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Research Article

Schur, Hermite-Hadamard, and Fejér Type Inequalities for the Class of Higher-Order Generalized Convex Functions

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The study of convex functions is an interesting area of research due to its huge applications in pure and applied mathematics special in optimization theory. The aim of this paper is to introduce and study a more generalized class of convex functions. We established Schur (S), Hermite-Hadamard (HH), and Fejér (F) type inequalities for introduced class of convex functions. The results presented in this paper extend and generalize many existing results of the literature.

1. Introduction

The study of convexity is very simple and ordinary concept and very important due to its massive applications in industry and business; convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. Problems faced in constrained control and estimation are convex. Geometrically, "the convex function is a real-valued function if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space" [1, 2].

The inequality theory is a best way to understand and apply convexity and the following HH inequality for a C-function g is most famous and most studied inequality in recent years;

$$g\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx \leq \frac{g(a_1)+g(a_2)}{2}. \quad (1)$$

There are many interesting generalizations for HH inequality, for example, see [3, 4]. Alp et al. [5] established q–HH inequalities and gave quantum estimates for midpoint type inequalities via convex and quasi–C-functions. Rahman et al. [6] presented certain fractional proportional integral inequalities via C-functions. Liu et al. [7] established the several HH type inequality via C-function for ψ–Riemann–Liouville fractional integrals. In [8], Farid et al. established certain inequalities for Riemann–Liouville fractional integrals in more general form via C-functions and proposed some applications of their established results. Tunc [9] studied h–C-functions and presented Ostrowski-type inequalities via h–C-functions. He also proposed applications of his established Ostrowski-type inequalities to special means. Set et al. [10] established the HH's inequality for some C-functions via fractional integrals. They also presented some other resulted results in the setting of fractional integrals.

Moreover, Avci, Kavurmaci, and Özdemir in [11] established the HH type for the class of s–C-functions in 2nd sense which is more generalized convexity. The exponentially m–C-functions were investigated by Rashid et al., in [12]. Different kinds of inequalities for exponentially m–C-functions were also established in the same paper. In [13], Baloch and Chu investigated harmonic C-functions and established Petrovic-type inequalities. Sun and Chu [14] studied g–C-functions and established inequalities for the generalized weighted mean values. Ali, Khan, and Khurshidi [15] investigated η–C-functions and established HH inequality for fractional integrals. Chu et al. [16] generalized the HH type inequalities for MT–C-functions and a variant of Jensen-type inequality for harmonic C-functions was given in [17].

The strongly C-function is one of the important functions of convexity, see [4, 18–21]. A function $g : I \longrightarrow \mathbb{R}$ is strongly C-function if

$$g(ta_1 + (1-t)a_2) \leq tg(a_1) + (1-t)g(a_2) - \mu t(1-t)(a_1 - a_2)^2 \tag{2}$$

holds $\forall a_1, a_2 \in L, t \in [0,1]$ and μ is modulus value.

Convexity is being generalized day by day using different methods, In this work, we discuss the convexity of higher order. For the generalization and different application of convexity, see for instance [22–24].

Convex function $g : I = [a_1, a_2] \subset \mathbb{R} \longrightarrow \mathbb{R}$ in the classical sense is introduced as

$$g(ta_1 + (1-t)a_2) \leq tg(a_1) + (1-t)g(a_2), \tag{3}$$

where $a_1, a_2 \in I$ and $t \in [0, 1]$.

A function g on close set L is said to be a higher-order strongly convex, if there exists a constant $\mu \geq 0$, such that

$$g(ta_1 + (1-t)a_2) \leq tg(a_1) + (1-t)g(a_2) - \mu\phi(t)\|a_2 - a_1\|^q, \tag{4}$$

where $a_1, a_2 \in L$, $q \geq 0$, $t \in [0, 1]$ and

$$\phi(t) = t(1-t). \tag{5}$$

If we put $q = 2$, then (4) becomes strongly C-function in classical sense with same $\phi(t)$ as defined in (5).

It is noticed that the function $\phi(.)$ in (5) is not modified. Characterizations of the higher-order strongly C-function discussed in [25] are not correct. Mohsen et al. [26] modify $\phi(.)$ for the higher-order strongly C-function. Noor and Noor [27] generalized the concept of higher-order strongly convex defined in [26] by higher-order strongly generalized C-functions. For more detailed survey, we refer the readers to [28] and references therein.

Saleem et al. [29] extended the concept of higher-order strongly generalized C-function by introducing generalized strongly p–C-functions of higher order. The generalized strongly modified (p, h)–C-functions of higher order are studied in this paper.

In this paper, we introduced the generalized strongly modified $(p\,; h)$–C-functions of higher order and establish some well-known inequalities. Motivation of this work is to improve and extend the existing results. As stated in the remarks of this paper, many existing results can be derived from our results.

This paper is organized as follows: Section 2 contains definitions, Section 3 contains some basic results, and Section 4 contains HH, F, and S type inequalities for the generalized strongly modified (p, h)–C-functions of higher order.

2. Definitions and Basic Results

The first part of this paper contains basic definitions of convexity.

Definition 1 (Additive). A function η is additive when $\eta(x_1, y_1) + \eta(x_2, y_2) = \eta(x_1 + x_2, y_1 + y_2) \forall x_1, x_2, y_1, y_2 \in \mathbb{R}$, see [30] for detail.

Definition 2 (Super multiplicative function) [31]. A mapping $g : I \subset \mathbb{R}$ is super multiplicative function when

$$g(a_1 a_2) \geq g(a_1)g(a_2), \forall a_1, a_2 \in I, t \in [0, 1]. \tag{6}$$

Definition 3. Non-negatively homogeneous. A function η is said to be non-negatively homogeneous if $\eta(\lambda a_1, \lambda a_2) = \lambda\, \eta(a_1, a_2) \forall a_1, a_2 \in \mathbb{R}$ and $\lambda \geq 0$.

The definition of h–C-function is given in [32], which was generalized by Noor et al. in [33] as modified h–C-function. Later on in [34], the class of generalized strongly modified h–C-function. The class of generalized strongly C-functions of higher order was introduced in [27] and generalized strongly p–C-functions of higher order were introduced in [29]. Motivated by these innovations, the extended version of higher order strongly convexity, which is generalized strongly modified (p, h)–C-functions of higher order, is introduced in this paper as follows:

Definition 4. Generalized strongly modified (p, h)–C-functions of higher order. A function $g : I \longrightarrow \mathbb{R}$ is said to be generalized strongly modified (p, h)– convex of higher order if

$$g\left(ta_1^p + (1-t)a_2^p\right)^{1/p} \leq g(a_2) + h(t)\eta(g(a_1), g(a_2)) - \mu\phi(t)\left\|a_2^p - a_1^p\right\|^q \tag{7}$$

holds for all $a_1, a_2 \in I$, $\mu \geq 0$, and $t \in [0, 1]$.

The class of all generalized strongly modified (p, h)–C-functions of higher order is denoted by GSMPHCF.

Remark 5.

(1) By taking $\mu = 0$, $p = 1$, and $\eta(a, b) = a - b$ in (7), we get the definition of modified h convex function

(2) By taking $\mu = 0, p = 1, \eta(a, b) = a - b$, and $h(t) = t$ in (7), we get the definition of C-function

(3) By taking $h(t) = t$ and $p = 1$ in (7), we get the definition of generalized strongly convex function

(4) By taking $h(t) = t$ in (7), we get the definition of generalized strongly p convex function

Hence, in the light of the above remark, our definition is more generalized than all exiting notions.

Now, we establish some basic properties.

Proposition 6. *Consider η is non-negatively homogeneous and additive and the functions $g, h : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are in GSMPHCF, then $a f + bg \forall a, b \in \mathbb{R}^+$ is also in GSMPHCF.*

Proof. It is cleared that $(a+b)\mu$ is always greater than μ because $a, b \in \mathbb{R}^+$ and $\mu \geq 0$ using this fact we can show that $af+bg$ $\forall a, b \in \mathbb{R}^+$ is in GSMPHCF. □

Proposition 7. *Consider the functions h_1, h_2 on L are non-negative and $h_2(t) \leqslant h_1(t)$. If g is generalized strongly modified (p, h_2)–C-function of higher order, then g is also generalized strongly modified (p, h_1)–C-function of higher order.*

Proof. The proof is straight forward. □

Proposition 8. *Suppose a class of functions $g_j : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be in GSMPHCF where $\lambda_j \in \mathbb{R}^+$ for every $j \in \mathbb{N}$ and η is non-negatively homogeneous and additive then their linear combination $f : \mathbb{R} \longrightarrow \mathbb{R}$ is also in GSMPHCF.*

Proof. Let the linear combination

$$f(x) = \sum_{j=1}^{m} \lambda_j g_j(x). \tag{8}$$

Set $x = (ta_1^p + (1-t)a_2^p)^{1/p}$, so

$$\begin{aligned} f\left(ta_1^p + (1-t)a_2^p\right)^{1/p} &= \sum_{j=1}^{m} \lambda_j g_j\left(ta_1^p + (1-t)a_2^p\right)^{1/p} \\ &\leq \sum_{j=1}^{m} \lambda_j g_j(a_2) + h(t) \sum_{j=1}^{m} \lambda_j \eta(g_i(a_1), g_i(a_2)) \\ &\quad - \sum_{j=1}^{m} \lambda_j \mu\phi(t) \left\| a_2^p - a_1^p \right\|^q \\ &\leq f(a_2) + h(t)\eta\left(\sum_{j=1}^{m} \lambda_j g_i(a_1), \sum_{j=1}^{m} \lambda_j g_i(a_2)\right) \\ &\quad - \mu\phi(t) \left\| a_2^p - a_1^p \right\|^q \\ &= f(a_2) + h(t)\eta(f(a_1), f(a_2)) - \mu\phi(t) \left\| a_2^p - a_1^p \right\|^q. \end{aligned} \tag{9}$$

In this proof, we have used a straight forward result that $\sum_{j=1}^{m} \lambda_j \mu$ is always greater than μ when $\lambda_j \in \mathbb{R}^+$ for every $j \in \mathbb{N}$. □

3. Hermite-Hadamard (HH), Fejér (F), and Schur (S) Type Inequalities

In this section, we present Hermite-Hadamard, Fejér, and Schur type inequalities for the new notion of convexity introduced in this paper.

3.1. S Type Inequality. Firstly, we present Schur type inequality.

Theorem 9. *Let $g : I \longrightarrow \mathbb{R}$ and h be the super multiplicative function, where $\eta : N \times N \longrightarrow M$ be a bi-function for appropriate $A, B \subseteq \mathbb{R}$. Then, for $a_1, a_2, a_3 \in L$, $a_1 < a_2 < a_3$ such that $a_3 - a_1, a_3 - a_2, a_2 - a_1 \in L$, the following inequality holds*

$$\begin{aligned} g(a_2)h\left(a_3^p - a_1^p\right) &\leq h\left(a_3^p - a_1^p\right)g(a_3) \\ &\quad + h\left(a_3^p - a_2^p\right)\eta(g(a_1), g(a_3)) \\ &\quad - \mu\phi\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right) \left\| a_1^p - a_3^p \right\|^q h\left(a_3^p - a_1^p\right) \end{aligned} \tag{10}$$

if and only if g is in GSMPHCF.

Proof. Let $a_1, a_2, a_3 \in L \subset \mathbb{R}$, such that $(a_3 - a_2)/(a_3 - a_1) \in (0,1) \subseteq L$, $(a_2 - a_1)/(a_3 - a_1) \in (0,1) \subseteq L$, and $((a_3 - a_2)/(a_3 - a_1)) + ((a_2 - a_1)/(a_3 - a_1)) = 1$, then

$$h\left(a_3^p - a_1^p\right) = h\left(\frac{a_3^p - a_1^p}{a_3^p - a_2^p}\left(a_3^p - a_2^p\right)\right) \geq h\left(\frac{a_3^p - a_1^p}{a_3^p - a_2^p}\right)h\left(a_3^p - a_2^p\right), \tag{11}$$

as h is super multiplicative.

Suppose $h(a_3^p - a_2^p) \geq 0$, by definition of g, we have

$$g(tx^p + (1-t)y^p)^{1/p} \leq g(y) + h(t)\eta(g(x), g(y)) - \mu\phi(t)\|y^p - x^p\|^q. \tag{12}$$

Inserting $((a_3^p - a_2^p)/(a_3^p - a_1^p)) = t$, $x = a_1$, and $y = a_3$ in inequality (12), we obtain,

$$\begin{aligned} &g\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}a_1^p + \left(1 - \frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right)a_3^p\right)^{1/p} \\ &\quad \leq g(a_3) + h\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right)\eta(g(a_1), g(a_3)) \\ &\quad\quad - \mu\phi\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right)\left\| a_3^p - a_1^p \right\|^q \\ &\quad \leq g(a_3) + \frac{h\left(a_3^p - a_2^p\right)}{h\left(a_3^p - a_1^p\right)}\eta(g(a_1), g(a_3)) \\ &\quad\quad - \mu\phi\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right)\left\| a_3^p - a_1^p \right\|^q. \end{aligned} \tag{13}$$

Multiplying (13) by $h(a_3^p - a_1^p)$ and using the fact that h is super multiplicative, we get

$$\begin{aligned} g(a_2)h\left(a_3^p - a_1^p\right) &\leq h\left(a_3^p - a_1^p\right)g(a_3) + h\left(a_3^p - a_2^p\right)\eta(g(a_1), g(a_3)) \\ &\quad - \mu\phi\left(\frac{\left(a_3^p - a_2^p\right)}{\left(a_3^p - a_1^p\right)}\right)\left\| a_3^p - a_1^p \right\|^q h\left(a_3^p - a_1^p\right) \end{aligned} \tag{14}$$

For conversely insert $a_1 = x$, $a_2 = (tx^p + (1-t)y^p)^{1/p}$, $a_3 = y$, and $(a_3^p - a_2^p)/(a_3^p - a_1^p) = t$ in inequality (10) and using the fact that h is super multiplicative function such that $h(t(y^p - x^p)) \geq h(y^p - x^p)h(t)$, we get

$$h(y^p - x^p)g(tx^p + (1-t)y^p)^{1/p} \leq h(y^p - x^p)g(y) + h(y^p - x^p)h(t)\eta(g(x), g(y)) - \mu\phi(t)\|y^p - x^p\|^q h(y^p - x^p), \tag{15}$$

implies

$$g(tx^p + (1-t)y^p)^{1/p} \leq g(y) + h(t)\eta(g(x), g(y)) - \mu\phi(t)\|y^p - x^p\|^q. \tag{16}$$

The prove is completed. □

Remark 10.

(1) For $h(t) = t$, (10) reduced to S type inequality for function defined in [29].

(2) For $\mu = 0$, $p = 1$, and $\eta(a, b) = a - b$, (10) reduced to S type inequality for modified h–C-function, see [33].

3.2. HH Type Inequality. Here we present Hermite–Hadamard type inequality.

Theorem 11. *Let $g : I \longrightarrow \mathbb{R}$ in GSMPHCF defined on $[a_1, a_2]$ with $a_1 < a_2$, then*

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} - h\left(\frac{1}{2}\right)M_\eta + \mu\phi\left(\frac{1}{2}\right)\|a_1^p - a_2^p\|^q\left(\frac{1 + (-1)^{q+1}}{2(q+1)}\right)^{1/p} \leq \frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g\left((x)dx \leq \frac{g(a_1) + g(a_2)}{2} + N_\eta\int_0^1 h(t)dt - \mu\|a_2^p - a_1^p\|^q \int_0^1 \phi(t)dt. \tag{17}$$

Proof. Choose $w = (ta_1^p + (1-t)a_2^p)^{1/p}$ and $z = ((1-t)a_1^p + ta_2^p)^{1/p}$, then

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} = g\left(\frac{w^p + z^p}{2}\right)^{1/p} \leq g(z) + h\left(\frac{1}{2}\right)\eta(g(w), g(z)) - \mu\phi\left(\frac{1}{2}\right)\|z^p - w^p\|^q. \tag{18}$$

Implies

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} \leq g((1-t)a_1^p + ta_2^p)^{1/p} + h\left(\frac{1}{2}\right)\eta\left(g(ta_1^p + (1-t)a_2^p)^{1/p}, g((1-t)a_1^p + ta_2^p)^{1/p}\right) - \mu(2t-1)^q\phi\left(\frac{1}{2}\right)\|a_2^p - a_1^p\|^q. \tag{19}$$

Integrating with respect to t on [0,1], we get

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} \leq \int_0^1 g((1-t)a_1^p + ta_2^p)^{1/p}dt + h\left(\frac{1}{2}\right)\int_0^1 \eta\left(g(ta_1^p + (1-t)a_2^p)^{1/p}, g((1-t)a_1^p + ta_2^p)^{1/p}\right)dt - \mu\phi\left(\frac{1}{2}\right)\|a_2^p - a_1^p\|^q\int_0^1 (2t-1)^q dt. \tag{20}$$

Implies

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} \leq \int_0^1 g((1-t)a_1^p + ta_2^p)^{1/p}dt + h\left(\frac{1}{2}\right)M_\eta - \mu\phi\left(\frac{1}{2}\right)\|a_2^p - a_1^p\|^q\left(\frac{1 + (-1)^{q+1}}{2(q+1)}\right)^{1/p}. \tag{21}$$

Putting $x = (1-t)a_1 + ta_2$, we get

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} \leq \frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1}g(x)dx + h\left(\frac{1}{2}\right)M_\eta - \mu\phi\left(\frac{1}{2}\right)\|a_2^p - a_1^p\|^q\left(\frac{1 + (-1)^{q+1}}{2(q+1)}\right)^{1/p}. \tag{22}$$

Implies

$$g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} - h\left(\frac{1}{2}\right)M_\eta + \mu\phi\left(\frac{1}{2}\right)\|a_2^p - a_1^p\|^q\left(\frac{1 + (-1)^{q+1}}{2(q+1)}\right)^{1/p} \leq \frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1}g((x)dx. \tag{23}$$

Now,

$$\int_0^1 g(ta_1^p + (1-t)a_2^p)^{1/p}dt \leq g(a_2) + \int_0^1 h(t)\eta(g(a_1), g(a_2))dt - \mu\|a_1^p - a_2^p\|^q\int_0^1 \phi(t)dt. \tag{24}$$

Taking $ta_1^p + (1-t)a_2^p)^{1/p} = x$, we get

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le g(a_2) + \eta(g(a_1), g(a_2))\int_0^1 h(t)dt - \mu\|a_1^p - a_2^p\|^q \int_0^1 \phi(t)dt = A. \quad (25)$$

Similarly,

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le g(a_1) + \eta(g(a_2), g(a_1))\int_0^1 h(t)dt - \mu\|a_1^p - a_2^p\|^q \int_0^1 \phi(t)dt = B. \quad (26)$$

From inequalities (25) and (26), we have

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le \min(A, B). \quad (27)$$

This implies

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le \frac{A+B}{2}.$$

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le \frac{g(a_1)+g(a_2)}{2} + \frac{\eta(g(a_1), g(a_2)) + \eta(g(a_2), g(a_1))}{2} \times \int_0^1 h(t)dt - \frac{\mu}{2}\|a_1^p - a_2^p\|^q \int_0^1 \phi(t)dt. \quad (28)$$

Implies

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} x^{p-1} g(x)dx \le \frac{g(a_1)+g(a_2)}{2} + N_\eta \int_0^1 h(t)dt - \mu\|a_2^p - a_1^p\|^q \int_0^1 \phi(t)dt. \quad (29)$$

Inequalities (23) and (29) assure to proof. □

Remark 12.

(1) For $p = 1, \mu = 0$, and $\eta(a, b) = a - b$, (17) reduced to HH type inequality for modified h–C-function, see [33].

(2) For $h(t) = t$, (17) reduced to HH type inequality for function defined in [29].

(3) For $\mu = 0, p = 1$, $\eta(a, b) = a - b$, and $h(t) = t$,(17) reduced to HH type inequality for C-function in classical sense

3.3. F Type Inequality. This subsection contains Fejér type inequality.

Lemma 13. *Consider g be in GSMPHCF and $\eta(x, y) = -\eta(y, x)$, then*

$$g\left(a_1^p + a_2^p - x^p\right)^{1/p} \le g(a_1) + g(a_2) - g(x) \forall x \in [a_1, a_2], \quad (30)$$

where $x = (ta_1^p + (1-t)a_2^p)^{1/p}$ and $t \in [0, 1]$.

Proof. Let $g : I \longrightarrow \mathbb{R}$ be in GSMPHCF then for $x = (ta_1^p + (1-t)a_2^p)^{1/p}$, we get

$$\begin{aligned} g\left(a_1^p + a_2^p - x^p\right)^{1/p} &= g\left((1-t)a_1^p + ta_2^p\right)^{1/p} \\ &\le g(a_1) + h(t)\eta(g(a_2), g(a_1)) - \mu\phi(t)\|a_1^p - a_2^p\|^q \\ &\le g(a_1) + g(a_2) - g(a_2) - h(t)\eta(g(a_1), g(a_2)) \\ &\quad - \mu\phi\left(t\|a_1^p - a_2^p\|^q + 2\mu\phi(t)\|a_1^p - a_2^p\|^q\right. \\ &\le g(a_1) + g(a_2) \\ &\quad - \left[g(a_2) + h(t)\eta(g(a_1), g(a_2) - \mu\phi(t)\|a_1^p - a_2^p\|^q\right] \\ &\le g(a_1) + g(a_2) - g(x). \end{aligned} \quad (31)$$

□

Theorem 14. *Let $g : [a_1, a_2] \longrightarrow \mathbb{R}$ be in GSMPHCF and $w : [a_1, a_2] \longrightarrow \mathbb{R}$ be integrable and symmetric w.r.t $a_1 + a_2/2$, where $w \ge 0$, then we have*

$$\begin{aligned} & g\left(\frac{a_1^p + a_2^p}{2}\right)^{1/p} \int_{a_1}^{a_2} x^{p-1} w(x)dx \\ &\quad + \mu\phi\left(\frac{1}{2}\right)\int_{a_1}^{a_2} x^{p-1}\|a_1^p + a_2^p - 2x^p\|^q \times w(x)dx \\ &\quad - h\left(\frac{1}{2}\right)\int_{a_1}^{a_2} x^{p-1}\eta\left(g\left(a_1^p + a_2^p - x^p\right)^{1/p}, g(x)\right)w(x)dx \\ &\le \int_{a_1}^{a_2} x^{p-1} g(x)w(x)dx \\ &\le \frac{(g(a_1) + g(a_2))}{2}\int_{a_1}^{a_2} x^{p-1}w(x)dx \\ &\quad + M_\eta \int_{a_1}^{a_2} x^{p-1} h\left(\frac{x^p - a_2^p}{a_1^p - a_2^p}\right)^{1/p} w(x)dx \\ &\quad - \mu\int_{a_1}^{a_2} x^{p-1}\phi\left(\frac{x^p - a_2^p}{a_1^p - a_2^p}\right)^{1/p} w(x)dx, \end{aligned} \quad (32)$$

where $M_\eta = [\eta(g(a_1), g(a_2)) + \eta(g(a_2), g(a_1))]/2$.

Proof. Let g be a generalized strongly modified (p,h)–C-function of higher order where $x=(ta_1^p+(1-t)a_2^p)^{1/p}$ and $y=(ta_2^p+(1-t)a_1^p)^{1/p}$, then

$$\begin{aligned}
g\left(\frac{a_1^p+a_2^p}{2}\right)^{1/p}\int_{a_1}^{a_2}x^{p-1}w(x)dx
&=\int_{a_1}^{a_2}x^{p-1}g\left(\frac{a_1^p+a_2^p-x^p+x^p}{2}\right)^{1/p}w(x)dx\\
&=\int_{a_1}^{a_2}x^{p-1}g\left(\frac{y^p+x^p}{2}\right)^{1/p}w(x)dx\\
&\le\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx+h\left(\frac{1}{2}\right)\int_{a_1}^{a_2}x^{p-1}\eta(g(y),g(x))w(x)dx\\
&\quad-\mu\phi\left(\frac{1}{2}\right)\int_{a_1}^{a_2}x^{p-1}\|x^p-y^p\|^q w(x)dx,
\end{aligned}\tag{33}$$

where $y^p=a_1^p+a_2^p-x^p$, so

$$\begin{aligned}
&g\left(\frac{a_1^p+a_2^p}{2}\right)^{1/p}\int_{a_1}^{a_2}x^{p-1}w(x)dx\\
&\quad+\mu\phi\left(\frac{1}{2}\right)\int_{a_1}^{a_2}x^{p-1}\left\|a_1^p+a_2^p-2x^p\right\|^q w(x)dx\\
&\quad-h\left(\frac{1}{2}\right)\int_{a_1}^{a_2}x^{p-1}\eta\left(g\left(a_1^p+a_2^p-x^p\right)^{1/p},g(x)\right)w(x)dx\\
&\le\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx.
\end{aligned}\tag{34}$$

Now, if $x=(ta_1^p+(1-t)a_2^p)^{1/p}$, then

$$\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx=\frac{(a_2^p-a_1^p)}{p}\int_0^1 g\left(ta_1^p+(1-t)a_2^p\right)^{1/p}w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt.\tag{35}$$

So,

$$\begin{aligned}
\frac{p}{a_2^p-a_1^p}\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx
&\le\int_0^1 g(a_2)w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt\\
&\quad+\eta(g(a_1),g(a_2))\int_0^1 h(t)w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt\\
&\quad-\mu\left\|a_2^p-a_1^p\right\|^q\int_0^1\phi(t)dt.
\end{aligned}\tag{36}$$

Similarly, if $x=(ta_2^p+(1-t)a_1^p)^{1/p}$, then

$$\begin{aligned}
\frac{p}{a_2^p-a_1^p}\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx
&\le\int_0^1 g(a_1)w\left(ta_2^p+(1-t)a_1^p\right)^{1/p}dt\\
&\quad+\eta g(a_2),g(a_1))\int_0^1 h(t)w\left(ta_2^p+(1-t)a_1^p\right)^{1/p}dt\\
&\quad-\mu\left\|a_2^p-a_1^q\right\|^p\int_0^1\phi(t)w\left(ta_2^p+(1-t)a_1^p\right)^{1/p}dt.
\end{aligned}\tag{37}$$

Adding inequalities (36) and (37), where w is symmetric, then

$$\begin{aligned}
\frac{2p}{a_2^p-a_1^p}\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx
&\le(g(a_1)+g(a_2))\int_0^1 w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt\\
&\quad+[\eta(g(a_1),g(a_2))+\eta(g(a_2),g(a_1))]\int_0^1 h(t)w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt\\
&\quad-2\mu\left\|a_2^p-a_1^p\right\|^q\int_0^1\phi(t)w\left(ta_1^p+(1-t)a_2^p\right)^{1/p}dt.
\end{aligned}\tag{38}$$

Now, for $x=(ta_1^p+(1-t)a_2^p)^{1/p}$, we have

$$\begin{aligned}
\int_{a_1}^{a_2}x^{p-1}g(x)w(x)dx&\le\frac{(g(a_1)+g(a_2))}{2}\int_{a_1}^{a_2}x^{p-1}w(x)dx\\
&\quad+\frac{[\eta(g(a_1),g(a_2))+\eta(g(a_2),g(a_1))]}{2}\int_{a_1}^{a_2}x^{p-1}h\\
&\quad\cdot\left(\frac{x^p-a_2^p}{a_1^p-a_2^p}\right)^{1/p}w(x)dx\\
&\quad-\mu\int_{a_1}^{a_2}x^{p-1}\phi\left(\frac{x^p-a_2^p}{a_1^p-a_2^p}\right)^{1/p}w(x)dx.
\end{aligned}\tag{39}$$

Inequalities (34) and (39) assure to proof. □

Remark 15.

(1) For $h(t)=t$, this F type inequality reduced to F type inequality for the function defined in [29].

(2) For $\mu=0, p=1$, and $\eta(a,b)=a-b$, this F Type inequality reduced to F type inequality for the function defined in [33].

(3) For $\mu=0$, $p=1$, $\eta(a,b)=a-b$, and $h(t)=t$, this F type inequality reduced to F type inequality for C-function in classical sense

4. Conclusions

In this paper, we introduced the notion of generalized strongly modified $(p\,;h)$–C-functions of higher order and established HH, F, and S type inequalities. Many existing results can be derived from our results. Our results are applicable in pure and applied mathematics especially in optimization theory. For applications point, we refer to the readers [35–38].

5. Future Directions

It will be interesting to introduce a new and more generalized version of the convexity. The inequalities established in this paper can be further extended for different versions of fractional integral operators.

Data Availability

All data required for this paper is included within this paper.

Conflicts of Interest

We do not have any competing interests.

Authors' Contributions

Y.M. wrote the final version of the paper and verified the results and arranged the funding for this paper. M.S.S. proposed the problem and supervised this work. I.B. proved the main results of the paper and Y.X. wrote the first version of the paper.

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