the remarkable accountability of the proposed methodology.

1. Introduction

A modern trend in contemporaneous information processing focuses on bipolar information. Bipolarity is important to identify between positive information which acts for what is guaranteed to be possible and negative information which acts for what is impossible or prohibited [1, 2]. This generic domain has recently contributed in several directions, including knowledge representation, preference modeling, multicriteria decision analysis, argumentation, and many others disciplines of science and technology [2–14]. Akram et al. [15] skillfully developed graphical representations for the comprehensive analysis of bipolar fuzzy information and presented novel decision-making techniques for the scrutinization of real-world complexities. Ali et al. [16] introduced competent attributes reduction algorithms for bipolar fuzzy relation decision systems to properly investigate sophisticated decision-making problems. Akram et al. [17] admirably designed multiskilled TOPSIS and ELECTRE-1 methods equipped with bipolar fuzzy information and patently demonstrated its empirical application in medical diagnosis.

The theory of bipolar fuzzy sets (BFSs) and its various generalizations were investigated by many researchers and practitioners as they render such brilliant mathematical tools that skillfully address the double-sided or bipolar human vague annotations [14, 18–21]. Akram et al. [22] launched a modified version of PROMETHEE method by employing bipolar fuzzy information for the determination of best green supplier in the supply chain management system. Al-Qudah and Hassan [23] introduced bipolar fuzzy soft expert sets by the amalgamation of BFSs and soft expert sets and defined algebraic operations of union, intersection, and complement for the proposed bipolar fuzzy soft expert sets. Akram et al. [24] presented an advanced trapezoidal bipolar fuzzy TOPSIS method and dexterously illustrated its application for the selection of the best construction project proposal.

All the contemporary approaches have a common limitation that arises because of their nonparameterized modeling capabilities. To overcome this problem, Molodtsov [25] developed an advanced novel theory of soft sets (S $_f$ Ss) that integrates parametrization for the specification of the expertise of rational alternatives. One of the most important applications that are mentioned in this theory is the evaluation of many practical problems in engineering, economics, medical science, social science, and in various other modern directions. Later, Maji [26] has further studied the theory of S $_f$ Ss for the proper examination of some complex realistic problems. Maji [27] explored some fundamental properties of S $_f$ Ss and also defined its algebraic operations inclusive of restricted and extended union, intersection, and difference. Chan [28], Zou and Xiao [29], Ali [30], Kong [31], Cagman and Enginoglu [32, 33], and many other pedagogues contributed to the development and improvement of the S $_f$ S theory. Moreover, Maji [34] combined the proficient theories of both S $_f$ Ss and fuzzy sets and developed an innovative concept of fuzzy soft sets (FS $_f$ Ss). Akram et al. [35] admirably proposed a hybrid theory of bipolar fuzzy N-soft sets and established the TOPSIS method under such a versatile environment for the robust inspection of its high potential applications.

Zadeh [36] proposed an entirely new theory of fuzzy sets for capturing the imprecision of obscure data with the assistance of a MF (MF) that can grab values from the closed unit interval. The traditional fuzzy sets are also called type-1 fuzzy sets [37]. The most important application in FS $_f$ Ss is the investigation of actual decision-making perplexities [38–40]. Later, Zadeh [41–43] introduced the notion of a type-2 fuzzy set (T2FS) in which the membership degree of each element is given by another type-1 fuzzy set defined over the interval [0,1]. It can also be generalized from type-1 FS $_f$ Ss to type-2 FS $_f$ Ss. Matsumoto and Tanaka [44, 45] discussed some elementary algebraic properties of T2FSs. Dubois and Prade [46] remarkably presented several notions and applications of T2FSs. Later, many researchers conducted a lot of theoretical work on the properties of T2FS and manifested its many pragmatic applications [47–50]. Type-2 fuzzy sets were applied in several domains, especially in decision-making [51], connection admission control [52], mobile robotic control [53], equalization of nonlinear fading channels [54–57], extraction of knowledge from questionnaire surveys [52, 58], forecasting of time-series [57, 59, 60], function approximation [58], learning linguistic membership grades [61], preprocessing radiographic images [62], relational databases [63], solving fuzzy relation equations [64], and transport scheduling [65]. Zhang and Zhang [66] considered type-2 FS $_f$ Ss with applications.

In several tricky practical applications, we often face an intricate situation during evaluation processes in which stipulated parameters are highly equivocal. It can be represented by fuzzy sets rather than exact numerical values, intervals, and interval-valued intuitionistic fuzzy numbers. Obviously, it is very difficult for the classical S $_f$ S and its contemporary extensions to deal with such arduous problems. Therefore, it is necessary to extend the soft set theory for the accommodation of problematic scenarios in which the specified evaluation parameters are totally ambiguous in nature. The purpose of this paper is to further generalize the concept of type-2 fuzzy sets and type-2 FS $_f$ Ss to BT2FSs and bipolar type-2 S $_f$ Ss, respectively. Now, we present the highly competent novel theories of BT2FSs and BT2FS $_f$ Ss and define some basic operations on them. Moreover, we also examine the applications of BT2FSs and BT2FS $_f$ Ss and weighted type-2 FS $_f$ Ss in decision-making problems.

The rest of this paper is organized as follows. Section 2 recollects some relevant studies of the literature. Section 3 explicates the novel theory of BT2FSs, its elementary properties, and some algebraic operations inclusive of union, intersection, and complement. Section 4 presents an advanced concept of BT2FS $_f$ Ss and discusses its basic algebraic operations along with some fundamental characteristics. Section 5 illustrates the applications of BT2FSs, BT2FS $_f$ Ss, and weighted BT2FS $_f$ Ss in decision-making problems. Section 6 explores some concluding remarks and future research directions.

2. Preliminaries

In this section, we shall review the concepts of S $_f$ S, fuzzy soft set, BFS and bipolar fuzzy soft set.

Definition 1 (see [25]). Let E be a set of parameters and let $\mathscr{H} \subseteq$ E. Let F be a mapping defined by

$$F : \mathscr{H} \longrightarrow P(\mathbb{Q}), \tag{1}$$

where $P(\mathbb{Q})$ is a collocation of all subsets of the initial universal set $\mathbb{Q}$. Then, the pair $(F, \mathscr{H})$ is a S $_f$ S.

Definition 2 (see [67, 68]). Let $\mathscr{H} \subseteq E$ be a subset of parameters set. Let $P_F(\mathbb{Q})$ be a collocation of all fuzzy subset of the initial universal set $\mathbb{Q}$ and let F be a mapping defined by

$$F : \mathscr{H} \longrightarrow P_F(\mathbb{Q}) \tag{2}$$

Then, we call the pair $(F, \mathscr{H})$ is a FS $_f$ S over $\mathbb{Q}$.

Definition 3. A BFS $\mathscr{I}$ on U is an object having the form

$$\mathscr{I} = \{(\mathfrak{q}, (\mu^+_{\mathscr{I}}(\mathfrak{q}), \mu^-_{\mathscr{I}}(\mathfrak{q}))) ; \mathfrak{q} \in U\}, \tag{3}$$

where $\mu^+_{\mathscr{I}} : U \longrightarrow [0, 1], \mu^-_{\mathscr{I}} : U \longrightarrow [-1, 0]$, $\mu^+_{\mathscr{I}}$ represent positive information and $\mu^-_{\mathscr{I}}$ represents negative information.

Definition 4. Let pair $(F_b, \mathscr{H})$ be a bipolar FS $_f$ S over initial universal set $\mathbb{Q}$, where $\mathscr{H} \subseteq E$ is a parameters set and F_b is a mapping such that

$$F_b : \mathscr{H} \longrightarrow BF(\mathbb{Q}), \tag{4}$$

where $BF(\mathbb{Q})$ illustrates all bipolar fuzzy subsets of $\mathbb{Q}$.

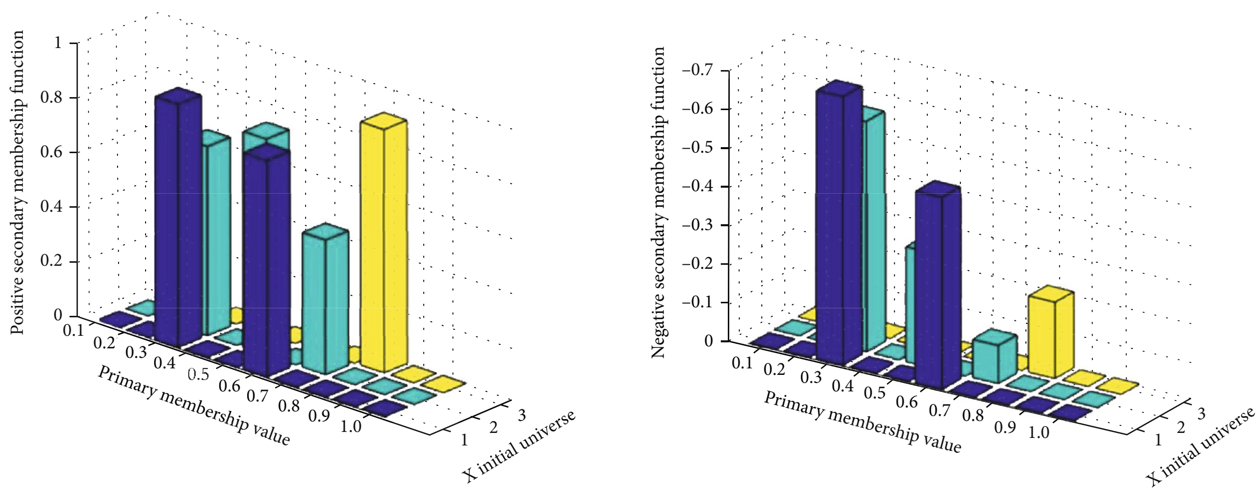

FIGURE 1: Bipolar type-2 fuzzy sets.

Definition 5. $\mathscr{H}$ is called a T2FS over initial universe $\mathbb{Q}$ if it is characterized by a secondary MF $\mu_{\mathscr{H}}(\mathfrak{q},\mathfrak{k})$: $\mathbb{Q}\times[0,1]\longrightarrow[0,1]$ where $\mathfrak{k}\in J_{\mathfrak{q}}$ is primary membership value of an element $\mathfrak{q}\in\mathbb{Q}$ such that $J_{\mathfrak{q}}:\mathbb{Q}\longrightarrow[0,1]$. Thus we define the set $\mathscr{H}$ by

$$\mathscr{H}=\left\{\left((\mathfrak{q},\mathfrak{k}),\mu_{\mathscr{H}}(\mathfrak{q},\mathfrak{k})\right);\forall\mathfrak{q}\in\mathbb{Q},\mathfrak{k}\in j_{\mathfrak{q}}\right\}, \quad (5)$$

or by

$$\mathscr{H}=\int_{\mathfrak{q}\in\mathbb{Q}}\left(\int_{\mathfrak{k}\in J_{\mathfrak{q}}\subseteq[0,1]}\mu_{\mathscr{H}}(\mathfrak{q},\mathfrak{k})/\mathfrak{k}\right)/\mathfrak{q}. \quad (6)$$

Take into account the symbol $\int\int$ denotes union over all $\mathfrak{q}$ and $\mathfrak{k}$, and + also denotes union [69].

3. Bipolar Type-2 Fuzzy Set

In this section, we present a new concept of BT2FS, its operations, and some of its basic fundamental properties.

Definition 6. Let $\mathbb{Q}$ be the initial universal set. Then $\mathscr{H}_b$ is called a BT2FS if it is characterized by a pair $(\mu^{+}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k}),\mu^{-}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k}))$, where $\mu^{+}_{\mathscr{H}_b}:\mathbb{Q}\times[0,1]\longrightarrow[0,1]$ and $\mu^{-}_{\mathscr{H}_b}:\mathbb{Q}\times[0,1]\longrightarrow[-1,0]$ represents positive and negative secondary MF of $\mathfrak{q}\in\mathbb{Q}$, respectively, and $\mathfrak{k}\in J_{\mathfrak{q}}\subseteq[0,1]$ denotes the primary membership value of an element $\mathfrak{q}\in\mathbb{Q}$ such that $\mathrm{J}_{\mathfrak{q}}:\mathbb{Q}\longrightarrow[0,1]$. Then, the set $\mathscr{H}_{\mathrm{b}}$ is defined by

$$\mathscr{H}_b=\int_{\mathfrak{q}\in\mathbb{Q}}\left(\frac{\int_{\mathfrak{k}\in J_{\mathfrak{q}}\subseteq[0,1]}\left(\mu^{+}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k}),\mu^{-}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\right)}{\mathfrak{k}}\right)/\mathfrak{q}. \quad (7)$$

The set of all BT2FSs in $\mathbb{Q}$ is denoted by BF_{T2}.

Remark 7. The notion $\int$ denotes union over all admissible $\mathfrak{q}$ and $\mathfrak{k}$, and + also denotes union [70].

To clarify the concept of BT2FS, let us consider the following example.

Example 1. Let $\mathbb{Q}=\{\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3\}$ be the initial universal set and let $\mathscr{H}_b$ be a BT2FS defined as

$$\mathscr{H}_b=\frac{(0.8,-0.5)/0.6+(0.9,-0.7)/0.3}{\mathfrak{q}_1}+\frac{(0.7,-0.6)/0.3+(0.8,-0.3)/0.5+(0.5,-0.1)/0.7}{\mathfrak{q}_2}+\frac{(0.9,-0.2)/0.8}{\mathfrak{q}_3}. \quad (8)$$

Graphical representation is shown in Figure 1.

Suppose $\mathscr{H}_b,\mathscr{J}_b\in BF_{T2}$ then

$$\mathscr{H}_b=\int_{\mathfrak{q}\in\mathbb{Q}}\left(\frac{\int_{\mathfrak{k}\in J_{\mathfrak{q}}\subseteq[0,1]}\left(\mu^{+}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k}),\mu^{-}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\right)}{\mathfrak{k}}\right)/\mathfrak{q},$$
$$\mathscr{J}_b=\int_{\mathfrak{q}\in\mathbb{Q}}\left(\frac{\int_{\nu\in J_{\mathfrak{q}}\subseteq[0,1]}\left(\mu^{+}_{\mathscr{J}_b}(\mathfrak{q},\nu),\mu^{-}_{\mathscr{J}_b}(\mathfrak{q},\nu)\right)}{\nu}\right)/\mathfrak{q}. \quad (9)$$

Definition 8. For two BT2FSs $\mathscr{H}_b$ *and* $\mathscr{J}_b$ of $\mathbb{Q}$, we say that $\mathscr{J}_b$ is a bipolar type-2 fuzzy (BT2F) subset of $\mathscr{H}_b$ if and only if

(i) $\nu\le\mathfrak{k}$,

(ii) $\mu^{+}_{\mathscr{J}_b}(\mathfrak{q},\nu)\le\mu^{+}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})$ for a positive secondary MF

(iii) $\mu^{-}_{\mathscr{J}_b}(\mathfrak{q},\nu)\ge\mu^{-}_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})$ for a negative secondary MF

where $\mathfrak{k}$ is a primary membership value of $\mathscr{H}_b$ and ν is a primary membership value of $\mathscr{J}_b$.

Example 2. Consider two sets of $\mathrm{BF}_{\mathrm{T2}}\mathscr{H}_b$ "used in Example 1" and $\mathscr{J}_b$ define

$$\mathscr{H}_b = \frac{(0.8,-0.4)/0.1 + (0.2,-0.5)/0.3}{\mathfrak{q}_1} + \frac{(0.4,-0.3)/0.2 + (0.6,-0.1)/0.3 + (0.3,-0.1)/0.5}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.3}{\mathfrak{q}_3}. \quad (10)$$

So, $\mathscr{I}_b$ is a BT2F subset of $\mathscr{H}_b$.

Remark 9. $\mathscr{H}_b$ and $\mathscr{I}_b$ two BF_{T2} sets of $\mathbb{Q}$ are said to be equal, if $\mathscr{H}_b$ is a BT2F subset of $\mathscr{I}_b$ and $\mathscr{I}_b$ is a BT2F subset of $\mathscr{H}_b$.

3.1. Operations of BF_{T2}. In this section, we dialogue three essential arithmetic operations inclusive of complement, union, and intersection of BF_{T2}.

(1) Complement: the complement of $\mathscr{H}_b$ a BF_{T2} is denoted $\mathscr{H}_b^c$ and defined by

$$\mathscr{H}_b^c = \frac{\int_{\mathfrak{q}\in\mathbb{Q}}\left(\int_{\mathfrak{k}\in J_\mathfrak{q}\subseteq[0,1]}\left(\mu^+_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k}),\mu^-_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\right)/1-\mathfrak{k}\right)}{\mathfrak{q}}. \quad (11)$$

(2) Union: let $\mathscr{H}_b, \mathscr{I}_b \in BF_{T2}$, so we defined the union of $\mathscr{H}_b$ and $\mathscr{I}_b$ by $\mathscr{J}_b$:

$$\mathscr{J}_b = \mathscr{H}_b \sqcup \mathscr{I}_b = \frac{\int_{\mathfrak{q}\in\mathbb{Q}}\left(\int_{w\in J_\mathfrak{q}\subseteq[0,1]}\left(\mu^+_{\mathscr{J}_b}(\mathfrak{q},w),\mu^-_{\mathscr{J}_b}(\mathfrak{q},w)\right)/w\right)}{\mathfrak{q}}, \quad (12)$$

where $w = \mathfrak{k}\vee\nu$; $\mu^+_{\mathscr{J}_b}(\mathfrak{q},w) = \mu^+_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\wedge\mu^+_{\mathscr{I}_b}(\mathfrak{q},\nu)$, $\mu^-_{\mathscr{J}_b}(\mathfrak{q},w) = \mu^-_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\vee\mu^-_{\mathscr{I}_b}(\mathfrak{q},\nu)$, $\vee$ is a maximum and $\wedge$ is a minimum.

(3) Intersection: let $\mathscr{H}_b, \mathscr{I}_b \in BF_{T2}$, so defined the intersection of $\mathscr{H}_b$ and $\mathscr{I}_b$ by $\mathscr{J}_b$:

$$\mathscr{J}_b = \mathscr{H}_b \sqcap \mathscr{I}_b = \frac{\int_{\mathfrak{q}\in\mathbb{Q}}\left(\int_{w\in J_\mathfrak{q}\subseteq[0,1]}\left(\mu^+_{\mathscr{J}_b}(\mathfrak{q},w),\mu^-_{\mathscr{J}_b}(\mathfrak{q},w)\right)/w\right)}{\mathfrak{q}}, \quad (13)$$

where $\mu^+_{\mathscr{J}_b}(\mathfrak{q},w) = \mu^+_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\wedge\mu^+_{\mathscr{I}_b}(\mathfrak{q},\nu)$; $\mu^-_{\mathscr{J}_b}(\mathfrak{q},w) = \mu^-_{\mathscr{H}_b}(\mathfrak{q},\mathfrak{k})\vee\mu^-_{\mathscr{I}_b}(\mathfrak{q},\nu)$ and where $w = \mathfrak{k}\wedge\nu$.

Example 3. Let us consider two BF_{T2} of the initial universe $\mathbb{Q}$ as

$$\begin{aligned}\mathscr{H}_b &= \frac{(0.8,-0.5)/0.6 + (0.9,-0.7)/0.3}{\mathfrak{q}_1} + \frac{(0.7,-0.6)/0.3 + (0.8,-0.3)/0.5 + (0.5,-0.1)/0.7}{\mathfrak{q}_2} + \frac{(0.9,-0.2)/0.8}{\mathfrak{q}_3},\\ \mathscr{I}_b &= \frac{(0.9,-0.6)/0.8 + (1.0,-0.8)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.9 + (0.1,-0.5)/0.4}{\mathfrak{q}_2} + \frac{(0.6,-0.3)/0.5 + (0.2,-0.8)/0.4}{\mathfrak{q}3},\end{aligned} \quad (14)$$

then

$$\begin{aligned}\mathscr{H}_b^c &= \frac{(0.8,-0.5)/0.4 + (0.9,-0.7)/0.7)}{\mathfrak{q}_1} + \frac{(0.7,-0.6)/0.7 + (0.8,-0.3)/0.5 + (0.5,-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.9,-0.2)/0.2}{\mathfrak{q}_3},\\ \mathscr{H}_b\sqcup\mathscr{I}_b &= \frac{(0.8,-0.5)/0.8 + (0.8,-0.5)/0.7 + (0.9,-0.6)/0.8 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.9 + (0.1,-0.5)/0.4 + (0.1,-0.3)/0.5 + (0.5,-0.1)/0.9 + (0.1,-0.1)/0.7)}{\mathfrak{q}_2}\\ &+ \frac{(0.6,-0.2)/0.8 + (0.2,-0.2)/0.8}{\mathfrak{q}_3}, = \frac{(0.9,-0.6)/0.8 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.9 + (0.1,-0.5)/0.4 + (0.1,-0.3)/0.5 + (0.1,-0.1)/0.7}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.8}{\mathfrak{q}_3},\\ \mathscr{H}_b\sqcap\mathscr{I}_b &= \frac{(0.8,-0.5)/0.6 + (0.8,-0.5)/0.6 + (0.9,-0.6)/0.3 + (0.9,-0.7)/0.3}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.3 + (0.1,-0.2)/0.3 + (0.6,-0.2)/0.5 + (0.1,-0.3)/0.4 + (0.5,-0.1)/0.7 +}{\mathfrak{q}2}\\ &+ \frac{(0.6,-0.2)/0.5 + (0.2,-0.2)/0.4}{\mathfrak{q}_3}, = \frac{(0.8,-0.5)/0.6 + (0.9,-0.7)/0.3}{\mathfrak{q}_1} + \frac{(0.6,-0.5)/0.3 + (0.6,-0.2)/0.5 + (0.1,-0.3)/0.4 + (0.5,-0.1)/0.7}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.5 + (0.2,-0.2)/0.4}{\mathfrak{q}_3},\end{aligned} \quad (15)$$

Table 1: Tabular representation of $\Gamma = (\tau, \mathscr{H}, \Lambda.)$.

$\mathbb{Q}$	$\xi_1, \Lambda_1 = 0.2$	$\xi_2, \Lambda_2 = 0.9$	$\xi_3, \Lambda_3 = 0.7$	$\xi_4, \Lambda_4 = 0.4$
$\mathfrak{q}_1$	$\frac{(0.30,-0.65)}{0.30} + \frac{(0.50,-0.75)}{0.60}$	$\frac{(0.50,-0.40)}{0.25}$	$\frac{(0.15,-0.25)}{0.15} + \frac{(0.40,-0.60)}{0.75}$	$\frac{(0.75,-0.60)}{0.85}$
$\mathfrak{q}_2$	$\frac{(0.45,-0.30)}{0.60} + \frac{(0.70,-0.25)}{0.85}$	$\frac{(0.55,-0.20)}{0.45} + \frac{(0.80,-0.35)}{0.75}$	$\frac{(0.70,-0.30)}{0.50}$	$\frac{(0.80,-0.30)}{0.80}$

3.2. Properties. Consider three BF_{T2} sets $\mathscr{H}_b$,$\mathscr{I}_b$, and $\mathscr{J}_b$

$$\begin{aligned}
&\mathscr{H}_b \sqcup \mathscr{H}_b = \mathscr{H}_b,\\
&\mathscr{H}_b \sqcap \mathscr{H}_b = \mathscr{H}_b,\\
&(\mathscr{H}_b \sqcup \mathscr{I}_b) \sqcup \mathscr{J}_b = \mathscr{H}_b \sqcup (\mathscr{I}_b \sqcup \mathscr{J}_b),\\
&(\mathscr{H}_b \sqcap \mathscr{I}_b) \sqcap = \mathscr{H}_b \sqcap (\mathscr{I}_b \sqcap \mathscr{J}_b),\\
&\mathscr{H}_b \sqcup (\mathscr{I}_b \sqcap \mathscr{J}_b) = (\mathscr{H}_b \sqcup \mathscr{I}_b) \sqcap (\mathscr{H}_b \sqcup \mathscr{J}_b),\\
&\mathscr{H}_b \sqcap (\mathscr{I}_b \sqcup \mathscr{J}_b) = (\mathscr{H}_b \sqcap \mathscr{I}_b) \sqcup (\mathscr{H}_b \sqcap \mathscr{J}_b),\\
&(\mathscr{H}_b \sqcup \mathscr{I}_b)^c = \mathscr{H}_b^c \sqcap \mathscr{I}_b^c,\\
&(\mathscr{H}_b \sqcap \mathscr{I}_b)^c = \mathscr{H}_b^c \sqcup \mathscr{I}_b^c,\\
&\mathscr{H}_b \sqcup \mathscr{I}_b = \mathscr{I}_b \sqcup \mathscr{H}_b,\\
&\mathscr{H}_b \sqcap \mathscr{I}_b = \mathscr{I}_b \sqcup \mathscr{H}_b.
\end{aligned} \tag{16}$$

Example 4. Reconsider Example 3 and let $\mathscr{J}_b \in BF_{T2}$ defined by

$$\mathscr{J}_b = \frac{(0.9,-0.3)/0.1 + (0.8,-0.1)/0.5}{\mathfrak{q}_1} + \frac{(0.6,-0.1)/0.8)}{\mathfrak{q}_2} + \frac{(0.3,-0.6)/1.0 + (0.4,-0.7)/0.2 + (0.5,-0.5)/0.7}{\mathfrak{q}_3}. \tag{17}$$

Thus, we have

$$\begin{aligned}
(\mathscr{H}_b \sqcup \mathscr{I}_b) \sqcup \mathscr{J}_b &= \left[\frac{(0.9,-0.6)/0.8 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.9 + (0.1,-0.5)/0.4 + (0.1,-0.3)/0.5 + (0.1,-0.1)/0.7}{\mathfrak{q}_2}\right.\\
&\quad \left. + \frac{(0.6,-0.2)/0.8}{\mathfrak{q}_3}\right] \sqcup \frac{(0.9,-0.3)/0.1 + (0.8,-0.1)/0.5 + ((0.6,-0.1)/0.8/\mathfrak{q}_2)}{\mathfrak{q}_1}\\
&\quad + \frac{(0.3,-0.6)/1.0 + (0.4,-0.7)/0.2 + (0.5,-0.5)/0.7}{\mathfrak{q}_3},\\
&= \frac{(0.9,-0.3)/0.8 + (0.9,-0.3)/0.7 + ((0.6,-0.1)/0.9 + (0.1,-0.1)/0.8/\mathfrak{q}_2)}{\mathfrak{q}_1} + \frac{(0.3,-0.2)/1.0 + (0.5,-0.2)/0.8}{\mathfrak{q}_3}
\end{aligned} \tag{18}$$

We also have

$$\begin{aligned}
\mathscr{H}_b \sqcup (\mathscr{I}_b \sqcup \mathscr{J}_b) &= \frac{(0.8,-0.5)/0.6 + (0.9,-0.7)/03}{\mathfrak{q}_1} + \frac{(.7,-0.6)/0.3 + (0.8,-0.3)/0.5 + (0.5,-.1)/0.7}{\mathfrak{q}_2}\\
&\quad + \frac{(0.9,-0.2)/0.8}{\mathfrak{q}_3} \sqcup \left[\frac{(0.9,-0.6)/0.8 + (1.0,-0.8)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.9 + (0.1,-0.5)/0.4}{\mathfrak{q}_2}\right.\\
&\quad + \frac{(0.6,-0.3)/0.5 + (0.2,-0.8)/0.4}{\mathfrak{q}3} \sqcup \frac{(0.9,-0.3)/0.1 + (0.8,-0.1)/0.5}{\mathfrak{q}_1} + \frac{(0.6,-0.1)/0.8}{\mathfrak{q}_2}\\
&\quad \left. + \frac{(0.3,-0.6)/1.0 + (0.4,-0.7)/0.2 + (0.5,-0.5)/0.7}{\mathfrak{q}_3}\right],\\
&= \frac{(0.9,-0.3)/0.8 + (0.9,-0.3)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.1)/0.9 + (0.1,-0.1)/0.8}{\mathfrak{q}_2} + \frac{(0.3,-0.2)/1.0 + (0.5,-0.2)/08}{\mathfrak{q}_3}.
\end{aligned} \tag{19}$$

Table 2: Tabular representation of $\Gamma = (\tau, \mathscr{H}, \Lambda)$ with weighted choice value.

$\mathbb{Q}$	$\xi_1, \Lambda_1 = 0.2$	$\xi_2, \Lambda_2 = 0.9$	$\xi_3, \Lambda_3 = 0.7$	$\xi_4, \Lambda_4 = 0.4$	Weighted choice value c_i
$\mathfrak{q}_1$	0	0	1	1	$c_1 = 1.1$
$\mathfrak{q}_2$	1	1	0	1	$c_2 = 1.40$

So, we obtain

$$(\mathscr{H}_b \sqcup \mathscr{I}_b) \sqcup \mathscr{J}_b = \mathscr{H}_b \sqcup (\mathscr{I}_b \sqcup \mathscr{J}_b)$$

$$(\mathscr{H}_b \sqcup \mathscr{I}_b)^c = \frac{(0.9,-0.6)/0.2 + (0.9,-0.7)/0.3}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.1 + (0.1,-0.5)/0.6 + (0.1,-0.3)/0.5 + (0.1,-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.2}{\mathfrak{q}_3},$$
$$\mathscr{H}_b^c \sqcap \mathscr{I}_b^c = \frac{(0.9,-0.6)/0.2 + (0.9,-0.7)/0.3}{\mathfrak{q}_1} + \frac{(0.6,-0.2)/0.1 + (0.1,-0.5)/0.6 + (0.1,-0.3)/0.5 + (0.1.-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.2}{\mathfrak{q}_3}, \quad (20)$$

Then, we deduce $(\mathscr{H}_b \sqcup \mathscr{I}_b)^c = \mathscr{H}_b^c \sqcap \mathscr{I}_b^c$.

$$(\mathscr{H}_b \sqcap \mathscr{I}_b)^c = \mathscr{H}_b^c \sqcup \mathscr{I}_b^c \quad (21)$$

Indeed

$$(\mathscr{H}_b \sqcap \mathscr{I}_b)^c = \frac{(0.8,-0.5)/0.4 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.5)/0.7 + (0.6,-0.2)/0.5 + (0.1,-0.3)/0.6 + (0.5,-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.5 + (0.2,-0.2)/0.6}{\mathfrak{q}_3},$$
$$\mathscr{H}_b^c \sqcup \mathscr{I}_b^c = \left[\frac{(0.8,-0.5)/0.4 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.7,-0.6)/0.7 + (0.8,-0.3)/0.5 + (0.5,-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.9,-0.2)/0.2}{\mathfrak{q}_3}\right.$$
$$\left. + \frac{(0.6,-0.3)/0.5 + (0.2,-0.8)/0.6}{\mathfrak{q}3}\right] = \frac{(0.8,-0.5)/0.4 + (0.9,-0.7)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.5)/0.7 + (0.6,-0.2)/0.5 + (0.1,-0.3)/0.6 + (0.5,-0.1)/0.3}{\mathfrak{q}_2} + \frac{(0.6,-0.2)/0.5 + (0.2,-0.2)/0.6}{\mathfrak{q}_3}. \quad (22)$$

Theorem 10. *If $\mathscr{H}_b$ BT2F is a subset of $\mathscr{I}_b$ and $\mathscr{I}_b$ BT2F a subset of $\mathscr{J}_b$, then $\mathscr{H}_b$ BT2F is a subset of $\mathscr{J}_b$.*

Proof. Suppose that $\mathscr{H}_b$ BT2F subset of $\mathscr{I}_b$, thus $\mathfrak{k} \le \nu$ for primary membership values and $\mu^+_{\mathscr{H}_b}(\mathfrak{q}, \mathfrak{k}) \le \mu^+_{\mathscr{I}_b}(\mathfrak{q}, \nu)$, $\mu^-_{\mathscr{H}_b}(\mathfrak{q}, \mathfrak{k}) \ge \mu^-_{\mathscr{I}_b}(\mathfrak{q}, \nu)$ for, respectively, positive and negative secondary membership functions of $\mathfrak{q} \in \mathbb{Q}$, and suppose that $\mathscr{I}_b$ BT2F subset of $\mathscr{J}_b$, thus $\nu \le w$ for primary membership values and $\mu^+_{\mathscr{I}_b}(\mathfrak{q}, \nu) \le \mu^+_{\mathscr{J}_b}(\mathfrak{q}, w)$,$\mu^-_{\mathscr{I}_b}(\mathfrak{q}, \nu) \ge \mu^-_{\mathscr{J}_b}(\mathfrak{q}, w)$ for, respectively, positive and negative secondary membership functions of $\mathfrak{q} \in \mathbb{Q}$. □

Therefore, $\mathfrak{k} \le w$,$\mu^+_{\mathscr{H}_b}(\mathfrak{q}, \mathfrak{k}) \le \mu^+_{\mathscr{J}_b}(\mathfrak{q}, w)$, and $\mu^-_{\mathscr{H}_b}(\mathfrak{q}, \mathfrak{k}) \ge \mu^-_{\mathscr{J}_b}(\mathfrak{q}, w)$.

Hence, $\mathscr{H}_b$ BT2F is a subset of $\mathscr{J}_b$.

Theorem 11. *If $\mathscr{H}_b$ BT2FS is equal to $\mathscr{I}_b$ and $\mathscr{I}_b$ BT2FS is equal to $\mathscr{J}_b$, then $\mathscr{H}_b$ BT2FS is equal to $\mathscr{J}_b$.*

Proof. Same proof as Theorem 10. □

4. Bipolar Type-2 Fuzzy Soft Set

Throughout this section, we call $\mathbb{Q}$ is an initial universe set, E is a set of all possible parameters under consideration with respect to $\mathbb{Q}$, $BF_{T2}(\mathbb{Q})$ is all BT2FSs of $\mathbb{Q}$, $\mathfrak{k}$,ν, and w be primary membership degrees, and $\mu^+_{\tau(\xi)}, \mu^-_{\tau(\xi)}$ are, respectively, the positive and negative secondary MF of $\mathbb{Q}$.

Definition 12. Let $\mathscr{H} \subseteq E$ be a set of parameters, and τ is a mapping given by

$$\tau : \mathscr{H} \longrightarrow BF_{T2}(\mathbb{Q}). \tag{23}$$

Then the pair $(\tau, \mathscr{H})$ is called a BT2FS $_f$ S over $\mathbb{Q}$, and $BFS_{T2}(\mathbb{Q})$ refers to all BT2FS $_f$ Ss over $\mathbb{Q}$.

So far a BT2FS $_f$ S is a parameterized family of BT2F subset of $\mathbb{Q}$.

For any $\xi \in \mathscr{H}, \tau(\xi)$ refers to the set of ξ-approximate element of the BT2FS $_f$ S $(\tau, \mathscr{H})$, it is actually a BT2FS on $\mathbb{Q}$, and it can be written as

$$\tau(\xi) = \frac{\int_{\mathfrak{q}\in\mathbb{Q}}\left(\int_{\mathfrak{k}\in J_{\mathfrak{q}}\subseteq[0,1]}\left(\mu^{+}_{\tau(\xi)}(\mathfrak{q},\mathfrak{k}), \mu^{-}_{\tau(\xi)}(\mathfrak{q},\mathfrak{k})\right)/\mathfrak{k}\right)}{\mathfrak{q}}, \tag{24}$$

where $\mathfrak{k}$ is the primary membership degree and $\mu^{+}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k}), \mu^{-}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k})$ are, respectively, the positive and negative secondary membership degree that object $\mathfrak{q}$ holds on parameter ξ.

Example 5. Consider a $(\tau, \mathscr{H}) \in BFS_{T2}(\mathbb{Q})$ over $\mathbb{Q}$, where $\mathbb{Q}$ is a set of four cars under the consideration of decision-maker to purchase $\mathbb{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4\}$, and $\mathscr{H}$ *is a* parameter set, where $\mathscr{H} = \{\xi_1, \xi_2, \xi_3\} = \{expensive, beautiful, comfortable\}$. The *BT2FS $_f$ S* $(\tau, \mathscr{H})$ describes the "attractiveness of the cars" to this decision-maker.

Assume that

$$\begin{aligned}
\tau(\xi_1) &= \frac{(0.3,-0.5)/0.1 + (1.0,-0.4)/0.2 + (0.7,-0.1)/0.3}{\mathfrak{q}_1} + \frac{(0.4,-0.2)/0.5 + (0.8,-1.0)/0.9}{\mathfrak{q}_2} + \\
&\quad \frac{(0.5,-0.2)/0.7}{\mathfrak{q}_3} + \frac{(0.3,-0.6)/0.4 + (0.8,-0.1)/0.5 + (0.2,-0.5)/0.6}{\mathfrak{q}_4}, \\
\tau(\xi_2) &= \frac{(0.3,-0.6)/0.1 + (0.6,-0.9)/0.4 + (0.5,-0.8)/0.7}{\mathfrak{q}_1} \\
&\quad + \frac{(0.5,-0.1)/0.8 + (0.2,-0.7)/0.1 + (0.6,-0.3)/0.2 + (0.9,-0.6)/0.6}{\mathfrak{q}_2} + \frac{(0.7,-0.2)/0.8}{\mathfrak{q}_3} + \frac{(0.9,-0.6)/0.4}{\mathfrak{q}_4}, \\
\tau(\xi_3) &= \frac{(0.2,-0.4)/0.6 + (0.8,-0.3)/0.8 + (0.6,-1.0)/0.9}{\mathfrak{q}_1} + \frac{(0.6,-0.8)/0.4 + (0.8,-0.1)/0.7}{\mathfrak{q}_2} \\
&\quad + \frac{(0.8,-0.4)/0.9}{\mathfrak{q}_3} + \frac{(0.7,-0.6)/0.3 + (0.1,-0.4)/0.6}{\mathfrak{q}_4}.
\end{aligned} \tag{25}$$

Definition 13. Let $\mathscr{H}, \mathscr{I} \subseteq E$. Let $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in \mathrm{BFS}_{T2}(\mathbb{Q})$. If $\mathscr{H} \subseteq \mathscr{I}$ and $\forall \xi \in \mathscr{H}, \tau(\xi) \subseteq \psi(\xi)$ then $(\tau, \mathscr{H})$ is called a bipolar type-2 fuzzy soft (BT2FS $_f$) subset of $(\psi, \mathscr{I})$.

Remark 14. For $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in BFS_{T2}(\mathbb{Q})$ are said to be equal, if and only if $(\tau, \mathscr{H})$ is *BT2FS $_f$* subset of $(\psi, \mathscr{I})$ and $(\psi, \mathscr{I})$ is BT2FS $_f$ subset of $(\tau, \mathscr{H})$.

4.1. Operations on $BFS_{T2}(\mathbb{Q})$. In this section, we debate five operations, namely, complement, union, intersection, AND, and OR of a $BFS_{T2}(\mathbb{Q})$. Let $E = \{\xi_1, \xi_2, \cdots, \xi_n\}$ be a parameter set, and defined $\neg E = \{\neg\xi_1, \neg\xi_2, \cdots, \neg\xi_n\}$ where $\neg\xi_i$ is not ξ_i and let $\mathscr{H}, \mathscr{I} \subseteq E$.

(1) Complement

The complement of BT2FS $_f$ S is defined by

$$(\tau, \mathscr{H})^c = (\tau^c, \neg\mathscr{H}), \tag{26}$$

where $\tau^c(\xi) = \neg(\tau(\neg\xi)$; for all $\xi \in \neg\mathscr{H}$.

(2) Union

Let $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in BFS_{T2}(\mathbb{Q})$, the union of $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ is $(\varphi, \mathscr{J}) \in BFS_{T2}(\mathbb{Q})$, and defined it for all $\xi \in \mathscr{J} = \mathscr{H} \cup \mathscr{I}$

$$\varphi(\xi) = \begin{cases} \tau(\xi) \text{ if} \xi \in \mathscr{H} - \mathscr{I}, \\ \psi(\xi) \text{if} \xi \in \mathscr{I} - \mathscr{H}, \\ \tau(\xi) \mathrm{o} \psi(\xi) \text{if} \xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \tag{27}$$

(3) Intersection

Let $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in BFS_{T2}(\mathbb{Q})$ with $\mathscr{H} \cap \mathscr{I} \neq \phi$, the intersection of $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ is $(\varphi, \mathscr{J}) \in BFS_{T2}(\mathbb{Q})$, defined by

$$\varphi(\xi) = \tau(\xi) \sqcap \psi(\xi) \quad \text{for all} \quad \xi \in \mathscr{J} = \mathscr{H} \cap \mathscr{I}. \tag{28}$$

(4) AND

$(\tau, \mathscr{H})$ AND $(\psi, \mathscr{I})$ is defined by $(\tau, \mathscr{H}) \wedge (\psi, \mathscr{I}) = (\varphi, \mathscr{H} \times \mathscr{I})$, where for all $(a, b) \in \mathscr{H} \times \mathscr{I}, \varphi(a, b) = \tau(a) \sqcap \psi(b)$.

(5) OR

$(\tau, \mathscr{H})$ OR $(\psi, \mathscr{I})$ is defined by $(\tau, \mathscr{H}) \vee (\psi, \mathscr{I}) = (\varphi, \mathscr{H} \times \mathscr{I})$, which is $\varphi(a, b) = \tau(a) \sqcup \psi(b)$ for all $(a, b) \in \mathscr{H} \times \mathscr{I}$.

Example 6. Let $E = \{\xi_1, \xi_2, \xi_3\}$ be a parameter set, and let $\mathscr{H} = \{\xi_1, \xi_2\}$ and $\mathscr{I} = \{\xi_2, \xi_3\}$ be subsets of E. Now consider two BT2FS $_f$ Ss $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ over $\mathbb{Q}$, where $\mathbb{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ and define $(\tau, \mathscr{H})$ by

$$\begin{aligned}\tau(\xi_1) &= \frac{(0.3,-0.5)/0.1 + (1.0,-0.4)/0.2 + (0.7,-0.1)/0.3}{\mathfrak{q}_1} + \frac{(0.4,-0.2)/0.5 + (0.8,-0.1)/0.9}{\mathfrak{q}_2} + \frac{(0.5,-0.2)/0.7}{\mathfrak{q}_3},\\ \tau(\xi_2) &= \frac{(0.3,-0.6)/0.1 + (0.6,-0.9)/0.4 + (0.5,-0.8)/0.7}{\mathfrak{q}_1} + \frac{(0.5,-0.1)/0.8 + (0.2,-0.7)/0.1 + (0.6,-0.3)/0.2 + (0.9,-0.6)/0.6}{\mathfrak{q}_2} + \frac{(0.7,-0.2)/0.8}{\mathfrak{q}_3}.\end{aligned} \tag{29}$$

and $(\psi, \mathscr{I})$ by

$$\begin{aligned}\psi(\xi_2) &= \frac{(0.2,-0.4)/0.6 + (0.8,-0.3)/0.8 + (0.6,-1.0)/0.9}{\mathfrak{q}_1} + \frac{(0.6,-0.8)/0.4 + (0.8,-0.1)/0.7}{\mathfrak{q}_2} + \frac{(0.8,-0.4)/0.9}{\mathfrak{q}_3},\\ \psi(\xi_3) &= \frac{(0.3,-0.6)/0.4 + (0.8,-0.1)/0.5 + (0.2,-0.5)/0.6}{\mathfrak{q}_1} + \frac{(0.9,-0.6)/0.4}{\mathfrak{q}_2} + \frac{(0.7,-0.6)/0.3 + (0.1,-0.4)/0.6}{\mathfrak{q}_3}.\end{aligned} \tag{30}$$

We will compute complement, union, intersection, AND, and OR of BFS_{T2} sets.

(i) The complement $(\tau, \mathscr{H})^c = (\tau^c, \neg\mathscr{H})$ of BFS_{T2} set $(\tau, \mathscr{H})$ is given

$$\begin{aligned}\tau(\neg\xi_1) &= \frac{(0.3,-0.5)/0.9 + (1.0,-0.4)/0.8 + (0.7,-0.1)/0.7}{\mathfrak{q}_1} + \frac{(0.4,-0.2)/0.5 + (0.8,-0.1)/0.1}{\mathfrak{q}_2} + \frac{(0.5,-0.2)/0.3}{\mathfrak{q}_3},\\ \tau(\neg\xi_2) &= \frac{(0.3,-0.6)/0.9 + (0.6,-0.9)/0.6 + (0.5,-0.8)/0.3}{\mathfrak{q}_1} + \frac{(0.5,-0.1)/0.2 + (0.2,-0.7)/0.9 + (0.6,-0.3)/0.8 + (0.9,-0.6)/0.4}{\mathfrak{q}_2} + \frac{(0.7,-0.2)/0.2}{\mathfrak{q}_3}.\end{aligned} \tag{31}$$

(ii) The union of $(\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}) = (\varphi, \mathscr{J})$, where $\mathscr{J} = \mathscr{H} \cup \mathscr{I} = \{\xi_1, \xi_2, \xi_3\}$ is as follows since $\xi_1 \in \mathscr{H} - \mathscr{I}$ then

$$\varphi(\xi_1) = \tau(\xi_1) = \frac{(0.3,-0.5)/0.1 + (1.0,-0.4)/0.2 + (0.7,-0.1)/0.3}{\mathfrak{q}_1} + \frac{(0.4,-0.2)/0.5 + (0.8,-0.1)/0.9}{\mathfrak{q}_2} + \frac{(0.5,-0.2)/0.7}{\mathfrak{q}_3}, \tag{32}$$

since $\xi_2 \in \mathscr{H} \cap \mathscr{I}$ then

$$\begin{aligned}\varphi(\xi_2) &= \tau(\xi_2) \sqcup \psi(\xi_2)\\ &= \frac{(0.2,-0.4)/0.6 + (0.6,-0.3)/0.8 + (0.6,-0.9)/0.9 + (0.2,-0.4)/0.7}{\mathfrak{q}_1} + \frac{(0.5,-0.1)/0.8 + (0.6,-0.7)/0.4 + (0.8,-0.1)/0.7 + (0.6,-0.6)/0.6}{\mathfrak{q}_2} + \frac{(0.7,-0.2)/0.9}{\mathfrak{q}_3},\end{aligned} \tag{33}$$

since $\xi_3 \in \mathscr{I} - \mathscr{H}$ then

$$\varphi(\xi_3) = \psi(\xi_3) = \frac{(0.3,-0.6)/0.4 + (0.8,-0.1)/0.5 + (0.2,-0.5)/0.6}{\mathfrak{q}_1} + \frac{(0.9,-0.6)/0.4}{\mathfrak{q}_2} + \frac{(0.7,-0.6)/0.3 + (0.1,-0.4)/0.6}{\mathfrak{q}_3}. \tag{34}$$

(iii) The intersection of $(\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I}) = (\sigma, D)$, where $D = \mathscr{H} \cap \mathscr{I} = \{\xi_2\}$ is given

$$\sigma(\xi_2) = \tau(\xi_2) \sqcap \psi(\xi_2) = \frac{(0.3,-0.6)/0.1 + (0.6,-0.9)/0.4 + (0.2,-0.4)/0.6 + (0.5,-0.8)/0.7}{\mathfrak{q}_1} + \frac{(0.6,-0.6)/0.4 + (0.5,-0.1)/0.7 + (0.2,-0.7)/0.1 + (0.6,-0.3)/0.2 + (0.8,-0.1)/0.6}{\mathfrak{q}_2} + \frac{(0.7,-0.2)/0.8}{\mathfrak{q}_3}. \tag{35}$$

(iv) A $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I}) = (\tau, \mathscr{H}) \wedge (\psi, \mathscr{I}) = (\varphi, \mathscr{H} \times \mathscr{I})$ are as follows:

$$\begin{aligned}
\varphi(\xi_1, \xi_2) = \tau(\xi_1) \sqcap \psi(\xi_2) &= \frac{(0.3,-0.5)/0.1 + (0.6,-0.4)/0.2 + (0.7,-0.1)/0.3}{q_1} + \frac{(0.6,-0.2)/0.4 + (0.4,-0.1)/0.5 + (0.8,-0.1)/0.7}{q_2} \\
&\quad + \frac{(0.5,-0.2)/0.7}{q_3}, \\
\varphi(\xi_1, \xi_3) = \tau(\xi_1) \sqcap \psi(\xi_3) &= \frac{(0.3,-0.5)/0.1 + (0.8,-0.4)/0.2 + (0.7,-0.1)/0.3}{q_1} + \frac{(0.8,-0.2)/0.4}{q_2} + \frac{(0.5,-0.2)/0.3 + (0.1,-0.2)/0.6}{q_3}, \\
\varphi(\xi_2, \xi_2) = \tau(\xi_2) \sqcap \psi(\xi_2) &= \frac{(0.3,-0.6)/0.1 + (0.6,-0.9)/0.4 + (0.2,-0.4)/0.6 + (0.5,-0.8)/0.7}{q_1} \\
&\quad + \frac{(0.6,-0.6)/0.4 + (0.5,-0.1)/0.7 + (0.2,-0.7)/0.1 + (0.6,-0.3)/0.2 + (0.8,-0.1)/0.6}{q_2} + \frac{(0.7,-0.2)/0.8}{q_3}, \\
\varphi(\xi_2, \xi_3) = \tau(\xi_2) \sqcap \psi(\xi_3) &= \frac{(0.3,-0.6)/0.1 + (0.6,-0.6)/0.4 + (0.5,-0.1)/0.5 + (0.2,-0.5)/0.6}{q_1} \\
&\quad + \frac{(0.9,-0.6)/0.4 + (0.2,-0.6)/0.1 + (0.6,-0.3)/0.2}{q_2} + \frac{(0.7,-0.2)/0.3 + (0.1,-0.2)/0.6}{q_3}.
\end{aligned} \tag{36}$$

(v) A $(\tau, \mathscr{H})$ or $(\psi, \mathscr{I}) = (\tau, \mathscr{H}) \vee (\psi, \mathscr{I}) = (\sigma, \mathscr{H} \times \mathscr{I})$ is as follows:

$$\begin{aligned}
\sigma(\xi_1, \xi_2) = \tau(\xi_1) \sqcup \psi(\xi_2) &= \frac{(0.2,-0.4)/0.6 + (0.8,-0.3)/0.8 + (0.6,-0.5)/0.9}{q_1} \\
&\quad + \frac{(0.4,-0.2)/0.5 + (0.4,-0.1)/0.7 + (0.8,-0.1)/0.9}{q_2} + \frac{(0.5,-0.2)/0.9}{q_3}, \\
\sigma(\xi_1, \xi_3) = \tau(\xi_1) \sqcup \psi(\xi_3) &= \frac{(0.3,-0.5)/0.4 + (0.8,-0.1)/0.5 + (0.2,-0.5)/0.6}{q_1} \\
&\quad + \frac{(0.4,-0.2)/0.5 + (0.8,-0.1)/0.9}{q_2} + \frac{(0.5,-0.2)/0.7}{q_3}, \\
\sigma(\xi_2, \xi_2) = \tau(\xi_2) \sqcup \psi(\xi_2) &= \frac{(0.2,-0.4)/0.6 + (0.6,-0.3)/0.8 + (0.6,-0.9)/0.9 + (0.2,-0.4)/0.7}{q_1} \\
&\quad + \frac{(0.5,-0.1)/0.8 + (0.6,-0.7)/0.4 + (0.8,-0.1)/0.7 + (0.6,-0.6)/0.6}{q_2} + \frac{(0.7,-0.2)/0.9}{q_3}, \\
\sigma(\xi_2, \xi_3) = \tau(\xi_2) \sqcup \psi(\xi_3) &= \frac{(0.3,-0.6)/0.4 + (0.6,-0.1)/0.5 + (0.2,-0.5)/0.6 + (0.5,-0.6)/0.7}{q_1} \\
&\quad + \frac{(0.5,-0.1)/0.8 + (0.6,-0.6)/0.4 + (0.9,-0.6)/0.6}{q_2} + \frac{(0.7,-0.2)/0.8}{q_3}.
\end{aligned} \tag{37}$$

Proposition 15. *Let $\mathscr{H}, \mathscr{I}, \mathscr{J} \subseteq E$ and let $(\tau, \mathscr{H}), (\psi, \mathscr{I})$, $(\varphi, \mathscr{J}) \in BFS_{T2}(\mathbb{Q})$. Then, the following results hold:*

(1) $(\tau, \mathscr{H}) \sqcup (\tau, \mathscr{H}) = (\tau, \mathscr{H})$ and $(\tau, \mathscr{H}) \sqcap (\tau, \mathscr{H}) = (\tau, \mathscr{H})$,

(2) $(\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}) = (\psi, \mathscr{I}) \sqcup (\tau, \mathscr{H})$ and $(\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I}) = (\psi, \mathscr{I}) \sqcap (\tau, \mathscr{H})$,

(3) $(\tau, \mathscr{H}) \sqcup ((\psi, \mathscr{I}) \sqcap (\varphi, \mathscr{J})) = ((\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I})) \sqcap (\varphi, \mathscr{J})$,

(4) $(\tau, \mathscr{H}) \sqcap ((\psi, \mathscr{I}) \sqcap (\varphi, \mathscr{J})) = ((\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I})) \sqcap (\varphi, \mathscr{J})$,

(5) $((\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I}))^c \subseteq (\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c$,

(6) $(\tau, \mathscr{H})^c \sqcap (\psi, \mathscr{I})^c \subseteq ((\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}))^c$.

Proof. Suppose that $(\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I}) = (\sigma, \gamma)$, where $\sigma(\xi) = \tau(\xi) \sqcap \psi(\xi)$; $\gamma = \mathscr{H} \cap \mathscr{I}$, so we deduce

$$((\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}))^c = (\sigma, \gamma)^c = (\sigma^c, \neg\gamma), \text{where} \quad \neg\gamma = \neg\mathscr{H} \cap \neg\mathscr{I}. \tag{38}$$

This imply

$$\sigma^c(\xi) = \tau^c(\xi) \sqcup \psi^c(\xi) \quad ; \quad \forall \xi \in \neg\gamma = \neg\mathscr{H} \cap \neg\mathscr{I}. \tag{39}$$

□

And if we suppose

$$(\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c = (\Omega, \delta), \text{where} \quad \delta = \neg\mathscr{H} \cup \neg\mathscr{I}, \tag{40}$$

then we obtain

$$\Omega(\xi) = \begin{cases} \tau^c(\xi) & \text{if} \xi \in \neg\mathscr{H} - \neg\mathscr{I}, \\ \psi^c(\xi) & \text{if} \xi \in \neg\mathscr{I} - \neg\mathscr{H}, \\ \tau^c(\xi) \sqcup \psi^c(\xi) & \text{if} \xi \in \neg\mathscr{H} \cap \neg\mathscr{I}. \end{cases} \tag{41}$$

Hence $((\tau, \mathscr{H}) \sqcap (\psi, \mathscr{I}))^c \subseteq (\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c$.

Suppose that $(\tau, \mathscr{H})^c \sqcap (\psi, \mathscr{I})^c = (\sigma, \gamma)$, where $\sigma(\xi) = \tau^c(\xi) \sqcap \psi^c(\xi)$ and $\gamma = \neg\mathscr{H} \cap \neg\mathscr{I}$.

And also suppose

$$(\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}) = (\Omega, \delta) \quad \text{where} \quad \delta = \mathscr{H} \cup \mathscr{I}, \tag{42}$$

this imply

$$\Omega(\xi) = \begin{cases} \tau(\xi) \text{ if } \xi \in \mathscr{H} - \mathscr{I}, \\ \psi(\xi) \text{ if} \xi \in \mathscr{I} - \mathscr{H}, \\ \tau(\xi) \sqcup \psi(\xi) \text{if} \xi \in \mathscr{H} \cap \mathscr{I}, \end{cases} \tag{43}$$

so, we deduce

$$((\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}))^c = (\Omega, \delta)^c = (\Omega^c, \neg\delta); \neg\delta = \neg\mathscr{H} \cup \neg\mathscr{I}. \tag{44}$$

Therefore

$$\Omega^c(\xi) = \begin{cases} \tau^c(\xi) \text{if } \xi \in \neg\mathscr{H} - \neg\mathscr{I}, \\ \psi^c(\xi) \text{ if} \xi \in \neg\mathscr{I} - \neg\mathscr{H}, \\ \tau^c(\xi) \sqcap \psi^c(\xi) \text{if} \xi \in \neg\mathscr{H} \cap \neg\mathscr{I}. \end{cases} \tag{45}$$

Hence, $(\tau, \mathscr{H})^c \sqcap (\psi, \mathscr{I})^c \subseteq ((\tau, \mathscr{H}) \sqcup (\psi, \mathscr{I}))^c$.

Theorem 16. *Let $\mathscr{H}, \mathscr{I}, \mathscr{J} \subseteq E$. For $(\tau, \mathscr{H}), (\psi, \mathscr{I}), (\varphi, \mathscr{J}) \in BFS_{T2}(\mathbb{Q})$. If $(\tau, \mathscr{H})$ is BT2FS $_f$ subset of $(\psi, \mathscr{I})$ and $(\psi, \mathscr{I})$ is BT2FS $_f$ subset of $(\varphi, \mathscr{J})$, thus $(\tau, \mathscr{H})$ is BT2FS $_f$ subset of $(\varphi, \mathscr{J})$.*

Proof. Assume that $(\tau, \mathscr{H})$ is a BT2FS $_f$ subset of $(\psi, \mathscr{I})$ over $\mathbb{Q}$, then we have

(i) $\mathscr{H} \subseteq \mathscr{I}$

(ii) $\forall \xi \in \mathscr{H}$; $\tau(\xi) \subseteq \psi(\xi)$

And also assume that $(\psi, \mathscr{I})$ is a BT2FS $_f$ subset of $(\varphi, \mathscr{J})$ over $\mathbb{Q}$, then we get:

(i) $\mathscr{I} \subset \mathscr{J}$

(ii) $\forall \xi \in \mathscr{I}$; $\psi(\xi) \subseteq \varphi(\xi)$

Hence, $(\tau, \mathscr{H})$ is a BT2FS $_f$ subset of $(\varphi, \mathscr{J})$ over $\mathbb{Q}$. □

Theorem 17. *Let $\mathscr{H}, \mathscr{I}, \mathscr{J} \subseteq E$. For $(\tau, \mathscr{H}), (\psi, \mathscr{I}), (\varphi, \mathscr{J}) \in BFS_{T2}(\mathbb{Q})$. If $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ are equal and at the same time $(\psi, \mathscr{I})$ and $(\varphi, \mathscr{J})$ are also equal, then $(\tau, \mathscr{H})$ and $(\varphi, \mathscr{J})$ are equal.*

Proof. Same proof as the previous theorem. □

Definition 18. A null BT2FS $_f$ S denoted $\phi_{(T2)_{\mathscr{H}}}$ is a BT2FS $_f$ S $(\tau, \mathscr{H})$ over $\mathbb{Q}$ which is for all $\xi \in \mathscr{H}$, for all $\mathfrak{q} \in \mathbb{Q}$:

$$\mu^+_\phi(\mathfrak{q}, \mathfrak{k}) = \begin{cases} 1; & \mathfrak{k} = 0, \\ 0; & \mathfrak{k} \neq 0, \end{cases} \mu^-_\phi(\mathfrak{q}, \mathfrak{k}) = \begin{cases} -1; & \mathfrak{k} = 0, \\ 0; & \mathfrak{k} \neq 0. \end{cases} \tag{46}$$

Definition 19. An absolute BT2FS $_f$ S denoted $\mathbb{Q}_{(T2)_{\mathscr{H}}}$ is a BT2FS $_f$ S $(\tau, \mathscr{H})$ over $\mathbb{Q}$ which is for all $\xi \in \mathscr{H}$, and for all $\mathfrak{q} \in \mathbb{Q}$:

$$\mu^+_{\mathbb{Q}}(\mathfrak{q}, \mathfrak{k}) = \begin{cases} 1; & \mathfrak{k} = 1, \\ 0; & \mathfrak{k} \neq 1, \end{cases} \mu^-_{\mathbb{Q}}(\mathfrak{q}, \mathfrak{k}) = \begin{cases} -1; & \mathfrak{k} = 1, \\ 0; & \mathfrak{k} \neq 1. \end{cases} \tag{47}$$

Theorem 20. *Let $(\tau, \mathscr{H})$ be a $BFS_{T2}(\mathbb{Q})$. Let $\Phi_{(T2)_{\mathscr{H}}}$ and $\mathbb{Q}_{(T2)_{\mathscr{H}}}$ be a null and absolute BT2FS $_f$ S over $\mathbb{Q}$, then*

(i) $\Phi^c_{(T2)_{\mathscr{H}}} = \mathbb{Q}_{(T2)_{\mathscr{H}}}$ *and* $\mathbb{Q}^c_{(T2)_{\mathscr{H}}} = \Phi_{(T2)_{\mathscr{H}}}$

(ii) $(\tau, \mathscr{H}) \sqcap \mathbb{Q}_{(T2)_{\mathscr{H}}} = (\tau, \mathscr{H})$,

(iii) $(\tau,\mathscr{H})\sqcup\Phi_{(T2)_{\mathscr{H}}}=(\tau,\mathscr{H})$

But $(\tau,\mathscr{H})\sqcup\mathbb{Q}_{(T2)_{\mathscr{H}}}\neq(\tau,\mathscr{H})$ and $(\tau,\mathscr{H})\sqcap\Phi_{(T2)_{\mathscr{H}}}\neq(\tau,\mathscr{H})$.

Proof.

(i) Suppose that $\Phi_{T2_{\mathscr{H}}}$ is a null BT2FS $_f$ S over $\mathbb{Q}$, and define for all $\xi\in\mathscr{H}$

$$\mu^{+}_{\phi}(\mathfrak{q},\mathfrak{k})=\begin{cases}1; & \mathfrak{k}=0,\\ 0; & \mathfrak{k}\neq 0,\end{cases}\quad \mu^{-}_{\phi}(\mathfrak{q},\mathfrak{k})=\begin{cases}-1; & \mathfrak{k}=0,\\ 0; & \mathfrak{k}\neq 0.\end{cases}\tag{48}$$

So, we get $(\Phi_{T2_{\mathscr{H}}})^{c}$ defined by

$$\mu^{+}_{(\phi)^c}(\mathfrak{q},\mathfrak{k})=\begin{cases}1; & \mathfrak{k}=0,\\ 0; & \mathfrak{k}\neq 0,\end{cases}\quad \mu^{-}_{(\phi)^c}(\mathfrak{q},\mathfrak{k})=\begin{cases}-1; & \mathfrak{k}=0,\\ 0; & \mathfrak{k}\neq 0.\end{cases}\tag{49}$$

Therefore $(\Phi_{T2_{\mathscr{H}}})^{c}=\mathbb{Q}_{T2_{\mathscr{H}}}$.
Same proof for $\mathbb{Q}^{c}_{(T2)_{\mathscr{H}}}=\Phi_{(T2)_{\mathscr{H}}}$.

(ii) Let $(\tau,\mathscr{H})$ be a $BFS_{T2}(\mathbb{Q})$. Let $\mathbb{Q}_{T2_{\mathscr{H}}}$ be an absolute BT2FS $_f$ S over $\mathbb{Q}$. So, we have

$$(\tau,\mathscr{H})\sqcap\mathbb{Q}_{T2_{\mathscr{H}}}=(\varphi,\mathscr{H})\text{ where }\varphi(\xi)=\tau(\xi)\sqcap\mathbb{Q}_{T2_{\mathscr{H}}}(\xi)=\tau(\xi).\tag{50}$$

Thus $(\tau,\mathscr{H})\sqcap\mathbb{Q}_{T2_{\mathscr{H}}}=(\tau,\mathscr{H})$.

(iii) Let $(\tau,\mathscr{H})$ be a $BFS_{T2}(\mathbb{Q})$. Let $\Phi_{T2_{\mathscr{H}}}$ be a null BT2FS $_f$ S over $\mathbb{Q}$. So, we have

$$\begin{aligned}(\tau,\mathscr{H})\sqcup\Phi_{T2_{\mathscr{H}}}&=(\varphi,\mathscr{H})\text{ where }\varphi(\xi)=\tau(\xi)\sqcap\mathbb{Q}_{T2_{\mathscr{H}}}(\xi)\\&=\tau(\xi).\end{aligned}\tag{51}$$

Thus, $(\tau,\mathscr{H})\sqcap\Phi_{T2_{\mathscr{H}}}=(\tau,\mathscr{H})$. □

Theorem 21. *Let $(\tau,\mathscr{H})$ and $(\psi,\mathscr{I})$ be the two $BFS_{T2}(\mathbb{Q})$, where $\mathscr{H},\mathscr{I}\in E$. Then the following equations hold:*

(i) $(\tau,\mathscr{H})\wedge(\psi,\mathscr{I}))^{c}=(\tau,\mathscr{H})^{c}\vee(\psi,\mathscr{I})^{c}$

(ii) $(\tau,\mathscr{H})\vee(\psi,\mathscr{I}))^{c}=(\tau,\mathscr{H})^{c}\wedge(\psi,\mathscr{I})^{c}$

Proof.

(i) Suppose that $(\tau,\mathscr{H})$ and $(\psi,\mathscr{I})$ are two $BFS_{T2}(\mathbb{Q})$, we have

$$\begin{aligned}(\tau,\mathscr{H})\wedge(\psi,\mathscr{I})&=(\varphi,\mathscr{H}\times\mathscr{I})\text{ define }\varphi\text{ by }\varphi(a,b)\\&=\tau(a)o\psi(b)\quad\forall(a,b)\in\mathscr{H}\times\mathscr{I}.\end{aligned}\tag{52}$$

Then $((\tau,\mathscr{H})\wedge(\psi,\mathscr{I}))^{c}=(\varphi,\mathscr{H}\times\mathscr{I})^{c}=(\varphi^{c},\neg\mathscr{H}\times\neg\mathscr{I})$ which$\varphi^{c}(\alpha,\beta)=(\tau(\alpha)\sqcup\psi(\beta))^{c}=\tau^{c}(\alpha)o\psi^{c}(\beta)\quad\forall(\alpha,\beta)\in\neg\mathscr{H}\times\neg\mathscr{I}$.
On the other hand, we have

$$\begin{aligned}(\tau,\mathscr{H})^{c}\vee(\psi,\mathscr{I})^{c}&=(\tau^{c},\neg\mathscr{H})\vee(\psi^{c},\neg\mathscr{I})\\&=(\Omega,\neg\mathscr{H}\times\neg\mathscr{I}),\text{ which}\quad\Omega(\alpha,\beta)\\&=\tau^{c}(\alpha)o\psi^{c}(\beta)\quad\forall(\alpha,\beta)\in\neg\mathscr{H}\times\neg\mathscr{I}.\end{aligned}\tag{53}$$

□

Therefore $((\tau,\mathscr{H})\wedge(\psi,\mathscr{I}))^{c}=(\tau,\mathscr{H})^{c}\vee(\psi,\mathscr{I})^{c}$.

(ii) Suppose that $(\tau,\mathscr{H})$ and $(\psi,\mathscr{I})$ are two $BFS_{T2}(\mathbb{Q})$, we have

$$\begin{aligned}&(\tau,\mathscr{H})\vee(\psi,\mathscr{I})=(\varphi,\mathscr{H}\times\mathscr{I})\text{ define}\\&\varphi\text{ by }\varphi(a,b)=\tau(a)o\psi(b)\quad\forall(a,b)\in\mathscr{H}\times\mathscr{I}.\end{aligned}\tag{54}$$

Then $((\tau,\mathscr{H})\vee(\psi,\mathscr{I}))^{c}=(\varphi,\mathscr{H}\times\mathscr{I})^{c}=(\varphi^{c},\neg\mathscr{H}\times\neg\mathscr{I})$ which $\varphi^{c}(\alpha,\beta)=(\tau(\alpha)o\psi(\beta))^{c}=\tau^{c}(\alpha)o\psi^{c}(\beta)\forall(\alpha,\beta)\in\neg\mathscr{H}\times\neg\mathscr{I}$.
On the other hand, we obtain

$$\begin{aligned}(\tau,\mathscr{H})^{c}\vee(\psi,\mathscr{I})^{c}&=(\tau^{c},\neg\mathscr{H})\vee(\psi^{c},\neg\mathscr{I})\\&=(\Omega,\neg\mathscr{H}\times\neg\mathscr{I}),\text{ which}\\\Omega(\alpha,\beta)&=\tau^{c}(\alpha)o\psi^{c}(\beta)\forall(\alpha,\beta)\in\neg\mathscr{H}\times\neg\mathscr{I}.\end{aligned}\tag{55}$$

Therefore, $((\tau,\mathscr{H})\vee(\psi,\mathscr{I}))^{c}=(\tau,\mathscr{H})^{c}\wedge(\psi,\mathscr{I})^{c}$.

Proposition 22. *Let $\mathscr{H}\in E$. Let two $BFS_{T2}(\mathbb{Q})$ sets $(\tau,\mathscr{H})$ and $(\psi,\mathscr{H})$ over $\mathbb{Q}$. Then*

(i) $((\tau,\mathscr{H})o(\psi,\mathscr{H}))^{c}=(\tau,\mathscr{H})^{c}o(\psi,\mathscr{H})^{c}$,

(ii) $((\tau,\mathscr{H})o(\psi,\mathscr{H}))^{c}=(\tau,\mathscr{H})^{c}o(\psi,\mathscr{H})^{c}$

Proof.

(i) Let two $(\tau,\mathscr{H}),(\psi,\mathscr{H})\in BFS_{T2}(\mathbb{Q})$. Then we get

$$(\tau,\mathscr{H})o(\psi,\mathscr{H})=(\varphi,\mathscr{H}),\tag{56}$$

since $\mathscr{H}\cap\mathscr{H}=\mathscr{H}$ and$\forall\xi\in\mathscr{H}\ \varphi(\xi)=\tau(\xi)\sqcap\psi(\xi)$.

Therefore, $((\tau,\mathscr{H})\sqcup(\psi,\mathscr{H}))^{c}=(\varphi,\mathscr{H})^{c}=(\varphi^{c},\neg\mathscr{H})$, where $\forall e\in\neg\mathscr{H}\,\varphi^{c}(e)=(\tau(e)\sqcap\psi(e))^{c}=\tau^{c}(e)\sqcup\psi^{c}(e)$.

On the other hand, we have

$$(\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{H})^c = (\tau^c, \neg\mathscr{H}) \sqcup (\psi^c, \neg\mathscr{H}) = (\Omega, \neg\mathscr{H}), \quad (57)$$

where $\Omega(e) = \tau^c(e) \sqcup \psi^c$; $\forall e \in \neg\mathscr{H}$.

(ii) In the same way.

As we noted earlier, the De Morgan's laws does not hold. Thus, we can define two conditions under union and intersection operations. □

Definition 23. For two BT2FS $_f$ Ss $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ with $\mathscr{H} \cap \mathscr{I} \neq \phi$ over $\mathbb{Q}$. Then the restricted union of $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ is the BT2FS $_f$ S $(\varphi, \mathscr{J})$ defined by

$$\forall \xi \in \mathscr{J} = \mathscr{H} \cap \mathscr{I} \ \varphi(\xi) = \tau(\xi) \sqcup_r \psi(\xi). \quad (58)$$

Definition 24. For two BT2FS $_f$ Ss $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ over $\mathbb{Q}$. Then the extended intersection of $(\tau, \mathscr{H})$ and $(\psi, \mathscr{I})$ is the BT2FS $_f$ S $(\varphi, \mathscr{J})$, where $\forall \xi \in \mathscr{J} = \mathscr{H} \cup \mathscr{I}$ defined by

$$\varphi(\xi) = \tau(\xi) \sqcup_e \psi(\xi) = \begin{cases} \tau(\xi) \text{if} \xi \in \mathscr{H} - \mathscr{I}, \\ \psi(\xi) \text{if} \xi \in \mathscr{I} - \mathscr{H}, \\ \tau(\xi) \sqcup \psi(\xi) \text{if} \xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \quad (59)$$

Theorem 25. *Let $\mathscr{H}, \mathscr{I} \subset E$. Let $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in BFS_{T2}(\mathbb{Q})$. Then*

(i) $((\tau, \mathscr{H}) \sqcup_e (\psi, \mathscr{I}))^c = (\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c$,

(ii) $((\tau, \mathscr{H}) \sqcup_r (\psi, \mathscr{I}))^c = (\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c$.

Proof.

(i) For $(\tau, \mathscr{H}), (\psi, \mathscr{I}) \in BFS_{T2}(\mathbb{Q})$. Then we have

$$(\tau, \mathscr{H}) \sqcup_e (\psi, \mathscr{I}) = \begin{cases} \tau(\xi) \text{ if } \xi \in \mathscr{H} - \mathscr{I}, \\ \psi(\xi) \text{ if } \xi \in \mathscr{I} - \mathscr{H}, \\ \tau(\xi) \text{o} \psi(\xi) \text{ if } \xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \quad (60)$$

Thus, we obtain

$$\begin{aligned} &((\tau, \mathscr{H}) \text{o}_e (\psi, \mathscr{I}))^c \\ &= \begin{cases} \tau^c(\neg\xi) \text{ if} \neg\xi \in \mathscr{H} - \mathscr{I}, \\ \psi^c(\neg\xi) \text{ if} \neg\xi \in \mathscr{I} - \mathscr{H}, \\ (\tau(\xi) \text{o} \psi(\xi))^c \text{ if } \xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \quad (61) \\ &= \begin{cases} \tau^c(\neg\xi) \text{if} \neg\xi \in \mathscr{H} - \mathscr{I}, \\ \psi^c(\neg\xi) \text{if} \neg\xi \in \mathscr{I} - \mathscr{H}, \\ \tau^c(\neg\xi) \text{o} \psi^c(\neg\xi) \text{if} \neg\xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} &(\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c \\ &= (\tau^c, \neg\mathscr{H}) \sqcup (\psi^c, \neg\mathscr{I}) \\ &= \begin{cases} \tau^c(\neg\xi) \text{ if} \neg\xi \in \mathscr{H} - \mathscr{I}, \\ \psi^c(\neg\xi) \text{ if} \neg\xi \in \mathscr{I} - \mathscr{H}, \\ \tau^c(\neg\xi) \text{o} \psi^c(\neg\xi) \text{ if} \neg\xi \in \mathscr{H} \cap \mathscr{I}. \end{cases} \end{aligned} \quad (62)$$

we conclude that

$$((\tau, \mathscr{H}) \sqcup_e (\psi, \mathscr{I}))^c = (\tau, \mathscr{H})^c \sqcup (\psi, \mathscr{I})^c. \quad (63)$$

(ii) Can be done in the same way.

□

5. Application of BFS_{T2} Set in Decision Making

Decision making is one of the most popular applications of S $_f$ S theory which has a lot of applications in many practical problems. Maji et al. introduced the idea of S $_f$ Ss for the evaluation of parameterized decision-making problems. Moreover, Roy and Maji presented an algorithm to solve the recognition problem by employing FS $_f$ Ss. Subsequently, Kong et al. gave a counterexample to illustrate a fact that the optimal choice could not be obtained in general by using Roy and Maji's algorithm and conferred a modified version. Feng et al. gave deeper insights into decision-making methods based on FS $_f$ Ss. They conferred an adjustable approach for fuzzy-soft-set-based decision making by employing level S $_f$ Ss. By generalizing the adjustable approach for fuzzy-soft-set-based decision making, Jiang et al. presented an adjustable approach for intuitionistic fuzzy-soft-set-based decision making and explore some illustrative examples.

In this section, we shall present an adjustable approach for BT2FS $_f$ S-based decision-making problems. This technique is based on the concept of level S $_f$ Ss.

Definition 26. Let $\mathscr{H} \subset E$. *Let* $(\tau, \mathscr{H})$ be a *BT2FS* $_f$ *S* over the initial universe $\mathbb{Q}$. For $\gamma \in [0, 1]$ and $\beta \in [0, 1]$, the (γ, β)-level (the threshold) *S* $_f$ *S* of $(\tau, \mathscr{H})$ *is* crisp soft set $T((\tau, \mathscr{H}), \gamma, \beta) = (\tau_{(\gamma,\beta),\mathscr{H})}$ for all $\xi \in \mathscr{H}$, defined by

$$\tau_{(\gamma,\beta)}(\xi) = \tau(\xi)^{\gamma}_{\beta} = \{\mathfrak{q} \in \mathbb{Q} \mid \mathfrak{k}(\mathfrak{q}) \geq \beta; \forall \mathfrak{k}(\mathfrak{q}) \in J^{\gamma}_{\mathfrak{q}}\}, \quad (64)$$

where $J^{\gamma}_{\mathfrak{q}} = \{\mathfrak{k} \mid (\mu^{+}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k}) - \mu^{-}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k})) \geq \gamma ; \mathfrak{k} \in J_{\mathfrak{q}}\}$.

Definition 27. Let $\mathscr{H} \subset E$. Let $(\tau, \mathscr{H})$ be a BFS_{T2} set over $\mathbb{Q}$. Let $\gamma : \mathscr{H} \longrightarrow [0, 1]$ and $\beta : \mathscr{H} \longrightarrow [0, 1]$ be two fuzzy sets in $\mathscr{H}$ which are two threshold fuzzy sets. Then, the level S $_f$ S of $(\tau, \mathscr{H})$ with respect to γ and β is crisp soft set $T((\tau, \mathscr{H}), \gamma,$

$\beta) = (\tau_{(\gamma,\beta),\mathscr{H}})$ *for all* $\xi \in \mathscr{H}$, *defined by*

$$\tau_{(\gamma,\beta)}(\xi) = \tau(\xi)^{\gamma}_{\beta} = \left\{\mathfrak{q} \in \mathbb{Q} \middle| \mathfrak{k}(\mathfrak{q}) \geq \beta(\xi); \forall \mathfrak{k}(\mathfrak{q}) \in J^{\gamma}_{\mathfrak{q}}\right\}, \quad (65)$$

where $J^{\gamma}_{\mathfrak{q}} = \{\mathfrak{k} \mid (\mu^{+}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k}) - \mu^{-}_{\tau(\xi)}(\mathfrak{q}, \mathfrak{k})) \geq \gamma(\xi)\ ; \mathfrak{k} \in J_{\mathfrak{q}}\}$.

Definition 28. Let $\mathscr{H} \subset E$ *be a parameter set. Let* $(\tau, \mathscr{H})$ *be a* BFS_{T2} *set over* $\mathbb{Q}$ *, and let* $\Lambda : \mathscr{H} \longrightarrow [0, 1]$ *be the weight function specifying the weight* $\Lambda_i = \Lambda(\xi_i)$ *for all* $\xi_i \in \mathscr{H}$*. Then, the triple* $(\tau, \mathscr{H}, \Lambda)$ *is a weighted BT2FS* $_f$ *S.*

5.1. Algorithm. Consider the following.

Step 1: Input a weighted BT2FS $_f$ S $\Gamma = (\tau, \mathscr{H}, \Lambda)$.

Step 2: Input two threshold fuzzy sets $\gamma : \mathscr{H} \longrightarrow [0, 1]$ and $\beta : \mathscr{H} \longrightarrow [0, 1]$ for decision making.

Step 3: Compute the level S $_f$ S $T((\tau, \mathscr{H}), \gamma, \beta)$ of Γ w.r.t the two threshold fuzzy sets γ and β.

Step 4: Present the level S $_f$ S $T((\tau, \mathscr{H}), \gamma, \beta)$ in tabular form. For all $\mathfrak{q}_i \in \mathbb{Q}$, compute the weighted choice value c of $\mathfrak{q}_i$, where $c = \sum_{\xi \in \mathscr{H}} \tau(\xi)(\mathfrak{q}_i).\Gamma(\xi)$.

Step 5: The optimal decision is to select $\mathfrak{q}_i$ if $c_k = max_{\mathfrak{q}_i \in \mathbb{Q}}\{c_i\}$.

Step 6: If k has more than one value then any one of h_k may be chosen.

Example 7. Suppose that a company is planning to increase its green area in the center of a desert city, so by raising a large number of trees and plants, it will improve generally. There are two ways for the cultivation of area, one is the traditional way and the other one is by groasis technology $\mathbb{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2\}$. *Let* there exist are four parameters $E = \{e_1, e_2, e_3, e_4\} = \{Temperature, Soilfertility, Humidity, Sunlight\}$. Assume that weight for the parameters in *E*: $\Lambda_1 = \Lambda(\xi_1) = 0.2, \Lambda_2 = \Lambda(\xi_2) = 0.9, \Lambda_3 = \Lambda(\xi_3) = 0.7$ *and* $\Lambda_4 = \Lambda(\xi_4) = 0.4$; and two threshold fuzzy sets $\gamma : E \longrightarrow [0, 1]$ and $\beta : E \longrightarrow [, 1]$ defined by

$$\begin{aligned}\gamma &= \frac{0.90}{\xi_1} + \frac{0.80}{\xi_2} + \frac{0.85}{\xi_3} + \frac{0.60}{\xi_4}\ ; \\ \beta &= \frac{0.75}{\xi_1} + \frac{0.60}{\xi_2} + \frac{0.55}{\xi_3} + \frac{0.40}{\xi_4}\end{aligned} \quad (66)$$

Consider bipolar type-2 FS $_f$ S as follows:

$$\begin{aligned}(\tau, E) = \Bigg\{ &\Bigg(e_1, \Bigg(\frac{((0.30,-0.65)/0.30) + ((0.50,-0.75)/0.60)}{\mathfrak{q}_1} \\ &+ \frac{((0.45,-0.30)/0.60) + (0.70,-0.25/0.85)}{\mathfrak{q}_2}\Bigg)\Bigg), \\ &\cdot \Bigg(e_2, \Bigg(\frac{0.50,-0.40/0.25}{\mathfrak{q}_1} \\ &+ \frac{(0.55,-0.20/0.45) + (0.80,-0.35/0.75)}{\mathfrak{q}_2}\Bigg)\Bigg), \\ &\cdot \Bigg(e_3, \Bigg(\frac{(0.15,-0.25/0.15) + (0.40,-0.60/0.75)}{\mathfrak{q}_1} \\ &+ \frac{0.70,-0.30/0.50}{\mathfrak{q}_2}\Bigg)\Bigg), \Bigg(e_4, \Bigg(\frac{0.75,-0.60/0.85}{\mathfrak{q}_1} \\ &+ \frac{0.80,-0.30/0.80}{\mathfrak{q}_2}\Bigg)\Bigg).\end{aligned} \quad (67)$$

The values are shown in Tables 1, 2

We deduced that weighted optimal choice value $max_{1 \leq i \leq 2}\{c_i\} = c_2$. Hence, $\mathfrak{q}_2$ be the required optimal choice.

6. Conclusion

In this research article, we have rapturously consummated our goal by establishing the ground-breaking incredible theories of BT2FS and BT2FS $_f$ S. These flexible models are highly dexterous for addressing a comprehensive range of ambiguity embedded in bipolar human interpretations. The newly developed BT2FS $_f$ S provides an outstanding mathematical framework that admirably integrates the specialities of novel BT2FS with the parametric nature of S $_f$ S. The proposed theories have extrapolated the contemporary unipolar fuzzy structures because of their fantastic representational and reasoning potentialities. Firstly, we have explicated the systematic definitions of presented notions and discussed some of their fundamental properties. We have thrown light on some elementary set-theoretical operations for these emerging theories inclusive of equality, subset, complement, union and intersection. Another noteworthy contribution of this study is the development of a highly proficient MCDM algorithm for the specification of the best pragmatic alternative on the grounds of multiple rational parameters. Finally, we have skilfully implemented the proposed methodology on real-life heuristic application to demonstrate the versatility of our developed strategy. The presented frameworks have an edge over the already existing notions, as they remarkably decipher all the faults and shortcomings of recent approaches by capturing both polarity and abstruseness of paradoxical data set. Admittedly, our proposed models have some difficulties because of their limited one-dimensional modeling capabilities. Thus in the future, we are envisioning to amplify our research study for the two dimensional vague information by presenting the excellent frameworks of complex BT2FS and complex BT2FS $_f$ S in order to design more advanced decision-making strategies. We intend to deliberate the scope of these decision-making approaches in the disciplines of artificial intelligence, medical sciences, logistics and aeronautical industries. Another goal is to unfold significant fusions of ELECTRE methods with the proposed theories to develop more practical decision-making approaches.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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