

Research Article

A Study on Some New Generalizations of Reversed Dynamic Inequalities of Hilbert-Type via Supermultiplicative Functions

M. Zakarya ^{1,2}, Ahmed I. Saied,³ Ghada ALNemer,⁴ H. A. Abd El-Hamid,⁵ and H. M. Rezk ⁶

¹King Khalid University, College of Science, Department of Mathematics, P.O. Box 9004, 61413 Abha, Saudi Arabia

²Department of Mathematics, Faculty of Science, Al-Azhar University, 71524 Assiut, Egypt

³Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

⁴Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

⁵Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

⁶Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt

Correspondence should be addressed to H. M. Rezk; haythamrezk@azhar.edu.eg

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In this article, we establish some new generalizations of reversed dynamic inequalities of Hilbert-type via supermultiplicative functions by applying reverse Hölder inequalities with Specht's ratio on time scales. We will generalize the inequalities by using a supermultiplicative function which the identity map represents a special case of it. Also, we use some algebraic inequalities such as the Jensen inequality and chain rule to prove the essential results in this paper. Our results (when $\mathbb{T} \ll \mathbb{N}$) are essentially new.

1. Introduction

In [1], Hardy established that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{w_m z_n}{m+n} \leq \frac{\pi}{\sin \pi/p} \left(\sum_{m=1}^{\infty} w_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} z_n^q \right)^{1/q}, \quad (1)$$

where $w_m, z_n \geq 0$ with $0 < \sum_{m=1}^{\infty} w_m^p < \infty, 0 < \sum_{n=1}^{\infty} z_n^q < \infty$ and $p > 1, 1/p + 1/q = 1$. In [2], Hardy showed that

$$\int_0^{\infty} \int_0^{\infty} \frac{f(\vartheta)g(y)}{\vartheta+y} d\vartheta dy \leq \frac{\pi}{\sin \pi/p} \left(\int_0^{\infty} f^p(\vartheta) d\vartheta \right)^{1/p} \left(\int_0^{\infty} g^q(y) dy \right)^{1/q}, \quad (2)$$

where f and g are measurable nonnegative functions such that $0 < \int_0^{\infty} f^p(\vartheta) d\vartheta < \infty$ and $0 < \int_0^{\infty} g^q(y) dy < \infty$. The constant $\pi/\sin(\pi/p)$ is the best possible.

In [3], Hölder proved that

$$\sum_{k=1}^n \zeta_k \gamma_k \leq \left(\sum_{k=1}^n \zeta_k^p \right)^{1/p} \left(\sum_{k=1}^n \gamma_k^q \right)^{1/q}, \quad (3)$$

where (ζ_k) and (γ_k) are positive sequences and $p, q > 1$ such that $1/p + 1/q = 1$. The integral form of (3) is

$$\int_w^z \psi(\tau) \omega(\tau) d\tau \leq \left(\int_w^z \psi^p(\tau) d\tau \right)^{1/p} \left(\int_w^z \omega^q(\tau) d\tau \right)^{1/q}, \quad (4)$$

where $\psi, \omega \in C((w, z), \mathbb{R}^+)$.

In [4], Zhao and Cheung showed that if $\psi(\zeta)$ and $\omega(\zeta)$ are nonnegative continuous functions and $\psi^{1/p}(\zeta)\omega^{1/q}(\zeta)$ is integrable on $[w, z]$, then

$$\left(\int_w^z \psi^p(\zeta) d\zeta \right)^{1/p} \left(\int_w^z \omega^q(\zeta) d\zeta \right)^{1/q} \leq_w^z S \left(\frac{Y\psi^p(\zeta)}{\vartheta\omega^q(\zeta)} \right) \psi(\zeta)\omega(\zeta) d\zeta, \quad (5)$$

with

$$\vartheta = \int_w^z \psi^p(\zeta) d\zeta, Y = \int_w^z \omega^q(\zeta) d\zeta, p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \quad (6)$$

where $S(\cdot)$ is the Specht's ratio function (see [5]) and defined by

$$S(l) := \frac{l^{1/(l-1)}}{e \log l^{1/(l-1)}}, l \neq 1, S(1) = 1. \quad (7)$$

Also, they proved that if $\psi, \omega \in C((w, z), \mathbb{R}^+)$ and $\gamma > 0$, then

$$\int_w^z \frac{\psi^{\gamma+1}(\zeta)}{\omega^\gamma(\zeta)} d\zeta \leq \frac{\left(\int_w^z S(G\psi^{\gamma+1}(\zeta)/F\omega^{\gamma+1}(\zeta)) \psi(\zeta) d\zeta \right)^{\gamma+1}}{\left(\int_w^z \omega(\zeta) d\zeta \right)^\gamma}, \quad (8)$$

where

$$\begin{aligned} G &= \int_w^z \omega(\zeta) d\zeta, \\ F &= \int_w^z \frac{\psi^{\gamma+1}(\zeta)}{\omega^\gamma(\zeta)} d\zeta. \end{aligned} \quad (9)$$

In addition to this, they proved the discrete case of (8) as follows:

$$\sum_{i=1}^n \frac{w_i^{m+1}}{z_i^m} \leq \frac{\sum_{i=1}^n S(zw_i^{m+1}/wz_i^{m+1}) w_i}{\left(\sum_{i=1}^n z_i \right)^m}, \quad (10)$$

where $z = \sum_{i=1}^n z_i$ and $w = \sum_{i=1}^n w_i^{m+1}/z_i^m$.

In [6], the authors proved the reverse Hilbert-type inequalities by using the Specht's ratio as follows: if $0 \leq p, q \leq 1$ and $\{\lambda_m\}, \{\psi_n\}$ are nonnegative and decreasing sequences of real numbers for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$ with $k, r \in \mathbb{N}$, then

$$\begin{aligned} \sum_{m=1}^k \sum_{n=1}^r \frac{S_{p,q,k,r,m,n} \left(\sum_{s=1}^m \lambda_s \right)^p \left(\sum_{t=1}^n \psi_t \right)^q}{(mn)^{1/2}} &\geq 2C(p, q, k, r) \\ &\cdot \left(\sum_{m=1}^k \left[\lambda_m \left(\sum_{s=1}^m \lambda_s \right)^{p-1} \right]^2 (k-m+1) \right)^{1/2} \\ &\times \left(\sum_{n=1}^r \left[\psi_n \left(\sum_{t=1}^n \psi_t \right)^{q-1} \right]^2 (r-n+1) \right)^{1/2}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} C(p, q, r, s) &= \frac{1}{2} p q (kr)^{1/2}, \\ S_{p,q,k,r,m,n} &= S \left(\frac{k \sum_{s=1}^m \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2}{\sum_{s=1}^k (k-s+1) \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2} \right) \\ &\times S \left(\frac{r \sum_{t=1}^n \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2}{\sum_{t=1}^r (r-t+1) \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2} \right) \\ &\times S \left(\frac{m \left[\lambda_u \left(\sum_{\tau=1}^u \lambda_\tau \right)^{p-1} \right]^2}{\sum_{s=1}^m \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2} \right) \\ &\times S \left(\frac{n \left[\psi_v \left(\sum_{\tau=1}^v \psi_\tau \right)^{q-1} \right]^2}{\sum_{t=1}^n \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2} \right), \end{aligned} \quad (12)$$

where

$$\begin{aligned} S \left(\frac{m \left[\lambda_u \left(\sum_{\tau=1}^u \lambda_\tau \right)^{p-1} \right]^2}{\sum_{s=1}^m \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2} \right) &= \max \left\{ S \left(\frac{m \left[\lambda_1 \left(\sum_{\tau=1}^1 \lambda_\tau \right)^{p-1} \right]^2}{\sum_{s=1}^m \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2} \right); S \left(\frac{m \left[\lambda_m \left(\sum_{\tau=1}^m \lambda_\tau \right)^{p-1} \right]^2}{\sum_{s=1}^m \left[\lambda_s \left(\sum_{\tau=1}^s \lambda_\tau \right)^{p-1} \right]^2} \right) \right\}, \\ S \left(\frac{n \left[\psi_v \left(\sum_{\tau=1}^v \psi_\tau \right)^{q-1} \right]^2}{\sum_{t=1}^n \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2} \right) &= \max \left\{ S \left(\frac{n \left[\psi_1 \left(\sum_{\tau=1}^1 \psi_\tau \right)^{q-1} \right]^2}{\sum_{t=1}^n \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2} \right); S \left(\frac{n \left[\psi_n \left(\sum_{\tau=1}^n \psi_\tau \right)^{q-1} \right]^2}{\sum_{t=1}^n \left[\psi_t \left(\sum_{\tau=1}^t \psi_\tau \right)^{q-1} \right]^2} \right) \right\}. \end{aligned} \quad (13)$$

In 1990, Ariño and Muckenhoupt [7] proved an inequality which maps from L_p to $L_p, 1 \leq p < \infty$ and contains one weighted function that they characterized such that the inequality

$$\int_0^\infty w(t) \left(\frac{1}{t} \int_0^t f(x) dx \right)^p dt \leq C \int_0^\infty w(t) f^p(t) dt, \quad (14)$$

holds for all nonnegative nonincreasing measurable function f on $(0, \infty)$ with a constant $C > 0$ independent on f (here $1 \leq p < \infty$). The characterization reduces to the condition that the function w satisfies

$$\int_t^\infty \frac{w(x)}{x^p} dx \leq \frac{B}{t^p} \int_0^t w(x) dx, \text{ for all } t \in (0, \infty) \text{ and } B > 0. \quad (15)$$

In 1993, Sinnamon [8] generalized (14) to map from L_p to $L_q, 0 < q < 1 < p$ which has two different weighted functions and characterized the weights such that the inequality

$$\left(\int_a^b u(x) \left(\int_a^x f(t) dt \right)^q dx \right)^{1/q} \leq C \left(\int_a^b v(x) f^p(x) dx \right)^{1/p}, \quad (16)$$

holds for all measurable $f \geq 0$ and the constant C is independent of f (here $0 < q < 1 < p$ and $1/r = 1/q - 1/p$). The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$\int_a^b \left(\int_x^b u(t) dt \right)^{r/p} \left(\int_a^x v^{1-p'}(t) dt \right)^{r/p'} u(x) dx = K < \infty, p' = \frac{p}{p-1}. \quad (17)$$

Also, many authors study the inequalities with weighted functions (see [9–16]).

For the discrete case of (14), in 2006, Bennett and Gross-Erdmann [11] characterized the weights such that the inequality

$$\sum_{n=1}^\infty w_n \left(\frac{1}{n} \sum_{k=1}^n g_k \right)^p \leq C \sum_{n=1}^\infty w_n g_n^p, 1 \leq p < \infty, \quad (18)$$

holds for all nonnegative nonincreasing sequence g_n and $C > 0$. The characterization reduces to the condition that the nonnegative sequence w_n satisfies

$$\sum_{k=n}^\infty \frac{w_k}{k^p} \leq \frac{B}{n^p} \sum_{k=1}^n w_k, \text{ for all } n \in \mathbb{N} \text{ and } B > 0. \quad (19)$$

In the last decades, a time scale theory has been discovered to unify the continuous calculus and discrete calculus. A time scale \mathbb{T} is defined as an arbitrary nonempty closed subset of the real numbers \mathbb{R} . Many authors proved

some dynamic inequalities on time scales which unify the inequalities in the continuous calculus and discrete calculus (see [17–26]). In particular, in 2021, Saker et al. [25] unified the inequalities (14) and (18) to be general inequality on time scales and proved that \mathbb{T} is a time scale with $a \in \mathbb{T}$ and $1 \leq p < \infty$. Furthermore, assume that f is a nonincreasing function and $\int_a^\infty w(t) f^p(t) \Delta t < \infty$. Suppose there is a constant $B > 0$ with

$$\int_t^\infty \frac{w(s)}{(\sigma(s) - a)^p} \Delta s \leq \frac{B}{(\sigma(t) - a)^p} \int_a^{\sigma(t)} w(s) \Delta s \text{ for all } t \in (a, \infty)_{\mathbb{T}}. \quad (20)$$

Then,

$$\int_a^\infty w(t) \left(\frac{1}{\sigma(t) - a} \int_a^{\sigma(t)} f(x) \Delta x \right)^p \Delta t \leq p^p (B + 1)^p \int_a^\infty w(t) f^p(t) \Delta t. \quad (21)$$

As special cases of (20), when $(\mathbb{T} \ll \mathbb{R}, \sigma(t) = t, a = 0)$, we obtain the inequality (14) and when $(\mathbb{T} \ll \mathbb{N}, \sigma(t) = t + 1, a = 1)$, we have the inequality (18).

Also, we can get new inequalities, for example, when $(\mathbb{T} \ll h\mathbb{N} : h > 0, a = h)$, where $\sigma(t) = t + h$, we get a new inequality in a new calculus of (21) in the form

$$\sum_{n=h}^\infty w_n \left(\frac{1}{n} \sum_{k=1}^n f_k \right)^p \leq p^p (B + 1)^p \sum_{n=h}^\infty w_n f_n^p, 1 \leq p < \infty, \quad (22)$$

For some reversed dynamic inequalities of Hilbert-type, Hölder-type, and Hardy-type on time scale, see the papers ([27–32]).

The goal of this manuscript is to use reverse Hölder inequalities with Specht’s ratio on \mathbb{T} to develop some new generalizations of reverse Hilbert-type inequalities via supermultiplicative functions on time scales.

The following is a breakdown of the paper’s structure. In Section 2, we cover some fundamentals of time scale theory as well as several time scale lemmas that will be useful in Section 3, where we prove our findings. As specific examples (when $\mathbb{T} = \mathbb{N}$), our major results yield (11) proved by Zhao and Cheung [6].

2. Preliminaries and Fundamental Axioms

For completeness, we recall the following concepts related to the notion of time scales. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [33, 34] which summarize and organize much of the time scale calculus. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} .

First, we define $\sigma(\tau) := \inf \{u \in \mathbb{T} : u > \tau\}$, $C_{rd}(\mathbb{T}, \mathbb{R})$ introduces the set of all such rd-continuous functions, and for any function $u : \mathbb{T} \rightarrow \mathbb{R}$, the notation $u^\sigma(\tau)$ denotes $u(\sigma(\tau))$.

The derivative of the product $u\omega$ and the quotient u/ω (where $\omega \omega^\sigma \neq 0$) of two differentiable functions u and ω

are given by

$$(u\bar{\omega})^\Delta = u^\Delta \bar{\omega} + u^\sigma \bar{\omega}^\Delta = u\bar{\omega}^\Delta + u^\Delta \bar{\omega}^\sigma, \left(\frac{u}{\bar{\omega}}\right)^\Delta = \frac{u^\Delta \bar{\omega} - u\bar{\omega}^\Delta}{\bar{\omega}\bar{\omega}^\sigma}. \quad (23)$$

The integration by parts formula on \mathbb{T} is

$$\int_{v_0}^v \lambda(\tau)\varphi^\Delta(\tau)\Delta\tau = [\lambda(\tau)\varphi(\tau)]_{v_0}^v - \int_{v_0}^v \lambda^\Delta(\tau)\varphi^\sigma(\tau)\Delta\tau. \quad (24)$$

The time scale chain rule ([9], Theorem 1.87) is

$$(\bar{\omega} \circ \varphi)^\Delta(\tau) = \bar{\omega}'(\varphi(\kappa))\varphi^\Delta(\tau), \text{ where } \kappa \in [\tau, \sigma(\tau)], \quad (25)$$

where it is supposed that $\bar{\omega} : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable.

Definition 1 (see [35]). A function $L : I \rightarrow \mathbb{R}^+$ is supermultiplicative if

$$L(\kappa\zeta) \geq L(\kappa)L(\zeta), \forall \kappa, \zeta \in I \subset \mathbb{R}. \quad (26)$$

When L is the identity map (i.e., $L(\zeta) = \zeta$), the inequality (26) holds with equality. L is said to be a submultiplicative function if the last inequality has the opposite sign.

Lemma 2. If $w \in \mathbb{T}$, $\lambda \in C_{rd}(\mathbb{T}, \mathbb{R})$, and $0 \leq \gamma \leq 1$, then

$$\left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^\gamma \geq \gamma \int_w^{\sigma(t)} \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta. \quad (27)$$

Proof. Using (25) on the term $\int_w^\vartheta \lambda(\tau)\Delta\tau$, we get

$$\left[\left(\int_w^\vartheta \lambda(\tau)\Delta\tau\right)^\gamma\right]^\Delta = \gamma \left(\int_w^\zeta \lambda(\tau)\Delta\tau\right)^{\gamma-1} \lambda(\vartheta), \zeta \in [\vartheta, \sigma(\vartheta)]. \quad (28)$$

Since $\zeta \leq \sigma(\vartheta)$, we have (note $0 \leq \gamma \leq 1$) that

$$\left(\int_w^\zeta \lambda(\tau)\Delta\tau\right)^{\gamma-1} \geq \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{\gamma-1}. \quad (29)$$

Substituting (29) into (28), we see that

$$\left[\left(\int_w^\vartheta \lambda(\tau)\Delta\tau\right)^\gamma\right]^\Delta \geq \gamma \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{\gamma-1} \lambda(\vartheta). \quad (30)$$

Integrating (30) over ϑ from w to $\sigma(t)$, we have

$$\int_w^{\sigma(t)} \left[\left(\int_w^\vartheta \lambda(\tau)\Delta\tau\right)^\gamma\right]^\Delta \Delta\vartheta \geq \gamma \int_w^{\sigma(t)} \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta. \quad (31)$$

That is,

$$\left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^\gamma \geq \gamma \int_w^{\sigma(t)} \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta, \quad (32)$$

which is (27). \square

Lemma 3 (Specht's ratio [5]). Let α, β be positive numbers, $p > 1$, and $1/p + 1/q = 1$. Then,

$$S\left(\frac{\alpha}{\beta}\right) \alpha^{1/p} \beta^{1/q} \geq \frac{\alpha}{p} + \frac{\beta}{q}, \quad (33)$$

where

$$S(t) = \frac{t^{1/(t-1)}}{e \log t^{1/(t-1)}}, t \neq 1. \quad (34)$$

The function $S(t)$ is strictly decreasing for $0 < t < 1$ and strictly increasing for $t > 1$. In addition, the following equations are true:

$$\begin{aligned} S(1) &= 1, \\ S(t) &= S\left(\frac{1}{t}\right) \forall t > 0. \end{aligned} \quad (35)$$

Lemma 4 ([16, when $\alpha = 1$]). Let $f, g \in C([w, z]_{\mathbb{T}}, \mathbb{R}^+)$ such that f^β, g^ν be Δ -integrable on $[w, z]_{\mathbb{T}}$. If $\beta > 1$ and $1/\beta + 1/\nu = 1$, then

$$\int_w^z S\left(\frac{Yf^\beta(\zeta)}{Xg^\nu(\zeta)}\right) f(\zeta)g(\zeta)\Delta\zeta \geq \left(\int_w^z f^\beta(\zeta)\Delta\zeta\right)^{1/\beta} \left(\int_w^z g^\nu(\zeta)\Delta\zeta\right)^{1/\nu}, \quad (36)$$

where $X = \int_w^z f^\beta(\zeta)\Delta\zeta$ and $Y = \int_w^z g^\nu(\zeta)\Delta\zeta$.

Lemma 5 (Jensen's inequality, alnamer). Assume that $\zeta_0, \zeta \in \mathbb{T}$ and $r_0, r \in \mathbb{R}$. If $\lambda \in C_{rd}([\zeta_0, \zeta]_{\mathbb{T}}, \mathbb{R})$, $\varphi \in C_{rd}([\zeta_0, \zeta]_{\mathbb{T}}, (r_0, r))$, and $\Psi : (r_0, r) \rightarrow \mathbb{R}$ is continuous and convex, then

$$\Psi\left(\frac{1}{\int_{\zeta_0}^\zeta \lambda(\tau)\Delta\tau} \int_{\zeta_0}^\zeta \lambda(\tau)\varphi(\tau)\Delta\tau\right) \leq \frac{1}{\int_{\zeta_0}^\zeta \lambda(\tau)\Delta\tau} \int_{\zeta_0}^\zeta \lambda(\tau)\Psi(\varphi(\tau))\Delta\tau. \quad (37)$$

The inequality (37) is reversed when Ψ is continuous and concave.

Lemma 6. Let $w \in \mathbb{T}, \lambda, \psi$ be positive and decreasing functions, f, g be positive and nondecreasing functions, and $0 \leq p, q \leq 1$. In addition, assume that ϕ, φ are positive, increasing, concave, and supermultiplicative functions. If $\beta > 1, \nu > 1$ with $1/\beta + 1/\nu = 1$, then

Proof. For $\vartheta \leq y$, we have

$$S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta \vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(\zeta)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \Delta \vartheta} \right) = \max \left\{ S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta \vartheta \right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \Delta \vartheta} \right); S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta \vartheta \right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \Delta \vartheta} \right) \right\}, \tag{38}$$

$$S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta \tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta \tau \right)^{q-1} \right] \Delta y} \right) = \max \left\{ S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(w) \left(\int_w^{\sigma(w)} \psi(\tau) \Delta \tau \right)^{q-1} \right]}{g^\nu(w) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta \tau \right)^{q-1} \right] \Delta y} \right); S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\xi) \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta \tau \right)^{q-1} \right]}{g^\nu(\xi) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta \tau \right)^{q-1} \right] \Delta y} \right) \right\}. \tag{39}$$

$$\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \leq \int_w^{\sigma(y)} \lambda(\tau) \Delta \tau, \tag{40}$$

and then (where $0 \leq p \leq 1$)

$$\left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \geq \left(\int_w^{\sigma(y)} \lambda(\tau) \Delta \tau \right)^{p-1}. \tag{41}$$

□

Since λ is decreasing, we have from (41) (where $\vartheta \leq y$) that

$$\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \geq \lambda(y) \left(\int_w^{\sigma(y)} \lambda(\tau) \Delta \tau \right)^{p-1}. \tag{42}$$

Using the facts that $\beta > 1$, ϕ is an increasing function and (42), we get

$$\phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq \phi^\beta \left[\lambda(y) \left(\int_w^{\sigma(y)} \lambda(\tau) \Delta \tau \right)^{p-1} \right], \tag{43}$$

and then, we obtain (where $\vartheta \leq y$ and f is nondecreasing) that

$$\frac{1}{f^\nu(\vartheta)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq \frac{1}{f^\nu(y)} \phi^\beta \left[\lambda(y) \left(\int_w^{\sigma(y)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]. \tag{44}$$

Thus, the function $1/f^\nu(\vartheta) \phi^\beta [\lambda(\vartheta) (\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau)^{p-1}]$ is decreasing. Therefore, we have for $w \leq \vartheta$ that

$$\frac{1}{f^\nu(w)} \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq \frac{1}{f^\nu(\vartheta)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right], \tag{45}$$

and then,

$$f^\nu(\vartheta) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq f^\nu(w) \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]. \tag{46}$$

Integrating the last inequality over ϑ from w to $\sigma(t)$, we have

$$\begin{aligned} & \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \int_w^{\sigma(t)} f^\nu(\vartheta) \Delta \vartheta \\ & \geq f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \Delta \vartheta, \end{aligned} \tag{47}$$

and then,

$$\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta \vartheta \right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \Delta \vartheta} \geq 1. \tag{48}$$

Since the function $1/f^\nu(\vartheta) \phi^\beta [\lambda(\vartheta) (\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau)^{p-1}]$ is decreasing, we have for $\vartheta \leq t$ that

$$\frac{1}{f^\nu(\vartheta)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq \frac{1}{f^\nu(t)} \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta \tau \right)^{p-1} \right], \tag{49}$$

and then,

$$f^\nu(t) \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta \tau \right)^{p-1} \right] \geq f^\nu(\vartheta) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta \tau \right)^{p-1} \right]. \tag{50}$$

By integrating (50) over ϑ from w to $\sigma(t)$, we have

$$\begin{aligned} f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta \\ \geq \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta. \end{aligned} \quad (51)$$

Thus,

$$\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \leq 1. \quad (52)$$

From (48) and (52), we observe that

$$\begin{aligned} \frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \\ \geq \dots \geq 1 \geq \dots \geq \frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta}. \end{aligned} \quad (53)$$

Since $S(\cdot)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, we get that one of

$$\begin{aligned} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right), \\ S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right), \end{aligned} \quad (54)$$

is maximum (where $S(1) = 1$), and it is in the form

$$\begin{aligned} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \\ = \max \left\{ S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right); S \right. \\ \left. \cdot \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right\}, \end{aligned} \quad (55)$$

which is (38). Similarly, with respect to ψ when $0 \leq q \leq 1$,

we get

$$\begin{aligned} S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \\ = \max \left\{ S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(w) \left(\int_w^{\sigma(w)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(w) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right); S \right. \\ \left. \cdot \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\xi) \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\xi) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right\}, \end{aligned} \quad (56)$$

which is (39).

Throughout the article, we will assume that the functions are nonnegative rd-continuous functions on $[w, \infty)_{\mathbb{T}} := [w, \infty) \cap \mathbb{T}$ and the integrals considered are assumed to exist. Now, we will present and justify our main findings.

3. Main Results

Theorem 7. Let $w \in \mathbb{T}$, $0 \leq p, q \leq 1$, and λ, ψ be positive and decreasing functions. Assume that $\omega, \bar{\omega}, \phi, \varphi$ are positive functions such that ϕ, φ are increasing, concave, and supermultiplicative functions with

$$\begin{aligned} A\phi \leq \omega \leq B\phi, \\ A\varphi \leq \bar{\omega} \leq B\varphi, \end{aligned} \quad (57)$$

where A, B are positive constants. If f, g are positive and non-decreasing functions and $\beta > 1, \nu > 1$ with $1/\beta + 1/\nu = 1$, then

$$\begin{aligned} \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi) P(t) Q(\xi) \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right] \\ \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^p f(t) \\ \times \bar{\omega} \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right] \\ \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^q g(\xi) \Delta t \Delta \xi \geq \nu \frac{A^2}{B^2} C(p, q, r, s, \nu) \\ \cdot \left(\int_w^{\sigma(r)} \omega^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta) \Delta\vartheta \right)^{1/\beta} \\ \times \left(\int_w^{\sigma(s)} \bar{\omega}^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] (\sigma(s) - y) \Delta y \right)^{1/\beta}, \end{aligned} \quad (58)$$

holds for all $r, s \in \omega, \infty)_{\mathbb{T}}$, where

$$\begin{aligned} \Phi(t, \xi) &= \frac{f(t)g(\xi)}{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)^{1/\nu} \left(\int_w^{\sigma(\xi)} g^\nu(y)\Delta y\right)^{1/\nu}} \\ &\times S \left(\frac{\left(\int_w^{\sigma(r)} f^\nu(\vartheta)\Delta\vartheta\right) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(t) \int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} (\sigma(r) - \vartheta)\Delta\vartheta} \right) \\ &\times S \left(\frac{\left(\int_w^{\sigma(s)} g^\nu(\vartheta)\Delta\vartheta\right) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{q-1}\right]}{g^\nu(\xi) \int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]} (\sigma(s) - y)\Delta y \right), \end{aligned} \tag{59}$$

$$P(t) = \int_w^{\sigma(t)} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\vartheta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta \right) f(\vartheta)\Delta\vartheta, \tag{61}$$

$$Q(\xi) = \int_w^{\sigma(\xi)} S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y)\Delta y\right) \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]}{g^\nu(y) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]} \Delta y \right) g(y)\Delta y, \tag{62}$$

with

$$C(p, q, r, s, \nu) = \frac{\phi(p)\phi(q)}{\nu} \left(\int_w^{\sigma(r)} f^\nu(\vartheta)\Delta\vartheta\right)^{1/\nu} \left(\int_w^{\sigma(s)} g^\nu(\vartheta)\Delta\vartheta\right)^{1/\nu}, \tag{60}$$

such that

$$\begin{aligned} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) &= \max \left\{ S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(w) \left(\int_w^{\sigma(w)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(w) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right); S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(t) \left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(t) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) \right\}, \\ S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y)\Delta y\right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{q-1}\right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]} \Delta y} \right) &= \max \left\{ S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y)\Delta y\right) \phi^\beta \left[\psi(w) \left(\int_w^{\sigma(w)} \psi(\tau)\Delta\tau\right)^{q-1}\right]}{g^\nu(w) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]} \Delta y} \right); S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y)\Delta y\right) \phi^\beta \left[\psi(\xi) \left(\int_w^{\sigma(\xi)} \psi(\tau)\Delta\tau\right)^{q-1}\right]}{g^\nu(\xi) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{q-1}\right]} \Delta y} \right) \right\}. \end{aligned} \tag{63}$$

Proof. Applying (27) with $\gamma = p$, we have

$$\left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^p \geq p \int_w^{\sigma(t)} \lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1} \Delta\vartheta. \tag{64}$$

□

Multiplying the last inequality by

$$S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) f(t), \tag{65}$$

we get (where f is nondecreasing) that

$$\begin{aligned} &S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) \left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^p f(t) \\ &\geq p \int_w^{\sigma(t)} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) \\ &\quad \times \lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1} f(t)\Delta\vartheta \\ &\geq p \int_w^{\sigma(t)} S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]} \Delta\vartheta} \right) \\ &\quad \times \lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1} f(\vartheta)\Delta\vartheta. \end{aligned} \tag{66}$$

Applying (36) on the right hand side of (72), we observe that

$$\begin{aligned}
 P(t)\omega & \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^v(\vartheta)\Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]}{f^v(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau \right)^p f(t) \\
 & \times Q(\xi)\omega \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^v(y)\Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau \right)^{q-1} \right]}{g^v(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau)\Delta\tau \right)^q g(\xi) \geq A^2 \phi(p) \left(\int_w^{\sigma(t)} f^v(\vartheta)\Delta\vartheta \right)^{1/v} \\
 & \cdot \left(\int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\vartheta \right)^{1/\beta} \\
 & \times \varphi(q) \left(\int_w^{\sigma(\xi)} g^v(y)\Delta y \right)^{1/v} \left(\int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] \Delta y \right)^{1/\beta}.
 \end{aligned} \tag{73}$$

Multiplying (73) by

$$\begin{aligned}
 \Phi(t, \xi) & = \frac{f(t)g(\xi)}{\left(\int_w^{\sigma(t)} f^v(\vartheta)\Delta\vartheta \right)^{1/v} \left(\int_w^{\sigma(\xi)} g^v(y)\Delta y \right)^{1/v}} \\
 & \times S \left(\frac{\left(\int_w^{\sigma(r)} f^v(\vartheta)\Delta\vartheta \right) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]}{f^v(t) \int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta)\Delta\vartheta} \right) \\
 & \times S \left(\frac{\left(\int_w^{\sigma(s)} g^v(\vartheta)\Delta\vartheta \right) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau \right)^{q-1} \right]}{g^v(\xi) \int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] (\sigma(s) - y)\Delta y} \right),
 \end{aligned} \tag{74}$$

and then, taking the integration over t from w to $\sigma(r)$ and the integration over ξ from w to $\sigma(s)$, we get to obtain

$$\begin{aligned}
 & \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi)P(t)Q(\xi) \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^v(\vartheta)\Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]}{f^v(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau)\Delta\tau \right)^p f(t) \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^v(y)\Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau \right)^{q-1} \right]}{g^v(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \left. \left. \left. \right] \right] \right],
 \end{aligned} \tag{75}$$

$$\begin{aligned}
 & \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau)\Delta\tau \right)^q g(\xi) \Big] \Delta t \Delta \xi \\
 & \geq A^2 \phi(p)\varphi(q) \int_w^{\sigma(r)} \left(\int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\vartheta \right)^{1/\beta} f(t) \\
 & \times S \left(\frac{\left(\int_w^{\sigma(r)} f^v(\vartheta)\Delta\vartheta \right) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]}{f^v(t) \int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta)\Delta\vartheta} \right) \Delta t \\
 & \times \int_w^{\sigma(s)} \left(\int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] \Delta y \right)^{1/\beta} g(\xi) \\
 & \times S \left(\frac{\left(\int_w^{\sigma(s)} g^v(\vartheta)\Delta\vartheta \right) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau)\Delta\tau \right)^{q-1} \right]}{g^v(\xi) \int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] (\sigma(s) - y)\Delta y} \right) \Delta \xi.
 \end{aligned} \tag{76}$$

Applying the integration by parts formula on the term

$$\int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta)\Delta\vartheta, \tag{77}$$

with $u(\vartheta) = (\sigma(r) - \vartheta)$ and $v^\Delta(\vartheta) = \phi^\beta[\lambda(\vartheta)(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau)^{p-1}]$, we get

$$\begin{aligned}
 & \int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta)\Delta\vartheta \\
 & = (\sigma(r) - \vartheta)v(\vartheta)|_w^{\sigma(r)} + \int_w^{\sigma(r)} v^\sigma(\vartheta)\Delta\vartheta,
 \end{aligned} \tag{78}$$

where $v(\vartheta) = \int_w^\vartheta \phi^\beta[\lambda(\theta)(\int_w^{\sigma(\theta)} \lambda(\tau)\Delta\tau)^{p-1}]\Delta\theta$ and then where $v(w) = 0$,

$$\begin{aligned}
 & \int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta)\Delta\vartheta \\
 & = \int_w^{\sigma(r)} \int_w^{\sigma(\vartheta)} \phi^\beta \left[\lambda(\theta) \left(\int_w^{\sigma(\theta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\theta \Delta\vartheta.
 \end{aligned} \tag{79}$$

Similarly, we see that

$$\begin{aligned}
 & \int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{q-1} \right] (\sigma(s) - y)\Delta y \\
 & = \int_w^{\sigma(s)} \int_w^{\sigma(y)} \phi^\beta \left[\psi(\theta) \left(\int_w^{\sigma(\theta)} \psi(\tau)\Delta\tau \right)^{q-1} \right] \Delta\theta \Delta y.
 \end{aligned} \tag{80}$$

Substituting (79) and (80) into (76), we observe that

$$\begin{aligned}
 & \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi) P(t) Q(\xi) \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^p f(t) \left. \right] \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^q g(\xi) \left. \right] \Delta t \Delta \xi \geq A^2 \phi(p) \varphi(q) \int_w^{\sigma(r)} \\
 & \cdot \left(\int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta \right)^{1/\beta} f(t) \\
 & \times S \left(\frac{\left(\int_w^{\sigma(r)} f^\nu(\vartheta) \Delta\vartheta \right) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\zeta}{f^\nu(t) \int_w^{\sigma(r)} \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\theta) \left(\int_w^{\sigma(\theta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\theta \Delta\vartheta} \right) \Delta t \\
 & \times \int_w^{\sigma(s)} \left(\int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y \right)^{1/\beta} g(\xi) \\
 & \times S \left(\frac{\left(\int_w^{\sigma(s)} g^\nu(\vartheta) \Delta\vartheta \right) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta\eta}{g^\nu(\xi) \int_w^{\sigma(s)} \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(\theta) \left(\int_w^{\sigma(\theta)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta\theta \Delta y} \right) \Delta \xi,
 \end{aligned} \tag{81}$$

and then, by applying (36), (79), and (80), we see that

$$\begin{aligned}
 & \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi) P(t) Q(\xi) \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^p f(t) \left. \right] \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^q g(\xi) \left. \right] \Delta t \Delta \xi \geq A^2 \phi(p) \varphi(q) \\
 & \times \left(\int_w^{\sigma(r)} f^\nu(\vartheta) \Delta\vartheta \right)^{1/\nu} \left(\int_w^{\sigma(s)} g^\nu(\vartheta) \Delta\vartheta \right)^{1/\nu} \\
 & \times \left(\int_w^{\sigma(r)} \int_w^{\sigma(\vartheta)} \phi^\beta \left[\lambda(\theta) \left(\int_w^{\sigma(\theta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\theta \Delta\vartheta \right)^{1/\beta} \\
 & \times \left(\int_w^{\sigma(s)} \int_w^{\sigma(y)} \phi^\beta \left[\psi(\theta) \left(\int_w^{\sigma(\theta)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta\theta \Delta y \right)^{1/\beta} \\
 & = A^2 \phi(p) \varphi(q) \left(\int_w^{\sigma(r)} f^\nu(\vartheta) \Delta\vartheta \right)^{1/\nu} \left(\int_w^{\sigma(s)} g^\nu(\vartheta) \Delta\vartheta \right)^{1/\nu} \\
 & \times \left(\int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta) \Delta\vartheta \right)^{1/\beta} \\
 & \times \left(\int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] (\sigma(s) - y) \Delta y \right)^{1/\beta},
 \end{aligned} \tag{82}$$

and thus, we have from (57) and (60) that

$$\begin{aligned}
 & \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi) P(t) Q(\xi) \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^p f(t) \left. \right] \\
 & \times \omega \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^q g(\xi) \left. \right] \Delta t \Delta \xi \geq \nu \frac{A^2}{B^2} C(p, q, r, s, \nu) \\
 & \cdot \left(\int_w^{\sigma(r)} \omega^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta) \Delta\vartheta \right)^{1/\beta} \\
 & \times \left(\int_w^{\sigma(s)} \omega^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] (\sigma(s) - y) \Delta y \right)^{1/\beta},
 \end{aligned} \tag{83}$$

which is (58).

Remark 8. If $A = B = 1$, then $\omega = \phi, \bar{\omega} = \varphi$ and then we have for $w \in \mathbb{T}, 0 \leq p, q \leq 1, \beta > 1, \nu > 1$ such that $1/\beta + 1/\nu = 1$ that

$$\begin{aligned}
 & \int_w^{\sigma(s)} \int_w^{\sigma(r)} \Phi(t, \xi) P(t) Q(\xi) \\
 & \times \phi \left[S \left(\frac{\left(\int_w^{\sigma(t)} f^\nu(\vartheta) \Delta\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_w^{\sigma(\zeta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_w^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] \Delta\vartheta} \right) \right. \\
 & \times \frac{1}{P(t)} \left(\int_w^{\sigma(t)} \lambda(\tau) \Delta\tau \right)^p f(t) \left. \right] \\
 & \times \varphi \left[S \left(\frac{\left(\int_w^{\sigma(\xi)} g^\nu(y) \Delta y \right) \phi^\beta \left[\psi(\eta) \left(\int_w^{\sigma(\eta)} \psi(\tau) \Delta\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_w^{\sigma(\xi)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] \Delta y} \right) \right. \\
 & \times \frac{1}{Q(\xi)} \left(\int_w^{\sigma(\xi)} \psi(\tau) \Delta\tau \right)^q g(\xi) \left. \right] \Delta t \Delta \xi \geq \nu C(p, q, r, s, \nu) \\
 & \cdot \left(\int_w^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_w^{\sigma(\vartheta)} \lambda(\tau) \Delta\tau \right)^{p-1} \right] (\sigma(r) - \vartheta) \Delta\vartheta \right)^{1/\beta} \\
 & \times \left(\int_w^{\sigma(s)} \phi^\beta \left[\psi(y) \left(\int_w^{\sigma(y)} \psi(\tau) \Delta\tau \right)^{q-1} \right] (\sigma(s) - y) \Delta y \right)^{1/\beta}.
 \end{aligned} \tag{84}$$

Remark 9. If $\mathbb{T} \ll \mathbb{R}$, $\sigma(s) = s$, and $\omega = 0$, the inequality (58) becomes

$$\begin{aligned} & \int_0^s \int_0^r \Phi(t, \xi) P(t) Q(\xi) \\ & \times \omega \left[S \left(\frac{\left(\int_0^t f^\nu(\vartheta) d\vartheta \right) \phi^\beta \left[\lambda(\zeta) \left(\int_0^\zeta \lambda(\tau) d\tau \right)^{p-1} \right]}{f^\nu(\zeta) \int_0^t \phi^\beta \left[\lambda(\vartheta) \left(\int_0^\vartheta \lambda(\tau) d\tau \right)^{p-1} \right] d\vartheta} \right) \right] \\ & \times \frac{1}{P(t)} \left(\int_0^t \lambda(\tau) d\tau \right)^p f(t) \\ & \times \omega \left[S \left(\frac{\left(\int_0^\xi g^\nu(y) dy \right) \phi^\beta \left[\psi(\eta) \left(\int_0^\eta \psi(\tau) d\tau \right)^{q-1} \right]}{g^\nu(\eta) \int_0^\xi \phi^\beta \left[\psi(y) \left(\int_0^y \psi(\tau) d\tau \right)^{q-1} \right] dy} \right) \right] \\ & \times \frac{1}{Q(\xi)} \left(\int_0^\xi \psi(\tau) d\tau \right)^q g(\xi) \geq \nu \frac{A^2}{B^2} C(p, q, r, s, \nu) \\ & \cdot \left(\int_0^r \omega^\beta \left[\lambda(\vartheta) \left(\int_0^\vartheta \lambda(\tau) d\tau \right)^{p-1} \right] (r - \vartheta) d\vartheta \right)^{1/\beta} \\ & \times \left(\int_0^s \omega^\beta \left[\psi(y) \left(\int_0^y \psi(\tau) d\tau \right)^{q-1} \right] (s - y) dy \right)^{1/\beta}. \end{aligned} \quad (85)$$

This inequality is new in continuous calculus.

Remark 10. As a special case of (84) when $\mathbb{T} \ll \mathbb{N}$, $\varphi(\vartheta) = \phi(\vartheta) = \vartheta$, $f(\vartheta) = g(\vartheta) = 1$, $\omega = 1$, and $\beta = \nu = 2$, we obtain (11) demonstrated in [36].

4. Conclusion and Future Work

In this paper, we studied some new generalizations of reversed dynamic inequalities of Hilbert-type via supermultiplicative functions on time scales by applying reverse Hölder inequalities and the Specht's ratio function. We generalized the inequality proved by Zhao and Cheung [6] by using a supermultiplicative function which the identity map represents a special case of it and also by using the power greater than 2 when we used different powers β , $\nu > 1$ with $1/\beta + 1/\nu = 1$, where we get the special case when $\beta = \nu = 2$. Also, we added some increasing functions for generalizing the results where we get the results when we take the identity function as a special case. In the future, we will continue to generalize more reversed dynamic inequalities of Hilbert-type by using Kantorovich's ratio and n-tuple fractional integral. In particular, such inequalities can be introduced by using fractional integrals and fractional derivatives of the Riemann-Liouville type on time scales. In addition to this, we may generalize these results to be with multidimensional Hilbert-type inequalities via supermultiplicative functions on time scales. It will also be very enjoyable to introduce such inequalities in quantum calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

Software and writing-original draft were contributed by H. M. Rezk, G. AlNemer, and A. I. Saied. Writing-review and editing was contributed by H. M. Rezk and M. Zakarya. All authors have read and agreed to the published version of the manuscript.

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