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Research Article

Fixed Point Results on Closed Ball in Convex Rectangular $b-$ Metric Spaces and Applications

Chaobo Li,[1] Yunan Cui ,[1] and Lili Chen[2]

[1]*Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China*
[2]*College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Yunan Cui; cuiya@hrbust.edu.cn

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In this paper, the concept of convex rectangular $b-$ metric spaces is introduced as a generalization of both convex metric spaces and rectangular $b-$ metric spaces. The purpose of this study is to indicate a way of generalizing Mann's iteration algorithm and a series of fixed point results in rectangular $b-$ metric spaces. Furthermore, certain examples are given to support the results. We also study well posedness of fixed point problems of some mappings in convex rectangular $b-$ metric spaces, and an application to the dynamic programming is entrusted to manifest the viability of the obtained results. Our results extend comparable results in the existing literature.

1. Introduction and Preliminaries

It is well known that fixed point theory has become an important field of mathematics due to its high degree of unity and wide application. No doubt that the most significant fundamental result of this theory is Banach contraction principle [1] which was published in 1922. Banach contraction principle proposes for the first time to use Picard iteration to approximate a fixed point, which not only proves the existence of the fixed point but also proves the uniqueness of the fixed point. Later in 1968, Kannan [2] studied a new type of contractive mapping. Since then, there have been many results related to mappings satisfy various types of contractive inequality, see for example [3–9].

In 2000, Branciari [10] developed the notion of a rectangular space as a generalization of normal metric space via substituting the triangle inequality with the quadrilateral inequality and extended Banach contraction principle to this space. Successively, George et al. [11] introduced the notion of a rectangular $b-$ metric space as a generalization of rectangular metric space and they also proved some fixed point results for contractive mappings. The concept of a convex structure and a convex metric space was introduced by Takahashi [12]. Later, Goebel and Kirk [13] studied some iterative processes for nonexpansive mappings in a hyperbolic metric space, and in 1988, Ding [14] found fixed points of quasicontraction mappings in convex metric spaces by Ishikawa's iteration scheme. However, iterative methods have received vast investigation for finding fixed points of nonexpansive mappings, see [15–17]. Particularly, in the process of the research on some fixed point problem, one of the most famous fixed point method is the Mann iteration [18, 19] as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \quad (1)$$

for some suitably chosen scalars $\alpha_n \in [0, 1]$. Due to [20], Mann iterative sequence $\{x_n\}$ converges weakly to a fixed point of T if the sequence $\{\alpha_n\} \in [0, 1]$ satisfies following conditions: $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$.

Very recently, Chen et al. [21] introduced the notion of a convex $b-$ metric space and extend Mann's algorithms directly to $b-$ metric spaces. After that, Asif et al. [22] investigate fixed point of single-valued Hardy-Roger's type $F-$ contraction globally as well as locally in a convex $b-$ metric space. Along the line, Chen et al. [23] introduce the concept of a convex graphical rectangular $b-$metric space and prove some fixed point theorems in this space. New some fixed point results on a closed ball can be seen in [22, 24–26].

In this work, we firstly introduce the concept of the convex rectangular $b-$metric spaces which is a combination of properties of rectangular $b-$metric spaces and convex metric spaces. However, we prove some fixed point theorems using generalized Mann's iteration algorithm and show concrete examples supporting our main results. In addition, we claim that fixed point problem is well posed and as an application, we apply our main results to solve the dynamic programming problem.

Some fundamental definitions related to our work are given below:

Definition 1. (See [11]). Let X be a nonempty set and the mapping $\mathrm{d}: \mathrm{X}\times\mathrm{X} \longrightarrow [0,\infty)$ satisfy

$\mathrm{d}(\mathrm{x},\mathrm{y})=0$ if and only if $\mathrm{x}=\mathrm{y}$ for all $\mathrm{x},\mathrm{y}\in\mathrm{X}$

$\mathrm{d}(\mathrm{x},\mathrm{y})=\mathrm{d}(\mathrm{y},\mathrm{x})$ for all $\mathrm{x},\mathrm{y}\in\mathrm{X}$

(3) There exists a real number such that $\mathrm{d}(\mathrm{x},\mathrm{y})\le \mathrm{s}[\mathrm{d}(\mathrm{x},\mathrm{u})+\mathrm{d}(\mathrm{u},\mathrm{v})+\mathrm{d}(\mathrm{v},\mathrm{y})]$ for all $\mathrm{x},\mathrm{y}\in\mathrm{X}$ and all distinct points $\mathrm{u},\mathrm{v}\in\mathrm{X}\setminus\{\mathrm{x},\mathrm{y}\}$

Then, d is called a rectangular $b-$metric on X, and (X,d) is called a rectangular $b-$metric space (R_bMS) with coefficient $s\ge 1$.

Remark 2. Note that every metric space is a rectangular metric space (RMS) (see [11]), and every RMS is a R_bMS with coefficient $s=1$.

Definition 3. (See [11]). Let (X,d) be a R_bMS, $\{x_n\}$ be a sequence in X and $x\in X$. Then,

(a) The sequence $\{x_n\}$ is said to be convergent in X to x, if for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $d(x_n,x)<\varepsilon$ for all $n>n_0$, and this fact is represented by $\lim_{\mathrm{n}\longrightarrow\infty}\mathrm{x}_\mathrm{n}=\mathrm{x}$

(b) The sequence $\{x_n\}$ is said to be a Cauchy sequence in X if for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ for all $n,m>n_0$, and this fact is represented by $\lim_{n,m\longrightarrow\infty} d(x_n,x_m)=0$

X is said to be a complete R_bMS if every Cauchy sequence in X converges to some $x\in X$

Definition 4. (See [12]). Let (X,d) be a metric space and $I=[0,1]$. A continuous function $w: X\times X\times[0,1]\longrightarrow X$ is said to be a convex structure on X if for each $x,y\in X$ and $\alpha\in I$,

$$d(u,w(x,y,\alpha))\le \alpha d(u,x)+(1-\alpha)d(u,y), \tag{2}$$

for all $u\in X$. A metric space (X,d) with a convex structure w is called a convex metric space.

2. Main Results

In this section, we introduce a generalization of both convex metric spaces and rectangular $b-$metric spaces, which we call convex rectangular $b-$metric spaces. We also establish some fixed point theorems arising from this metric space.

Definition 5. Let (X,d) be a R_bMS with constant $s\ge 1$. If a mapping $w: X\times X\times[0,1]\longrightarrow X$ satisfies

$$d(u,w(x,y,\alpha))\le \alpha d(u,x)+(1-\alpha)d(u,y), \tag{3}$$

for all $x,y,u\in X$ and $\alpha\in[0,1]$, then (X,d,w) is said to be a convex rectangular $b-$metric space (CR_bMS).

Definition 6. Let $(\mathrm{X},\mathrm{d},\mathrm{w})$ be a CR_bMS and $T: X\longrightarrow X$ be a mapping. Let $\{x_n\}$ be the sequence generated by Mann's iterative procedure involving the mapping T, as follows:

$$x_{n+1}=w(x_n,Tx_n\,;\alpha_n), \tag{4}$$

where $\alpha_n\in[0,1]$ and $x_0\in X$ are the initial value.

Definition 7. If $s=1$ in Definition 5, we call the resultant space to be a convex rectangular metric space (CRMS), which is, indeed, the RMS with a convex structure w.

Next, we see some specific examples of CR_bMS.

Example 8. Let $X=\mathbb{R}$. For any $x,y\in X$, we define the metric $d: X\times X\longrightarrow[0,+\infty)$ by $d(x,y)=|x-y|^r$ and $r\ge 1$. Notice that, for any $a,b,c\in[0,+\infty)$ and $1\le r<\infty$, then the convex of the function $f(x)=x^r(x>0)$ implies that

$$\left(\frac{a+b+c}{3}\right)^r\le\frac{a^r+b^r+c^r}{3}\,(a,b,c>0). \tag{5}$$

Then, for any distinct points $u,v\in X\setminus\{x,y\}$, we have

$$\begin{aligned} d(x,y)&=|x-y|^r=|x-u+u-v+v-y|^r\\ &\le[|x-u|+|u-v|+|v-y|]^r\\ &\le 3^{r-1}[|x-u|^r+|u-v|^r+|v-y|^r]\\ &=3^{r-1}[d(x,u)+d(u,v)+d(v,y)]. \end{aligned} \tag{6}$$

Hence, (X,d) is a R_bMS with $s=3^{r-1}$. For any $x,y\in X$, let $w: X\times X\times\{1/2\}\longrightarrow X$ be a mapping defined by

$$w(x,y\,;\alpha)=\alpha x+(1-\alpha)y,\ \alpha=\frac{1}{2}. \tag{7}$$

Now, we verify that w satisfies inequality (3). In fact, for any $x,y,u\in X$, we can see that

$$\begin{aligned} d(u,w(x,y\,;\alpha))&=|u-[\alpha x+(1-\alpha)y]|^r\\ &\le 2^{r-1}[\alpha^r|u-x|^r+(1-\alpha)^r|u-y|^r]\\ &=\alpha d(u,x)+(1-\alpha)d(u,y). \end{aligned} \tag{8}$$

Therefore, (X, d, w) is a $\mathrm{CR}_b\mathrm{MS}$ with $s = 3^{r-1}$. Note that (X, d, w) is not a metric space as follows:

$$d(1, 3) = 2^2 > d(1, 2) + d(2, 3) = 2, \tag{9}$$

for we take $r = 2$. Moreover, (X, d, w) is a CRMS when we let $r = 1$, and it shows that $\mathrm{CR}_b\mathrm{MS}$ reduces to a CRMS for $s = 1$.

Example 9. Let $X = \mathbb{R}$. For any $x, y \in X$, we define the metric $d : X \times X \longrightarrow [0,+\infty)$ by $d(x, y) = |x - y|^2$. From Example 8, it follows that (X, d) is a $R_b\mathrm{MS}$ with $s = 3$. For any $x, y \in X$, let $w : X \times X \times [0, 1] \longrightarrow X$ be a mapping defined by

$$w(x, y ; \alpha) = \alpha x + (1 - \alpha)y. \tag{10}$$

For any $x, y, u \in X$, we obtain that

$$\begin{aligned} d(u, w(x, y ; \alpha)) &= |u - [\alpha x + (1 - \alpha)y]|^2 \\ &\le [\alpha|u - x| + (1 - \alpha)|u - y|]^2 \\ &= \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + 2\alpha(1 - \alpha)|u - x||u - y| \\ &\le \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + 2\alpha(1 - \alpha)\frac{|u - x|^2 + |u - y|^2}{2} \\ &= \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + \alpha(1 - \alpha)\{d(u, x) + d(u, y)\} \\ &= \alpha d(u, x) + (1 - \alpha)d(u, y). \end{aligned} \tag{11}$$

Therefore, (X, d, w) is a $\mathrm{CR}_b\mathrm{MS}$ with $s = 3$, but not a CRMS.

Example 10. Let $X = [0, 2]$, $d : X \times X \longrightarrow [0,+\infty)$, such that $d(x, y) = d(y, x)$ and

$$d(x, y) = \begin{cases} 0 \text{ if } x = y, \\ 2a \text{ if } x, y \in [0, 1), \\ \frac{1}{2}a \text{ otherwise,} \end{cases} \tag{12}$$

where $a > 0$ is a constant. Then, (X, d) is a $R_b\mathrm{MS}$ with coefficient $s = 4/3$. The mapping $w : X \times X \times [0, 1] \longrightarrow X$ is defined by $w(x, y ; \alpha) = 2 - \alpha xy$, $\alpha = 1/4$, and then

$$d(u, w(x, y ; \alpha)) \le \frac{1}{4}d(u, x) + \frac{3}{4}d(u, y). \tag{13}$$

So, (X, d, w) is a $\mathrm{CR}_b\mathrm{MS}$ with coefficient $s = 4/3$, but not a CRMS.

Definition 11. Let (X, d, w) be a $\mathrm{CR}_b\mathrm{MS}$ with constant $s \ge 1$, x_0 is some element in X, and $\varepsilon > 0$, and then the set $B_\varepsilon\bar{[}x_0]$ $= \{x \in X : \mathrm{d}(x_0, x) \le \varepsilon\}$ is called a closed ball in X.

In the paper [3], George et al. proved Banach contraction principle in complete $R_b\mathrm{MS}$ by means of Picard iteration. Now, we will show Banach contraction principle for complete $\mathrm{CR}_b\mathrm{MS}$ using generalized Mann's iteration algorithm.

Theorem 12. *Let (X, d, w) be a complete $\mathrm{CR}_b\mathrm{MS}$ with constant $s \ge 1$ and $T : X \longrightarrow X$ be a mapping satisfying*

$$d(Tx, Ty) \le \beta d(x, y), \tag{14}$$

for all $x, y \in X$, where $\beta \in [0, 1)$. Let the sequence $\{x_n\}$ generated by the Mann iterative process and $x_0 \in X$ such that $d(x_0, Tx_0) = M < \infty$. If $\beta \in [0, (1/2s^2)]$ and $\alpha_n \in [0, (sr/2s^3 + s - 2)]$ (r is an arbitrary positive real number and $r < 1$), then T has a unique fixed point in X. Moreover, the sequence $\{x_n\} \subseteq B_\varepsilon\bar{[}x_0]$ and $x_n \longrightarrow x^* \in B_\varepsilon\bar{[}x_0]$ as $n \longrightarrow \infty$, if the following inequality holds:

$$d(x_0, Tx_0) \le \beta(1 - s\beta)\varepsilon. \tag{15}$$

Proof. Without loss of generality, we suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Indeed, If $x_n = x_{n+1}$, then $x_n = w(x_n, Tx_n ; \alpha_n)$. We conclude that

$$d(x_n, Tx_n) = d(w(x_n, Tx_n ; \alpha_n), Tx_n) \le \alpha_n d(x_n, Tx_n), \tag{16}$$

and it shows $d(x_n, Tx_n) = 0$; then, x_n is a fixed point of T, and the proof is finished. It follows from Definition 5 and Definition 6,

$$d(x_n, x_{n+1}) = d(x_n, w(x_n, Tx_n ; \alpha_n)) \le (1 - \alpha_n)d(x_n, Tx_n). \tag{17}$$

□

Now, we consider the following two cases:

Case 13. If $x_n \ne Tx_{n-1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, Tx_n) &= d(w(x_{n-1}, Tx_{n-1} ; \alpha_{n-1}), Tx_n) \\ &\le \alpha_{n-1}d(x_{n-1}, Tx_n) + (1 - \alpha_{n-1})d(Tx_{n-1}, Tx_n) \\ &\le \alpha_{n-1}d(x_{n-1}, Tx_n) + (1 - \alpha_{n-1})\beta d(x_{n-1}, x_n) \\ &\le s\alpha_{n-1}[d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n)] \\ &\quad + (1 - \alpha_{n-1})^2\beta d(x_{n-1}, Tx_{n-1}) \\ &\le s\alpha_{n-1}[(1 - \alpha_{n-1})d(x_{n-1}, Tx_{n-1}) \\ &\quad + \alpha_{n-1}d(x_{n-1}, Tx_{n-1}) + (1 - \alpha_{n-1})\beta d(x_{n-1}, Tx_{n-1})] \\ &\quad + (1 - \alpha_{n-1})^2\beta d(x_{n-1}, Tx_{n-1}) \\ &\le [s\alpha_{n-1}(1 + (1 - \alpha_{n-1})\beta) \\ &\quad + (1 - \alpha_{n-1})^2\beta]d(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{18}$$

Let $\lambda_{n-1} = s\alpha_{n-1}[1 + (1 - \alpha_{n-1})\beta] + (1 - \alpha_{n-1})^2\beta$, with the assumption $0 \le \beta \le 1/2s^2$ and $0 \le \alpha_n \le sr/2s^3 + s - 2$, and we obtain that

$$\begin{aligned}\lambda_{n-1} &\le s\alpha_{n-1}\left(1+(1-\alpha_{n-1})\frac{1}{2s^2}\right)+(1-\alpha_{n-1})^2\frac{1}{2s^2}\\ &= s\alpha_{n-1}+\frac{\alpha_{n-1}}{2s}-\frac{\alpha_{n-1}^2}{2s}+\frac{1}{2s^2}-\frac{\alpha_{n-1}}{s^2}+\frac{\alpha_{n-1}^2}{2s^2}\\ &= \left(\frac{1}{2s^2}-\frac{1}{2s}\right)\alpha_{n-1}^2+\left(s+\frac{1}{2s}-\frac{1}{s^2}\right)\alpha_{n-1}+\frac{1}{2s^2}\\ &\le \left(s+\frac{1}{2s}-\frac{1}{s^2}\right)\alpha_{n-1}+\frac{1}{2s^2}=\left(\frac{2s^3+s-2}{2s^2}\right)\\ &\quad\times\frac{sr}{2s^3+s-2}+\frac{1}{2s}\le\frac{r+1}{2s}\le\frac{r+1}{2}.\end{aligned} \tag{19}$$

Hence,

$$d(x_n, Tx_n)\le\lambda_{n-1}d(x_{n-1}, Tx_{n-1})\le\frac{r+1}{2}d(x_{n-1}, Tx_{n-1}). \tag{20}$$

Case 14. If $x_n = Tx_{n-1}$ for some $n\in\mathbb{N}$, we have

$$\begin{aligned}d(x_n, Tx_n) &= d(Tx_{n-1}, Tx_n)\le\beta d(x_{n-1}, x_n)\\ &\le\beta(1-\alpha_n)d(x_{n-1}, Tx_{n-1})\\ &\le\frac{1}{2s^2}d(x_{n-1}, Tx_{n-1}).\end{aligned} \tag{21}$$

Denote that $\lambda = r+1/2<1$, and it follows from (20) and (21) that

$$d(x_n, Tx_n)\le\lambda d(x_{n-1}, Tx_{n-1}), \text{for } all\ n\in\mathbb{N}, \tag{22}$$

which implies that $\{d(x_n, Tx_n)\}$ is a decreasing sequence of nonnegative reals. Hence, there exists $\gamma\ge 0$ such that

$$\lim_{n\longrightarrow\infty} d(x_n, Tx_n)=\gamma. \tag{23}$$

We will show that $\gamma=0$. Suppose that $\gamma>0$, letting $n\longrightarrow\infty$ in inequality (22), we obtain

$$\gamma\le\lambda\gamma, \tag{24}$$

a contradiction. Hence, we get that $\gamma=0$. Furthermore, we have

$$d(x_n, x_{n+1})\le(1-\alpha_n)d(x_n, Tx_n), \tag{25}$$

which shows that $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1})=0$. Also, we can assume $x_n\ne x_{n+p}$ for any $p>1$. Indeed, if $x_n = x_{n+p}$, then using the inequality (21), we have

$$d(x_n, Tx_n)=d(x_{n+p}, Tx_{n+p})\le\lambda^{p-1}d(x_n, Tx_n), \tag{26}$$

in which shows that $d(x_n, Tx_n)=0$ and $x_n = Tx_n$, and then x_n is a fixed point, and the proof is finished. Next, we shall prove $\lim_{n\longrightarrow\infty} d(x_n, x_{n+2})=0$ for all $n\in\mathbb{N}$. In order to do it, we will consider the following two cases:

Case 15. If $x_{n+2}\ne Tx_n$ for all $n\in\mathbb{N}$, then we have

$$\begin{aligned}d(x_n, x_{n+2}) &\le s[d(x_n, Tx_n)+d(Tx_n, Tx_{n+2})+d(Tx_{n+2}, x_{n+2})]\\ &\le s[d(x_n, Tx_n)+\beta d(x_n, x_{n+2})+d(Tx_{n+2}, x_{n+2})],\end{aligned} \tag{27}$$

which establishes that

$$d(x_n, x_{n+2})\le\frac{s}{1-s\beta}[d(x_n, Tx_n)+d(x_{n+2}, Tx_{n+2})]\left(as\ 0\le\beta<\frac{1}{2s^2}\right). \tag{28}$$

Case 16. If there exist some $n\in\mathbb{N}$ such that $x_{n+2}=Tx_n$, then

$$d(x_n, x_{n+2})\le d(x_n, Tx_n). \tag{29}$$

It follows from (28) and (29) that

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+2})=0. \tag{30}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence by contradiction. Assume that there exists $\varepsilon_0>0$ and the subsequences $\{x_{\theta(k)}\}$ and $\{x_{\eta(k)}\}$ of x_n such for $\theta(k)>\eta(k)>k$ with $d(x_{\theta(k)}, x_{\eta(k)})\ge\varepsilon_0$, $d(x_{\theta(k)-1}, x_{\eta(k)})<\varepsilon_0$. On the one hand,

$$\begin{aligned}\varepsilon_0 &\le d\left(x_{\theta(k)}, x_{\eta(k)}\right)\le s\Big[d\left(x_{\theta(k)}, x_{\eta(k)+1}\right)\\ &\quad+d\left(x_{\eta(k)+1}, x_{\eta(k)+2}\right)+d\left(x_{\eta(k)+2}, x_{\eta(k)}\right)\Big],\end{aligned} \tag{31}$$

taking the limit superior in above inequality as $k\longrightarrow\infty$, and we conclude

$$\frac{\varepsilon_0}{s}\le\limsup_{k\longrightarrow\infty} d\left(x_{\theta(k)}, x_{\eta(k)+1}\right). \tag{32}$$

On the other hand, let $x_{\eta(k)}\ne x_{\eta(k)+2}\ne x_{\theta(k)-1}\ne x_{\eta(k)+1}$ and $Tx_{\eta(k)}\ne x_{\eta(k)}\ne Tx_{\theta(k)-1}\ne x_{\eta(k)+1}$, and we have

$$\begin{aligned}d\left(x_{\theta(k)}, x_{\eta(k)+1}\right) &= d\left(w\left(x_{\theta(k)-1}, Tx_{\theta(k)-1}; \alpha_{\theta(k)-1}\right), x_{\eta(k)+1}\right)\\ &\le\alpha_{\theta(k)-1}d\left(x_{\theta(k)-1}, x_{\eta(k)+1}\right)\\ &\quad+\left(1-\alpha_{\theta(k)-1}\right)d\left(Tx_{\theta(k)-1}, x_{\eta(k)+1}\right)\\ &\le\alpha_{\theta(k)-1}s\Big[d\left(x_{\theta(k)-1}, x_{\eta(k)}\right)\\ &\quad+d\left(x_{\eta(k)}, x_{\eta(k)+2}\right)+d\left(x_{\eta(k)+2}, x_{\eta(k)+1}\right)\Big]\\ &\quad+\left(1-\alpha_{\theta(k)-1}\right)s\Big[d\left(Tx_{\theta(k)-1}, Tx_{\eta(k)}\right)\\ &\quad+d\left(Tx_{\eta(k)}, x_{\eta(k)}\right)+d\left(x_{\eta(k)}, x_{\eta(k)+1}\right)\Big]\\ &\le\left(\alpha_{\theta(k)-1}s+\left(1-\alpha_{\theta(k)-1}\right)s\beta\right)d\left(x_{\theta(k)-1}, x_{\eta(k)}\right)\\ &\quad+\alpha_{\theta(k)-1}s\Big[d\left(x_{\eta(k)}, x_{\eta(k)+2}\right)+d\left(x_{\eta(k)+2}, x_{\eta(k)+1}\right)\Big]\\ &\quad+\left(1-\alpha_{\theta(k)-1}\right)s\Big[d\left(Tx_{\eta(k)}, x_{\eta(k)}\right)+d\left(x_{\eta(k)}, x_{\eta(k)+1}\right)\Big],\end{aligned} \tag{33}$$

by taking the limit superior on both sides of above the inequality as $k \longrightarrow \infty$, and we get

$$\begin{aligned}\frac{\varepsilon_0}{s} &\le \limsup_{k\longrightarrow\infty} d\left(x_{\theta(k)}, x_{\eta(k)+1}\right)\\ &\le \left(\alpha_{\theta(k)-1}s + \left(1-\alpha_{\theta(k)-1}\right)s\beta\right)\varepsilon_0\\ &\le \left(\alpha_{\theta(k)-1}s + \frac{1}{2s}\left(1-\alpha_{\theta(k)-1}\right)\right)\varepsilon_0\\ &\le \left(\alpha_{\theta(k)-1}s + \frac{1}{2s} - \frac{\alpha_{\theta(k)-1}}{2s}\right)\varepsilon_0\\ &\le \left(\frac{1}{2s^2-1} \times \left(s - \frac{1}{2s}\right) + \frac{1}{2s}\right)\varepsilon_0\\ &= \left(\frac{r+1}{2}\right)\frac{\varepsilon_0}{s} < \frac{1}{s}\varepsilon_0,\end{aligned} \tag{34}$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since the space (X, d, w) is complete, there exists $x^* \in X$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$. We shall show that x^* is a fixed point of T. Applying the rectangular inequality, we obtain that

$$\begin{aligned}d(x^*, Tx^*) &\le s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)]\\ &\le sd(x^*, x_n) + sd(x_n, x_{n+1})\\ &\quad + s[\alpha_n d(x_n, Tx^*) + (1-\alpha_n)d(Tx_n, Tx^*)]\\ &\le sd(x^*, x_n) + sd(x_n, x_{n+1})\\ &\quad + s^2\alpha_n[d(x_n, x_{n+1}) + d(x_{n+1}, x^*) + d(x^*, Tx^*)]\\ &\quad + s(1-\alpha_n)\beta d(x_n, x^*),\end{aligned} \tag{35}$$

since $s^2\alpha_n < 1$, and then

$$\begin{aligned}d(x^*, Tx^*) \le \frac{1}{1-s^2\alpha_n}&\{sd(x^*, x_n) + sd(x_n, x_{n+1})\\ &+ s^2\alpha_n[d(x_n, x_{n+1}) + d(x_{n+1}, x^*)]\\ &+ s(1-\alpha_n)\beta d(x_n, x^*)\},\end{aligned} \tag{36}$$

letting $n \longrightarrow \infty$, and we deduce $d(x^*, Tx^*) = 0$ which implies $Tx^* = x^*$. Thus, x^* is a fixed point of T. Suppose that $x^*, y^* \in X$ are two distinct fixed points of T, that is, $Tx^* = x^*$ and $Ty^* = y^*$. Then,

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le \beta d(x^*, y^*), \tag{37}$$

which is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. Thus, T has a unique fixed point. Next, we proceed to show that the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. In order to complete it, we will use mathematical induction. Thanks to Definition 5 and Definition 6, we obtain

$$\begin{aligned}d(x_0, x_1) &= d(x_0, w(x_0, Tx_0; \alpha_0))\\ &\le (1-\alpha_0)d(x_0, Tx_0)\\ &\le (1-\alpha_0)\beta(1-s\beta)\varepsilon < \varepsilon,\end{aligned} \tag{38}$$

which implies $d(x_0, x_1) < \varepsilon$; therefore, $x_1 \in B_\varepsilon[x_0]$. Suppose $x_2, x_3, \cdots, x_m \in B_\varepsilon[x_0]$, observe from above proof, we get $d(x_n, Tx_n) \le \lambda^n d(x_0, Tx_0)$ for all $n \in \mathbb{N}$. It is easy to see that $\beta(1-s\beta) \le 1/4s$. Now, we can assume that $x_{m+1} \ne x_m$. If $Tx_0 \ne Tx_m \ne x_0 \ne x_{m+1}$, then

$$\begin{aligned}d(x_0, x_{m+1}) &\le s[d(x_0, Tx_0) + d(Tx_0, Tx_m) + d(Tx_m, x_{m+1})]\\ &\le s[\beta(1-s\beta)\varepsilon + \beta d(x_0, x_m) + \alpha_m d(Tx_m, x_m)]\\ &\le s[\beta(1-s\beta)\varepsilon + \beta\varepsilon + \alpha_m\lambda^m\beta(1-s\beta)\varepsilon]\\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned} \tag{39}$$

We also need to distinguish the following four cases:

Case 17. If $x_0 = Tx_0$, then we have

$$\begin{aligned}d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m))\\ &\le \alpha_m d(x_0, x_m) + (1-\alpha_m)d(Tx_0, Tx_m)\\ &\le \alpha_m\varepsilon + (1-\alpha_m)\beta\varepsilon < \varepsilon.\end{aligned} \tag{40}$$

Case 18. if $x_0 = Tx_m$, then we have

$$\begin{aligned}d(x_0, x_{m+1}) &= d(Tx_m, w(x_m, Tx_m; \alpha_m))\\ &\le \alpha_m d(Tx_m, x_m)\\ &\le \alpha_m\lambda^m d(Tx_0, x_0)\\ &\le \alpha_m\lambda^m\beta(1-s\beta)\varepsilon < \varepsilon.\end{aligned} \tag{41}$$

Case 19. if $x_{m+1} = Tx_0$, then we have

$$d(x_0, x_{m+1}) = d(x_0, Tx_0) \le \beta(1-\beta)\varepsilon < \varepsilon. \tag{42}$$

Case 20. if $x_{m+1} = Tx_m$, then we have

$$\begin{aligned}d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m))\\ &\le \alpha_m d(x_0, x_m) + (1-\alpha_m)d(x_0, Tx_m)\\ &= \alpha_m d(x_0, x_m) + (1-\alpha_m)d(x_0, x_{m+1}),\end{aligned} \tag{43}$$

which implies

$$d(x_0, x_{m+1}) \le d(x_0, x_m) \le \varepsilon. \tag{44}$$

Finally, by above cases, we prove that $d(x_0, x_{m+1}) \le \varepsilon$ which show that $x_{m+1} \in B_\varepsilon[x_0]$. Hence, by induction $x_n \in B_\varepsilon[x_0]$, therefore, we conclude that $x_n \in B_\varepsilon[x_0]$ for all $n \in \mathbb{N}$. As every closed ball in a complete metric space is complete, so $x_n \longrightarrow x^* \in B_\varepsilon[x_0]$, as $n \longrightarrow \infty$.

The following example illustrates the above theorem.

Example 21. Let $X = R^+ \cup \{0\}$ and $Tx = x/5$ for all $x \in X$. For any $x, y \in X$, we define $d : X \times X \longrightarrow [0,+\infty)$ by $d(x, y) = (x-y)^2$. The mapping $w : X \times X \times [0, 1] \longrightarrow X$ is defined by

$$w(x, y; \alpha) = \alpha x + (1-\alpha)y, x, y \in X. \tag{45}$$

Set $x_{n+1}=w(x_n, Tx_n\,;\alpha_n)$ and $\alpha_n=1/2s^2+2$. If $\beta=1/2s^2+1$, then $x_n\in B_\varepsilon\bar{[}x_0]$ and T have a unique fixed point in $B_\varepsilon\bar{[}x_0]$.

Proof. It is easy to see that (X,d) is a CR$_b$MS with $s=3$. In addition, for any $x,y,u\in X$, we have

$$\begin{aligned} d(u,w(x,y\,;\alpha_n)) &= [\alpha_n(u-x)+(1-\alpha_n)(u-y)]^2 \\ &\le \alpha_n^2(u-x)^2+(1-\alpha_n)^2(u-y)^2 \\ &\quad +2\alpha_n(1-\alpha_n)(u-x)(u-y) \\ &\le \alpha_n^2(u-x)^2+(1-\alpha_n)^2(u-y)^2 \\ &\quad +\alpha_n(1-\alpha_n)\left((u-x)^2+(u-y)^2\right) \\ &\le \alpha_n(u-x)^2+(1-\alpha_n)(u-y)^2. \end{aligned} \tag{46}$$

So, (X,d,w) is a CR$_b$MS with $s=3$. It is not difficult to see that T satisfies

$$d(Tx,Ty)=\frac{1}{25}d(x,y)\le\beta d(x,y), \tag{47}$$

for $\beta=1/18$. According to $x_{n+1}=w(x_n,Tx_n\,;\alpha_n)$, we have $x_{n+1}=1/20x_n+19/20Tx_n$, since $Tx=x/5$, and we obtain

$$x_{n+1}=\frac{1}{20}x_n+\frac{19}{20}\times\frac{1}{5}x_n, \tag{48}$$

that is, $x_{n+1}=6/25x_n$, then

$$x_n=\frac{6}{25}x_{n-1}, x_{n-1}=\frac{6}{25}x_{n-2},\cdots,x_1=\frac{6}{25}x_0, \tag{49}$$

And we obtain

$$x_n=\left(\frac{6}{25}\right)^n x_0, Tx_n=\frac{1}{5}\times\left(\frac{6}{25}\right)^n x_0, \tag{50}$$

while $n\longrightarrow\infty$, getting $x_n\longrightarrow 0\in X$ and $Tx_n\longrightarrow 0\in X$. Hence, 0 is a fixed point of T in X. Suppose $x^*,y^*\in X$ are two distinct fixed points of T, then we have $d(x^*,y^*)=d(Tx^*,Ty^*)=1/25d(x^*,y^*)$ which shows that $d(x^*,y^*)=0$, that is, $x^*=y^*$. Thus, T has a unique fixed point in X. Let $\varepsilon=x_0^2/\beta(1-s\beta)>0$, then $\beta(1-s\beta)\varepsilon=x_0^2\ge d(x_0,Tx_0)=16/25x_0^2$. For all $n\in\mathbb{N}$, we obtain $d(x_0,x_n)=(x_0-(6/25)^n x_0)^2<x_0^2<\varepsilon$, and this means that the sequence $\{x_n\}\subseteq B_\varepsilon\bar{[}x_0]$. Furthermore, $d(x_0,0)=x_0^2<\varepsilon$, that is, $0\in B_\varepsilon\bar{[}x_0]$. □

Now, we prove the Kannan type fixed point theorem for a complete CR$_b$MS, which extends the results in the paper [3], replacing Picard's iteration algorithm by Mann's iteration algorithm.

Theorem 22. *Let (X,d,w) be a CR$_b$MS with constant $s\ge 1$ and the mapping $T:X\longrightarrow X$ be defined by*

$$d(Tx,Ty)\le k[d(x,Tx)+d(y,Ty)], \tag{51}$$

for all $x,y\in X$, and $k\in[0,1/2)$. Let the sequence $\{x_n\}$ generated by the Mann iterative process and $x_0\in X$ such that $d(x_0,Tx_0)=M<\infty$. If $k\in[0,(1/3s)]$ and $\alpha_n\in[0,(1/us^2)]$ (u is an arbitrary real number and $u>5$), then T has a unique fixed point in X. Moreover, the sequence $\{x_n\}\subseteq B_\varepsilon\bar{[}x_0]$ and $x_n\longrightarrow x^*\in B_\varepsilon\bar{[}x_0]$ as $n\longrightarrow\infty$, if the following inequality holds:

$$d(x_0,Tx_0)\le k(1-sk)\varepsilon. \tag{52}$$

Proof. Without loss of generality, we suppose that $x_n\ne x_{n+1}$ for all $n\ge\mathbb{N}$. Indeed, If $x_n=x_{n+1}$, that is, $x_n=w(x_n,Tx_n\,;\alpha_n)$. Then, we have

$$d(x_n,Tx_n)=d(w(x_n,Tx_n\,;\alpha_n),Tx_n)\le\alpha_n d(x_n,Tx_n), \tag{53}$$

and it shows that $d(x_n,Tx_n)=0$ and $x_n=Tx_n$, which means that x_n is a fixed point of T, and the proof is finished. Thanks to Definition 5 and Definition 6, we have

$$d(x_n,x_{n+1})=d(x_n,w(x_n,Tx_n\,;\alpha_n))\le(1-\alpha_n)d(x_n,Tx_n). \tag{54}$$

□

Now, we have the following two cases:

Case 23. If $x_n\ne Tx_{n-1}$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned} d(x_n,Tx_n) &= d(w(x_{n-1},Tx_{n-1}\,;\alpha_{n-1}),Tx_n) \\ &\le \alpha_{n-1}d(x_{n-1},Tx_n)+(1-\alpha_{n-1})d(Tx_{n-1},Tx_n) \\ &\le s\alpha_{n-1}[d(x_{n-1},x_n)+d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)] \\ &\quad +(1-\alpha_{n-1})d(Tx_{n-1},Tx_n) \\ &\le s\alpha_{n-1}[(1-\alpha_{n-1})d(x_{n-1},Tx_{n-1})+\alpha_{n-1}d(x_{n-1},Tx_{n-1}) \\ &\quad +kd(x_{n-1},Tx_{n-1})+kd(x_n,Tx_n)] \\ &\quad +(1-\alpha_{n-1})k[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)] \\ &\le [s\alpha_{n-1}+s\alpha_{n-1}k+(1-\alpha_{n-1})k]d(x_{n-1},Tx_{n-1}) \\ &\quad +[s\alpha_{n-1}k+(1-\alpha_{n-1})k]d(x_n,Tx_n), \end{aligned} \tag{55}$$

which establishes that

$$\begin{aligned} &[1-(1-\alpha_{n-1})k-s\alpha_{n-1}k)]d(x_n,Tx_n) \\ &\quad\le [s\alpha_{n-1}+s\alpha_{n-1}k+(1-\alpha_{n-1})k]d(x_{n-1},Tx_{n-1}). \end{aligned} \tag{56}$$

Notice that $(1-\alpha_{n-1})k+s\alpha_{n-1}k)<1$, then we have

$$d(x_n,Tx_n)\le\frac{s\alpha_{n-1}+s\alpha_{n-1}k+(1-\alpha_{n-1})k}{1-(1-\alpha_{n-1})k-s\alpha_{n-1}k}d(x_{n-1},Tx_{n-1}). \tag{57}$$

Since $u > 5$, we conclude that

$$\frac{s\alpha_{n-1} + s\alpha_{n-1}k + (1-\alpha_{n-1})k}{1 - s\alpha_{n-1}k - (1-\alpha_{n-1})k} \leq \frac{(1/us) + \left(1/3us^2\right) + (1/3s)}{1 - 1/3us^2 - 1/3s} \leq \frac{u+4}{2u-1} < 1; \tag{58}$$

Case 24. If $x_n = Tx_{n-1}$ for some $n \in \mathbb{N}$, then

$$d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) \leq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)], \tag{59}$$

and this implies that

$$(1-k)d(x_n, Tx_n) \leq kd(x_{n-1}, Tx_{n-1}). \tag{60}$$

Since $0 \leq k \leq 1/3s$, then we get

$$d(x_n, Tx_n) \leq \frac{k}{1-k} d(x_{n-1}, Tx_{n-1}). \tag{61}$$

Noticing that

$$\frac{k}{1-k} \leq \frac{1}{2} \leq \frac{u+4}{2u-1}. \tag{62}$$

Let $\lambda_u = u + 4/2u - 1$, it is clear that $\lambda_u < 1$, and for any $n \in \mathbb{N}$, we obtain the following inequality:

$$d(x_n, Tx_n) \leq \lambda_u d(x_{n-1}, Tx_{n-1}), \text{for} all n \in \mathbb{N}, \tag{63}$$

and it implies that $\{d(x_n, Tx_n)\}$ is a decreasing sequence of nonnegative reals. Hence, there exists $\gamma \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(x_n, Tx_n) = \gamma. \tag{64}$$

We will show that $\gamma = 0$. Suppose that $\gamma > 0$. Letting $n \longrightarrow \infty$ in inequality (63), we obtain

$$\gamma \leq \lambda_u \gamma, \tag{65}$$

a contradiction. Hence, we get that $\gamma = 0$. Moreover, we have

$$d(x_n, x_{n+1}) \leq (1-\alpha_n) d(x_n, Tx_n), \tag{66}$$

which shows that $\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 0$. Next, we shall prove that $\lim_{n \longrightarrow \infty} d(x_n, x_{n+2}) = 0$. In order to do it, we will consider the following two cases:

Case 25. if $x_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$, then we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &\leq d(x_n, w(x_{n+1}, Tx_{n+1}; \alpha_{n+1})) \\ &\leq \alpha_{n+1} d(x_n, x_{n+1}) + (1-\alpha_{n+1}) d(x_n, Tx_{n+1}) \\ &\leq \alpha_{n+1} d(x_n, x_{n+1}) + (1-\alpha_{n+1}) s[d(x_n, Tx_n) \\ &\quad + d(Tx_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})] \\ &\leq \alpha_{n+1} d(x_n, x_{n+1}) + (1-\alpha_{n+1}) s[d(x_n, Tx_n) \\ &\quad + \alpha_n d(Tx_n, x_n) + d(x_{n+1}, Tx_{n+1})]. \end{aligned} \tag{67}$$

Hence,

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \alpha_{n+1} d(x_n, x_{n+1}) \\ &\quad + (1-\alpha_{n+1}) s[(1+\alpha_n) d(Tx_n, x_n) + d(x_{n+1}, Tx_{n+1})]. \end{aligned} \tag{68}$$

Case 26. If there exist some $n \in \mathbb{N}$ such that $x_{n+1} = Tx_n$, then we get

$$\begin{aligned} d(x_n, x_{n+2}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+3}) + d(x_{n+3}, x_{n+2})] \\ &= s[d(x_n, x_{n+1}) + d(x_{n+1}, w(x_{n+2}, Tx_{n+2}; \alpha_{n+2})) \\ &\quad + d(x_{n+3}, x_{n+2})] \\ &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2} d(x_{n+1}, x_{n+2}) \\ &\quad + (1-\alpha_{n+2}) d(x_{n+1}, Tx_{n+2}) + (1-\alpha_{n+2}) d(Tx_{n+2}, x_{n+2})] \\ &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2} d(x_{n+1}, x_{n+2}) \\ &\quad + (1-\alpha_{n+2}) kd(x_n, Tx_n) + (1-\alpha_{n+2}) kd(x_{n+2}, Tx_{n+2}) \\ &\quad + (1-\alpha_{n+2}) d(Tx_{n+2}, x_{n+2})]. \end{aligned} \tag{69}$$

Hence,

$$\begin{aligned} d(x_n, x_{n+2}) &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2} d(x_{n+1}, x_{n+2}) \\ &\quad + (1-\alpha_{n+2}) kd(x_n, Tx_n) + (1-\alpha_{n+2}) kd(x_{n+2}, Tx_{n+2}) \\ &\quad + (1-\alpha_{n+2}) d(Tx_{n+2}, x_{n+2})]. \end{aligned} \tag{70}$$

It follows from (68) and (70) that

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+2}) = 0. \tag{71}$$

Next, we will claim that $\{x_n\}$ is a Cauchy sequence by contradiction. Assume there exists $\varepsilon_0 > 0$ and the subsequences $\{x_{\theta(k)}\}$ and $\{x_{\eta(k)}\}$ of $\{x_n\}$ such for $\theta(k) > \eta(k) > k$ with $d(x_{\theta(k)}, x_{\eta(k)}) \geq \varepsilon_0, d(x_{\theta(k)-1}, x_{\eta(k)}) < \varepsilon_0$. On the one hand,

$$\begin{aligned} \varepsilon_0 \leq d\left(x_{\theta(k)}, x_{\eta(k)}\right) &\leq s\Big[d\left(x_{\theta(k)}, x_{\eta(k)+1}\right) \\ &\quad + d\left(x_{\eta(k)+1}, x_{\eta(k)+2}\right) + d\left(x_{\eta(k)+2}, x_{\eta(k)}\right)\Big], \end{aligned} \tag{72}$$

taking the limit superior in above inequality as $k \longrightarrow \infty$, and we get

$$\frac{\varepsilon_0}{s} \le \limsup_{k\longrightarrow\infty} d\left(x_{\theta(k)}, x_{\eta(k)+1}\right). \tag{73}$$

On the other hand, let $x_{\eta(k)} \ne x_{\eta(k)+2} \ne x_{\theta(k)-1} \ne x_{\eta(k)+1}$ and $Tx_{\eta(k)} \ne x_{\eta(k)} \ne Tx_{\theta(k)-1} \ne x_{\eta(k)+1}$, and we have

$$\begin{aligned} d\left(x_{\theta(k)}, x_{\eta(k)+1}\right) &= d\left(w\left(x_{\theta(k)-1}, Tx_{\theta(k)-1}; \alpha_{\theta(k)-1}\right), x_{\eta(k)+1}\right) \\ &\le \alpha_{\theta(k)-1} d\left(x_{\theta(k)-1}, x_{\eta(k)+1}\right) \\ &\quad + \left(1-\alpha_{\theta(k)-1}\right) d\left(Tx_{\theta(k)-1}, x_{\eta(k)+1}\right) \\ &\le \alpha_{\theta(k)-1} s\left[d\left(x_{\theta(k)-1}, x_{\eta(k)}\right)\right. \\ &\quad \left. + d\left(x_{\eta(k)}, x_{\eta(k)+2}\right) + d\left(x_{\eta(k)+2}, x_{\eta(k)+1}\right)\right] \\ &\quad + \left(1-\alpha_{\theta(k)-1}\right) s\left[d\left(Tx_{\theta(k)-1}, Tx_{\eta(k)}\right)\right. \\ &\quad \left. + d\left(Tx_{\eta(k)}, x_{\eta(k)}\right) + d\left(x_{\eta(k)}, x_{\eta(k)+1}\right)\right] \\ &\le \alpha_{\theta(k)-1} s\left[d\left(x_{\theta(k)-1}, x_{\eta(k)}\right)\right. \\ &\quad \left. + d\left(x_{\eta(k)}, x_{\eta(k)+2}\right) + d\left(x_{\eta(k)+2}, x_{\eta(k)+1}\right)\right] \\ &\quad + \left(1-\alpha_{\theta(k)-1}\right) s\left[kd\left(x_{\theta(k)-1}, Tx_{\theta(k)-1}\right)\right. \\ &\quad + kd\left(x_{\eta(k)}, Tx_{\eta(k)}\right) \\ &\quad \left. + d\left(Tx_{\eta(k)}, x_{\eta(k)}\right) + d\left(x_{\eta(k)}, x_{\eta(k)+1}\right)\right]. \end{aligned} \tag{74}$$

We obtain

$$\frac{\varepsilon_0}{s} \le \limsup_{k\longrightarrow\infty} d\left(x_{\theta(k)}, x_{\eta(k)+1}\right) \le \frac{1}{us}\varepsilon_0 < \frac{1}{s}\varepsilon_0, \tag{75}$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since the space (X, d, w) is complete, there exists $x^* \in X$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$. We shall show that x^* is a fixed point of T. Applying the rectangular inequality, we obtain that

$$\begin{aligned} d(x^*, Tx^*) &\le s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &\le sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &\quad + s[\alpha_n d(x_n, Tx^*) + (1-\alpha_n) d(Tx_n, Tx^*)] \\ &\le sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &\quad + s^2\alpha_n[d(x_n, x_{n+1}) + d(x_{n+1}, x^*) + d(x^*, Tx^*)] \\ &\quad + s(1-\alpha_n)\{kd(x_n, Tx_n) + kd(x^*, Tx^*)\}, \end{aligned} \tag{76}$$

since $s^2\alpha_n + s(1-\alpha_n)k < 1$, and then

$$\begin{aligned} d(x^*, Tx^*) \le \frac{1}{1 - s^2\alpha_n - s(1-\alpha_n)k} &\{sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &+ s^2\alpha_n[d(x_n, x_{n+1}) + d(x_{n+1}, x^*)] \\ &+ s(1-\alpha_n)kd(x_n, Tx_n)\}, \end{aligned} \tag{77}$$

letting $n \longrightarrow \infty$, and we deduce $d(x^*, Tx^*) = 0$ which implies $Tx^* = x^*$. Thus, x^* is a fixed point of T. Suppose that $x^*, y^* \in X$ are two distinct fixed points of T, that is, $Tx^* = x^*$, $Ty^* = y^*$. Then,

$$0 < d(x^*, y^*) = d(Tx^*, Ty^*) \le k[d(x^*, Tx^*) + d(y^*, Ty^*)] = 0, \tag{78}$$

which is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Thus, T has a unique fixed point. Finally, we will prove the iteration sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. In order to complete it, we will use mathematical induction. Choose $x_0 \in X$, and we have

$$\begin{aligned} d(x_0, x_1) &= d(x_0, w(x_0, Tx_0; \alpha_0)) \\ &\le (1-\alpha_0) d(x_0, Tx_0) \\ &\le (1-\alpha_0)\beta(1 - s\beta)\varepsilon < \varepsilon, \end{aligned} \tag{79}$$

which implies $d(x_0, x_1) < \varepsilon$; therefore, $x_1 \in B_\varepsilon[x_0]$. Suppose $x_2, x_3, \cdots, x_m \in B_\varepsilon[x_0]$. It is easy to see that $s[k(1-sk) < 2/9$ s. Without loss of generality, we can assume that $x_{m+1} \ne x_m$. If $Tx_0 \ne Tx_m \ne x_0 \ne x_{m+1}$, then

$$\begin{aligned} d(x_0, x_{m+1}) &\le s[d(x_0, Tx_0) + d(Tx_0, Tx_m) + d(Tx_m, x_{m+1})] \\ &\le s\left[k(1-sk)\varepsilon + k^2(1-sk)\varepsilon + k\lambda_u^m d(x_0, Tx_0)\right. \\ &\quad \left. + \alpha_n \lambda_u^m d(Tx_0, x_0)\right] \\ &\le s\left[k(1-sk)\varepsilon + k^2(1-sk)\varepsilon + \lambda_u^m k^2(1-sk)\right. \\ &\quad \left. + \alpha_m \lambda^m k(1-sk)\varepsilon\right] \\ &\le \frac{2\varepsilon}{9} + \frac{2\varepsilon}{27} + \frac{2\varepsilon}{27} + \frac{2\varepsilon}{9} < \varepsilon. \end{aligned} \tag{80}$$

We also need to distinguish the following four cases:

Case 27. If $x_0 = Tx_0$, then we have

$$\begin{aligned} d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\ &\le \alpha_m d(x_0, x_m) + (1-\alpha_m) d(Tx_0, Tx_m) \\ &\le \alpha_m\varepsilon + (1-\alpha_m)k[d(x_0, Tx_0) + d(x_m, Tx_m)] \\ &\le \alpha_m\varepsilon + 2k(1 - sk1 - \alpha_m)k\varepsilon < \varepsilon. \end{aligned} \tag{81}$$

Case 28. If $x_0 = Tx_m$, then we have

$$d(x_0, x_{m+1}) = d(Tx_m, w(x_m, Tx_m; \alpha_m)) \le \alpha_m d(Tx_m, x_m) \le \alpha_m \lambda_u^m d(Tx_0, x_0) \le \alpha_m \lambda_u^m k(1 - sk)\varepsilon < \varepsilon. \quad (82)$$

Case 29. If $x_{m+1} = Tx_0$, then we have

$$d(x_0, x_{m+1}) = d(x_0, Tx_0) \le k(1 - sk)\varepsilon < \varepsilon. \quad (83)$$

Case 30. If $x_{m+1} = Tx_m$, then we have

$$\begin{aligned} d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\ &\le \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, Tx_m) \\ &= \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, x_{m+1}), \end{aligned} \quad (84)$$

which implies

$$d(x_0, x_{m+1}) \le d(x_0, x_m) \le \varepsilon. \quad (85)$$

Finally, by above cases, we prove that $d(x_0, x_{m+1}) \le \varepsilon$, which show that $x_{m+1} \in B_\varepsilon[x_0]$. Hence, by induction $x_n \in B_\varepsilon[x_0]$. Therefore, we conclude that $x_n \in B_\varepsilon[x_0]$ for all $n \in \mathbb{N}$. As every closed ball in a complete metric space is complete, so $x^* \in B_\varepsilon[x_0]$, as $n \longrightarrow \infty$.

Next, we give the following example to illustrate above theorem.

Example 31. Let $X = \mathbb{R}^+ \cup \{0\}$ and the mapping $T : X \longrightarrow X$ such that

$$Tx = \begin{cases} 0, \mathit{if} x \in \left[0, \sqrt{2}\right), \\ \dfrac{1}{2x}, \text{ if } x \in \left[\sqrt{2}, +\infty\right), \end{cases} \quad (86)$$

for any $x, y \in X$. Let us define the metric $d : X \times X \longrightarrow X$ by the formula $d(x, y) = (x - y)^2$ as well as the mapping $w : X \times X \times [0, 1] \longrightarrow X$ by the formula $w(x, y; \alpha) = \alpha x + (1 - \alpha) y$. Choose $x_0 \ge 0$ to be the initial value and $x_{n+1} = w(x_n, Tx_n; \alpha_n)$, where $\alpha_n = 1/49$. If $k = 1/9$, then $x_n \in B_\varepsilon[x_0]$, and T has a unique fixed point in $B_\varepsilon[x_0]$.

Proof. It is easy to see that (X, d, w) is a $\mathrm{CR}_b\mathrm{MS}$ with $s = 3$. We claim that T satisfies inequality

$$d(Tx, Ty) \le k[d(x, Tx) + d(y, Ty)], \quad (87)$$

for any $x, y \in X$. Next, we will consider the four cases:

(a) If $x, y \in [0, \sqrt{2})$, then it is easy to see that inequality (87) holds

(b) If $x \in [0, \sqrt{2})$ and $y \in [\sqrt{2}, +\infty)$, then

$$\begin{aligned} d(Tx, Ty) &- \frac{1}{9}[d(x, Tx) + d(y, Ty)] \\ &= \left(\frac{1}{2y}\right)^2 - \frac{1}{9}\left[x^2 + \left(y - \frac{1}{2y}\right)^2\right] \\ &\le \left(\frac{1}{2y}\right)^2 - \frac{1}{9}\left(y - \frac{1}{2y}\right)^2 \le 0, \end{aligned} \quad (88)$$

which implies that

$$d(Tx, Ty) \le \frac{1}{9}[d(x, Tx) + d(y, Ty)], \quad (89)$$

holds for any $x \in [0, \sqrt{2})$ and $y \in [\sqrt{2}, +\infty)$.

(c) If $x \in [\sqrt{2}, +\infty)$ and $y \in [0, \sqrt{2})$, then, similarly to case (b), we can also get that inequality (87) holds

(d) If $x, y \in [\sqrt{2}, +\infty)$, then

$$\begin{aligned} d(Tx, Ty) &- \frac{1}{9}[d(x, Tx) + d(y, Ty)] \\ &= \frac{1}{4}\left(\frac{1}{x} - \frac{1}{y}\right)^2 - \frac{1}{9}\left[\left(x - \frac{1}{2x}\right)^2 + \left(y - \frac{1}{2y}\right)^2\right] \\ &= \frac{1}{4}\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{2}{xy}\right) - \frac{1}{9}\left(x^2 + y^2 + \frac{1}{4x^2} + \frac{1}{4y^2} - 2\right) \\ &= \frac{8}{36}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) - \frac{1}{9}\left(x^2 + y^2 + \frac{1}{2xy} - 2\right) \\ &\le \frac{8}{36}\left(\frac{1}{2} + \frac{1}{2}\right) - \frac{1}{9}\left(2 + 2 + \frac{1}{4} - 2\right) \\ &= \frac{8}{36} - \frac{9}{36} < 0, \end{aligned} \quad (90)$$

which shows that

$$d(Tx, Ty) < \frac{1}{9}[d(x, Tx) + d(y, Ty)] \quad (91)$$

holds for all $x, y \in [\sqrt{2}, +\infty)$.

Summarizing, inequality, (87) holds for all $x, y \in X$. Next, we will show that T has a unique fixed point in X. In order to do it, we will consider the following two cases:

(i) If $x_0 < \sqrt{2}$, then

$$
\begin{aligned}
&Tx_0 = 0,\\
&x_1 = \frac{1}{49}x_0 + \frac{48}{49}Tx_0 = \frac{1}{49}x_0, Tx_1 = 0,\\
&x_2 = \frac{1}{49}x_1 + \frac{48}{49}Tx_1 = \left(\frac{1}{49}\right)^2 x_0, Tx_2 = 0,\\
&\cdots\\
&x_n = \frac{1}{49}x_{n-1} + \frac{48}{49}Tx_{n-1} = \left(\frac{1}{49}\right)^n x_0.
\end{aligned} \tag{92}
$$

Obviously, $x_n \longrightarrow 0$ as $n \longrightarrow \infty$,

(ii) If $x_0 \geq \sqrt{2}$, then

$$
\begin{aligned}
Tx_0 &= \frac{1}{2x_0},\\
x_1 &= \frac{1}{49}x_0 + \frac{48}{49}Tx_0,\\
\frac{x_1}{x_0} &= \frac{1}{49} + \frac{48}{49} \times \frac{1}{2x_0^2} \leq \frac{13}{49}.
\end{aligned} \tag{93}
$$

If $0 \leq x_1 < \sqrt{2}$, then $Tx_1 = 0$. From the case (i), it follows that $x_n \longrightarrow 0$ as $n \longrightarrow \infty$. If $x_1 \geq \sqrt{2}$, then $x_2/x_1 = 1/49 + 48/49 \times (1/2x_1^2) \leq 13/49$. From the above procedure, without loss of generality, we can assume that $x_{n-1} \geq \sqrt{2}$. Then, we obtain

$$
\begin{aligned}
\frac{x_n}{x_{n-1}} &= \frac{1}{49} + \frac{48}{49} \times \frac{1}{2x_1^2} \leq \frac{13}{49},\\
\frac{x_n}{x_0} &= \frac{x_1}{x_0} \times \frac{x_2}{x_1} \times \cdots \times \frac{x_n}{x_{n-1}} \leq \left(\frac{13}{49}\right)^n,
\end{aligned} \tag{94}
$$

which implies that $x_n \leq (13/49)^n x_0$.

Hence, $\lim_{n\longrightarrow\infty} x_n = 0$, where 0 is a fixed point of T. Actually, 0 is a unique fixed point of T in . Indeed, suppose that $y^* \in [\sqrt{2},+\infty)$ is a fixed point of T, then $Ty^* = y^*$, that is, $y^* = Ty^* = 1/2y^*$, which implies $y^* = \sqrt{2}/2 < \sqrt{2}$, a contradiction. Thus, T has a unique fixed point in . Let $\varepsilon = x_0^2/k(1 - sk) > 0$, then $k(1 - sk)\varepsilon = x_0^2 \geq d(x_0, Tx_0)$. For all $n \in \mathbb{N}$, from above proof, we can obtain $x_n \leq (13/49)^n x_0$, then $d(x_0, x_n) = (x_0 - (13/49)^n x_0)^2 < x_0^2 < \varepsilon$, and this means that the sequence $\{x_n\} \subseteq \bar{B}_\varepsilon[x_0]$. Furthermore, $d(x_0, 0) = x_0^2 < \varepsilon$, that is, $0 \in \bar{B}_\varepsilon[x_0]$, and the proof is finished. □

The concept of well posedness is very important in many fields of mathematics and has evoked much interest to several researchers [27–29].

Definition 32. (see [26]). Let (X, d) be a metric space and T be a self-map. The fixed point problem of T is said to be well posed if

T has a unique fixed point $x^* \in X$

(2) For any sequence $\{y_n\}$ in X with $\lim_{n\longrightarrow\infty} d(y_n, Ty_n) = 0$, we have $\lim_{n\longrightarrow\infty} d(y_n, x^*) = 0$

We next study the well posedness of the fixed point problem of T in complete CR_bMS.

Theorem 33. *Let (X, d, w) be a CR_bMS with constant $s \geq 1$ and all the hypotheses of Theorem 12 hold. If the constant $0 < \alpha < 1$, then fixed point problem of T is well posed.*

Proof. Let x^* is a unique fixed point of T and assume $\{y_n\}$ be a sequence in X such that $\lim_{n\longrightarrow\infty} d(y_n, Ty_n) = 0$. Because of uniqueness of the fixed point of T, for all $n \in \mathbb{N}$, we can assume that $y_n \neq Ty_n$. If $y_n = w(y_n, Ty_n ; \alpha)$ for some $\alpha \in (0, 1)$, $n \in \mathbb{N}$, then

$$
\begin{aligned}
d(y_n, x^*) &= d(w(y_n, Ty_n ; \alpha), x^*)\\
&\leq \alpha d(y_n, x^*) + (1 - \alpha)d(Ty_n, x^*)\\
&\leq (\alpha + (1 - \alpha)\beta)d(y_n, x^*),
\end{aligned} \tag{95}
$$

since $\alpha + (1 - \alpha)\beta < 1$, and we get $d(y_n, x^*) = 0$. Due to $\alpha > 0$, it is not difficult to see that $Ty_n \neq w(y_n, Ty_n ; \alpha)$, indeed, if not,

$$
d(y_n, Ty_n) = d(y_n, w(x_n, Ty_n ; \alpha)) \leq (1 - \alpha)d(y_n, Ty_n), \tag{96}
$$

a contradiction. Therefore, let us assume that $y_n \neq Ty_n \neq w(y_n, Ty_n ; \alpha)$, and then

$$
\begin{aligned}
d(y_n, x^*) &\leq s[d(y_n, w(y_n, Ty_n ; \alpha)) + d(w(y_n, Ty_n ; \alpha), Ty_n) + d(Ty_n, x^*)]\\
&\leq s(1 - \alpha)d(y_n, Ty_n) + s\alpha d(y_n, Ty_n) + s\beta d(y_n, x^*),
\end{aligned} \tag{97}
$$

combining with $1 - s\beta > 0$, and we obtain

$$
d(y_n, x^*) \leq \frac{s}{1 - s\beta} d(y_n, Ty_n), \tag{98}
$$

which implies $\lim_{n\longrightarrow\infty} d(y_n, x^*) = 0$, which completes the proof. □

Theorem 34. *Let (X, d, w) be a CR_bMS with constant $s \geq 1$ and all the hypotheses of Theorem 22 hold. If the constant $0 < \alpha < 1$, then fixed point problem of T is well posed.*

Proof. Let x^* be a unique fixed point of T and a sequence y_n in sequence in X such that $\lim_{n\longrightarrow\infty} d(y_n, Ty_n) = 0$. Without loss of generality, let $y_n \neq x^*$, for all $n \in \mathbb{N}$. By the help of uniqueness of the fixed point of T, then we have $y_n \neq Ty_n$. If $y_n = w(y_n, Ty_n ; \alpha)$ for some $\alpha \in (0, 1)$, $n \in \mathbb{N}$, then

$$\begin{aligned} d(y_n, x^*) &= d(w(y_n, Ty_n; \alpha), x^*) \\ &\leq \alpha d(y_n, x^*) + (1-\alpha) d(Ty_n, x^*) \\ &\leq \alpha d(y_n, x^*) + (1-\alpha) k\{d(y_n, Ty_n) + d(x^*, Tx^*)\} \\ &\leq \alpha d(y_n, x^*) + (1-\alpha) k d(y_n, Ty_n). \end{aligned} \tag{99}$$

Hence,

$$d(y_n, x^*) \leq k d(y_n, Ty_n), \tag{100}$$

and we conclude that $\lim_{n\longrightarrow\infty} d(y_n, x^*) = 0$. Due to $\alpha > 0$, it is not difficult to see that $Ty_n \neq w(y_n, Ty_n; \alpha)$, indeed, if not,

$$d(y_n, Ty_n) = d(y_n, w(y_n, Ty_n; \alpha)) \leq (1-\alpha) d(y_n, Ty_n), \tag{101}$$

a contradiction. Therefore, let us assume that $y_n \neq Ty_n \neq w(y_n, Ty_n; \alpha)$, and then

$$\begin{aligned} d(y_n, x^*) &\leq s[d(y_n, w(y_n, Ty_n; \alpha)) + d(w(y_n, Ty_n; \alpha), Ty_n) + d(Ty_n, x^*)] \\ &\leq s(1-\alpha) d(y_n, Ty_n) + s\alpha d(y_n, Ty_n) + sk\{d(y_n, Ty_n) + d(x^*, Tx^*)\}, \end{aligned} \tag{102}$$

combining with $1 - s\beta > 0$, and we obtain

$$d(y_n, x^*) \leq (s + sk) d(y_n, Ty_n), \tag{103}$$

which implies $\lim_{n\longrightarrow\infty} d(y_n, x^*) = 0$, which completes the proof. □

3. Applications

In this section, we will apply our result to solving the following functional equation arising in dynamic programming:

$$p(u) = \sup_{v\in B}\{f(u, v) + G(u, v, p(\varphi(u, v)))\}, \tag{104}$$

for all $u \in A$, where $f: A \times B \longrightarrow R$, $\varphi: A \times B \longrightarrow A$, and $G: A \times B \times R \longrightarrow R$. We assume that C and D are Banach spaces, $A \subseteq C$ is a state space, and $B \subseteq D$ is a decision space. Precisely, see also [30, 31]. Let $X = R(A)$ denote the set of all bounded real-valued functions on A and the norm $\|\cdot\|$ defined as $\|x\| = \sup_{u\in A}|x(u)|$ for all $x \in X$. Clearly, $(X, \|\cdot\|)$ is a Banach space. Moreover, we can define a rectangular $b-$metric d by

$$d(x, y) = \sup_{u\in A}|x(u) - y(u)|^2, \tag{105}$$

for all $x, y \in X$. Since $(X, \|\cdot\|)$ is complete, we deduce that (X, d) is a complete rectangular $b-$metric space with $s = 3$. In order to show the existence of a solution of equation (104), we consider the operator $T: X \longrightarrow X$ of the form

$$T(x)(u) = \sup_{v\in B}\{f(u, v) + G(u, v, x(\varphi(u, v)))\}, \tag{106}$$

for all $u \in A$ and $x \in X$. We will prove the following theorem.

Theorem 35. *Let $T: X \longrightarrow X$ be given by (106). Suppose that the following hypotheses hold:*

(A1) $f: A \times B \longrightarrow R$ and $G: A \times B \times R \longrightarrow R$ are bounded functions;

(A2) There exists $a > 0$, for all $u \in A$, $v \in B$ and $x, y \in X$, such that

$$|G_1(u, v, x(u)) - G_2(u, v, y(u))| \leq a|x(u) - y(u)|. \tag{107}$$

Then, the functional equation (104) has a bounded solution.

Proof. Obviously, T is well defined, since f and G are bounded. That is, $Tx \in X$ and operator T are well defined. Then, from (A2), we have

$$\begin{aligned} |Tx(u) - Ty(u)|^2 &= \left|\sup_{v\in B}\{f(u, v) + G(u, v, x(\varphi(u, v)))\}\right. \\ &\quad \left. - \sup_{v\in B}\{f(u, v) + G(u, v, y(\varphi(u, v)))\}\right|^2 \\ &\leq \sup_{v\in B}|G(u, v, x(\varphi(u, v))) - G(u, v, y(\varphi(u, v)))|^2 \\ &\leq a^2 \sup_{v\in B}|x(u) - y(u)|^2. \end{aligned} \tag{108}$$

Let $0 \leq a \leq 1/3\sqrt{2}$; thus, all the conditions of Theorem 12 are fulfilled, and there exists a fixed point $x^* \in X$ of T such that $Tx^* = x^*$. In other words,

$$x^*(u) = \sup_{v\in B}\{f(u, v) + G(u, v, x^*(\varphi(u, v)))\}, \tag{109}$$

for all $u \in A$. This completes the proof. □

Example 36. Consider the functional equation

$$x(u) = \sup_{v\in[0,1]}\left\{\sin(u + v) + \ln\left(1 + uv + \frac{1}{6}x(uv)\right)\right\} \tag{110}$$

for $u \in [0, 2]$. We let $A = [0, 2]$, $B = [0, 1]$. $f: A \times B \longrightarrow R$ is defined by $f(u, v) = \sin(u + v)$, $\varphi: A \times B \longrightarrow A$ is defined by $\varphi(u, v) = uv$, and

$$G: A \times B \times R \longrightarrow R \tag{111}$$

is defined by $G(u, v, x) = \ln(1 + uv + 1/6x)$ for $x \in X$. It is not difficult to see that f and G are bounded functions. Moreover,

$$\begin{aligned}
&|G(u, v, x(\varphi(u, v))) - G(u, v, y(\varphi(u, v)))|^2 \\
&\quad = |\ln(1 + uv + x) - \ln(1 + uv + x)|^2 \\
&\quad = \left|\ln \frac{1 + uv + 1/6x + 1/6y - 1/6y}{1 + uv + 1/6y}\right|^2 \\
&\quad = \left|\ln\left(1 + \frac{1/6x - 1/6y}{1 + uv + 1/6y}\right)\right|^2 \\
&\quad \le \left|\ln\left(1 + \frac{1}{6}x - \frac{1}{6}y\right)\right|^2 \\
&\quad \le \frac{1}{36}|x - y|^2.
\end{aligned} \tag{112}$$

Thus, all the conditions of Theorem 35 are fulfilled. Hence, functional equation (110) has a solution $x^*(u) \in R(A)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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