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Research Article

Analytical Investigation of Nonlinear Fractional Harry Dym and Rosenau-Hyman Equation via a Novel Transform

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We use a new integral transform approach to solve the fractional Harry Dym equation and fractional Rosenau-Hyman equation in this work. The Elzaki transform and the integral transformation are combined in the suggested method (ET). To handle two nonlinear problems, we first construct the Elzaki transforms of the Caputo fractional derivative (CFD) and Atangana-Baleanu fractional derivative (ABFD). The ultimate purpose of this study is to find an error analysis that demonstrates that our final result converges to the exact and approximate result. The convergent series form solution demonstrates the method's efficiency in resolving several types of fractional differential equations. Furthermore, the solutions obtained in this study agree well with the exact solutions; thus, this strategy is powerful and efficient as an alternate way for obtaining approximate solutions to both linear and nonlinear fractional differential equations.

1. Introduction

Fractional calculus FC history dates back 300 years. FC originated with Leibniz's usage of the nth derivative notation in his papers in 1695. L'Hopital raises a query from Leibniz about the result of his nth derivative notation if the order of "n" is 1 /2 [1]. Many phenomena in engineering and other fields can be effectively represented by models based on fractional calculus, that is, the theory of fractional derivatives and integrals of fractional noninteger order. Respectable interest in fractional calculus has been utilised in several studies in recent years, such as regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, viscoelasticity, electrical circuits, electro-analytical chemistry, biology, and control theory [2–5].

Due to their prevalence in a wide range of applications and accurate description of nonlinear processes, researchers are increasingly focusing on fractional order differential equations, particularly fractional partial differential equations (FPDEs). FPDEs are the most common mathematical tools used to simulate diverse physical phenomena in applied sciences such as physics, engineering, and other social sciences. Many applications of science and engineering, including as material sciences, biology, chemistry, fluid dynamics, chemical kinetics, and many other physical processes, use modelling in the form of FPDE systems [6–10]. For the solution of fractional-order PDE problems, different analytical and numerical methodologies have been developed in the literature. The numerical schemes are a finite difference scheme with nonuniform time steps [11–13], a higher order numerical scheme [14], an implicit finite-difference scheme [15], a compact difference scheme [16], Adomian decomposition method [17], homotopy analysis transform method [18], fractional-order reduced differential transform method [19], variational iteration method [20], natural transform decomposition method [21], Elzaki transform decomposition method [22], iterative methods [23–25], and much more [26–30]. The abovementioned techniques have the straight forward implementations to both linear and nonlinear FDEs.

In the present study, we implement the Elzaki transform in connection with the CFD and ABC operators to solve two nonlinear problems. We consider fractional Harry Dym equation and fractional Rosenau-Hyman equation of the form

$$D_\tau^\rho \psi(\upsilon,\tau) = \psi^3(\upsilon,\tau)\psi_{\upsilon\upsilon\upsilon}(\upsilon,\tau), \tag{1}$$

having initial source

$$\psi(\upsilon,0) = \left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{2/3}, \tag{2}$$

and

$$D_\tau^\rho \psi(\upsilon,\tau) = \psi(\upsilon,\tau)\psi_{\upsilon\upsilon\upsilon}(\upsilon,\tau) + \psi(\upsilon,\tau)\psi_\upsilon(\upsilon,\tau) + 3\psi_\upsilon(\upsilon,\tau)\psi_{\upsilon\upsilon}(\upsilon,\tau), \tag{3}$$

having initial source

$$\psi(\upsilon,0) = -\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right). \tag{4}$$

The Harry Dym is a crucial dynamical equation that is used in a variety of physical systems. The Harry Dym equation was initially published in Kruskal and Moser [31] and is credited to Harry Dym in an unpublished study from 1973-1974. It denotes a system in which dispersion and nonlinearity are inextricably linked. Harry Dym is a totally integrable nonlinear evolution equation that obeys an infinite number of conversion rules but lacks the Painleve property. The Harry Dym equation is closely related to the Korteweg-de Vries equation, and this equation has been used to hydrodynamic problems [32]. The Sturm-Liouville operator is linked to the Lax pair of the Harry Dym equation. This operator is spectrally transformed into the Schrodinger operator by the Liouville transformation [33]. Rosenau and Hyman [34] found the Rosenau-Hyman equation, which arises in the creation of patterns in liquid drops with compaction solutions. The Rosenau-Hyman equation compact on investigations is useful in applied sciences and mathematical physics [35–38].

The following is how the rest of the paper is structured: we begin with basic preliminaries and definitions of fractional calculus in Section 2. The proposed method's general methodology is introduced in Section 3. Section 4 focuses on applying the approach to a set of test problems, using graphs and tables to demonstrate the technique's efficiency. The discussion and conclusion of this work were delivered in Section 5.

2. Preliminaries

In this section, we mention the following basic definitions of fractional calculus.

Definition 1. The fractional derivative in Caputo manner (CFD) is given as [39]

$${}_0^C D_\tau^\rho(\kappa(\tau)) = \begin{cases} \dfrac{1}{\Gamma(m-\rho)}\displaystyle\int_0^\tau \frac{\kappa^m(\eta)}{(\tau-\eta)^{\rho+1-m}}d\eta, & m-1<\rho<m, \\ \dfrac{d^m}{d\tau^m}\kappa(\tau), & \rho = m. \end{cases} \tag{5}$$

Definition 2. The Atangana-Baleanu Caputo operator (ABC) is defined as [40]

$${}_m^{\mathrm{ABC}} D_\tau^\rho(\kappa(\tau)) = \frac{N(\rho)}{1-\rho}\int_m^\tau \kappa'(\eta)E_\rho\left[-\frac{\rho(\tau-\eta)^\rho}{1-\rho}\right]d\eta, \tag{6}$$

where $\kappa \in H^1(\alpha,\beta), \beta>\alpha, \rho\in[0,1]$. A normalisation function equal to 1 when $\rho=0$ and $\rho=1$ is represented by $N(\rho)$ in Eq. (6).

Definition 3. The fractional integral operator in ABC manner is given as [40]

$${}_m^{\mathrm{ABC}} I_\tau^\rho(\kappa(\tau)) = \frac{1-\rho}{N(\rho)}\kappa(\tau) + \frac{\rho}{\Gamma(\rho)N(\rho)}\int_m^\tau \kappa(\eta)(\tau-\eta)^{\rho-1}d\eta. \tag{7}$$

Definition 4. For exponential function in set A, the Elzaki transform is given as [41, 42]

$$A = \left\{\kappa(\tau):\ \exists G, p_1, p_2>0, |\kappa(\tau)| < Ge^{|\tau|/p_j}, \text{if } \tau\in(-1)^j\times[0,\infty)\right\}. \tag{8}$$

G is a finite number, but p_1 and p_2 may be finite or infinite for a function selected in the set.

Definition 5. The Elzaki transform of $\kappa(\tau)$ is given as [42]

$$\mathscr{E}\{\kappa(\tau)\}(\mu) = \tilde{U}(\mu) = \mu\int_0^\infty e^{-\tau/\mu}\kappa(\tau)d\tau, \tag{9}$$

where $\tau\ge 0, p_1\le\mu\le p_2$.

Theorem 6. *(Elzaki transformation convolution theorem, [43]) The following equality holds:*

$$\mathscr{E}\{\kappa * \nu\} = \frac{1}{\mu}\mathscr{E}(\kappa)\mathscr{E}(\nu), \tag{10}$$

where $\mathscr{E}\{.\}$ represents Elzaki transform.

Definition 7. The Elzaki transform of ${}_0^C D_\tau^\rho(\kappa(\tau))$ CFD operator is as [44]

$$\mathscr{E}\left\{{}_0^C D_\tau^\rho(\kappa(\tau))\right\}(\mu) = \mu^{-\rho}\tilde{U}(\mu) - \sum_{k=0}^{m-1}\mu^{2-\rho+k}\kappa^k(0), \tag{11}$$

where $m-1<\rho<m$.

Theorem 8. *The Elzaki transform of ${}_m^{ABC}D_\tau^\rho(\kappa(\tau))$ ABC operator is as*

$$\mathscr{E}\left\{{}_m^{ABC}D_\tau^\rho(\kappa(\tau))\right\}(\mu) = \frac{N(\rho)\mu}{\rho\mu^\rho + 1-\rho}\left(\frac{\tilde{U}(\mu)}{\mu} - \mu\kappa(0)\right), \tag{12}$$

where $\mathscr{E}\{\kappa(\tau)\}\mu = \tilde{U}(\mu)$.

Proof. From Definition 2, we get

$$\mathscr{E}\{{}^{ABC}_{m}D^{\rho}_{\tau}(\kappa(\tau))\}(\mu) = \mathscr{E}\left\{\frac{N(\rho)}{1-\rho}\int_0^{\tau}\kappa'(\eta)E_{\rho}\left[-\frac{\rho(\tau-\eta)^{\rho}}{1-\rho}\right]d\eta\right\}(\mu). \tag{13}$$

Then, from Elzaki transform definition and its convolution, we obtain

$$\begin{aligned}\mathscr{E}\{{}^{ABC}_{m}D^{\rho}_{\tau}(\kappa(\tau))\}(\mu) &= \mathscr{E}\left\{\frac{N(\rho)}{1-\rho}\int_0^{\tau}\kappa'(\eta)E_{\rho}\left[-\frac{\rho(\tau-\eta)^{\rho}}{1-\rho}\right]d\eta\right\}\\ &= \frac{N(\rho)}{1-\rho}\frac{1}{\mu}\mathscr{E}\left\{\kappa'(\eta)\right\}\mathscr{E}\left\{E_{\rho}\left[-\frac{\rho\tau^{\rho}}{1-\rho}\right]d\eta\right\}\\ &= \frac{N(\rho)}{1-\rho}\left[\frac{\tilde{U}(\mu)}{\mu}-\mu\kappa(0)\right]\left[\int_0^{\infty}e^{-1/\mu}E_{\rho}\left[-\frac{\rho\tau^{\rho}}{1-\rho}\right]d\tau\right]\\ &= \frac{N(\rho)\mu}{\rho\mu^{\rho}+1-\rho}\left[\frac{\tilde{U}(\mu)}{\mu}-\mu\kappa(0)\right].\end{aligned} \tag{14}$$

□

3. Description of the Technique via a New Integral Transform

In this part, we presented the general methodology used in this article to solve fractional nonlinear PDE as

$$\begin{gathered}D^{\rho}_{\tau}\psi(\upsilon,\tau)+L(\psi(\upsilon,\tau))+N(\psi(\upsilon,\tau))=\theta(\upsilon,\tau),\\ (\upsilon,\tau)\in[0,1]\times[0,T],\kappa-1<\rho<\kappa,\end{gathered} \tag{15}$$

with initial source

$$\frac{\partial^{z}\psi}{\partial\tau^{z}}(\upsilon,0)=\kappa_z(\upsilon), z=0,1,\cdots,\kappa-1, \tag{16}$$

and the boundary sources

$$\psi(0,\tau)=\gamma_0(\tau),\psi(\upsilon,\tau)=\gamma_1(\tau),\tau\ge 0, \tag{17}$$

Here, known functions are $\kappa_z, \theta, \gamma_0$, and γ_1. In Eq. (15), $D^{\rho}_{\tau}\psi(\upsilon,\tau)$ represents the Caputo or ABC fractional derivatives whereas $L(.)$ and $N(.)$ are linear and nonlinear terms. (1)-(2) and (3)-(4) represent the problems to be solved. By means of Elzaki transform of CFD in Eq. (11) and ABC in Eq. (12), we take $E\{\psi(\upsilon,\tau)\}(\mu)=\tilde{\zeta}(\upsilon,\mu)$ in Eq. (15). Thus, by means of Caputo fractional derivative, we get

$$\tilde{\zeta}(\upsilon,\mu)=\mu^{\rho}\left(\widetilde{\theta}(\upsilon,\mu)-\mathscr{E}[L(\psi(\upsilon,\tau))+N(\psi(\upsilon,\tau))]\right)+\mu^2\psi(\upsilon,0). \tag{18}$$

Also by means of ABC derivative, we get

$$\tilde{\zeta}(\upsilon,\mu)=\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\left(\widetilde{\theta}(\upsilon,\mu)-\mathscr{E}[L(\psi(\upsilon,\tau))+N(\psi(\upsilon,\tau))]\right)+\mu^2\psi(\upsilon,0). \tag{19}$$

Here, $\mathscr{E}[\theta(\upsilon,\tau)]=\widetilde{\theta}(\upsilon,\mu)$. Now by taking the Elzaki transform of the boundary conditions, we obtain

$$\mathscr{E}[\gamma_0(\tau)]=\tilde{\zeta}(0,\mu), \mathscr{E}[\gamma_1(\tau)]=\tilde{\zeta}(1,\mu), \mu\ge 0. \tag{20}$$

We get the solution of Eqs. (15)-(17) by means of perturbation technique

$$\tilde{\zeta}(\upsilon,\mu)=\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}_{\mathscr{E}}(\upsilon,\mu), \mathscr{E}=0,1,2,\cdots. \tag{21}$$

In Eq. (15), the nonlinear terms are calculated as

$$N[\psi(\upsilon,\tau)]=\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau), \tag{22}$$

and the terms $\upsilon_{\mathscr{E}}(\upsilon,\tau)$ are taken in [45] as

$$\upsilon_{\mathscr{E}}(\psi_0,\psi_1,\cdots,\psi_{\mathscr{E}})=\frac{1}{\mathscr{E}!}\frac{\partial^{\mathscr{E}}}{\partial\omega^{\mathscr{E}}}\left[N\left(\sum_{i=0}^{\infty}\omega^i\psi_i\right)\right]_{\lambda=0}, \mathscr{E}=0,1,2,\cdots. \tag{23}$$

For Caputo operator, the solution is determined as by putting Eqs. (21) and (22) into Eq. (18),

$$\begin{aligned}\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}(\upsilon,\mu)=-\mathscr{X}\mu^{\rho}\left(\mathscr{E}\left[L\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(\upsilon,\tau)\right)+\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right]\right)\\ +\mu^{\rho}\left(\widetilde{\theta}(\upsilon,\mu)\right)+\mu^2\psi(\upsilon,0).\end{aligned} \tag{24}$$

Also for Atangana-Baleanu operator, the solution is determined as by putting Eqs. (21) and (22) into Eq. (19),

$$\begin{aligned}\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}(\upsilon,\mu)=-\mathscr{X}\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\left(\mathscr{E}\left[L\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(\upsilon,\tau)\right)+\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right]\right)\\ +\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\left(\widetilde{\theta}(\upsilon,\mu)\right)+\mu^2\psi(\upsilon,0).\end{aligned} \tag{25}$$

Then, by solving (24) and (25) in terms of $\mathscr{X}$, the given Caputo homotopies are obtained:

$$\begin{aligned}\mathscr{X}^0 &: \tilde{\zeta}_0(\upsilon,\mu)=\mu^{\rho}\left(\widetilde{\theta}(\upsilon,\mu)\right)+\mu^2\psi(\upsilon,0),\\ \mathscr{X}^1 &: \tilde{\zeta}_1(\upsilon,\mu)=-\mu^{\rho}\mathscr{E}[L(\psi_0(\upsilon,\tau))+\varphi_0(\upsilon,\tau)],\\ \mathscr{X}^2 &: \tilde{\zeta}_2(\upsilon,\mu)=-\mu^{\rho}\mathscr{E}[L(\psi_1(\upsilon,\tau))+\varphi_1(\upsilon,\tau)],\\ &\vdots\\ \mathscr{X}^{n+1} &: \tilde{\zeta}_{n+1}(\upsilon,\mu)=-\mu^{\rho}\mathscr{E}[L(\psi_n(\upsilon,\tau))+\varphi_n(\upsilon,\tau)].\end{aligned} \tag{26}$$

In addition, the ABC homotopies are obtained as given:

$$\begin{aligned}
\mathscr{X}^0 &: \tilde{\zeta}_0(\upsilon,\mu) = \left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\widetilde{\theta}(\upsilon,\mu) + \mu^2\psi(\upsilon,0),\\
\mathscr{X}^1 &: \tilde{\zeta}_1(\upsilon,\mu) = -\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}[L(\psi_0(\upsilon,\tau)) + \varphi_0(\upsilon,\tau)],\\
\mathscr{X}^2 &: \tilde{\zeta}_2(\upsilon,\mu) = -\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}[L(\psi_1(\upsilon,\tau)) + \varphi_1(\upsilon,\tau)],\\
&\vdots\\
\mathscr{X}^{n+1} &: \tilde{\zeta}_{n+1}(\upsilon,\mu) = -\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}[L(\psi_n(\upsilon,\tau)) + \varphi_n(\upsilon,\tau)].
\end{aligned} \tag{27}$$

When $\mathscr{X} \longrightarrow 1$, we get Eqs. (26) and (27) approximate solution for Eqs. (24) and (25) as

$$\Delta_n(\upsilon,\mu) = \sum_{\sigma=0}^{n}\tilde{\zeta}_\sigma(\upsilon,\mu). \tag{28}$$

Now by taking inverse ET of Eq. (28), we get the approximate solution of Eq. (15)

$$\psi(\upsilon,\mu) \cong \psi_n(\upsilon,\tau) = \mathscr{E}^{-1}|\{\rho_n(\upsilon,\mu)\}. \tag{29}$$

4. Applications

In this part, we will solve problems in Eqs. (1)-(4) by implementing Elzaki transform. First, we implement Elzaki transform technique in combination with Caputo derivative to solve problem (1) having initial source (2). By taking the Elzaki transform, we get

$$\tilde{\zeta}(\upsilon,\mu) = \mu^\rho\mathscr{E}\left[\psi^3(\upsilon,\tau)\psi_{\upsilon\upsilon\upsilon}(\upsilon,\tau)\right] + \mu^2\psi(\upsilon,0). \tag{30}$$

Now applying Elzaki perturbation transform technique in Eq. (30), we obtain

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}_{\mathscr{E}}(\upsilon,\mu) = +\mu^2\psi(\upsilon,0). \tag{31}$$

On taking Elzaki inverse transform of Eq. (31), we get

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(\upsilon,\mu) = \mathscr{X}\mathscr{E}^{-1}\left[\mu^\rho\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right)\right]\right] + \mathscr{E}^{-1}\left[\mu^2\psi(\upsilon,0)\right]. \tag{32}$$

In Eq. (43), the $\upsilon_{\mathscr{E}}(.)$ denotes the nonlinear terms given in Eq. (24),

$$\begin{aligned}
\varphi_0(\psi) &= \psi_0^3(\psi_0)_{\upsilon\upsilon\upsilon},\\
\varphi_1(\psi) &= \psi_0^3(\psi_1)_{\upsilon\upsilon\upsilon} + 3\psi_0^2\psi_1(\psi_0)_{\upsilon\upsilon\upsilon},\\
&\vdots .
\end{aligned} \tag{33}$$

Thus by considering powers of $\mathscr{X}$, we get Caputo operator solution as

$$\begin{aligned}
\mathscr{X}^0 &: \psi_0(\upsilon,\tau) = \mathscr{E}^{-1}\left[\mu^2\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{2/3}\right] = \left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{2/3},\\
\mathscr{X}^1 &: \psi_1(\upsilon,\tau) = \mathscr{E}^{-1}[\mu^\rho\mathscr{E}[L(\varphi_0(\upsilon,\tau))]] = -b^{3/2}\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{-1/3}\frac{\tau^\rho}{\Gamma(\rho+1)},\\
\mathscr{X}^2 &: \psi_2(\upsilon,\tau) = \mathscr{E}^{-1}[\mu^\rho\mathscr{E}[L(\varphi_1(\upsilon,\tau))]] = -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{-4/3}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},\\
&\vdots .
\end{aligned} \tag{34}$$

The series form solution of the problem is given as

$$\psi(\upsilon,\tau) = \left(\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{2/3} - b^{3/2}\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{-1/3}\frac{\tau^\rho}{\Gamma(\rho+1)} - \frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}\upsilon\right)^{-4/3}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)} + \cdots\right), \tag{35}$$

which gives the solution at $(\rho = 1)$ as $\left(a - 3\sqrt{b}/2(\upsilon + b\tau)\right)^{2/3}$.

Now, we implement Elzaki transform technique in combination with Atangana-Baleanu operator to solve same problem. By taking the Elzaki transform, we get

$$\tilde{\zeta}(\upsilon,\mu) = \left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}\left[\psi^3(\upsilon,\tau)\psi_{\upsilon\upsilon\upsilon}(\upsilon,\tau)\right] + \mu^2\psi(\upsilon,0). \tag{36}$$

Now applying Elzaki perturbation transform technique to (36), we obtain

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}_{\mathscr{E}}(\upsilon,\mu) = \mathscr{X}\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right)\right] + \mu^2\psi(\upsilon,0). \tag{37}$$

On taking Elzaki inverse transform of Eq. (37), we get

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(\upsilon,\tau) = \mathscr{X}\mathscr{E}^{-1}\left[\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right)\right]\right] + \mathscr{E}^{-1}\left[\mu^2\psi(\upsilon,0)\right]. \tag{38}$$

In Eq.(38), $v_{\mathscr{E}}(.)$ denotes the nonlinear terms given in Eq. (23). By repeating the same process for nonlinear terms, we obtain the following terms:

$$
\begin{aligned}
\mathscr{X}^0 &: \psi_0(v,\tau) = \mathscr{E}^{-1}\left[\mu^2\left(a - \frac{3\sqrt{b}}{2}v\right)^{2/3}\right] = \left(a - \frac{3\sqrt{b}}{2}v\right)^{2/3}, \\
\mathscr{X}^1 &: \psi_1(v,\tau) = \mathscr{E}^{-1}\left[\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}[\varphi_0(v,\tau)]\right] = \left(\frac{-b^{3/2}\left(a - \left(3\sqrt{b}/2\right)v\right)^{-1/3}}{N(\rho)}\right)\left(\frac{\rho\tau^\rho}{\Gamma(\rho+1)} + 1 - \rho\right), \\
\mathscr{X}^2 &: \psi_2(v,\tau) = \mathscr{E}^{-1}\left[\left(\frac{\rho\mu^\rho + 1 - \rho}{N(\rho)}\right)\mathscr{E}[\varphi_1(v,\tau)]\right] = \left(\frac{-b^3/2\left(a - \left(3\sqrt{b}/2\right)v\right)^{-4/3}}{N^2(\rho)}\right)\left(\frac{(\rho\tau^\rho)^2}{\Gamma(2\rho+1)} + \frac{2\rho(1-\rho)\tau^\rho}{\Gamma(\rho+1)} + (1-\rho)^2\right), \\
&\vdots .
\end{aligned}
\tag{39}
$$

Thus, the approximate solution by means of ABC operator is given as

$$
\begin{aligned}
\psi(v,\tau) = \sum_{\sigma=0}^{n}\psi_\sigma(v,\tau) &= \left(a - \frac{3\sqrt{b}}{2}v\right)^{2/3} + \left(\frac{-b^{3/2}\left(a - \left(3\sqrt{b}/2\right)v\right)^{-1/3}}{N(\rho)}\right)\left(\frac{\rho\tau^\rho}{\Gamma(\rho+1)} + 1 - \rho\right) \\
&+ \left(\frac{-b^3/2\left(a - \left(3\sqrt{b}/2\right)v\right)^{-4/3}}{N^2(\rho)}\right)\left(\frac{(\rho\tau^\rho)^2}{\Gamma(2\rho+1)} + \frac{2\rho(1-\rho)\tau^\rho}{\Gamma(\rho+1)} + (1-\rho)^2\right) + \cdots,
\end{aligned}
\tag{40}
$$

which gives the solution at $(\rho = 1)$ as $\left(a - 3\sqrt{b}/2(v + b\tau)\right)^{2/3}$.

Second, we implement Elzaki transform technique in combination with Caputo derivative to solve problem (3) having initial source (4). By taking the Elzaki transform, we get

$$
\tilde{\zeta}(v,\mu) = \mu^\rho\mathscr{E}[\psi(v,\tau)\psi_{vvv}(v,\tau) + \psi(v,\tau)\psi_v(v,\tau) + 3\psi_v(v,\tau)\psi_{vv}(v,\tau)] + \mu^2\psi(v,0). \tag{41}
$$

Now applying Elzaki perturbation transform technique in Eq. (41), we obtain

$$
\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}_{\mathscr{E}}(v,\mu) = \mathscr{X}\mu^\rho\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(v,\tau)\right)\right] + \mu^2\psi(v,0). \tag{42}
$$

On taking Elzaki inverse transform of Eq. (42), we get

$$
\begin{aligned}
\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(v,\mu) = \mathscr{X}\mathscr{E}^{-1}&\left[\mu^\rho\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(v,\tau)\right)\right]\right] \\
&+ \mathscr{E}^{-1}\left[\mu^2\psi(v,0)\right].
\end{aligned}
\tag{43}
$$

Table 1: Comparison of absolute errors of proposed method solution at various fractional-orders with $a, b = 1$ for problem 1.

τ	υ	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1(\text{ETM}_{\text{CFD}})$	$\rho = 1(\text{ETM}_{\text{ABC}})$
	0.2	3.0035700000E-04	2.0035600000E-04	1.0035400000E-04	3.5300000000E-07	3.5300000000E-07
	0.4	3.0033900000E-04	2.0033800000E-04	1.0033600000E-04	3.3500000000E-07	3.3500000000E-07
0.01	0.6	3.0031800000E-04	2.0031700000E-04	1.0031500000E-04	3.1400000000E-07	3.1400000000E-07
	0.8	3.0029400000E-04	2.0029300000E-04	1.0029100000E-04	2.9000000000E-07	2.9000000000E-07
	1	3.0026800000E-04	2.0026700000E-04	1.0026500000E-04	2.6400000000E-07	2.6400000000E-07
	0.2	3.0071500000E-04	2.0071200000E-04	1.0071000000E-04	7.0700000000E-07	7.0700000000E-07
	0.4	3.0067800000E-04	2.0067500000E-04	1.0067300000E-04	6.7000000000E-07	6.7000000000E-07
0.02	0.6	3.0063600000E-04	2.0063300000E-04	1.0063100000E-04	6.2800000000E-07	6.2800000000E-07
	0.8	3.0058900000E-04	2.0058600000E-04	1.0058400000E-04	5.8100000000E-07	5.8100000000E-07
	1	3.0053500000E-04	2.0053200000E-04	1.0053000000E-04	5.2700000000E-07	5.2700000000E-07
	0.2	3.0107100000E-04	2.0106700000E-04	1.0106400000E-04	1.0600000000E-06	1.0600000000E-06
	0.4	3.0101600000E-04	2.0101200000E-04	1.0100900000E-04	1.0050000000E-06	1.0050000000E-06
0.03	0.6	3.0095300000E-04	2.0094900000E-04	1.0094600000E-04	9.4200000000E-07	9.4200000000E-07
	0.8	3.0088200000E-04	2.0087800000E-04	1.0087500000E-04	8.7100000000E-07	8.7100000000E-07
	1	3.0080100000E-04	2.0079700000E-04	1.0079400000E-04	7.9000000000E-07	7.9000000000E-07
	0.2	3.0142700000E-04	2.0142200000E-04	1.0141800000E-04	1.4130000000E-06	1.4130000000E-06
	0.4	3.0135400000E-04	2.0134900000E-04	1.0134500000E-04	1.3400000000E-06	1.3400000000E-06
0.04	0.6	3.0127000000E-04	2.0126500000E-04	1.0126100000E-04	1.2560000000E-06	1.2560000000E-06
	0.8	3.0117600000E-04	2.0117100000E-04	1.0116700000E-04	1.1620000000E-06	1.1620000000E-06
	1	3.0106700000E-04	2.0106200000E-04	1.0105800000E-04	1.0530000000E-06	1.0530000000E-06
	0.2	3.0178400000E-04	2.0177900000E-04	1.0177300000E-04	1.7670000000E-06	1.7670000000E-06
	0.4	3.0169200000E-04	2.0168700000E-04	1.0168100000E-04	1.6750000000E-06	1.6750000000E-06
0.05	0.6	3.0158700000E-04	2.0158200000E-04	1.0157600000E-04	1.5700000000E-06	1.5700000000E-06
	0.8	3.0146900000E-04	2.0146400000E-04	1.0145800000E-04	1.4520000000E-06	1.4520000000E-06
	1	3.0133300000E-04	2.0132800000E-04	1.0132200000E-04	1.3160000000E-06	1.3160000000E-06

In Eq. (43), $\upsilon_{\mathscr{E}}(.)$ denotes the nonlinear terms given in Eq. (24),

$$\begin{aligned}\varphi_0(\psi) &= \psi_0(\psi_0)_\upsilon + 3(\psi_0)_\upsilon(\psi_0)_{\upsilon\upsilon} + \psi_0(\psi_0)_{\upsilon\upsilon\upsilon},\\ \varphi_1(\psi) &= \psi_1(\psi_0)_\upsilon + \psi_0(\psi_1)_\upsilon + 3(\psi_1)_\upsilon(\psi_0)_{\upsilon\upsilon} + 3(\psi_0)_\upsilon(\psi_1)_{\upsilon\upsilon} + \psi_1(\psi_0)_{\upsilon\upsilon\upsilon} + \psi_0(\psi_1)_{\upsilon\upsilon\upsilon},\\ &\vdots\end{aligned} \quad (44)$$

Thus by considering powers of $\mathscr{X}$, we get Caputo operator solution as

$$\begin{aligned}\mathscr{X}^0 &: \psi_0(\upsilon,\tau) = \mathscr{E}^{-1}\left[\mu^2\left(-\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right)\right)\right] = -\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right),\\ \mathscr{X}^1 &: \psi_1(\upsilon,\tau) = \mathscr{E}^{-1}[\mu^\rho\mathscr{E}[L(\varphi_0(\upsilon,\tau))]] = -\frac{2}{3}c^2\sin\left(\frac{\upsilon}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)},\\ \mathscr{X}^2 &: \psi_2(\upsilon,\tau) = \mathscr{E}^{-1}[\mu^\rho\mathscr{E}[L(\varphi_1(\upsilon,\tau))]] + \mathscr{E}^{-1}[\mu^\rho\mathscr{E}[\upsilon_1(\upsilon,\tau)]] = \frac{1}{3}c^3\cos\left(\frac{\upsilon}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},\\ &\vdots\end{aligned} \quad (45)$$

The series form solution of the problem is given as

$$\psi(\upsilon,\tau) = \left(-\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right) - \frac{2}{3}c^2\sin\left(\frac{\upsilon}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)} + \frac{1}{3}c^3\cos\left(\frac{\upsilon}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)} + \cdots\right), \quad (46)$$

which gives the solution at $(\rho = 1)$ as, $-8/3c\cos^2(1/4(\upsilon - c\tau))$.

Now, we implement Elzaki transform technique in combination with Atangana-Baleanu operator to solve same problem.

Table 2: Comparison of absolute errors of proposed method solution at various fractional-orders with $c = 0.5$ for problem 2.

τ	υ	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1(\mathrm{ETM}_{\mathrm{CFD}})$	$\rho = 1(\mathrm{ETM}_{\mathrm{ABC}})$
	0.2	5.0265200000E-04	3.3503800000E-04	1.6749500000E-04	2.0000000000E-08	2.0000000000E-08
	0.4	1.0002620000E-03	6.6670800000E-04	3.3329700000E-04	2.1000000000E-08	2.1000000000E-08
0.01	0.6	1.4878770000E-03	9.9171600000E-04	4.9576800000E-04	1.9000000000E-08	1.9000000000E-08
	0.8	1.9606260000E-03	1.3068160000E-03	6.5328500000E-04	1.9000000000E-08	1.9000000000E-08
	1	2.4137850000E-03	1.6088580000E-03	8.0427600000E-04	1.8000000000E-08	1.8000000000E-08
	0.2	5.0535100000E-04	3.3681900000E-04	1.6839800000E-04	8.2000000000E-08	8.2000000000E-08
	0.4	1.0055720000E-03	6.7019200000E-04	3.3503200000E-04	8.2000000000E-08	8.2000000000E-08
0.02	0.6	1.4957440000E-03	9.9686700000E-04	4.9831800000E-04	8.0000000000E-08	8.0000000000E-08
	0.8	1.9709720000E-03	1.3135820000E-03	6.5662400000E-04	7.7000000000E-08	7.7000000000E-08
	1	2.4265060000E-03	1.6171720000E-03	8.0837000000E-04	7.4000000000E-08	7.4000000000E-08
	0.2	5.0779200000E-04	3.3845200000E-04	1.6925200000E-04	1.8600000000E-07	1.8600000000E-07
	0.4	1.0103250000E-03	6.7333600000E-04	3.3662600000E-04	1.8400000000E-07	1.8400000000E-07
0.03	0.6	1.5027620000E-03	1.0014910000E-03	5.0063600000E-04	1.7900000000E-07	1.7900000000E-07
	0.8	1.9801840000E-03	1.3196410000E-03	6.5964500000E-04	1.7300000000E-07	1.7300000000E-07
	1	2.4378200000E-03	1.6246040000E-03	8.1206200000E-04	1.6500000000E-07	1.6500000000E-07
	0.2	5.1008200000E-04	3.4000300000E-04	1.7008800000E-04	3.3200000000E-07	3.3200000000E-07
	0.4	1.0147360000E-03	6.7627600000E-04	3.3814400000E-04	3.2700000000E-07	3.2700000000E-07
0.04	0.6	1.5092490000E-03	1.0057910000E-03	5.0282100000E-04	3.1800000000E-07	3.1800000000E-07
	0.8	1.9886830000E-03	1.3252570000E-03	6.6247400000E-04	3.0700000000E-07	3.0700000000E-07
	1	2.4482470000E-03	1.6314820000E-03	8.1550800000E-04	2.9300000000E-07	2.9300000000E-07
	0.2	5.1227100000E-04	3.4150300000E-04	1.7092100000E-04	5.1800000000E-07	5.1800000000E-07
	0.4	1.0189040000E-03	6.7907600000E-04	3.3961600000E-04	5.1100000000E-07	5.1100000000E-07
0.05	0.6	1.5153560000E-03	1.0098610000E-03	5.0491600000E-04	4.9800000000E-07	4.9800000000E-07
	0.8	1.9966670000E-03	1.3305580000E-03	6.6517100000E-04	4.8000000000E-07	4.8000000000E-07
	1	2.4580280000E-03	1.6379590000E-03	8.1877900000E-04	4.5700000000E-07	4.5700000000E-07

By taking the Elzaki transform, we get

$$\tilde{\zeta}(\upsilon,\mu) = \left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\mathscr{E}[\psi(\upsilon,\tau)\psi_{\upsilon\upsilon\upsilon}(\upsilon,\tau) + \psi(\upsilon,\tau)\psi_{\upsilon}(\upsilon,\tau) + 3\psi_{\upsilon}(\upsilon,\tau)\psi_{\upsilon\upsilon}(\upsilon,\tau)] + \mu^2\psi(\upsilon,0). \tag{47}$$

Now applying Elzaki perturbation transform technique to Eq. (47), we obtain

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\tilde{\zeta}_{\mathscr{E}}(\upsilon,\mu) = \mathscr{X}\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right)\right] + \mu^2\psi(\upsilon,0). \tag{48}$$

On taking Elzaki inverse transform of Eq. (48), we get

$$\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\psi_{\mathscr{E}}(\upsilon,\tau) = \mathscr{X}\mathscr{E}^{-1}\left[\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\mathscr{E}\left[\left(\sum_{\mathscr{E}=0}^{\infty}\mathscr{X}^{\mathscr{E}}\varphi_{\mathscr{E}}(\upsilon,\tau)\right)\right]\right] + \mathscr{E}^{-1}\left[\mu^2\psi(\upsilon,0)\right]. \tag{49}$$

In Eq. (38), $\upsilon_{\mathscr{E}}(.)$ denotes the nonlinear terms given in Eq. (23). By repeating the same process for nonlinear terms, we obtain the following terms:

$$\begin{aligned}
\mathscr{X}^0 &: \psi_0(\upsilon,\tau) = \mathscr{E}^{-1}\left[\mu^2\left(-\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right)\right)\right] = \left(-\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right)\right),\\
\mathscr{X}^1 &: \psi_1(\upsilon,\tau) = \mathscr{E}^{-1}\left[\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\mathscr{E}[\varphi_0(\upsilon,\tau)]\right] = \left(\frac{-2/3c^2\sin(\upsilon/2)}{N(\rho)}\right)\left(\frac{\rho\tau^{\rho}}{\Gamma(\rho+1)}+1-\rho\right),\\
\mathscr{X}^2 &: \psi_2(\upsilon,\tau) = \mathscr{E}^{-1}\left[\left(\frac{\rho\mu^{\rho}+1-\rho}{N(\rho)}\right)\mathscr{E}[\varphi_1(\upsilon,\tau)]\right] = \left(\frac{1/3c^3\cos(\upsilon/2)}{N^2(\rho)}\right)\left(\frac{(\rho\tau^{\rho})^2}{\Gamma(2\rho+1)}+\frac{2\rho(1-\rho)\tau^{\rho}}{\Gamma(\rho+1)}+(1-\rho)^2\right),\\
&\vdots
\end{aligned} \tag{50}$$

Table 3: Comparison of the exact and proposed method solution at various values of ρ with $a, b = 1$ for problem 1.

τ	υ	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1$(approx)	$\rho = 1$(exact)
	0.2	2.391919	2.392019	2.392119	2.392219	2.392219
	0.4	2.260796	2.260896	2.260996	2.261096	2.261096
0.01	0.6	2.125753	2.125853	2.125953	2.126053	2.126054
	0.8	1.986275	1.986375	1.986475	1.986575	1.986576
	1	1.841714	1.841814	1.841914	1.842015	1.842014
	0.2	2.391918	2.392018	2.392118	2.392218	2.392218
	0.4	2.260795	2.260895	2.260995	2.261095	2.261096
0.02	0.6	2.125752	2.125852	2.125952	2.126052	2.126053
	0.8	1.986274	1.986374	1.986474	1.986574	1.986575
	1	1.841713	1.841813	1.841913	1.842013	1.842014
	0.2	2.391917	2.392017	2.392117	2.392217	2.392218
	0.4	2.260794	2.260894	2.260994	2.261094	2.261095
0.03	0.6	2.125751	2.125851	2.125951	2.126051	2.126052
	0.8	1.986273	1.986373	1.986473	1.986573	1.986574
	1	1.841712	1.841812	1.841912	1.842012	1.842013
	0.2	2.391916	2.392016	2.392116	2.392216	2.392217
	0.4	2.260793	2.260893	2.260993	2.261093	2.261094
0.04	0.6	2.125750	2.125850	2.125950	2.126051	2.126052
	0.8	1.986272	1.986372	1.986472	1.986572	1.986573
	1	1.841711	1.841811	1.841911	1.842011	1.842012
	0.2	2.391915	2.392015	2.392115	2.392215	2.392216
	0.4	2.260792	2.260892	2.260992	2.261093	2.261094
0.05	0.6	2.125749	2.125849	2.125949	2.126050	2.126051
	0.8	1.986271	1.986371	1.986471	1.986572	1.986573
	1	1.841710	1.841810	1.841910	1.842011	1.842012

Thus, the approximate solution by means of ABC operator is given as

$$\begin{aligned}\psi(\upsilon,\tau) = \sum_{\sigma=0}^{n} \psi_\sigma(\upsilon,\tau) &= \left(-\frac{8}{3}c\cos^2\left(\frac{\upsilon}{4}\right)\right) + \left(\frac{-2/3c^2 \sin(\upsilon/2)}{N(\rho)}\right)\left(\frac{\rho\tau^\rho}{\Gamma(\rho+1)} + 1 - \rho\right) \\ &+ \left(\frac{1/3c^3 \cos(\upsilon/2)}{N^2(\rho)}\right)\left(\frac{(\rho\tau^\rho)^2}{\Gamma(2\rho+1)} + \frac{2\rho(1-\rho)\tau^\rho}{\Gamma(\rho+1)} + (1-\rho)^2\right) + \cdots,\end{aligned} \tag{51}$$

which gives the solution at $(\rho = 1)$ as, $-8/3c\cos^2(1/4(\upsilon - c\tau))$.

5. Results and Discussion

In this article, a detailed investigation of error analysis between exact and approximate solutions, as stated by Tables 1 and 2, has been conducted with greater accuracy. In table, calculating the absolute error at various fractional-orders demonstrates the simplicity and accuracy of the provided method. The error analysis between the exact and approximate solutions is shown in Tables 1 and 2, indicating that the series solution quickly converges to a small value. Also, in Tables 3 and 4, we show the numerical simulation of the proposed method solution. As a result, we will only use the third order of the series solution throughout the numerical evolution. The correctness of the error analytical result will be increased by inserting more terms of approximation solution. Figures 1 and 2 depict the

Table 4: Comparison of the exact and proposed method solution at various values of ρ with $c = 0.5$ for problem 2.

τ	υ	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1$(approx)	$\rho = 1$(exact)
	0.2	-1.330522	-1.330354	-1.330186	-1.330019	-1.330019
	0.4	-1.321077	-1.320744	-1.320410	-1.320077	-1.320077
0.01	0.6	-1.305094	-1.304598	-1.304102	-1.303606	-1.303606
	0.8	-1.282732	-1.282079	-1.281425	-1.280772	-1.280772
	1	-1.254215	-1.253410	-1.252605	-1.251801	-1.251801
	0.2	-1.330541	-1.330372	-1.330204	-1.330036	-1.330035
	0.4	-1.321116	-1.320780	-1.320445	-1.320110	-1.320110
0.02	0.6	-1.305151	-1.304652	-1.304154	-1.303656	-1.303656
	0.8	-1.282808	-1.282150	-1.281493	-1.280837	-1.280837
	1	-1.254307	-1.253498	-1.252689	-1.251881	-1.251881
	0.2	-1.330560	-1.330390	-1.330221	-1.330052	-1.330052
	0.4	-1.321153	-1.320816	-1.320480	-1.320143	-1.320143
0.03	0.6	-1.305208	-1.304706	-1.304205	-1.303705	-1.303705
	0.8	-1.282882	-1.282221	-1.281561	-1.280902	-1.280901
	1	-1.254399	-1.253585	-1.252773	-1.251961	-1.251961
	0.2	-1.330579	-1.330409	-1.330239	-1.330069	-1.330069
	0.4	-1.321191	-1.320852	-1.320514	-1.320176	-1.320176
0.04	0.6	-1.305263	-1.304760	-1.304257	-1.303754	-1.303754
	0.8	-1.282955	-1.282291	-1.281629	-1.280966	-1.280966
	1	-1.254489	-1.253672	-1.252856	-1.252041	-1.252041
	0.2	-1.330597	-1.330426	-1.330256	-1.330085	-1.330085
	0.4	-1.321228	-1.320888	-1.320549	-1.320209	-1.320209
0.05	0.6	-1.305318	-1.304813	-1.304308	-1.303803	-1.303803
	0.8	-1.283028	-1.282361	-1.281696	-1.281031	-1.281031
	1	-1.254578	-1.253758	-1.252939	-1.252121	-1.252120

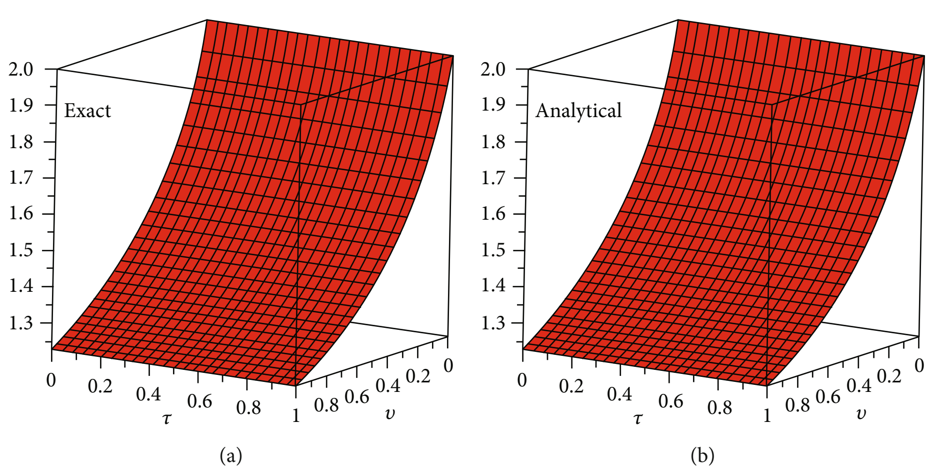

Figure 1: The graphical layout of the exact solution, proposed method solution at $\rho = 1$ and at various fractional orders of $\rho = 1, 0.8, 0.6, 0.4$ with $a, b = 1$ for problem 1.

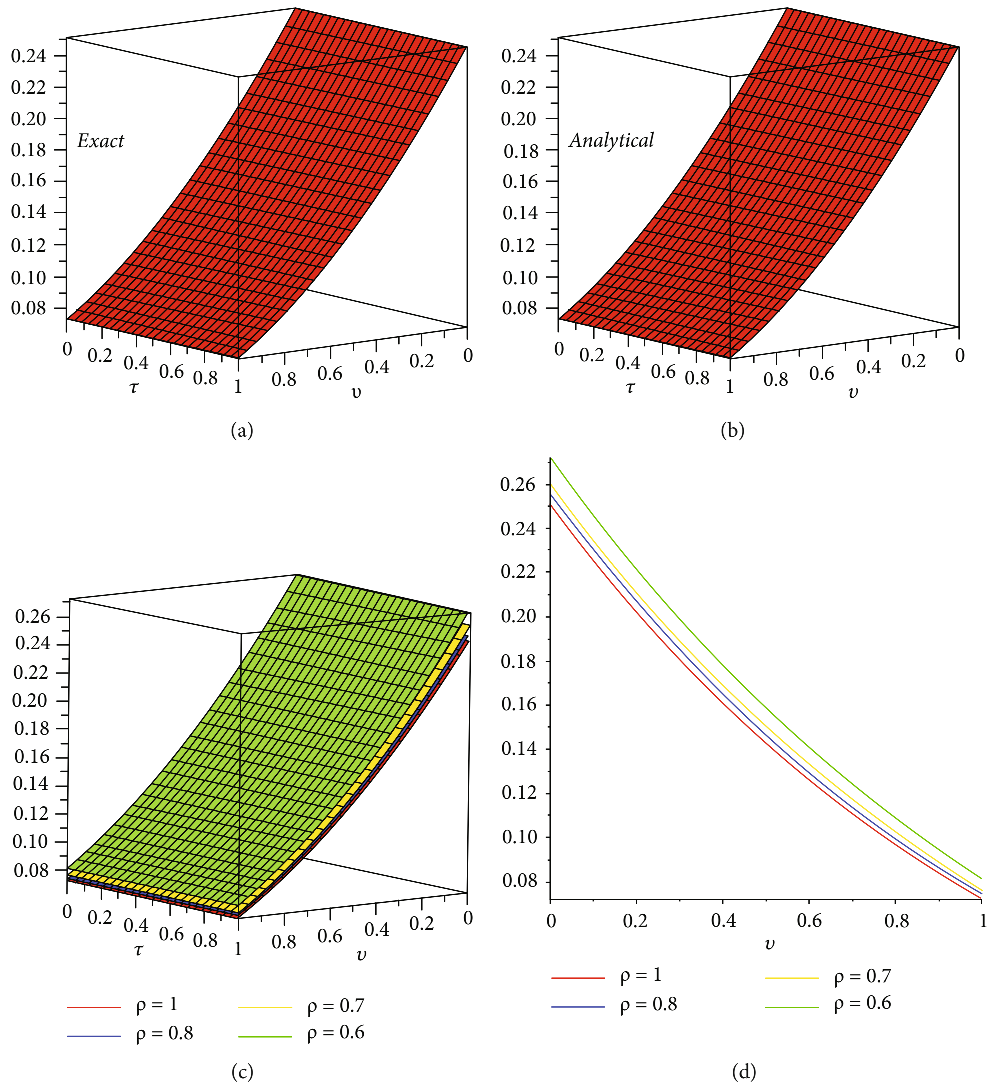

Figure 2: The graphical layout of the exact solution, proposed method solution at $\rho = 1$ and at various fractional orders of $\rho = 1,0.8,0.6,0.4$ with $c = 0.5$ for problem 2.

behaviour of the exact and proposed approach solutions and describe the properties of the approximate solution. We also present the proposed approach solution at different fractional-orders for a better understanding of the problems characteristics. We concluded that the recommended technique solution was in good agreement with the exact solution based on the tables and graphs.

6. Conclusion

The main goal of this study is to use an efficient technique to determine the solution to the fractional Harry Dym equation and fractional Rosenau-Hyman equation. The proposed method is used in addition to two fractional derivatives: Caputo fractional derivative and Atangana-Baleanu fractional derivative. Tables and figures are used to specify the results of the comparative solution. The tables and figures show that the suggested technique solution and the exact result have a better understanding. From the derived results, it shows the reliability of the algorithm, and it is greatly suitable for nonlinear fractional partial differential equation.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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