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Research Article

On Some Numerical Radius Inequalities Involving Generalized Aluthge Transform

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Let S be any bounded linear operator defined on a complex Hilbert space $\mathscr{H}$. In this paper, we present some numerical radius inequalities involving the generalized Aluthge transform to attain upper bounds for numerical radius. Numerical computations are carried out for some particular cases of generalized Aluthge transform.

1. Introduction

Mathematical inequalities play an essential role in developing various areas of pure and applied mathematics. The usefulness of mathematical inequalities is to estimate the solutions of real-life problems in engineering and other fields of science. In mathematics, particularly in functional analysis, the study of numerical radius inequalities has become the attention of many researchers due to the applications of numerical radius in operator theory and numerical analysis, etc. (see [1–4]). Various mathematicians have developed number of numerical radius inequalities to estimate the upper and lower bounds for numerical radius. It is interesting for researchers to get the refinements and generalization of these inequalities. The aim of this paper is to study the generalization and refinements of existing inequalities for numerical radius. Now, we recall some notions to proceed our work.

Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathscr{H}$. For $S \in \mathscr{B}(\mathscr{H})$, the usual operator norm is defined as

$$\|S\| = \sup \{\|Sx\| : x \in \mathscr{H}, \|x\| = 1\}, \tag{1}$$

and the numerical radius is defined as

$$w(S) = \sup \{|\langle Sx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}. \tag{2}$$

It is well known that the numerical radius defines an equivalent operator norm on $\mathscr{B}(\mathscr{H})$, and for $S \in \mathscr{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|S\| \le w(S) \le \|S\|. \tag{3}$$

Many authors worked on the refinement of inequality (3) (see [5–7]). Kittaneh developed the following upper bound of numerical radius:

$$w(S) \le \frac{1}{2}\left(\|S\| + \left\|S^2\right\|^{1/2}\right), \tag{4}$$

which is a refinement of inequality (3) (see [5]). For $S \in \mathscr{B}(\mathscr{H})$ having polar decomposition $S = U|S|$ where U is a partial isometry and $|S| = (S^* S)^{1/2}$, the Aluthge transform is defined as

$$\tilde{S} = |S|^{1/2} U |S|^{1/2}, \tag{5}$$

see [8]. Yamazaki developed an upper bound of numerical radius involving Aluthge transform given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\tilde{S}\right)\right), \tag{6}$$

which is an improvement of bounds (3) and (4) (see [9]). Bhunia et al. developed a bound of the numerical radius given by

$$w^2(S) \leq \frac{1}{4}\left(\|S\|^2 + w^2\left(\tilde{S}\right) + w\left(|S|\tilde{S} + \tilde{S}|S|\right)\right), \tag{7}$$

and proved that it is a refinement of bound (6) (see [10]). Okubo introduced a generalization of Aluthge transform which is defined as

$$\widetilde{S_\lambda} = |S|^\lambda U |S|^{1-\lambda}, \tag{8}$$

for $\lambda \in [0, 1]$, known as λ-Aluthge transform (see [11]). After that, a number of numerical radius inequalities were established involving λ-Aluthge transform (see [12–14]). Abu Omar and Kittaneh using λ-Aluthge transform generalized bound (6) given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\widetilde{S_\lambda}\right)\right), \tag{9}$$

see [12]. Shebrawi and Bakherad introduced another generalization of Aluthge transform which is defined as

$$\widetilde{S_{f,g}} = f(|S|)Ug(|S|), \tag{10}$$

where f and g are nonnegative and continuous functions such that $f(x)g(x) = x (x \geq 0)$, known as generalized Aluthge transform. The authors generalized inequality (9) given by

$$w(S) \leq \frac{1}{2}\left(\|S\| + w\left(\widetilde{S_{f,g}}\right)\right), \tag{11}$$

$$w(S) \leq \frac{1}{4}\left\|f^2(|S|) + g^2(|S|)\right\| + \frac{1}{2} w\left(\widetilde{S_{f,g}}\right), \tag{12}$$

see [15].

In this paper, we establish some new inequalities of the numerical radius using generalized Aluthge transform. Specifically, we generalize inequality (7) and improve the inequalities (3), (4), and (12). Some examples of operators are presented for which the bounds of numerical radius are computed from these inequalities for some choices of f,g in (10).

2. Main Results

Now, we recall a lemma that will be used to achieve our goals.

Lemma 1 (see [9]). *Let $S \in \mathscr{B}(\mathscr{H})$; then, for $\theta \in \mathbb{R}$, we have*

$$w(S) = \sup_{\theta \in \mathbb{R}} \|H_\theta\| = \sup_{\theta \in \mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}S\right)\right\|, \tag{13}$$

where $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$.

Polarization identity: [15] For each $x_1, y_1 \in \mathscr{H}$, we have

$$\begin{aligned}\langle x_1, y_1\rangle = \frac{1}{4}\big(&\|x_1 + y_1\|^2 - \|x_1 - y_1\|^2 \\ &+ i\|x_1 + iy_1\|^2 - i\|x_1 - iy_1\|^2\big).\end{aligned} \tag{14}$$

Now, we establish an inequality of numerical radius which is a generalization of inequality (7) and a refinement of inequality (12).

Theorem 2. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then,*

$$w^2(S) \leq \frac{1}{16}\|Q\|^2 + \frac{1}{4} w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8} w\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right), \tag{15}$$

where $Q = (f(|S|))^2 + (g(|S|))^2$ and f,g is nonnegative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$.

Proof. Let $S = U|S|$ be the polar decomposition of S. Then, by polarization identity, we have

$$\begin{aligned}\left\langle e^{i\theta}Sx, x\right\rangle &= \left\langle e^{i\theta}U|S|x, x\right\rangle = \left\langle e^{i\theta}Ug(|S|)f(|S|)x, x\right\rangle \\ &= \left\langle e^{i\theta}f(|S|)x, g(|S|)U^*x\right\rangle \\ &= \frac{1}{4}\left(\left\|e^{2i\theta}f(|S|)x + g(|S|)U^*x\right\|^2\right. \\ &\quad - \left\|e^{2i\theta}f(|S|)x - g(|S|)U^*x\right\|^2 \\ &\quad + i\left\|e^{2i\theta}f(|S|)x + ig(|S|)U^*x\right\|^2 \\ &\quad \left. - i\left\|e^{2i\theta}f(|S|)x - ig(|S|)U^*x\right\|^2\right).\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}&\mathrm{Re}\left\langle e^{i\theta}Sx, x\right\rangle \\ &\quad = \frac{1}{4}\left\|e^{2i\theta}f(|S|) + g(|S|)U^*x\right\|^2 \\ &\qquad - \frac{1}{4}\left\|e^{2i\theta}f(|S|) - g(|S|)U^*x\right\|^2 \\ &\quad \leq \frac{1}{4}\left\|e^{2i\theta}f(|S|) + g(|S|)U^*\right\|^2 \\ &\quad = \frac{1}{4}\left\|\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)^*\right\| \\ &\quad = \frac{1}{4}\left\|\left(e^{2i\theta}f(|S|) + g(|S|)U^*\right)\left(e^{-2i\theta}f(|S|) + Ug(|S|)\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Big\| (f(|S|))^2 + (g(|S|))^2 + e^{2i\theta} f(|S|) U g(|S|) \\
&\quad + e^{-2i\theta} g(|S|) U^* f(|S|) \Big\| \\
&= \frac{1}{4}\Big\| (f(|S|))^2 + (g(|S|))^2 + e^{2i\theta}\widetilde{S_{f,g}} + e^{-2i\theta}\left(\widetilde{S_{f,g}}\right)^* \Big\| \\
&= \frac{1}{4}\Big\| Q + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right) \Big\| \\
&= \frac{1}{4}\Big\| \left(Q + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)\right)^2 \Big\|^{1/2} \\
&= \frac{1}{4}\Big\| Q^2 + 4\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)^2 + 2Q\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right) \\
&\quad + 2\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right) Q \Big\|^{1/2} \\
&= \frac{1}{4}\Big\| Q^2 + 4\,\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)^2 \\
&\quad + 2\,\mathrm{Re}\left(e^{2i\theta}\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right)\right) \Big\|^{1/2} \\
&\le \frac{1}{4}\Big(\|Q\|^2 + 4\Big\|\mathrm{Re}\left(e^{2i\theta}\widetilde{S_{f,g}}\right)\Big\|^2 \\
&\quad + 2\Big\|\mathrm{Re}\left(e^{2i\theta}\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right)\right)\Big\| \Big)^{1/2}.
\end{aligned} \tag{17}
$$

Now taking supremum over $\theta \in \mathbb{R}$ in the last inequality and then applying Lemma 1, we obtain

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8}w\left(Q\widetilde{S_{f,g}} + \widetilde{S_{f,g}}Q\right), \tag{18}
$$

as desired. □

Theorem 2 includes some particular cases of generalized Aluthge transform for different choices of continuous functions f and g in (10) as follows.

Corollary 3. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_e\right) + \frac{1}{8}w\left(Q\widetilde{S}_e + \widetilde{S}_e Q\right), \tag{19}
$$

where $Q = \left(e^{|S|^\lambda}\right)^2 + \left(|S|e^{-|S|^\lambda}\right)^2$ *and* $\widetilde{S}_e = e^{|S|^\lambda} U |S| e^{|S|^{-\lambda}}$.

Corollary 4. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_e\right) + \frac{1}{8}w\left(Q\widetilde{S}_e + \widetilde{S}_e Q\right), \tag{20}
$$

where $Q = \left(|S|e^{-|S|^\lambda}\right)^2 + \left(e^{|S|^\lambda}\right)^2$ *and* $\widetilde{S}_e = |S| e^{|S|^{-\lambda}} U e^{|S|^\lambda}$.

Corollary 5. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, for* $\lambda \in [0, 1]$, *we have*

$$
w^2(S) \le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S}_\lambda\right) + \frac{1}{8}w\left(Q\widetilde{S}_\lambda + \widetilde{S}_\lambda Q\right), \tag{21}
$$

where $Q = |S|^{2\lambda} + |S|^{2(1-\lambda)}$. *In particular,*

$$
w^2(S) \le \frac{1}{4}\left(\|S\|^2 + w^2\left(\tilde{S}\right) + w\left(|S|\tilde{S} + \tilde{S}|S|\right)\right). \tag{22}
$$

Remark 6. By using the inequality

$$
w(YA + AY^*) \le 2\|Y\| w(A), \tag{23}
$$

for all $Y, A \in \mathscr{B}(\mathscr{H})$ (see [16]) in inequality (15) obtained in Theorem 2, we have

$$
\begin{aligned}
w^2(S) &\le \frac{1}{16}\|Q\|^2 + \frac{1}{4}w^2\left(\widetilde{S_{f,g}}\right) + \frac{1}{8}\left(2\|Q\| w\left(\widetilde{S_{f,g}}\right)\right) \\
&= \left(\frac{1}{4}\|Q\| + \frac{1}{2}w\left(\widetilde{S_{f,g}}\right)\right)^2.
\end{aligned} \tag{24}
$$

Hence,

$$
w(S) \le \frac{1}{4}\|Q\| + \frac{1}{2}w\left(\widetilde{S_{f,g}}\right), \tag{25}
$$

where $Q = (f(|S|))^2 + (g(|S|))^2$. Thus, inequality (15) obtained in Theorem 2 is better than inequality (12).

Remark 7. For continuous functions f and g in (10), if $\widetilde{S_{f,g}} = 0$, then inequality (15) becomes

$$
w(S) \le \frac{1}{4}\left\| (f(|S|))^2 + (g(|S|))^2 \right\|. \tag{26}
$$

In particular, if we take $f(|S|) = |S|^{1/2}$ and $g(|S|) = |S|^{1/2}$, for this choice of f and g if $\widetilde{S_{f,g}} = 0$, then inequality (15) becomes $w(S) \le 1/2\|S\|$, and combined with inequality (3), we get $w(S) = 1/2\|S\|$.

Theorem 8. Let $S \in \mathscr{B}(\mathscr{H})$. *Then, we have*

$$
\begin{aligned}
w^4(S) \le \frac{1}{4}&\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2 \\
&+ \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2,
\end{aligned} \tag{27}
$$

where $P = S^*S + SS^*$ *and* f,g *is nonnegative continuous functions defined on* $[0, \infty)$ *such that* $f(t)g(t) = t$.

Proof. Since $H_\theta = (e^{i\theta}S + e^{-i\theta}S^*)/2$ for all $\theta \in \mathbb{R}$, then we have

$$
\begin{aligned}
{H_\theta}^2 &= \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + SS^* + S^*S\right) \\
&= \frac{1}{4}\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2} + P\right),
\end{aligned} \tag{28}
$$

which yields

$$\begin{aligned}
H_\theta{}^4 &= \frac{1}{16}\left(\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2}\right)^2 + e^{2i\theta}\left(S^2P + PS^2\right) + e^{-2i\theta}\left(S^{*2}P + PS^{*2}\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}S^2 + e^{-2i\theta}S^{*2}\right)^2 + 2\left(\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}U|S|U|S| + e^{-2i\theta}|S|U^*|S|U^*\right)^2 + 2\left(\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}Ug(|S|)f(|S|)Ug(|S|)f(|S|) + e^{-2i\theta}f(|S|)g(|S|)U^*f(|S|)g(|S|)U^*\right)^2 + 2\left(\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right)\\
&= \frac{1}{16}\left(\left(e^{2i\theta}Ug(|S|)\widetilde{S_{f,g}}f(|S|) + e^{-2i\theta}f(|S|)\widetilde{S_{f,g}}^{\,*}g(|S|)U^*\right)^2 + 2\left(\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\right).
\end{aligned} \tag{29}$$

□

Since H_θ is self-adjoint, so $\|H_\theta{}^4\| = \|H_\theta\|^4$. Hence, using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned}
\|H_\theta\|^4 &\le \frac{1}{16}\left(\left(\left\|e^{2i\theta}Ug(|S|)\widetilde{S_{f,g}}f(|S|) + e^{-2i\theta}f(|S|)\widetilde{S_{f,g}}^{\,*}g(|S|)U^*\right\|\right)^2 + 2\left\|\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&\le \frac{1}{16}\left(\left(\left\|Ug(|S|)\widetilde{S_{f,g}}f(|S|)\right\| + \left\|f(|S|)\widetilde{S_{f,g}}^{\,*}g(|S|)U^*\right\|\right)^2 + 2\left\|\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&\le \frac{1}{16}\left(\left(\|U\|\|g(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|f(|S|)\| + \|f(|S|)\|\left\|\widetilde{S_{f,g}}^{\,*}\right\|\|g(|S|)\|\|U^*\|\right)^2 + 2\left\|\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
&= \frac{1}{16}\left(4\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2 + 2\left\|\operatorname{Re}\left(e^{2i\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right),
\end{aligned} \tag{30}$$

where the equality holds because $\|S\| = \|S^*\|$. Now taking supremum over $\theta \in \mathbb{R}$ in last equality, then applying Lemma 1, yields

$$w^4(S) \le \frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|\|g(|S|)\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|. \tag{31}$$

For different choices of f and g in (10), we obtain the following inequalities of numerical radius from Theorem 8.

Corollary 9. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$w^4(S) \le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|\left\||S|e^{-|S|^\lambda}\right\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2, \tag{32}$$

where $P = S^*S + SS^*$ *and* $\widetilde{S_e} = e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 10. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$w^4(S) \le \frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|\left\||S|e^{-|S|^\lambda}\right\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2, \tag{33}$$

where $P = S^*S + SS^*$ *and* $\widetilde{S_e} = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 11. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0, 1]$*, we have*

$$w^4(S) \le \frac{1}{4}\left(\|S\|\left\|\widetilde{S_\lambda}\right\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2, \tag{34}$$

where $P = S^*S + SS^*$*. In particular,*

$$w^4(S) \le \frac{1}{4}\left(\|S\|\|\tilde{S}\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2. \tag{35}$$

Remark 12. It is easy to check that $\|\tilde{S}\| \le \|S^2\|^{1/2}$ (see [9] for details). Using the following inequality

$$w(YA + AY^*) \le 2\|Y\|w(A), \tag{36}$$

for all $Y, A \in \mathscr{B}(\mathscr{H})$ (see [14]), Corollary 11 yields

$$\begin{aligned}
w^4(S) &\le \frac{1}{4}\left(\|S\|\|\tilde{S}\|\right)^2 + \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2\\
&\le \frac{1}{4}\left(\|S\|\|S^2\|^{1/2}\right)^2 + \frac{1}{4}w\left(S^2\right)\|P\| + \frac{1}{16}\|P\|^2\\
&\le \frac{1}{4}\left(\|S\|\|S^2\|^{1/2}\right)^2 + \frac{1}{4}\|S^2\|\|P\| + \frac{1}{16}\|P\|^2\\
&= \left(\frac{1}{2}\|S\|\|S^2\|^{1/2} + \frac{1}{4}\|P\|\right)^2.
\end{aligned} \tag{37}$$

We know that $\|S^*S+SS^*\|\le\|S^2\|+\|S\|^2$ (see [17]). Hence,

$$\begin{aligned}\frac{1}{2}\|S\|\left\|S^2\right\|^{1/2}+\frac{1}{4}\|S^*S+SS^*\| &\le \frac{1}{2}\|S\|\left\|S^2\right\|^{1/2}+\frac{1}{4}\left\|S^2\right\|+\frac{1}{4}\|S\|^2 \\ &= \left(\frac{1}{2}\left\|S^2\right\|^{1/2}+\frac{1}{2}\|S\|\right)^2.\end{aligned} \tag{38}$$

Thus, the bound given in Corollary 11 is better than bound (4).

Remark 13. If $\widetilde{S_{f,g}}=0$ in inequality (27) obtained in Theorem 8 for different choices of f and g in (10), then inequality (27) becomes

$$w^4(S)\le\frac{1}{8}w\left(S^2P+PS^2\right)+\frac{1}{16}\|P\|^2. \tag{39}$$

If $S^2=0$ and $\widetilde{S_{f,g}}=0$ are equivalent conditions, then inequality (27) becomes

$$w^2(S)=\frac{1}{4}\|P\|. \tag{40}$$

Theorem 14. Let $S\in\mathscr{B}(\mathscr{H})$**.** *Then, we have*

$$w^3(S)\le\frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|+w\left(S^2S^*+S^*S^2+SS^*S\right)\right), \tag{41}$$

where f,g is nonnegative continuous functions defined on $[0,\infty)$ such that $f(t)g(t)=t$.

Proof. Since $H_\theta=(e^{i\theta}S+e^{-i\theta}S^*)/2$ for all $\theta\in\mathbb{R}$, then we have

$${H_\theta}^2=\frac{1}{4}\left(e^{2i\theta}S^2+e^{-2i\theta}S^{*2}+SS^*+S^*S\right), \tag{42}$$

which implies

$$\begin{aligned}{H_\theta}^3&=\frac{1}{8}\left(\left(e^{i\theta}S+e^{-i\theta}S^*\right)\left(e^{2i\theta}S^2+e^{-2i\theta}S^{*2}+SS^*+S^*S\right)\right)\\ &=\frac{1}{8}\left(e^{3i\theta}S^3+e^{-3i\theta}S^{*3}+2\operatorname{Re}\left(e^{i\theta}\left(S^2S^*+S^*S^2+SS^*S\right)\right)\right).\end{aligned} \tag{43}$$

□

In the last equality, $\operatorname{Re}(e^{i\theta}(S^2S^*+S^*S^2+SS^*S))=(e^{i\theta}(S^*S^2+S^{*2}S+S^*SS^*)+e^{-i\theta}(S^2S^*+S^*S^2+SS^*S))/2$. Hence,

$$\begin{aligned}\|H_\theta\|^3&\le\frac{1}{8}\left\|e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2f(|S|)\right\|\\&\quad+\left\|e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^*\right)^2g(|S|)U^*\right\|\\&\quad+\left\|2\operatorname{Re}\left(e^{i\theta}\left(S^2S^*+S^*S^2+SS^*S\right)\right)\right\|\\&\le\frac{1}{8}\left(\|U\|\|g(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|f(|S|)\|\right.\\&\quad+\|f(|S|)\|\left\|\widetilde{S_{f,g}}^*\right\|^2\|g(|S|)\|\|U^*\|\\&\quad\left.+2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^*+S^*S^2+SS^*S\right)\right)\right\|\right)\\&=\frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right.\\&\quad\left.+\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^*+S^*S^2+SS^*S\right)\right)\right\|\right).\end{aligned} \tag{44}$$

The first inequality holds because $\|A_1+A_2\|\le\|A_1\|+\|A_2\|$ where $A,A_1,A_2\in\mathscr{B}(\mathscr{H})$, the second inequality holds because $\|A_1A_2\|\le\|A_1\|\|A_2\|$ and $\|A^n\|\le\|A\|^n\forall n\in\mathbb{N}$, and the third equality holds because $\|A\|=\|A^*\|$ and $\|U\|=\|U^*\|=1$. Now taking supremum over $\theta\in\mathbb{R}$ in above equality the using Lemma 1, we obtain

$$w^3(S)\le\frac{1}{4}\left(\|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|+w\left(S^2S^*+S^*S^2+SS^*S\right)\right), \tag{45}$$

as desired.

Corollary 15. Let $S\in\mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda\in[0,1]$, we have*

$$w^3(S)\le\frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|+w\left(S^2S^*+S^*S^2+SS^*S\right)\right), \tag{46}$$

where $\widetilde{S_e}=e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 16. Let $S\in\mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda\in[0,1]$, we have*

$$w^3(S)\le\frac{1}{4}\left(\left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|+w\left(S^2S^*+S^*S^2+SS^*S\right)\right), \tag{47}$$

where $\widetilde{S_e}=|S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 17. Let $S\in\mathscr{B}(\mathscr{H})$**.** *Then, for $\lambda\in[0,1]$, we have*

$$w^3(S)\le\frac{1}{4}\left(\|S\|\left\|\widetilde{S_\lambda}\right\|^2+w\left(S^2S^*+S^*S^2+SS^*S\right)\right). \tag{48}$$

In particular,

$$w^3(S) \le \frac{1}{4}\left(\|S\| \|\tilde{S}\|^2 + w\left(S^2 S^* + S^* S^2 + S S^* S\right) \right). \tag{49}$$

Remark 18. Yan et al. proved that

$$w^2(S) \le \frac{1}{2} \|f(|S|)\| \left\| \widetilde{S_{f,g}} \right\| \|g(|S|)\| + \frac{1}{4} \|S^* S + S S^*\|, \tag{50}$$

see [18]. Inequality (41) obtained in Theorem 14 gives better bounds of numerical radius of S for different choices of f and g in (10) when

$$S = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}. \tag{51}$$

Then, $S = U|S|$ is a polar decomposition of S, where

$$|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \tag{52}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{53}$$

is partial isometry.

Bounds (41) and (50) are computed for some choices of f and g in (10) for the given S in Table 1, whereas the numerical radius of S is

$$w(S) = 2.9154. \tag{54}$$

The spectral radius of an operator is defined as

$$r(S) = \sup_{\|x\|=1} \{|\lambda| : \lambda \in \sigma(S)\}, \tag{55}$$

where r denotes the spectral radius. For further inforamtion on spectral radius, see [19]. The following theorem will be used to develop the next inequality of numerical radius.

Theorem 19 (see [17]). *Let $M_1, M_2, N_1, N_2 \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} r(M_1 N_1 + M_2 N_2) &\le \frac{1}{2}(w(N_1 M_1) + w(N_2 M_2)) \\ &\quad + \frac{1}{2}\sqrt{w(N_1 M_1) - w(N_2 M_2) + 4\|N_1 M_2\| \|N_2 M_1\|}. \end{aligned} \tag{56}$$

Table 1: Bounds (41) and (50) for different choices of f and g in (10).

(f, g)	Bound (50)	Bound (41)
$\left(\|S\|^{1/2}, \|S\|^{1/2}\right)$	4.2640	3.2655
$\left(\|S\|^{1/7}, \|S\|^{6/7}\right)$	4.0704	3.0724
$\left(e^{\|S\|^{1/10}}, \|S\|e^{-\|S\|^{1/10}}\right)$	4.3582	3.3622
$\left(e^{\|S\|^{1/10}}, \|S\|e^{-\|S\|^{1/10}}\right)$	4.2055	3.2031

Theorem 20. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then,*

$$\begin{aligned} w^3(S) \le{}& \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^{\,3}\right) + \|f(|S|)\| \left\|\widetilde{S_{f,g}}\right\|^2 \|g(|S|)\| \right) \\ &+ \frac{1}{4} w\left(S^2 S^* + S^* S^2 + S S^* S\right), \end{aligned} \tag{57}$$

where f, g is nonnegative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$.

Proof. Since $H_\theta = (e^{i\theta} S + e^{-i\theta} S^*)/2$ for all $\theta \in \mathbb{R}$, then we have

$$H_\theta{}^2 = \frac{1}{4}\left(e^{2i\theta} S^2 + e^{-2i\theta} S^{*2} + S S^* + S^* S\right), \tag{58}$$

which implies

$$\begin{aligned} H_\theta{}^3 &= \frac{1}{8}\left(e^{i\theta} S + e^{-i\theta} S^*\right)\left(e^{2i\theta} S^2 + e^{-2i\theta} S^{*2} + S S^* + S^* S\right) \\ H_\theta{}^3 &= \frac{1}{8}\Bigg(e^{3i\theta} U g(|S|) \left(\widetilde{S_{f,g}}\right)^2 f(|S|) \\ &\quad + e^{-3i\theta} f(|S|) \left(\widetilde{S_{f,g}}^{\,*}\right)^2 g(|S|) U^* \\ &\quad + 2\,\mathrm{Re}\left(e^{i\theta}\left(S^2 S^* + S^* S^2 + S S^* S\right)\right)\Bigg). \end{aligned} \tag{59}$$

□

Now by using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} \|H_\theta\|^3 \le \frac{1}{8}\Bigg(&\Bigg\| e^{3i\theta} U g(|S|) \left(\widetilde{S_{f,g}}\right)^2 f(|S|) \\ &+ e^{-3i\theta} f(|S|) \left(\widetilde{S_{f,g}}^{\,*}\right)^2 g(|S|) U^* \Bigg\| \\ &+ 2\left\|\mathrm{Re}\left(e^{i\theta}\left(S^2 S^* + S^* S^2 + S S^* S\right)\right)\right\| \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left(r\left(e^{3i\theta}Ug(|S|)\left(\widetilde{S_{f,g}}\right)^2 f(|S|)\right.\right. \\
&\quad \left. + e^{-3i\theta}f(|S|)\left(\widetilde{S_{f,g}}^*\right)^2 g(|S|)U^*\right) \\
&\quad \left. + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) \qquad (60)\\
&= \frac{1}{8}\left(r(M_1N_1 + M_2N_2)\right. \\
&\quad \left. + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right),
\end{aligned}
$$

where $M_1 = e^{3i\theta}Ug(|S|)(\widetilde{S_{f,g}})^2$, $N_1 = f(|S|)$, $M_2 = e^{-3i\theta}f(|S|)(\widetilde{S_{f,g}}^*)^2$, and $N_2 = g(|S|)U^*$; the first equality holds for Hermitian operator satisfying $r(A) = \|A\|$. Now applying Theorem 19 on last equality with $w(\alpha A) = |\alpha|w(A)$ and $w(A) = w(A^*)$, we obtain

$$
\begin{aligned}
\|H_\theta\|^3 &\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^3\right)\right. \\
&\quad + \frac{1}{2}\sqrt{4\|(f(|S|))^2\|\left\|\left(\widetilde{S_{f,g}}^*\right)^2\right\|\|(g(|S|))^2\|\left\|\widetilde{S_{f,g}}^2\right\|} \\
&\quad \left. + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) \\
&\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^3\right)\right. \\
&\quad + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\left\|\widetilde{S_{f,g}}^*\right\|^2\|g(|S|)\|^2\left\|\widetilde{S_{f,g}}\right\|^2} \\
&\quad \left. + 2\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|\right) \\
&\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^3\right) + \|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right) \\
&\quad + \frac{1}{4}\left\|\operatorname{Re}\left(e^{i\theta}\left(S^2S^* + S^*S^2 + SS^*S\right)\right)\right\|.
\end{aligned}
\qquad (61)
$$

Taking supremum over $\theta \in \mathbb{R}$ in last inequality, then applying Lemma 1, we obtain

$$
\begin{aligned}
w^3(S) &\le \frac{1}{8}\left(w\left(\widetilde{S_{f,g}}^3\right) + \|f(|S|)\|\left\|\widetilde{S_{f,g}}\right\|^2\|g(|S|)\|\right) \\
&\quad + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right).
\end{aligned}
\qquad (62)
$$

Corollary 21. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$
\begin{aligned}
w^3(S) &\le \frac{1}{8}\left(w\left(\widetilde{S_e}^3\right) + \left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|\right) \\
&\quad + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right),
\end{aligned}
\qquad (63)
$$

where $\widetilde{S_e} = e^{|S|^\lambda}U|S|e^{-|S|^\lambda}$.

Corollary 22. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$
\begin{aligned}
w^3(S) &\le \frac{1}{8}\left(w\left(\widetilde{S_e}^3\right) + \left\|e^{|S|^\lambda}\right\|\left\|\widetilde{S_e}\right\|^2\left\||S|e^{-|S|^\lambda}\right\|\right) \\
&\quad + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right),
\end{aligned}
\qquad (64)
$$

where $\widetilde{S_e} = |S|e^{|S|^{-\lambda}}Ue^{|S|^\lambda}$.

Corollary 23. Let $S \in \mathscr{B}(\mathscr{H})$**.** *Then, for* $\lambda \in [0,1]$ *we have*

$$
w^3(S) \le \frac{1}{8}\left(w\left(\widetilde{S_\lambda}^3\right) + \|S\|\left\|\widetilde{S_\lambda}\right\|^2\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right). \qquad (65)
$$

In particular,

$$
w^3(S) \le \frac{1}{8}\left(w\left(\tilde{S}^3\right) + \|S\|\|\tilde{S}\|^2\right) + \frac{1}{4}w\left(S^2S^* + S^*S^2 + SS^*S\right). \qquad (66)
$$

Remark 24. It is easy to observe that inequality (57) obtained in Theorem 20 is better than inequality (41).

Now, we exhibit some examples where numerical radius bounds are computed from inequalities (15), (27), (41), and (57) for some choices of pair f, g in (10) and for a given operator S.

Example 1. Given

$$
S = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (67)
$$

Then, S = U|S| is a polar decomposition of S, where

$$
|S| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad (68)
$$

and

$$
U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (69)
$$

is partial isometry.

Table 2: Bounds (15), (27), (41), and (57) for different choices of f and g in (10).

(f, g)	Bound (15)	Bound (27)	Bound (41)	Bound (57)
$(\|S\|^{1/2}, \|S\|^{1/2})$	2.2704	2.3596	2.4662	2.3811
$(\|S\|^{1/3}, \|S\|^{2/3})$	2.3596	2.3264	2.4346	2.3332
$(\|S\|^{1/4}, \|S\|^{3/4})$	2.4817	2.3244	2.4327	2.3632
$(\|S\|^{1/5}, \|S\|^{4/5})$	2.5826	2.3510	2.4579	2.3462
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	8.7223	2.4234	2.5284	2.3854
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	5.3389	2.3492	2.4563	2.3459
$\left(e^{\|S\|^{1/4}}, \|S\|e^{-\|S\|^{1/4}}\right)$	4.4307	2.3277	2.4311	2.3314
$\left(e^{\|S\|^{1/5}}, \|S\|e^{-\|S\|^{1/5}}\right)$	4.0333	2.3380	2.4456	2.3394

Table 3: Bounds (15), (27), (41), and (57) for different choices of f and g in (10).

(f, g)	Bound (15)	Bound (27)	Bound (41)	Bound (57)
$(\|S\|^{1/2}, \|S\|^{1/2})$	1.5047	1.5065	1.4500	1.3880
$(\|S\|^{1/3}, \|S\|^{2/3})$	1.5416	1.5425	1.4901	1.4059
$(\|S\|^{1/4}, \|S\|^{3/4})$	1.5754	1.5637	1.5130	1.4432
$(\|S\|^{1/5}, \|S\|^{4/5})$	1.6026	1.5769	1.5276	1.4513
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	4.7336	1.5272	1.4726	1.4004
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	3.6923	1.5335	1.4794	1.4250
$\left(e^{\|S\|^{1/4}}, \|S\|e^{-\|S\|^{1/4}}\right)$	3.3318	1.5585	1.5069	1.4399
$\left(e^{\|S\|^{1/5}}, \|S\|e^{-\|S\|^{1/5}}\right)$	3.1529	1.5729	1.5231	1.4488

Bounds (15), (27), (41), and (57) are computed for some choices of f and g in (10) for the given S in Table 2, whereas the numerical radius of S is

$$w(S) = 2.0565. \tag{70}$$

Example 2. Let

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}. \tag{71}$$

Then, $S = U|S|$ is a polar decomposition of S, where

$$|S| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{72}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{73}$$

Bounds (15), (27), (41), and (57) are computed for some choices of f and g in (10) for the given S in Table 3, whereas the numerical radius of S is

$$w(S) = 1.3662. \tag{74}$$

3. Conclusion

From the results of this paper, we conclude that the inequalities of numerical radius involving generalized Aluthge transform have variety of upper bounds for numerical radius due to the choice of f, g in generalized Aluthge transform (10). The inequalities (15), (27), (41) and (57) obtained in Theorem 2, Theorem 8, Theorem 20, and Theorem 14 are new and generalized upper bounds for numerical radius. These generalized upper bounds can be useful to find better bounds of numerical radius already existing in literature for some choices of f, g in generalized Aluthge transform (10) and certain operators. It is proved that inequality (15) of Theorem 2 generalizes inequality (7) and improves inequality (12) for any choice of f, g in (10). Inequality (27) of Theorem 8 is sharper than inequality (12) for the choice of $f(t) = g(t) = t^{1/2}$ in (10). Inequality (41) of Theorem 14 is better than inequality (57) of Theorem 20. But for inequality (57) of Theorem 20, we can find such matrix and pairs of f, g for which the inequality of Theorem 20 can give better bound of numerical radius available in literature. To support theoretical investigations, some examples are presented where numerical radius and its upper bounds are computed for the pairs f, g in generalized Aluthge transform. Examples 1 and 2 show that there is no comparison between the bounds obtained from the inequalities (15), (27), and (41) of Theorem 2, Theorem 8, and Theorem 20; however, generalized Aluthge transform has choices of the pair f, g in (10) for which better upper bounds can be computed for certain operators.

Data Availability

There is no data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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