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Research Article

Fixed Point Results for $(\alpha, \perp_{\mathcal{F}})$-Contractions in Orthogonal $\mathcal{F}$-Metric Spaces with Applications

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The main objective of this article is to introduce the notion of $(\alpha, \perp_{\mathcal{F}})$-contraction in the setting of orthogonal $\mathcal{F}$-metric space and present some new fixed point results in such newly introduced space. We also furnish an example to manifest the originality of the obtained results. As application of our foremost result, we look into the solution of a nonlinear fractional differential equation.

1. Introduction and Preliminaries

One of the most important part in the theory of fixed points is the underlying space as well as the contractive mapping and contractive condition. In 1905, a French mathematician Maurice Fréchet gave the study of metric space which plays a virtual role in the pioneer result in this theory. In the last few decadess, many researchers have presented interesting generalizations of metric spaces. Most of the generalizations are made by making some changes in the triangle inequality of the original definition. Some of these well-known generalizations of metric space are b-metric space due to Czerwik [1], rectangular metric space due to Branciari [2], and JS-metric space due to Jleli and Samet [3]. After all such generalizations, Jleli and Samet [4] gave a compulsive extension of a metric space which is known as $\mathcal{F}$-metric space in this way.

Suppose $\mathcal{F}$ be the set of functions $\xi : (0,+\infty) \longrightarrow \mathbb{R}$ satisfying

$(\mathcal{F}_1) 0 < \hbar_1 < \hbar_2 \Longrightarrow \xi(\hbar_1) \leq \xi(\hbar_2)$

$(\mathcal{F}_2)$ for all $\{\hbar_n\} \subseteq \mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\hbar_n = 0$ if and only if $\lim_{n\longrightarrow\infty}\xi(\hbar_n) = -\infty$

Definition 1 (see [4]). Let $\mathfrak{R}$ be nonempty set, and let $\tau : \mathfrak{R} \times \mathfrak{R} \longrightarrow 0,+\infty)$. Suppose that there exists $(\xi, \alpha) \in \mathcal{F} \times 0,+\infty)$ such that

(D_1) $(\hbar, \varsigma) \in \mathfrak{R} \times \mathfrak{R}$, $\tau(\hbar, \varsigma) = 0$ if and lnly if $\hbar = \varsigma$

(D_2) $\tau(\hbar, \varsigma) = \tau(\varsigma, \hbar)$, for all $\hbar, \varsigma \in \mathfrak{R}$

(D_3) for all $(\hbar, \varsigma) \in \mathfrak{R} \times \mathfrak{R}$, and $(\hbar_i)_{i=1}^{N} \subset \mathfrak{R}$, with $(\hbar_1, \hbar_N) = (\hbar, \varsigma)$, we have

$$\tau(\hbar, \varsigma) > 0 \Longrightarrow \xi(\tau(\hbar, \varsigma)) \leq \xi\left(\sum_{i=1}^{N-1} \tau(\hbar_i, \hbar_{i+1})\right) + \alpha, \quad (1)$$

for all $N \geq 2$. Then, $(\mathfrak{R}, \tau)$ is said to be an $\mathcal{F}$-metric space ($\mathcal{F}$-MS).

Example 1 (see [4]). Let $\mathfrak{R} = \mathbb{R}$. Then, $\tau : \mathfrak{R} \times \mathfrak{R} \longrightarrow 0,+\infty)$ defined by

$$\tau(\hbar, \varsigma) = \begin{cases} (\hbar - \varsigma)^2 \text{ if } (\hbar, \varsigma) \in 0, 3] \times 0, 3], \\ |\hbar - \varsigma| \text{ if } (\hbar, \varsigma) \in 0, 3] \times 0, 3], \end{cases} \quad (2)$$

with $\xi(t) = \ln(t)$ and $\alpha = \ln(3)$ is an $\mathcal{F}$-metric.

Definition 2 (see [4]). Let $(\mathfrak{R}, \tau)$ be an $\mathcal{F}$-MS.

Let $\{\hbar_n\}$ be a sequence in $\mathfrak{R}$. Then, $\{\hbar_n\}$ is said to be an $\mathcal{F}$-convergent to $\hbar \in \mathfrak{R}$ if $\{\hbar_n\}$ is convergent to $\hbar$ with regard to an $\mathcal{F}$-metric τ.

(i) A sequence $\{\hbar_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\longrightarrow\infty} \tau(\hbar_n, \hbar_m) = 0. \tag{3}$$

For more particulars in this way, we mention the researchers to [5–9].

On the other hand, Gordji et al. [10] initiated the conception of the orthogonal set (in short, $\mathcal{O}$-set).

Definition 3. Let $\mathfrak{R}$ be a non empty and $\perp \subseteq \mathfrak{R} \times \mathfrak{R}$. If $\mathfrak{R}$ with the binary relation $\perp$ satisfies the condition:

$$\text{There exists } \hbar_0[(\text{for all } \varsigma \in \mathfrak{R}, \varsigma\perp\hbar_0) \text{ or } (\text{for all } \varsigma \in \mathfrak{R}, \hbar_0\perp\varsigma)], \tag{4}$$

then, it is said to be an orthogonal set. Moreover, $\hbar_0$ is said to be an orthogonal point. We represent this $\mathcal{O}$-set by $(\mathfrak{R}, \perp)$.

Example 2 (see [10]). Let $\mathfrak{R} = \mathbb{Z}$. Define $\perp$ on $\mathcal{U}$ by $i\perp j$ if there exists $l \in \mathbb{Z}$ such that $i = lj$. It is very simple to note that $0\perp j$, for all $j \in \mathbb{Z}$. Thus, $(\mathfrak{R},\perp)$ is an O-set.

Definition 4 (see [10]). Let $(\mathfrak{R}, \perp)$ be an $\mathcal{O}$-set. A sequence $\{\hbar_\mathrm{n}\}$ is called orthogonal sequence (in essence $\mathcal{O}$-sequence) if

$$(\text{for all } n \in \mathbb{N}, \hbar_n\perp\hbar_{n+1}) \text{ or } (\text{for all } n \in \mathbb{N}, \hbar_{n+1}\perp\hbar_n). \tag{5}$$

In this direction, these researchers [11–14] utilized $\mathcal{O}$-sets in different ways to obtain their results.

Very recently, Kanwal et al. [15] combined both concepts and introduced the notion of orthogonal $\mathcal{F}$-metric space in this way.

Definition 5 (see. [15]). Let $(\mathfrak{R},\perp)$ be an $\mathcal{O}$-set and τ be an $\mathcal{F}$-metric on $\mathfrak{R}$. Then, triplet $(\mathfrak{R},\perp,\tau)$ is claimed to be an orthogonal $\mathcal{F}$-metric space (OF-MS).

Example 3 (see [15]). Let $\mathfrak{R} = [0, 1]$. Define $\mathcal{F}$-metric τ by

$$\tau(\hbar, \varsigma) = \begin{cases} e^{(|\hbar-\varsigma|)}, \text{ if } \hbar \neq \varsigma, \\ 0, \text{ if } \hbar = \varsigma, \end{cases} \tag{6}$$

for all $\hbar, \varsigma \in \mathfrak{R}, \xi(t) = -1/t, t > 0$ and $\alpha = 1$. Define $\hbar\perp\varsigma$ if $\hbar\varsigma \leq \hbar$ or $\hbar\varsigma \leq \varsigma$. Then, for all $\hbar \in \mathfrak{R}$, $0\perp\varsigma$, so $(\mathfrak{R}, \perp)$ is an $\mathcal{O}$-set. Then, $(\mathfrak{R},\perp,\tau)$ is an OF-MS.

From now onwards, we represent $(\mathfrak{R}, \perp)$ as an $\mathcal{O}$-set and $(\mathfrak{R},\perp,\tau)$ an OF-MS.

Definition 6 (see [15]). A mapping $\mathcal{V}:(\mathfrak{R},\perp,\rho) \longrightarrow (\mathfrak{R},\perp,\rho)$ is professed to be orthogonality continuous (in short $\perp$-continuous) at $\hbar \in \mathfrak{R}$ if for each $\mathcal{O}$-sequence $\{\hbar_\mathrm{n}\}$ in $\mathfrak{R}$ if $\hbar_\mathrm{n} \longrightarrow \hbar$, then $\mathcal{V}\hbar_\mathrm{n} \longrightarrow \mathcal{V}\hbar$. Also $\mathcal{V}$ is $\perp$-continuous on $\mathfrak{R}$ if $\mathcal{V}$ is $\perp$-continuous in each $\hbar \in \mathfrak{R}$.

Definition 7 (see [15]). A set $\mathfrak{R}$ of $(\mathfrak{R},\perp,\tau)$ is professed to be orthogonally $\mathcal{F}$-complete (in short $\mathcal{O}$-$\mathcal{F}$-complete) if every Cauchy O-sequence is $\mathcal{F}$-convergent.

In 2012, Samet et al. [16] gave the conception of α-admissible in this way.

Definition 8. A mapping $\mathcal{V}$ is called α-admissible whenever $\alpha(\hbar, \varsigma) \geq 1$ implies $\alpha(\mathcal{V}\hbar, \mathcal{V}\varsigma) \geq 1$.

Later on, Ramezani [17] extended the above concept to orthogonal set as follows.

Definition 9. A mapping $\mathcal{V}$ is called orthogonally α-admissible whenever $\hbar\perp\varsigma$ and $\alpha(\hbar, \varsigma) \geq 1$ implies $\alpha(\mathcal{V}\hbar, \mathcal{V}\varsigma) \geq 1$.

We give the following property: (OH) $\alpha(\hbar, \varsigma) \geq 1$ for any $\hbar, \varsigma \in \{\hbar^* \in \mathfrak{R} \sim \hbar^* = \mathcal{V}\hbar^*\}$.

On the other side, Wardowski [18] introduced the following new notion of F-contraction in 2012.

Definition 10. Let $(\mathfrak{R}, \tau)$ be a metric space. A mapping $\mathcal{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be a F-contraction if there exists $\eth>0$ such that for $\hbar, \varsigma \in \mathfrak{R}$:

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) > 0 \Longrightarrow \eth+\mathfrak{I}(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)) \leq \mathfrak{I}(\tau(\hbar, \varsigma)), \tag{7}$$

where $\mathfrak{I} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a mapping satisfying $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, and $(\mathcal{F}_3)$, that is,

$(\mathcal{F}_3)$ there exists $0 < r < 1$ such that $\lim_{\hbar\longrightarrow 0^+} \hbar^r \mathfrak{I}(\hbar) = 0$.

We represent Ψ, the set of the functions $\mathfrak{I} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying $(\mathcal{F}_1)$-$(\mathcal{F}_3)$. Later on, Secelean [19] and Piri and Kumam [20] replaced the conditions $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$ with some weaker symmetrical conditions repectively. Popescu and Stan [21] obtained two new fixed point results for F-contractions with these weaker symmetrical conditions. Many authors [22–29] used the concept of F-contraction introduced by Wardowski in order to define and prove new results on fixed points in complete metric spaces.

In this article, we introduce the notion of $(\alpha, \perp_{\mathcal{F}})$-contraction in the background of orthogonal $\mathcal{F}$-metric space (OF-MS) and obtain some new results in such newly introduced space. We also furnish an example to manifest the originality of the obtained results. As application of our foremost result, we look into the solution of a nonlinear fractional differential equation.

2. Main Results

Definition 11. Let $(\mathfrak{R},\perp,\tau)$ be an OF-MS. A mapping $\mathcal{V} :\mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be $(\alpha, \perp_{\mathcal{F}})$-contraction if there exists $\mathfrak{I} \in \Psi, \alpha : \mathfrak{R} \times \mathfrak{R} \longrightarrow [1,\infty)$ and $\eth>0$ such that

$$\begin{aligned} &\text{for all } \hbar, \varsigma \in \mathfrak{R}, \hbar\perp\varsigma, \tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \neq 0 \\ &\quad\Longrightarrow \eth+\alpha(\hbar, \varsigma)\mathfrak{I}(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)) \leq \mathfrak{I}(\tau(\hbar, \varsigma)). \end{aligned} \tag{8}$$

Theorem 12. *Let $(\mathfrak{R},\perp,\tau)$ be an $\mathcal{O}$-complete OF-MS and $\mathcal{V}: \mathfrak{R} \longrightarrow \mathfrak{R}$ is an $(\alpha, \perp_{\mathcal{F}})$-contraction such that following assertions hold:*

(i) $\mathcal{V}$ is $\perp$-preserving, that is, $\hbar\perp\varsigma$ implies $\mathcal{V}\hbar\perp\mathcal{V}\varsigma$

(ii) $\mathcal{V}$ is orthogonally α-admissible

(iii) there exists $\hbar_0 \in \mathfrak{R}$ such that $\hbar_0\perp\mathcal{V}\hbar_0$ and $\alpha(\hbar_0, \mathcal{V}\hbar_0) \geq 1$

(iv) $\mathcal{V}$ is $\perp$-continuous

Then, there exists $\hbar^* \in \mathfrak{R}$ such that $\mathcal{V}\hbar^* = \hbar^*$. Furthermore, if $\mathfrak{R}$ has the property (OH), then this fixed point is unique.

Proof. By assertion (iii), there exists $\hbar_0 \in \mathfrak{R}$ such that $\hbar_0\perp\mathcal{V}\hbar_0$ and $\alpha(\hbar_0, \mathcal{V}\hbar_0) \geq 1$. Let the sequence $\{\hbar_n\}$ is defined as

$$\hbar_1 = \mathcal{V}\hbar_0, \cdots, \hbar_{n+1} = \mathcal{V}\hbar_n = \mathcal{V}^{n+1}\hbar_0, \tag{9}$$

for all $n \geq 0$. As $\mathcal{V}$ is $\perp$-preserving, so $\{\hbar_n\}$ is an O-sequence in $\mathfrak{R}$. As $\mathcal{V}$ is orthogonally α-admissible, so it follows that $\alpha(\mathcal{V}\hbar_n, \mathcal{V}\hbar_{n+1}) > 1$, for all $n \geq 0$. If $\hbar_n = \hbar_{n+1}$, for any $n \in \mathbb{N} \cup \{0\}$, it is then evident that $\hbar_n$ is a fixed point of $\mathcal{V}$. So we suppose that $\hbar_n$=$\hbar_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. Hence we have $\tau(\mathcal{V}\hbar_{n-1}, \mathcal{V}\hbar_n) = \tau(\hbar_n, \hbar_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. As $\mathcal{V}$ is $\perp$-preserving, we get

$$\hbar_n\perp\hbar_{n+1} \text{ or } \hbar_{n+1}\perp\hbar_n, \tag{10}$$

for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{\hbar_n\}$ is an $\mathcal{O}$-sequence. Hence, we presume that

$$0 < \tau(\hbar_n, \mathcal{V}\hbar_n) = \tau(\mathcal{V}\hbar_{n-1}, \mathcal{V}\hbar_n), \tag{11}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (8) and (11), we get

$$\begin{aligned}\eth+\mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) &= \eth+\mathfrak{F}(\tau(\mathcal{V}\hbar_{n-1}, \mathcal{V}\hbar_n)) \\ &\leq \eth+\alpha(\hbar_{n-1}, \hbar_n)\mathfrak{F}(\tau(\mathcal{V}\hbar_{n-1}, \mathcal{V}\hbar_n)) \\ &\leq \mathfrak{F}(\tau(\hbar_{n-1}, \hbar_n)),\end{aligned} \tag{12}$$

hence,

$$\eth+\mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) \leq \mathfrak{F}(\tau(\hbar_{n-1}, \hbar_n)), \tag{13}$$

for all $n \in \mathbb{N} \cup \{0\}$. Consequently, we have

$$\begin{aligned}\mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) &\leq \mathfrak{F}(\tau(\hbar_{n-1}, \hbar_n))-\eth \\ &\leq \mathfrak{F}(\tau(\hbar_{n-2}, \hbar_{n-1})) - 2\eth \\ &\leq \vdots \leq \mathfrak{F}(\tau(\hbar_0, \hbar_1)) - n\eth,\end{aligned} \tag{14}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now by applying $n \longrightarrow \infty$ and by $(\mathcal{F}_2)$, we have

$$\lim_{n\longrightarrow\infty} \mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) = -\infty \Leftrightarrow \lim_{n\longrightarrow\infty} \tau(\hbar_n, \hbar_{n+1}) = 0. \tag{15}$$

From the condition $(\mathcal{F}_3)$, there exists $0 < r < 1$ such that

$$\lim_{n\longrightarrow\infty} [\tau(\hbar_n, \hbar_{n+1})]^r \mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) = 0. \tag{16}$$

From (14) and (16), we have

$$\begin{aligned}&[\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{F}(\tau(\hbar_n, \hbar_{n+1})) - [\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{F}(\tau(\hbar_0, \hbar_1)) \\ &\quad\leq [\tau(\hbar_n, \hbar_{n+1})]^r[\mathfrak{F}(\tau(\hbar_0, \hbar_1)) - n\tau] - [\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{F}(\tau(\hbar_0, \hbar_1)) \\ &\quad\leq -n\tau s^n d_b(\hbar_n, \hbar_{n+1})]^r \leq 0.\end{aligned} \tag{17}$$

Taking $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} n\tau(\hbar_n, \hbar_{n+1})^r = 0. \tag{18}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\tau(\hbar_n, \hbar_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{19}$$

for all $n > n_1$. This yields

$$\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \tag{20}$$

for $m > n$. Since $\sum_{i=n}^{\infty} 1/i^{1/r}$ is convergent, so there exists $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} < \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} < \delta, \tag{21}$$

for $n > N$. Hence, by (21) and $(\mathcal{F}_1)$, we get

$$\xi\left(\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1})\right) \leq \xi\left(\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}\right) < \xi(\varepsilon) - a, \tag{22}$$

$m > n \geq N$. Using (D_3) and (22), we get

$$\begin{aligned}\tau(\hbar_n, \hbar_m) > 0, m > n \geq N &\Longrightarrow \xi(\tau(\hbar_n, \hbar_m)) \\ &\leq \xi\left(\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1})\right) + a < \xi(\varepsilon),\end{aligned} \tag{23}$$

which, from $(\mathcal{F}_1)$, gives that

$$\tau(\hbar_n, \hbar_m) < \varepsilon, m > n \geq N. \tag{24}$$

Hence, $\{\hbar_n\}$ is a Cauchy O-sequence in $(\mathfrak{R},\perp,\tau)$. The $\mathcal{O}$-completeness of $(\mathfrak{R},\perp,\tau)$ guarantees that there exists $\hbar^* \in \mathfrak{R}$ such that $\lim_{n\longrightarrow\infty}\hbar_n \longrightarrow \hbar^*$. Now, we prove that $\hbar^*$ is fixed point of $\mathcal{V}$. By $\perp$-continuity of $\mathcal{V}$ gives $\mathcal{V}\hbar_n \longrightarrow \mathcal{V}\hbar^*$ as n$\longrightarrow\infty$. Thus,

$$\mathcal{V}\hbar^* = \lim_{n\longrightarrow\infty} \mathcal{V}\hbar_n = \lim_{n\longrightarrow\infty} \hbar_{n+1} = \hbar^*. \tag{25}$$

Thus, $\hbar^*$ is a fixed point of $\mathscr{V}$. Lastly, we assume that $\hbar' = \mathscr{V}\hbar'$ is once more fixed point of $\mathscr{V}$ such that $\hbar' \neq \hbar^*$. From (OH), we have $\hbar^* \perp \hbar'$ or $\hbar' \perp \hbar^*$ and $\alpha(\hbar^*, \hbar') \geq 1$. Thus, from (8), we have

$$\begin{aligned} Ↄ+\Im\left(\tau\left(\hbar^*, \hbar'\right)\right) &= Ↄ+\Im\left(\tau\left(\mathscr{V}\hbar^*, \mathscr{V}\hbar'\right)\right) \\ &\leq Ↄ+\alpha\left(\hbar^*, \hbar'\right)\Im\left(\tau\left(\mathscr{V}\hbar^*, \mathscr{V}\hbar'\right)\right) \\ &\leq \Im\left(\tau\left(\hbar', \hbar^*\right)\right), \end{aligned} \tag{26}$$

which implies that

$$\Im\left(\tau\left(\hbar^*, \hbar'\right)\right) \leq \Im\left(\tau\left(\hbar', \hbar^*\right)\right) - Ↄ < \Im\left(\tau\left(\hbar', \hbar^*\right)\right), \tag{27}$$

that is a contradiction. Thus, $\hbar' = \hbar^*$. Hence, the fixed point is unique.

□

Corollary 13. *Let* $(\mathfrak{R}, \perp, \tau)$ *be an* $\mathcal{O}$*-complete OF-MS. If* $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ *is* $\perp$*-preserving and* $\perp$*-continuous satisfying*

$$\text{for all } \hbar, \varsigma \in \mathfrak{R}, \hbar \perp \varsigma, \tau(\mathscr{V}\hbar, \mathscr{V}\varsigma) \neq 0 \Longrightarrow Ↄ+\Im(\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma)) \leq \Im(\tau(\hbar, \varsigma)), \tag{28}$$

then, there exists $\hbar^* \in \mathfrak{R}$ *such that* $\mathscr{V}\hbar^* = \hbar^*$ *which is unique.*

Example 4. Define the sequence $\{\mu_n\}$ as follows:

$$\begin{aligned} \mu_1 &= \ln(1), \\ \mu_2 &= \ln(1+4), \\ &\vdots \\ \mu_n &= \ln(1+4+7+\cdots+(3n-2)) = \ln\left(\frac{n(3n-1)}{2}\right), \end{aligned} \tag{29}$$

for all $n \in \mathbb{N}$. Let $\mathfrak{R} \ll \{\mu_n : n \in \mathbb{N}\}$ equipped with the $\mathcal{F}$ metric defined by

$$\tau(\hbar, \varsigma) = \begin{cases} e^{|\hbar-\varsigma|}, \text{if } \hbar = \varsigma, \\ 0, \text{if } \hbar = \varsigma, \end{cases} \tag{30}$$

with $\xi(t) = -1/t$ and $a = 1$. For all $\mu_n, \mu_m \in \mathfrak{R}$, define $\mu_n \perp \mu_m$ if and only if $(m \geq 2 \wedge n = 1)$. Hence, $(\mathfrak{R}, \perp, \tau)$ is an O-complete O$\mathcal{F}$-MS. Define $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ by

$$\mathscr{V}(\mu_n) = \begin{cases} \mu_1, \text{if } n = 1, \\ \mu_{n-1}, \text{if } n > 1, \end{cases} \tag{31}$$

and $\alpha : \mathfrak{R} \times \mathfrak{R} \longrightarrow 1, +\infty)$ by

$$\alpha(\mu_n, \mu_m) = \begin{cases} 1, \text{if } \mu_n \neq \mu_m, \\ 0, \text{if } \mu_n = \mu_m. \end{cases} \tag{32}$$

Clearly,

$$\lim_{n \longrightarrow \infty} \frac{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_1))}{\tau(\mu_n, \mu_1)} = 1, \tag{33}$$

then $\mathscr{V}$ is not a contraction in the sense of [18].

It is easy to check that $\mathscr{V}$ is $\perp$-continuous and $\mathscr{V}$ is $\perp$-preserving. Let the mapping $\Im : (0, \infty) \longrightarrow \mathbb{R}$ defined by

$$\Im(t) = \ln t + t, t > 0. \tag{34}$$

It is easy to show that $\Im \in \Psi$. Now, to prove $\mathscr{V}$ is an $\perp_{\mathcal{F}}$-contraction, that is

$$\begin{aligned} \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow Ↄ+ &\ln \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \\ &+ \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \\ &\leq \ln \tau(\mu_n, \mu_m) + \tau(\mu_n, \mu_m), \end{aligned} \tag{35}$$

for Ↄ>0. The above condition is equivalent to

$$\begin{aligned} \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow &e^{Ↄ+\ln \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) + \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m))} \\ &\leq e^{\ln \tau(\mu_n, \mu_m) + \tau(\mu_n, \mu_m)}. \end{aligned} \tag{36}$$

So, we have to check that

$$\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow \frac{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m))}{\tau(\mu_n, \mu_m)} e^{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) - \tau(\mu_n, \mu_m)} \leq e^{-Ↄ}. \tag{37}$$

For every $m \in \mathbb{N}, m \geq 2$, we have

$$\begin{aligned} \tau(\mathscr{V}(\mu_m), \mathscr{V}(\mu_1)) \neq 0 \Longrightarrow &\frac{\tau(\mathscr{V}(\mu_m), \mathscr{V}(\mu_1))}{\tau(\mu_m, \mu_1)} e^{\tau(\mathscr{V}(\mu_m), \mathscr{V}(\mu_1)) - \tau(\mu_m, \mu_1)} \\ &\leq e^{-Ↄ} \frac{\tau(\mu_{m-1}, \mu_1)}{\tau(\mu_m, \mu_1)} e^{\tau(\mu_{m-1}, \mu_1) - \tau(\mu_m, \mu_1)} \\ &= \frac{e^{\mu_{m-1} - \mu_1}}{e^{\mu_m - \mu_1}} e^{e^{\mu_{m-1} - \mu_1} - e^{\mu_m - \mu_1}} \\ &= \frac{(m-1)(3m-4)}{m(3m-1)} e^{-6m+4} < e^{-1}. \end{aligned} \tag{38}$$

Thus, the inequality (8) is satisfied with $Ↄ = 1 > 0$. Hence, $\mathscr{V}$ is an $(\alpha, \perp_{\mathcal{F}})$-contraction. Thus, Theorem 12 implies that $\mu = \ln(1)$ is a unique fixed point of $\mathscr{V}$.

Example 5. Define the sequence $\{\mu_n\}$ as follows:

$$\begin{aligned}\mu_1 &= \log_2 1,\\ \mu_2 &= \log_2 1 + \log_2 2,\\ &\vdots\\ \mu_n &= \log_2 1 + \log_2 2 + \cdots + \log_2 n = \log_2 n!,\end{aligned} \tag{39}$$

for all $n \in \mathbb{N}$. Let $\mathfrak{R} \ll \{\mu_n : n \in \mathbb{N}\}$ equipped with the $\mathscr{F}$ metric defined by

$$\tau(\hbar, \varsigma) = \begin{cases} 2^{|\hbar-\varsigma|}, \text{ if } \hbar \neq \varsigma, \\ 0, \text{ if } \hbar = \varsigma, \end{cases} \tag{40}$$

with $\xi(t) = -1/t$ and $a = 1$. For all $\mu_n, \mu_m \in \mathfrak{R}$, define $\mu_n \perp \mu_m$ if and only if $(m \geq 2 \wedge n = 1)$. Hence, $(\mathfrak{R},\perp,\tau)$ is an $\mathcal{O}$-complete OF-MS. Define $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ by

$$\mathscr{V}(\mu_n) = \begin{cases} \mu_1, \text{ if } n = 1, \\ \mu_{n-1}, \text{ if } n > 1. \end{cases} \tag{41}$$

Clearly,

$$\lim_{n\longrightarrow\infty} \frac{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_1))}{\tau(\mu_n, \mu_1)} = 1, \tag{42}$$

then $\mathscr{V}$ is not a contraction in the sense of [18].

It is easy to check that $\mathscr{V}$ is $\perp$-continuous and $\mathscr{V}$ is $\perp$-preserving. Let the mapping $\mathfrak{J} : (0,\infty) \longrightarrow \mathbb{R}$ defined by

$$\mathfrak{J}(t) = \ln t + t, t > 0. \tag{43}$$

It is easy to show that $\mathfrak{J} \in \Psi$. Now, to prove $\mathscr{V}$ is an $\perp_{\mathscr{F}}$-contraction, that is,

$$\begin{aligned}\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow \beth + \ln \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \\ + \tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \\ \leq \ln \tau(\mu_n, \mu_m) + \tau(\mu_n, \mu_m),\end{aligned} \tag{44}$$

for $\beth>0$. The above condition is equivalent to

$$\begin{aligned}\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow e^{\beth + \ln \tau(\mathscr{V}(\mu_n),\mathscr{V}(\mu_m)) + \tau(\mathscr{V}(\mu_n),\mathscr{V}(\mu_m))} \\ \leq e^{\ln \tau(\mu_n,\mu_m) + \tau(\mu_n,\mu_m)}.\end{aligned} \tag{45}$$

So, we have to check that

$$\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m)) \neq 0 \Longrightarrow \frac{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m))}{\tau(\mu_n, \mu_m)} e^{\tau(\mathscr{V}(\mu_n),\mathscr{V}(\mu_m)) - \tau(\mu_n,\mu_m)} \leq e^{-\beth}. \tag{46}$$

For every $m \in \mathbb{N}, m \geq 2$, we have

$$\begin{aligned}&\frac{\tau(\mathscr{V}(\mu_m), \mathscr{V}(\mu_1))}{\tau(\mu_m, \mu_1)} e^{\tau(\mathscr{V}(\mu_m),\mathscr{V}(\mu_1)) - \tau(\mu_m,\mu_1)} \\ &= \frac{\tau(\mu_{m-1}, \mu_1)}{\tau(\mu_m, \mu_1)} e^{\tau(\mu_{m-1},\mu_1) - \tau(\mu_m,\mu_1)} \\ &= \frac{2^{\mu_{m-1}-\mu_1}}{2^{\mu_m-\mu_1}} e^{2^{\mu_{m-1}-\mu_1} - 2^{\mu_m-\mu_1}} \\ &= \frac{(m-1)!}{m!} e^{-(m-1)(m+1)!} < e^{-1}.\end{aligned} \tag{47}$$

For every $m \in \mathbb{N}, m > n > 1$, we have

$$\begin{aligned}&\frac{\tau(\mathscr{V}(\mu_n), \mathscr{V}(\mu_m))}{\tau(\mu_n, \mu_m)} e^{\tau(\mathscr{V}(\mu_n),\mathscr{V}(\mu_m)) - \tau(\mu_n,\mu_m)} \\ &= \frac{\tau(\mu_{n-1}, \mu_{m-1})}{\tau(\mu_n, \mu_m)} e^{\tau(\mu_{n-1},\mu_{m-1}) - \tau(\mu_n,\mu_m)} \\ &= \frac{2^{\mu_{m-1}-\mu_{n-1}}}{2^{\mu_m-\mu_n}} e^{2^{\mu_{m-1}-\mu_{n-1}} - 2^{\mu_m-\mu_n}} \\ &= \frac{(m-1)!}{(n-1)!} \times \frac{(n)!}{(m)!} e^{((m-1)!/(n-1)!) - ((n)!/(m)!)} \\ &= \frac{n}{m} e^{(((m-1)!)/((n-1)!))((n-m)/n)} < e^{-1}.\end{aligned} \tag{48}$$

Thus, the inequality (8) is satisfied with $\beth = 1 > 0$. Thus, Corollary 13 implies that $\mu = \log_2 1$ is a unique fixed point of $\mathscr{V}$.

Theorem 14. *Let $(\mathfrak{R}, \tau)$ be an $\mathscr{F}$-complete $\mathscr{F}$-metric space, $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ be an α-admissible mapping and there exist $\mathfrak{J} \in \Psi$ and $\alpha : \mathfrak{R} \times \mathfrak{R} \longrightarrow 1,\infty)$ such that*

$$\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma) \neq 0 \Longrightarrow \beth + \alpha(\hbar, \varsigma)\mathfrak{J}(\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma)) \leq \mathfrak{J}(\tau(\hbar, \varsigma)), \tag{49}$$

for all $\hbar, \varsigma \in \mathfrak{R}$ and $\beth>0$. Assume that there exists $\hbar_0 \in \mathfrak{R}$ such that $\alpha(\hbar_0, \mathscr{V}\hbar_0) \geq 1$, then, there exists $\hbar^* \in \mathfrak{R}$ such that $\mathscr{V}\hbar^* = \hbar^*$. Furthermore, if $\mathfrak{R}$ has the property (OH), then this fixed point is unique.

Proof. Suppose that

$$\hbar \perp \varsigma \text{ if and only if } \tau(\mathscr{V}\hbar, \mathscr{V}\varsigma) \neq 0. \tag{50}$$

Fix $\hbar_0 \in \mathfrak{R}$. Since $\mathscr{V}$ satisfies the inequality (49), for all $\varsigma \in \mathfrak{R}, \hbar_0 \perp \varsigma$. This implies that $(\mathfrak{R}, \perp)$ is $\mathcal{O}$-set. Now, it is obviously that $\mathfrak{R}$ is O-complete and $\mathscr{V}$ is $\perp$-continuous and $\perp$-preserving. Thu, by applying Theorem 12, $\mathscr{V}$ has a unique fixed point in $\mathfrak{R}$.

□

Corollary 15 (see [30]). *Let $(\mathfrak{R}, \tau)$ be an $\mathscr{F}$-complete $\mathscr{F}$-metric space, $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ be an α-admissible mapping and there exist $\mathfrak{J} \in \Psi$ such that*

$$\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma) \neq 0 \Longrightarrow \beth + \mathfrak{J}(\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma)) \leq \mathfrak{J}(\tau(\hbar, \varsigma)), \tag{51}$$

for all $\hbar, \varsigma \in \mathfrak{R}$ and $\beth>0$, then, there exists unique $\hbar^* \in \mathfrak{R}$ such that $\mathscr{V}\hbar^* = \hbar^*$.

If we take $\xi(t) = \ln(t)$, for $t > 0$ and $\alpha = 1$ in Definition 5, then OF-MS reduces to $\mathcal{O}$-metric space and we get the following result.

Theorem 16. *Let $(\mathfrak{R},\perp,\tau)$ be an $\mathcal{O}$-complete metric space and $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ is an $(\alpha, \perp_{\mathscr{F}})$-contraction such that following assertions hold:*

(i) $\mathscr{V}$ is $\perp$-preserving, that is, $\hbar\perp\varsigma$ implies $\mathscr{V}\hbar\perp\mathscr{V}\varsigma$

(ii) $\mathscr{V}$ is orthogonally α-admissible

(iii) there exists $\hbar_0 \in \mathfrak{R}$ such that $\hbar_0\perp\mathscr{V}\hbar_0$ and $\alpha(\hbar_0, \mathscr{V}\hbar_0) \geq 1$

(iv) $\mathscr{V}$ is $\perp$-continuous

Then, there exists $\hbar^* \in \mathfrak{R}$ such that $\mathscr{V}\hbar^* = \hbar^*$. Furthermore, if $\mathfrak{R}$ has the property (OH), then this fixed point is unique.

If we take $\alpha(\hbar, \varsigma) = 1$, for all $\hbar, \varsigma \in \mathfrak{R}$ in Theorem 16, then we get the following result of Mani et al. [31].

Corollary 17. *Let $(\mathfrak{R},\perp,\tau)$ be an $\mathcal{O}$-complete metric space, $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ and there exist $\mathfrak{I} \in \Psi$ such that*

$$\begin{aligned} &\text{for all } \hbar, \varsigma \in \mathfrak{R}, \hbar\perp\varsigma, \tau(\mathscr{V}\hbar, \mathscr{V}\varsigma) \neq 0 \\ &\quad\Longrightarrow \beth+\alpha(\hbar, \varsigma)\mathfrak{I}(\tau(\mathscr{V}\hbar, \mathscr{V}\varsigma)) \leq \mathfrak{I}(\tau(\hbar, \varsigma)). \end{aligned} \tag{52}$$

Assume that the following conditions hold:

(i) $\mathscr{V}$ is $\perp$-preserving, that is, $\hbar\perp\varsigma$ implies $\mathscr{V}\hbar\perp\mathscr{V}\varsigma$

(ii) $\mathscr{V}$ is orthogonally α-admissible

(iii) there exists $\hbar_0 \in \mathfrak{R}$ such that $\hbar_0\perp\mathscr{V}\hbar_0$ and $\alpha(\hbar_0, \mathscr{V}\hbar_0) \geq 1$

(iv) $\mathscr{V}$ is $\perp$-continuous

Then, there exists $\hbar^* \in \mathfrak{R}$ such that $\mathscr{V}\hbar^* = \hbar^*$. Furthermore, if $\mathfrak{R}$ has the property (OH), then this fixed point is unique.

2.1. Periodic Point Result. Let $(\mathfrak{R},\perp,\tau)$ be an OF-MS and $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ such that $\hbar = \mathscr{V}\hbar$, then $\hbar = \mathscr{V}^n\hbar$, for every $n \in \mathbb{N}$. Although, the converse of this reality is not correct generally. The mapping $\mathscr{V} : \mathfrak{R} \longrightarrow \mathfrak{R}$ fulfilling

$$\mathrm{Fix}(\mathscr{V}) = \mathrm{Fix}(\mathscr{V}^n), \tag{53}$$

for all $n \in \mathbb{N}$ is called to have property P.

Definition 18. Let $(\mathfrak{R},\perp,\tau)$ be an OF-MS and $\mathscr{V}:\mathfrak{R} \longrightarrow \mathfrak{R}$. The set

$$O(\hbar) = \left\{\hbar, \mathscr{V}\hbar, \mathscr{V}^2\hbar, \cdots, \mathscr{V}^n\hbar, \cdots\right\}, \tag{54}$$

is said to be an orbit of $\mathfrak{R}$. A mapping $\mathscr{V} : (\mathfrak{R},\perp,\tau) \longrightarrow (\mathfrak{R},\perp,\tau)$ is said to be an orbitally $\mathcal{O}$-continuous at $\hbar$ if for each $\mathcal{O}$-sequence $\{\mathscr{V}^n\hbar\}$ in $\mathfrak{R}$, $\lim_{n\longrightarrow\infty}\mathscr{V}^n\hbar = \varsigma \Longrightarrow \lim_{n\longrightarrow\infty}\mathscr{V}^{n+1}\hbar = \mathscr{V}\varsigma$. A mapping $\mathscr{V}:\mathfrak{R} \longrightarrow \mathfrak{R}$ is orbitally continuous on $\mathfrak{R}$ if $\mathscr{V}$ is orbitally $\mathcal{O}$-continuous at all $\hbar \in \mathfrak{R}$.

Theorem 19. *Let $(\mathfrak{R},\perp,\tau)$ be an $\mathcal{O}$-complete OF-MS and $\mathscr{V} : (\mathfrak{R},\perp,\tau) \longrightarrow (\mathfrak{R},\perp,\tau)$. Suppose that there exist $\mathfrak{I} \in \Psi$ and $\beth>0$ such that*

$$\begin{aligned} &\text{for all } \hbar \in \mathfrak{R} \text{ with } \mathscr{V}\hbar\perp\mathscr{V}^2\hbar\left[\tau\left(\mathscr{V}\hbar, \mathscr{V}^2\hbar\right)\right] > 0, \\ &\beth+\mathfrak{I}\left(\tau\left(\mathscr{V}\hbar, \mathscr{V}^2\hbar\right)\right) \leq \mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)), \end{aligned} \tag{55}$$

and $\mathscr{V}$ is $\perp$-preserving. Then, $\mathscr{V}$ has the property P supplied that $\mathscr{V}$ is orbitally continuous on $\mathfrak{R}$.

Proof. Let $\varepsilon > 0$ be fixed and there exists $\xi : (0,+\infty) \longrightarrow \mathbb{R}$ such that (D_3) is satisfied for $\alpha \geq 0$. Then, by $(\mathscr{F}_2)$, there exists $\delta > 0$ such that

$$0 < \hbar < \delta \text{ implies } \xi(\hbar) < \xi(\varepsilon) - a. \tag{56}$$

Now, we manifest that $\mathrm{Fix}(\mathscr{V}) \neq \varnothing$. Define $\{\hbar_n\}$ in $\mathfrak{R}$ such that $\hbar_{n+1} = \mathscr{V}^n\hbar$. If there exists $n_0 \in \mathbb{N}$, such that $\hbar_{n_0+1} = \hbar_{n_0}$, then $\hbar_{n_0}$ is a fixed point of $\mathscr{V}$. Now, we assume that $\tau(\hbar_n, \hbar_{n+1}) > 0$, for all $n \in \mathbb{N}$. Using inequality (14), we get

$$\begin{aligned} \beth+\mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) &= \beth+\mathfrak{I}\left(\tau\left(\mathscr{V}\hbar_{n-1}, \mathscr{V}^2\hbar_{n-1}\right)\right) \\ &\leq \mathfrak{I}(\tau(\hbar_{n-1}, \mathscr{V}\hbar_{n-1})). \end{aligned} \tag{57}$$

This implies that

$$\begin{aligned} \mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) &\leq \mathfrak{I}(\tau(\hbar_{n-1}, \mathscr{V}\hbar_{n-1}))-\beth \\ &\leq \mathfrak{I}(\tau(\hbar_{n-2}, \mathscr{V}\hbar_{n-2})) - 2\beth\cdots \\ &\leq \mathfrak{I}(\tau(\hbar_0, \hbar_1)) - n\beth. \end{aligned} \tag{58}$$

Thus, we have

$$\mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) \leq \mathfrak{I}(\tau(\hbar_0, \hbar_1)) - n\beth, \tag{59}$$

for all $n \in \mathbb{N}$. Now, by applying the limit as $n \longrightarrow \infty$ and using $(\mathscr{F}_2)$, we have

$$\lim_{n\longrightarrow\infty} \mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) = -\infty \Leftrightarrow \lim_{n\longrightarrow\infty} \tau(\hbar_n, \hbar_{n+1}) = 0. \tag{60}$$

From the condition $(\mathscr{F}_3)$, there exists $0 < r < 1$ such that

$$\lim_{n\longrightarrow\infty} [\tau(\hbar_n, \hbar_{n+1})]^r \mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) = 0. \tag{61}$$

From (59) and (61), we have

$$\begin{aligned} &[\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{I}(\tau(\hbar_n, \hbar_{n+1})) - [\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{I}(\tau(\hbar_0, \hbar_1)) \\ &\quad\leq [\tau(\hbar_n, \hbar_{n+1})]^r[\mathfrak{I}(\tau(\hbar_0, \hbar_1)) - n\tau] - [\tau(\hbar_n, \hbar_{n+1})]^r\mathfrak{I}(\tau(\hbar_0, \hbar_1)) \\ &\quad\leq -n\tau s^n d_b(\hbar_n, \hbar_{n+1})]^r \leq 0. \end{aligned} \tag{62}$$

Taking $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} n\tau(\hbar_n, \hbar_{n+1})^r = 0. \tag{63}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\tau(\hbar_n, \hbar_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{64}$$

for all $n > n_1$. This yields

$$\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}, \tag{65}$$

for $m > n$. Since $\sum_{i=n}^{\infty} 1/i^{1/r}$ is convergent, so there exists $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} < \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} < \delta, \tag{66}$$

for $n > N$. Hence, by (64) and $(\mathscr{F}_1)$, we get

$$\xi\left(\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1})\right) \leq \xi\left(\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}\right) < \xi(\varepsilon) - a, \tag{67}$$

$m > n \geq N$. Using (D_3) and (67), we get

$$\begin{aligned} \tau(\hbar_n, \hbar_m) > 0, m > n \geq N \Longrightarrow \xi(\tau(\hbar_n, \hbar_m)) \\ \leq \xi\left(\sum_{i=n}^{m-1} \tau(\hbar_i, \hbar_{i+1})\right) + a < \xi(\varepsilon), \end{aligned} \tag{68}$$

which, from $(\mathscr{F}_1)$, gives that

$$\tau(\hbar_n, \hbar_m) < \varepsilon, m > n \geq N. \tag{69}$$

Hence, $\{\hbar_n\}$ is a Cauchy $\mathcal{O}$-sequence in $(\mathfrak{R}, \perp, \tau)$. As $\{\mathscr{V}^n \hbar_0 : n \in \mathbb{N}\} \subseteq O(\hbar_0) \subseteq \mathfrak{R}$ and $\mathfrak{R}$ is $\mathcal{O}$-complete, there exists $\hbar \in \mathfrak{R}$ such that

$$\lim_{n\longrightarrow\infty} \mathscr{V}^n \hbar_0 = \hbar. \tag{70}$$

Now, by the orbital $\perp$-continuity of $\mathscr{V}$, we have

$$\hbar = \lim_{n\longrightarrow\infty} \mathscr{V}^{n-1} \hbar_0 = \mathscr{V} \hbar. \tag{71}$$

Thus, $\mathscr{V}$ has a fixed point and $\mathrm{Fix}(\mathscr{V}^n) = \mathrm{Fix}(\mathscr{V})$ is true for $n = 1$. Suppose $n > 1$. We assume on the contrary that $\hbar \in \mathrm{Fix}(\mathscr{V}^n)$ but $\hbar \notin \mathrm{Fix}(\mathscr{V})$, then $\tau(\hbar, \mathscr{V}\hbar) = \omega > 0$. Now,

$$\begin{aligned} \mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)) &= \mathfrak{I}\left(\tau\left(\mathscr{V}\left(\mathscr{V}^{n-1}\hbar\right), \mathscr{V}^2\left(\mathscr{V}^{n-1}\hbar\right)\right)\right) \\ &\leq \mathfrak{I}\left(\tau\left(\mathscr{V}^{n-1}\hbar, \mathscr{V}^n\hbar\right)\right) - \daleth \\ &\leq \mathfrak{I}\left(\tau\left(\mathscr{V}^{n-2}\hbar, \mathscr{V}^{n-1}\hbar\right)\right) - 2\daleth \cdots \\ &\leq \mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)) - n\daleth. \end{aligned} \tag{72}$$

Thus, we have

$$\mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)) \leq \mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)) - n\daleth, \tag{73}$$

for all $n \in \mathbb{N}$. Taking $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} \mathfrak{I}(\tau(\hbar, \mathscr{V}\hbar)) = -\infty, \tag{74}$$

which, from $(\mathscr{F}_2)$ we get

$$\tau(\hbar, \mathscr{V}\hbar) = 0, \tag{75}$$

which is contradiction. So, $\hbar \in \mathscr{V}\hbar$.

□

3. Applications

Consider a nonlinear differential equation of fractional order

$$^{C}D^{\eta}(\hbar(t)) = f(t, \hbar(t)). \tag{76}$$

$(0 < t < 1, 1 < \eta \leq 2)$ via the integral boundary conditions

$$\hbar(0) = 0, \hbar(1) = \int_0^{\pi} \hbar(s)ds, (0 < \pi < 1), \tag{77}$$

where $^{C}D^{\eta}$ represents the Caputo fractional derivative of order η given by

$$^{C}D^{\eta}f(t) = \frac{1}{\Gamma(j-\eta)} \int_0^t (t-s)^{j-\eta-1} f^j(s)ds, \tag{78}$$

$(j - 1 < \eta < j, j = [\eta] + 1)$, where $[\eta]$ represents the integer part of the real number η and $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. We take

$$\mathfrak{R} \ll \{\hbar : \hbar \in C([0, 1], \mathbb{R})\}, \tag{79}$$

with supremum norm $\|\hbar\|_{\infty} = \sup_{t\in 0,1]} |\hbar(t)|$. Thus, $(\mathfrak{R}, \|\hbar\|_{\infty})$ is a Banach space. Recall, the Riemann–Liouville fractional integral of order η is given by

$$I^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s)ds, \text{with } \eta > 0. \tag{80}$$

Lemma 20 (see [15]). *The Banach space $(\mathfrak{R}, \|\cdot\|_{\infty})$ equipped with the $\mathscr{F}$-metric d given by*

$$d(\hbar, \varsigma) = \|\hbar - \varsigma\|_{\infty} = \sup_{t\in 0,1]} |\hbar(t) - \varsigma(t)|, \tag{81}$$

and orthogonal relation $\hbar \perp \varsigma \Leftrightarrow \hbar\varsigma \geq 0$, where $\hbar, \varsigma \in \mathfrak{R}$, is an orthogonal $\mathscr{F}$-metric space.

Theorem 21. *Suppose that*

(i) there exists $\daleth \in 1, \infty)$ such that

$$|f(t,\hbar)-f(t,\varsigma)| \le \frac{\Gamma(\eta+1)}{5} e^{-\mathfrak{D}}|\hbar-\varsigma|, \tag{82}$$

for all $t \in 0, 1]$

(ii) there exists $\mathscr{L} : (\mathfrak{R},\perp,d) \longrightarrow (\mathfrak{R},\perp,d)$ is defined by

$$\begin{aligned}\mathscr{V}\hbar(t) = {} & \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\hbar(s))ds \\ & - \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} f(s,\hbar(s))ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\hbar(m))dm\right)ds,\end{aligned} \tag{83}$$

for all $\hbar, \varsigma \in \mathfrak{R}$ such that $\hbar(t)\varsigma(t) \ge 0$,where $(0 < \lambda < 1)$. Then, (76) has at least one solution

Proof. It is well known that $\hbar \in \mathfrak{R}$ is a solution of (76) iff $\hbar \in \mathfrak{R}$ is a solution of the integral equation

$$\begin{aligned}\hbar(t) = {} & \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\hbar(s))ds \\ & - \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} f(s,\hbar(s))ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\hbar(m))dm\right)ds.\end{aligned} \tag{84}$$

Then, problem (76) is equivalent to find $\hbar \in \mathfrak{R}$ which is a fixed point of V . Suppose that a relation on $\mathfrak{R}$, by

$$\hbar \perp \varsigma \Leftrightarrow \hbar(t)\varsigma(t) \ge 0, \tag{85}$$

for all $t \in 0, 1]$. With this relation, $\mathfrak{R}$ is orthogonal because for all $\hbar \in \mathfrak{R}$, there exists $\varsigma(t) = 0$, for all $t \in 0, 1]$ such that $\hbar(t)\varsigma(t) = 0$. We examine

$$d(\hbar,\varsigma) = \|\hbar-\varsigma\|_\infty = \sup_{t\in 0,1]} |\hbar(t)-\varsigma(t)|, \tag{86}$$

for all $\hbar, \varsigma \in \mathfrak{R}$. So, the triplet $(\mathfrak{R},\perp,d)$ is a complete OF-MS. It is clear from the definition that $\mathscr{V}$ is $\perp$-continuous. We first prove that $\mathscr{V}$ is $\perp$-preserving. Let $\hbar(t)\perp\varsigma(t)$, for all $t \in 0, 1]$. Now, we get

$$\begin{aligned}\mathscr{V}\hbar(t) = {} & \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\hbar(s))ds \\ & - \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} f(s,\hbar(s))ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\hbar(m))dm\right)ds > 0,\end{aligned} \tag{87}$$

which yields that $\mathscr{V}\hbar(t)\perp\mathscr{V}\varsigma(t)$, i.e., $\mathscr{V}$ is $\perp$-preserving. Next, for all $t \in 0, 1]$,$\hbar(t)\perp\varsigma(t)$, we have

$$\begin{aligned}|\mathscr{V}\hbar(t)-\mathscr{V}\varsigma(t)| = {} & \left| \begin{array}{l} \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\hbar(s))ds - \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} f(s,\hbar(s))ds \\ \quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\hbar(m))dm\right)ds \\ - \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\varsigma(s))ds + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} f(s,\varsigma(s))ds \\ \quad - \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\varsigma(m))dm\right)ds \end{array} \right| \\ \le {} & \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1} |f(s,\hbar(s))-f(s,\varsigma(s))|ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} |f(s,\hbar(s))-f(s,\kappa(s))|ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} |f(m,\varsigma(m))-f(m,\hbar(m))|dm\right)ds,\end{aligned} \tag{88}$$

which implies that

$$
\begin{aligned}
&|\mathcal{J}\hbar(t)-\mathcal{J}\varsigma(t)| \\
&\quad\le \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1}\frac{\Gamma(\eta+1)}{5}e^{-\pi}|\hbar(s)-\varsigma(s)|ds \\
&\qquad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1}\frac{\Gamma(\eta+1)}{5}e^{-\pi}|\hbar(s)-\varsigma(s)|ds \\
&\qquad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s |s-m|^{\eta-1}\frac{\Gamma(\eta+1)}{5}e^{-\pi}|\varsigma(m)-\hbar(m)|dm\right)ds \\
&\quad\le \frac{\Gamma(\eta+1)}{5}e^{-\eth}\|\hbar(s)-\varsigma(s)\|_\infty \sup_{t\in(0,1)}\begin{pmatrix} \dfrac{1}{\Gamma(\eta)}\displaystyle\int_0^t |t-s|^{\eta-1}ds \\ +\dfrac{2t}{(2-\lambda^2)\Gamma(\eta)}\displaystyle\int_0^1 (1-s)^{\eta-1}ds \\ +\dfrac{2t}{(2-\lambda^2)\Gamma(\eta)}\displaystyle\int_0^\lambda\int_0^s |s-m|^{\eta-1}dmds \end{pmatrix} \\
&\quad\le e^{-\eth}\|\hbar-\varsigma\|_\infty.
\end{aligned} \tag{89}
$$

From the above inequality, we obtain that

$$d(\mathcal{V}\hbar,\mathcal{V}\varsigma)\le e^{-\eth}d(\hbar,\varsigma). \tag{90}$$

Taking natural log function on both sides, we have

$$\ln\left(d(\mathcal{V}\hbar,\mathcal{V}\varsigma)\right)\le \ln\left(e^{-\eth}d(\hbar,\varsigma)\right), \tag{91}$$

that is

$$\ln\left(d(\mathcal{V}\hbar,\mathcal{V}\varsigma)\right)\le \ln\left(e^{-\eth}\right)+\ln\left(d(\hbar,\varsigma)\right), \tag{92}$$

that is

$$\ln\left(d(\mathcal{V}\hbar,\mathcal{V}\varsigma)\right)\le -\eth+\ln\left(d(\hbar,\varsigma)\right). \tag{93}$$

Thus,

$$\eth+\ln\left(d(\mathcal{V}\hbar,\mathcal{V}\varsigma)\right)\le \ln\left(d(\hbar,\varsigma)\right), \tag{94}$$

where $\eth>1$. Now, consider $\mathfrak{J}:\mathbb{R}^+\longrightarrow\mathbb{R}$ given by $\mathfrak{J}(t)=\ln(t)$ for each $t>0$, then $\mathfrak{J}\in\Psi$. Thus,

$$\eth+\mathfrak{J}(d(\mathcal{V}\hbar,\mathcal{V}\varsigma))\le \mathfrak{J}(d(\hbar,\varsigma)), \tag{95}$$

for all $\hbar,\varsigma\in\mathfrak{R}$ and $d(\mathcal{V}\hbar,\mathcal{V}\varsigma)>0$.

Now, let $(\hbar_n)$ be a Cauchy O-sequence converging in $\mathfrak{R}$. Thus, we get $\hbar_n(t)\hbar_{n+1}(t)\ge 0$, for all $t\in 0,1]$ and for $n\in\mathbb{N}$. □

We will have two possible cases to discuss:

Case 1. If $\hbar_n(t)\ge 0$, for all $n\in\mathbb{N}$ and $t\in 0,1]$. Then, for all $t\in 0,1]$, there exists a sequence of positives that converges to $\hbar(t)$. Thus, we obtain $\hbar(t)\ge 0$, for all $t\in 0,1]$, i.e., $\hbar_n(t)\perp\hbar(t)$, for all $n\in\mathbb{N}$ and $t\in 0,1]$.

Case 2. If $\hbar_n(t)\le 0$, for all $n\in\mathbb{N}$, has to be thrown out. Hence, by Corollary 13, Equation (76) has a unique solution.

4. Conclusion

In this article, we introduced the notion of $(\alpha,\perp_{\mathcal{F}})$-contraction in the context of orthogonal $\mathcal{F}$-metric space and established some new fixed point theorems in this newly introduced space. We have given soeme examples to manifest the authenticity of the obtained results. As application of our foremost result, we looked into the solution of a nonlinear fractional differential equation.

In this direction, our future work will focus on studying the fixed points for $(\alpha,\perp_{\mathcal{F}})$-contraction in the context of orthogonal $\mathcal{F}$-metric space endowed with a graph.

Data Availability

No such data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[2] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Universitatis Debreceniensis*, vol. 57, pp. 31–37, 2000.

[3] M. Jleli and B. Samet, "A generalized metric space and related fixed point theorems," *Fixed point theory and Applications*, vol. 2015, no. 1, p. 14, 2015.

[4] M. Jleli and B. Samet, "A new generalization of the Banach contraction principle," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 2014.

[5] L. A. Alnaser, D. Lateef, H. A. Fouad, and J. Ahmad, "Relation theoretic contraction results in $\mathcal{F}$-metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 12, no. 5, pp. 337–344, 2019.

[6] S. A. Al-Mezel, J. Ahmad, and G. Marino, "Fixed point theorems for generalized $(\alpha\beta-\psi)$-contractions in $\mathcal{F}$-metric spaces with applications," *Mathematics*, vol. 8, no. 4, 2020.

[7] M. Alansari, S. S. Mohammed, and A. Azam, "Fuzzy fixed point results in $\mathcal{F}$-metric spaces with applications," *Journal of Function Spaces*, vol. 2020, Article ID 5142815, 11 pages, 2020.

[8] D. Lateef and J. Ahmad, "Dass and Gupta's fixed point theorem in $\mathcal{F}$-metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 12, no. 6, pp. 405–411, 2019.

[9] A. Hussain and T. Kanwal, "Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results," *Transactions of A. Razmadze Mathematical Institute*, vol. 172, no. 3, pp. 481–490, 2018.

[10] M. E. Gordji, D. Rameani, M. De La Sen, and Y. J. Cho, "On orthogonal sets and Banach fixed point theorem," *Fixed Point Theory and Applications*, vol. 18, no. 2, p. 578, 2017.

[11] Z. Ahmadi, R. Lashkaripour, and H. Baghani, "A fixed point problem with constraint inequalities via a contraction in incomplete metric spaces," *Filomat*, vol. 32, no. 9, pp. 3365–3379, 2018.

[12] H. Baghani and M. Ramezani, "Coincidence and fixed points for multivalued mappings in incomplete metric spaces with applications," *Univerzitet u Nišu*, vol. 33, no. 1, pp. 13–26, 2019.

[13] A. C. M. Ran and M. C. B. Reuring, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[14] K. Javed, H. Aydi, F. Uddin, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy *b*-metric spaces with an application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.

[15] T. Kanwal, A. Hussain, H. Baghani, and M. De La Sen, "New fixed point theorems in orthogonal F -metric spaces with application to fractional differential equation," *Symmetry*, vol. 12, no. 5, 2020.

[16] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α -ψ-contractive type mappings," *Nonlinear Analysis*, vol. 75, no. 4, pp. 2154–2165, 2012.

[17] M. Ramezani, "Orthogonal metric space and convex contractions," *International Journal of Nonlinear Analysis and Applications*, vol. 6, pp. 127–132, 2015.

[18] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed point Theory and Applications*, vol. 2012, no. 1, 2012.

[19] N. A. Secelean, "Iterated function systems consisting of F-contractions," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[20] H. Piri and P. Kumam, "Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 2014.

[21] O. Popescu and G. Stan, "Two fixed point theorems concerning F-contraction in complete metric spaces," *Symmetry*, vol. 12, no. 1, 2020.

[22] M. Abbas, B. Ali, and S. Romaguera, "Coincidence points of generalized multivalued (f, L) -almost F -contraction with applications," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 919–934, 2015.

[23] M. Abbas, L. Ćirić, B. Damjanović, and M. A. Khan, "Coupled coincidence and common fixed point theorems for hybrid pair of mappings," *Fixed Point Theory and Applications*, vol. 2012, no. 1, p. 11, 2012.

[24] J. Ahmad, J. Al-Rawashdeh, and A. Azam, "New fixed point theorems for generalized F-contractions in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2015, no. 1, 2015.

[25] N. Hussain, J. Ahmad, and A. Azam, "On Suzuki-Wardowski type fixed point theorems," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1095–1111, 2015.

[26] M. A. Kutbi, J. Ahmad, A. Azam, and A. S. Al-Rawashdeh, "Generalized common fixed point results via greatest lower bound property," *Journal of Applied Mathematics*, vol. 2014, Article ID 265865, 11 pages, 2014.

[27] A. Latif, M. Abbas, and A. Hussain, "Coincidence best proximity point of F_g -weak contractive mappings in partially ordered metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 5, pp. 2448–2457, 2016.

[28] H. K. Nashine, D. Gopal, D. Jain, and A. Al-Rawashdeh, "Solutions of initial and boundary value problems via F-contraction mappings in metric-like space," *International Journal of Nonlinear Analysis and Applications*, vol. 9, pp. 129–145, 2018.

[29] H. K. Nashine, Z. Kadelburg, and R. P. Agarwal, "Existence of solutions of integral and fractional differential equations using α-type rational F-contractions in metric-like spaces," *Kyungpook Mathematical Journal*, vol. 58, pp. 651–675, 2018.

[30] A. Asif, M. Nazam, M. Arshad, and S. O. Kim, "F-Metric, F -contraction and common fixed-point theorems with applications," *Mathematics*, vol. 7, no. 7, 2019.

[31] G. Mani, A. J. Gnanaprakasam, N. Kausar, and M. Munir, "Orthogonal F-contraction mapping on O-complete metric space with applications," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 21, no. 3, pp. 243–250, 2021.