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Research Article

The Analysis of the Fractional-Order Navier-Stokes Equations by a Novel Approach

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This article introduces modified semianalytical methods, namely, the Shehu decomposition method and q-homotopy analysis transform method, a combination of decomposition method, the q-homotopy analysis method, and the Shehu transform method to provide an approximate method analytical solution to fractional-order Navier-Stokes equations. Navier-Stokes equations are widely applied as models for spatial effects in biology, ecology, and applied sciences. A good agreement between the exact and obtained solutions shows the accuracy and efficiency of the present techniques. These results reveal that the suggested methods are straightforward and effective for engineering sciences models.

1. Introduction

Many investigations have provided unique solutions that meet the needs of various fields, making fractional differential equation resolution lucrative mathematical models. Caputo and Riemann-Liouville derivatives were the most acceptable method to model the different natural processes that required noninteger derivatives. However, their limits led to the search for additional derivatives. Singular kernels exist in both fractional derivatives of Riemann-Liouville and Caputo. Furthermore, the constant's Riemann-Liouville derivative does not equal zero. To outcome the problem of the unique kernel, Caputo and Fabrizio [1] suggested without a singular kernel of the fractional derivative operator. Through several applications, Caputo Fabrizio has proven to be an efficient operator [2–5]. By using the Mittag-Leffler function, the authors suggested an actual new fractional derivative in [6]. The Atangana-Baleanu Riemann derivative (Riemann-Liouville sense) is one, whereas the Atangana-Baleanu Caputo derivative is the other (Caputo sense). The Atangana-Baleanu Caputo and Riemann derivatives feature all fractional derivative properties except the semigroup property, additionally to the fact that the kernel is nonsingular and nonlocal. However, these new fractional operators have recently been identified in a newly constructed fractional operator categorization [7–10].

Fluid mechanics, mathematical biology, viscoelasticity, electrochemistry, life sciences, and physics all use fractional-order partial differential equations (PDEs) to explain various nonlinear complex systems [11–14]. For example, fractional derivatives can be used to describe nonlinear seismic oscillations [15], and fractional derivatives can be used to overcome the assumption's weakness in the fluid-dynamic traffic model [16]. Furthermore, based on actual results, [17, 18] offer fractional partial differential equations for the propagation of shallow-water waves and seepage flow in porous media. In most circumstances, obtaining the right behavior of fractional differential equations is extremely difficult. Much work has gone into developing strategies for calculating the approximate behavior of these kinds of equations. The fractional model is the most potential candidate in nanohydrodynamics, where the continuum assumption fails miserably. The homotopy perturbation Sumudu transform method [19], homotopy perturbation method [20], Adomian decomposition method [21], homotopy analysis method [22], fractional reduced differential transform method [23, 24], and variation iteration method [25] have all been proposed in recent years for solving fractional PDEs.

The Navier-Stokes equation, which depicts the viscous fluid's flow motion, was developed by Navier in 1822. They used numerous physical processes as examples of fluid movements, such as blood flow, ocean currents, liquid flow in pipelines, and airflow around airplane arms. Due to its nonlinear character, it can only be solved exactly in a few circumstances. In these situations, we must consider a simple flow pattern configuration and make assumptions about the fluid's state. The central equation of viscous fluid flow movement, known as the Navier-Stokes equation, was published in 1822 [26]. This equation depicts a few projections, including sea streams, fluid flow in channels, blood flow, and wind current around an airship's wings. There are numerous methods for solving fractional-order Navier-Stokes equations in the literature. El-Shahed and Salem published the first fractional version of the Navier-Stokes equation in 2005 [27]. Kumar et al. [28] used a mixture of homotopy perturbation method and the Laplace transform to solve a nonlinear fractional Navier-Stokes problem analytically. Ragab et al. [29] and Ganji et al. [30] used the homotopy analysis method to solve the same Navier-Stokes equation. For the solution of fractional Navier-Stokes equations, Birajdar [31] and Maitama [32] used the Adomian decomposition method. Kumar et al. [33] used the Adomian decomposition method and Laplace transform algorithms to obtain the analytical answer of the fractional Navier-Stokes problem. In contrast, Jena et al. [34] used the Laplace transform and finite Hankel transform algorithm to solve the same equation. The current paper uses the homotopy perturbation transform method to provide a precise or approximate solution to the stated problem.

Hashim et al. were the first to introduce the homotopy analysis method (HAM) [35, 36]. A continuous mapping is created in HAM by constructing it from a preliminary calculation to get close to the right solution of the considered equation. To make such a continuous mapping, an auxiliary linear operator is chosen, and the convergence of the series solution is ensured through an auxiliary parameter. The effect of higher-order wave dispersion can be investigated using time-fractional Korteweg-De Vries equations. The Korteweg-De Vries-Burgers equation describes the waves on shallow water surfaces. The nonlocal property of the fractional Korteweg-De Vries [37, 38] equation is its strength.

The Adomian decomposition technique [39, 40] is a well-known systematic technique for solving deterministic or stochastic operator equations, such as integral equations, integro-differential equations, ordinary differential equations, and partial differential equations. In real-world implementations in engineering and applied sciences, the Adomian decomposition method is vital for approximation analytic solutions and numeric simulations. It enables us to solve nonlinear initial value problems and boundary value problems without relying on nonphysical assumptions such as linearization, perturbation, and beliefs, estimating the starting function or a set of fundamental terms [41, 42]. Furthermore, because Green's functions are difficult to determine, the Adomian decomposition method does not necessitate their use, which would complicate such analytic calculations in most cases. The accuracy of the approximate analytical answers obtained can be verified using direct substitution [43] emphasized the ADM's benefits over Picard's iterated method. More benefits of the Adomian decomposition method over the variational iteration approach were highlighted in [44, 45]. Adomian polynomials, which are tailored to the specific nonlinearity to solve nonlinear operator equations, are essential ideas.

2. Preliminaries Concepts

Definition 1. The Sumudu transformation is achieved across the function set [46, 47]

$$A = \left\{ v(\mathfrak{I}) \colon \exists N, \tau_1, \tau_2 > 0, |v(\mathfrak{I})| < N e^{(|\mathfrak{I}|/\tau_i)}, \text{if } \mathfrak{I} \in (-1)^i \times [0, \infty) \right\}, \tag{1}$$

by

$$S[f(\mathfrak{I})] = G(u) = \int_0^\infty f(u\mathfrak{I}) e^{(-\mathfrak{I})} \mathrm{d}\mathfrak{I}, u \in (-\tau_1, \tau_2). \tag{2}$$

Many researchers have identified and applied this transform to models in various scientific areas and [46, 48]. The link between Laplace and Sumudu transformations is demonstrated in the next theorem.

Theorem 2. *Let G and F be the Laplace and the Sumudu transformations of $f(\mathfrak{I}) \in A$. Then [49],*

$$G(u) = \frac{F(1/u)}{u}. \tag{3}$$

The Shehu transformation was developed in [50] generalizes the Laplace and the Sumudu integral transformations; they have applied it to the analysis of ODEs and PDEs.

Definition 3. The Shehu transformation is achieved over the set A by the following [50]:

$$\mathbb{H}[f(\mathfrak{I})] = V(s, u) = \int_0^\infty e^{(-s\mathfrak{I}/u)} f(\mathfrak{I}) d\mathfrak{I}. \tag{4}$$

It is evident that the Shehu transformation is linear as the Laplace and Sumudu transforms. The function Mittag-Leffler $E\delta(\mathfrak{I})$ is a straightforward generalized form of the exponential series. For $\delta = 1$, we get $E_\delta(\mathfrak{I}) = e^{(\mathfrak{I})}$. It is expressed as [51]

$$E_\delta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + 1)}, \delta \in \mathbb{C}, \operatorname{Re}(\delta) > 0. \tag{5}$$

Definition 4. Let $f \in H^1(a, b), b > a$; then, for $\delta \in (0, 1)$, the fractional derivative of Atangana-Baleanu in the sense

Caputo is define as [6]

$$ {}_{a}^{ABC}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J})) = \frac{B(\delta)}{1-\delta}\int_{a}^{\mathfrak{J}} f'(x)E_{\delta}\left(-\delta\frac{(\mathfrak{J}-x)^{\delta}}{1-\delta}\right)dx. \quad (6) $$

Definition 5. Let $f \in H^1(a,b), b > a$; then, for $\delta \in (0,1)$, the fractional derivative of Atangana-Baleanu in the sense Riemann-Liouville is expressed as [6]

$$ {}_{a}^{ABR}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J})) = \frac{B(\delta)}{1-\delta}\frac{d}{d\mathfrak{J}}\int_{a}^{\mathfrak{J}} f(x)E_{\delta}\left(-\delta\frac{(\mathfrak{J}-x)^{\delta}}{1-\delta}\right)dx. \quad (7) $$

Under the constraints $B(0) = B(1) = 1$, $B(\delta)$ is a normalising function.

Theorem 6. *The Laplace transformation of fractional derivative Atangana-Baleanu in sense of Caputo is define as [6]*

$$ \mathbb{L}\left\{{}_{0}^{ABC}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right\}(s) = \frac{B(\delta)}{1-\delta}\frac{s^{\delta}F(s) - s^{\delta-1}f(0)}{s^{\delta}+\delta/1-\delta}, \quad (8) $$

and the Laplace transformation of the fractional derivative Atangana-Baleanu in sense of Riemann-Liouville is define as

$$ \mathbb{L}\left\{{}_{0}^{ABC}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right\}(s) = \frac{B(\delta)}{1-\delta}\frac{s^{\delta}F(s)}{s^{\delta}+\delta/1-\delta}. \quad (9) $$

3. Main Solutions

In what follows, we suppose that $f \in H^1(a,b), b > a, \delta \in (0,1)$ and $f(\mathfrak{J}) \in A$.

Theorem 7. *The fractional derivative of Atangana-Baleanu Sumudu transformation in Caputo sense is define as [8]*

$$ S\left({}_{0}^{ABC}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1-\delta+\delta u^{\delta}}(G(u) - f(0)). \quad (10) $$

Proof. Applying (8) and (3), we achieve

$$ \begin{aligned} S\left({}_{0}^{ABC}D_{t}^{\delta}(f(\mathfrak{J}))\right) &= \frac{1}{u}\left(\frac{B(\delta)}{1-\delta}\frac{(1/u)^{\delta}F(1/u) - (1/u)^{\delta-1}f(0)}{(1/u)^{\delta}+\delta/1-\delta}\right) \\ &= \left(\frac{1}{u}\right)^{\delta}\frac{B(\delta)}{1-\delta}\frac{G(u)-f(0)}{(1/u)^{\delta}+\delta/1-\delta}. \end{aligned} \quad (11) $$

Then, we achieve the desired outcome

$$ S\left({}_{0}^{ABC}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1-\delta+\delta u^{\delta}}(G(u) - f(0)). \quad (12) $$

□

Theorem 8. *The Sumudu transformation of fractional derivative Atangana-Baleanu in sense of Riemann-Liouville is define as [8]*

$$ S\left({}_{0}^{ABR}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1-\delta+\delta u^{\delta}}G(u). \quad (13) $$

Proof. Using (9) and (3), we obtain

$$ \begin{aligned} S\left\{{}_{\delta}^{ABR}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right\}(s) &= \frac{1}{u}\frac{B(\delta)}{1-\delta}\frac{(1/u)^{\delta}F(1/u)}{(1/u)^{\delta}+\delta/1-\delta} \\ &= \frac{1}{u}\frac{B(\delta)}{1-\delta}\frac{(1/u)^{\delta}uG(u)}{(1/u)^{\delta}+\delta/1-\delta} \\ &= \frac{B(\delta)}{1-\delta}\frac{G(u)}{1+\delta/1-\delta u^{\delta}}. \end{aligned} \quad (14) $$

Then, we achieve the desired outcome

$$ S\left({}_{0}^{ABR}D_{\mathfrak{J}}^{\delta}(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1-\delta+\delta u^{\delta}}G(u). \quad (15) $$

The Sumudu and Shehu transformations are demonstrated in the following theorem. □

Theorem 9. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{J}) \in A$. Then [8],*

$$ V(s,u) = \frac{u}{s}G\left(\frac{u}{s}\right). \quad (16) $$

Proof. If $f(\mathfrak{J}) \in A$, then

$$ \mathrm{V}(\mathrm{s},\mathrm{u}) = \int_{0}^{\infty} e^{(-\mathrm{s}\mathfrak{J}/\mathrm{u})}\mathrm{f}(\mathfrak{J})\mathrm{d}\mathfrak{J}. \quad (17) $$

If we set $\tau = s\mathfrak{J}/u(\mathfrak{J} = u\tau/s)$, then the right hand side can be represent as

$$ \mathrm{V}(\mathrm{s},\mathrm{u}) = \int_{0}^{\infty} e^{(-\tau)}\mathrm{f}\left(\frac{\mathrm{u}}{\mathrm{s}}\tau\right)\frac{\mathrm{u}}{\mathrm{s}}\mathrm{d}\tau = \frac{\mathrm{u}}{\mathrm{s}}\int_{0}^{\infty} e^{(-\tau)}\mathrm{f}\left(\frac{\mathrm{u}}{\mathrm{s}}\tau\right)\mathrm{d}\tau. \quad (18) $$

The right side integral is obviously $G(u/s)$, thus obtaining (16). It is obvious that

$$ \mathrm{V}(\mathrm{s},1) = \frac{1}{\mathrm{s}}\mathrm{G}\left(\frac{1}{\mathrm{s}}\right) = \mathrm{F}(\mathrm{s}), \quad (19) $$

where $F(s)$ is the Laplace transformation of $f(\mathfrak{J})$. □

The following significant properties are achieved by applying the relationship among Sumudu and Shehu transformations (16).

Theorem 10. *The Shehu transformation of $\mathfrak{J}^{x-1}$ is [8]*

$$ V(s,u) = \Gamma(x)\left(\frac{u}{s}\right)^{x}, x > 0. \quad (20) $$

Proof. When $x > 0$, the Sumudu of $\mathfrak{J}^{x-1}$ is define as

$$G(u) = \Gamma(x)u^{x-1}, \tag{21}$$

where $\Gamma(x)$ is the Gamma function expressed as

$$\Gamma(x) = \int_0^\infty \mathfrak{J}^{x-1} e^{-\mathfrak{J}} d\mathfrak{J}. \tag{22}$$

Then, by applying (16), we get the achieved solution. □

Theorem 11. *Let $\delta, \omega \in \mathbb{C}$, with* Re $(\delta) > 0$. *Shehu transform of $E_\delta(\omega\mathfrak{J}^\delta)$ is given by [8]*

$$H\left(E_\delta\left(\omega\mathfrak{J}^\delta\right)\right) = \frac{u}{s}\left(1 - \omega\left(\frac{u}{s}\right)^\delta\right)^{-1}. \tag{23}$$

Proof. Based on the reference [52], we get

$$S\left(E_\delta\left(\omega\mathfrak{J}^\delta\right)\right) = \left(1 - \omega u^\delta\right)^{-1}, \tag{24}$$

and then, by applying (16), we get

$$H\left(E_\delta\left(\omega\mathfrak{J}^\delta\right)\right) = \left(\frac{u}{s}\right)\left(1 - \omega\left(\frac{u}{s}\right)^\delta\right)^{-1}. \tag{25}$$

□

Theorem 12. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{J}) \in A$. Then, the fractional derivative of Atangana-Baleanu Shehu transform in sense of Caputo is define as* $\mathbb{H}$

$$H\left({}_0^{ABC}D_{\mathfrak{J}}^\delta(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \tag{26}$$

Proof. Applying (10) and the relationship among Shehu and Sumudu transformations, we achieve

$$\begin{aligned} H\left({}_0^{ABC}D_{\mathfrak{J}}^\delta(f(\mathfrak{J}))\right) &= \frac{u}{s}\frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(G\left(\frac{u}{s}\right) - f(0)\right) \\ &= \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \end{aligned} \tag{27}$$

□

Theorem 13. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{J}) \in A$. Then, the fractional derivative of Atangana-Baleanu Shehu transform in sense of Riemann-Liouville is define as*

$$H\left({}_0^{ABR}D_{\mathfrak{J}}^\delta(f(\mathfrak{J}))\right) = \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \tag{28}$$

Proof. Applying (13) and the relationship among Shehu and Sumudu transformations (16), we achieve

$$\begin{aligned} H\left({}_0^{ABR}D_{\mathfrak{J}}^\delta(f(\mathfrak{J}))\right) &= \frac{u}{s}\frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}G\left(\frac{u}{s}\right) \\ &= \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}V(s,u). \end{aligned} \tag{29}$$

□

4. The Procedure of SDM

In this section, we describe the SDM procedure for fractional PDEs.

$$\begin{aligned} D_{\mathfrak{J}}^\delta\mu(\chi,\mathfrak{J}) + \mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{P}_1(\chi,\mathfrak{J}) = 0, \\ D_{\mathfrak{J}}^\delta\nu(\chi,\mathfrak{J}) + \mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{P}_2(\chi,\mathfrak{J}) = 0, 0 < \delta \le 1, \end{aligned} \tag{30}$$

with initial condition

$$\mu(\delta,0) = g_1(\chi), \quad \nu(\delta,0) = g_2(\chi), \tag{31}$$

where $D_{\mathfrak{J}}^\delta = \partial^\delta/\partial\mathfrak{J}^\delta$ is the Caputo derivative of fractional-order δ, $\mathscr{R}_1$ and $\mathscr{R}_2$ and $\mathscr{N}_1$ and $\mathscr{N}_2$ are linear and nonlinear terms, respectively, and $\mathscr{P}_1$ and $\mathscr{P}_2$ are source functions.

Applying the Shehu transform to Equation (30),

$$\begin{aligned} S\left[D_{\mathfrak{J}}^\delta\mu(\chi,\mathfrak{J})\right] + S[\mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{P}_1(\chi,\mathfrak{J})] = 0, \\ S\left[D_{\mathfrak{J}}^\delta\nu(\chi,\mathfrak{J})\right] + S[\mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{P}_2(\chi,\mathfrak{J})] = 0. \end{aligned} \tag{32}$$

Applying the Shehu transformation of differentiation property, we have

$$\begin{aligned} S[\mu(\chi,\mathfrak{J})] &= \frac{u}{s}\mu(\chi,0) + \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S[\mathscr{P}_1(\chi,\mathfrak{J})] \\ &\quad - \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S\{\mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu)\}], \\ S[\nu(\chi,\mathfrak{J})] &= \frac{u}{s}\nu(\chi,0) + \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S[\mathscr{P}_2(\chi,\mathfrak{J})] \\ &\quad - \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S\{\mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu)\}]. \end{aligned} \tag{33}$$

SDM defines the result of infinite series $\mu(\chi, \mathfrak{I})$ and $\nu(\chi, \mathfrak{I})$.

$$\mu(\chi, \mathfrak{I}) = \sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{I}), \quad \nu(\chi, \mathfrak{I}) = \sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{I}). \tag{34}$$

The nonlinear functions defined by Adomian polynomials $\mathcal{N}_1$ and $\mathcal{N}_2$ are expressed as

$$\mathcal{N}_1(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{A}_m, \mathcal{N}_2(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{B}_m. \tag{35}$$

The Adomian polynomials can be expressed as

$$\begin{aligned} \mathcal{A}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial \lambda^m}\left\{\mathcal{N}_1\left(\sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k\right)\right\}\right]_{\lambda=0}, \\ \mathcal{B}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial \lambda^m}\left\{\mathcal{N}_2\left(\sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k\right)\right\}\right]_{\lambda=0}. \end{aligned} \tag{36}$$

Putting Equations (34) and (36) into (33) gives

$$\begin{aligned} S\left[\sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{I})\right] &= \frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{I})\} \\ &\quad - \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} \\ &\quad \cdot S\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{A}_m\right\}, \\ S\left[\sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{I})\right] &= \frac{u}{s}\nu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{I})\} \\ &\quad - \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} \\ &\quad \cdot S\left\{\mathcal{R}_2\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{B}_m\right\}. \end{aligned} \tag{37}$$

Using the inverse Shehu transform of Equation (37),

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{I}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{I})\}\right] \\ &\quad - S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{A}_m\right\}\right], \\ \sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{I}) &= S^{-1}\left[\frac{u}{s}\nu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{I})\}\right] \\ &\quad - S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\left\{\mathcal{R}_2\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{B}_m\right\}\right], \end{aligned} \tag{38}$$

and we expressed the following terms:

$$\begin{aligned} \mu_0(\chi, \mathfrak{I}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{I})\}\right], \\ \nu_0(\chi, \mathfrak{I}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{I})\}\right], \\ \mu_1(\chi, \mathfrak{I}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{R}_1(\mu_0, \nu_0) + \mathcal{A}_0\}\right], \\ \nu_1(\chi, \mathfrak{I}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{R}_2(\mu_0, \nu_0) + \mathcal{B}_0\}\right]. \end{aligned} \tag{39}$$

The general for $m \geq 1$ is given by

$$\begin{aligned} \mu_{m+1}(\chi, \mathfrak{I}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{R}_1(\mu_m, \nu_m) + \mathcal{A}_m\}\right], \\ \nu_{m+1}(\chi, \mathfrak{I}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\{\mathcal{R}_2(\mu_m, \nu_m) + \mathcal{B}_m\}\right], \end{aligned} \tag{40}$$

5. Solution of SDM

Example 1. Consider the two dimensional fractional-order Navier-Stokes equation

$$\begin{aligned} D_{\mathfrak{I}}^{\delta}(\mu) + \mu\frac{\partial \mu}{\partial \chi} + \mu\frac{\partial \mu}{\partial \xi} &= \rho\left[\frac{\partial^2 \mu}{\partial \chi^2} + \frac{\partial^2 \mu}{\partial \xi^2}\right] + q, \\ D_{\mathfrak{I}}^{\delta}(\nu) + \mu\frac{\partial \nu}{\partial \chi} + \nu\frac{\partial \nu}{\partial \xi} &= \rho\left[\frac{\partial^2 \nu}{\partial \chi^2} + \frac{\partial^2 \nu}{\partial \xi^2}\right] - q, \end{aligned} \tag{41}$$

with initial conditions

$$\begin{cases} \mu(\chi, \xi, 0) = -\sin(\chi + \xi), \\ \nu(\chi, \xi, 0) = \sin(\chi + \xi). \end{cases} \tag{42}$$

Using Shehu transform of Equation (41), we have

$$S\left\{\frac{\partial^\delta \mu(\chi, \xi, \mathfrak{I})}{\partial \mathfrak{I}^\delta}\right\} = -S\left[\mu\frac{\partial \mu}{\partial \chi} + \mu\frac{\partial \mu}{\partial \xi} - \rho\left[\frac{\partial^2 \mu}{\partial \chi^2} + \frac{\partial^2 \mu}{\partial \xi^2}\right] + q\right],$$

$$S\left\{\frac{\partial^\delta \nu(\chi, \xi, \mathfrak{I})}{\partial \mathfrak{I}^\delta}\right\} = -S\left[\mu\frac{\partial \nu}{\partial \chi} + \nu\frac{\partial \nu}{\partial \xi} - \rho\left[\frac{\partial^2 \nu}{\partial \chi^2} + \frac{\partial^2 \nu}{\partial \xi^2}\right] - q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\mu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\mu(\chi,\xi,0)\right\} = -S\left[\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q\right],$$
$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\nu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\nu(\chi,\xi,0)\right\} = -S\left[\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q\right]. \tag{43}$$

The above equations can be written as

$$S\{\mu(\chi,\xi,\mathfrak{T})\} = \frac{u}{s}\{\mu(\chi,\xi,0)\} - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q\right],$$
$$S\{\nu(\chi,\xi,\mathfrak{T})\} = \frac{u}{s}\{\nu(\chi,\xi,0)\} - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q\right]. \tag{44}$$

Using inverse Shehu transform, we have

$$\mu(\chi,\xi,\mathfrak{T}) = \mu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S[q]\right] - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left(\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right)\right\}\right],$$
$$\nu(\chi,\xi,\mathfrak{T}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S[q]\right] - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left(\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right)\right\}\right]. \tag{45}$$

Suppose that the unknown functions $\mu(\chi,\xi,\mathfrak{T})$ and $\nu(\chi,\xi,\mathfrak{T})$ infinite series solution are as follows:

$$\mu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}),$$
$$\nu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}). \tag{46}$$

Note that $\mu\mu_{\chi} = \sum m = 0^{\infty}\mathscr{A}_m$, $\nu\mu_{\xi} = \sum m = 0^{\infty}\mathscr{B}_m$, $\mu\nu_{\chi} = \sum m = 0^{\infty}\mathscr{C}m$, and $\nu\nu_{\xi} = \sum_{m=0}^{\infty}\mathscr{D}_m$ are the Adomian polynomials, and the nonlinear terms were described. Applying such terms, Equation (45) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}) = \mu(\chi,\xi,0) + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right) + \rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$
$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right) + \rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{47}$$

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\xi^2}\right\}\right]\right],$$
$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\xi^2}\right\}\right]\right]. \tag{48}$$

According to Equation (36), the Adomian polynomials can be expressed as

$$\mathscr{A}_0 = \mu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{A}_1 = \mu_0\frac{\partial\mu_1}{\partial\chi}+\mu_1\frac{\partial\mu_0}{\partial\chi},$$
$$\mathscr{B}_0 = \nu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{B}_1 = \nu_0\frac{\partial\mu_1}{\partial\xi}+\nu_1\frac{\partial\mu_0}{\partial\xi},$$

$$\mathscr{C}_0 = \mu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{C}_1 = \mu_0\frac{\partial\nu_1}{\partial\chi}+\mu_1\frac{\partial\nu_0}{\partial\chi},$$
$$\mathscr{D}_0 = \nu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_1 = \nu_0\frac{\partial\nu_1}{\partial\chi}+\nu_1\frac{\partial\nu_0}{\partial\chi}. \tag{49}$$

Thus, we can easy achieve the recursive relationship Equation (48).

$$\mu_0(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right],$$
$$\nu_0(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]. \tag{50}$$

For $m=0$,

$$\mu_1(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi)\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right],$$
$$\nu_1(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi)\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]. \tag{51}$$

For $m=1$,

$$\mu_2(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^\delta}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right],$$
$$\nu_2(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^\delta}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]. \tag{52}$$

For $m=2$,

$$\mu_3(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right.$$
$$\left.+\frac{\delta^2(1-\delta)\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}\right],$$

$$\nu_3(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right.$$
$$\left.+\frac{\delta^2(1-\delta)\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}\right] \vdots \tag{53}$$

In the same method, the remaining μ_m and $\nu_m (m \geq 3)$ components of the SDM solution can be obtained seamlessly. Consequently, we describe the series of alternative solutions as

$$\mu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi) = \mu_0(\chi,\xi)+\mu_1(\chi,\xi)$$
$$+\mu_2(\chi,\xi)+\mu_3(\chi,\xi)+\cdots,$$

$$\nu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi) = \nu_0(\chi,\xi)+\nu_1(\chi,\xi)$$
$$+\nu_2(\chi,\xi)+\nu_3(\chi,\xi)+\cdots,$$

$$\mu(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]$$
$$+\sin(\chi+\xi)\frac{2\rho}{\mathrm{B}(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]$$
$$-\sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^\delta}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]$$
$$+\sin(\chi+\xi)\frac{(2\rho)^3}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right.$$
$$\left.+\frac{\delta^2(1-\delta)\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}\right]-\cdots,$$

$$\nu(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]$$
$$-\sin(\chi+\xi)\frac{2\rho}{\mathrm{B}(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right]$$
$$+\sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^\delta}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]$$
$$-\sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{I}^\delta}{\Gamma(\delta+1)}\right.$$
$$\left.+\frac{\delta^2(1-\delta)\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{I}^{2\delta+1}}{\Gamma(2\delta+2)}\right]+\cdots. \tag{54}$$

The exact result of Equation (41) at $\delta=1$ and $q=0$ is as follows:

$$\mu(\chi,\xi,\mathfrak{I}) = -e^{-2\rho\mathfrak{I}}\sin(\chi+\xi),$$
$$\nu(\chi,\xi,\mathfrak{I}) = e^{-2\rho\mathfrak{I}}\sin(\chi+\xi). \tag{55}$$

Example 2. Consider the two dimensional fractional-order Navier-Stokes equation

$$D_{\mathfrak{I}}^{\delta}(\mu)+\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi} = \rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q,$$
$$D_{\mathfrak{I}}^{\delta}(\nu)+\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi} = \rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q, \tag{56}$$

with the initial conditions

$$\begin{cases}\mu(\chi,\xi,0) = -e^{\chi+\xi},\\ \nu(\chi,\xi,0) = e^{\chi+\xi}.\end{cases} \tag{57}$$

Using Shehu transform of Equation (56), we have

$$S\left\{\frac{\partial^\delta\mu}{\partial\mathfrak{I}^\delta}\right\} = S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$S\left\{\frac{\partial^\delta\nu}{\partial\mathfrak{I}^\delta}\right\} = S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\mu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\mu(\chi,\xi,0)\right\} = S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\nu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\nu(\chi,\xi,0)\right\} = S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right]. \tag{58}$$

The above equations can be written as

$$S\{\mu(\chi,\xi,\mathfrak{T})\} = \frac{u}{s}\{\mu(\chi,\xi,0)\} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$S\{\nu(\chi,\xi,\mathfrak{T})\} = \frac{u}{s}\{\nu(\chi,\xi,0)\} = \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right]. \tag{59}$$

Applying inverse Shehu transformation, we get

$$\mu(\chi,\xi,\mathfrak{T}) = \mu(\chi,\xi,0) + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$

$$\nu(\chi,\xi,\mathfrak{T}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{60}$$

Suppose that the unknown functions $\mu(\chi,\xi,\mathfrak{T})$ and $\nu(\chi,\xi,\mathfrak{T})$ infinite series result as follows:

$$\mu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}), \qquad \nu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}). \tag{61}$$

Note that $\mu\mu_\chi = \sum m = 0^\infty \mathscr{A}_m$, $\nu\mu_\xi = \sum m = 0^\infty \mathscr{B}_m$, $\mu\nu_\chi = \sum m = 0^\infty \mathscr{C} m$, and $\nu\nu_\xi = \sum_{m=0}^{\infty}\mathscr{D}_m$ are the Adomian polynomials, and the nonlinear terms were described. Applying such terms, Equation (60) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}) = \mu(\chi,\xi,0) + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{62}$$

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\xi^2}\right\}\right]\right]. \tag{63}$$

According to Equation (36), the Adomian polynomials can be expressed as

$$\mathscr{A}_0 = \mu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{A}_1 = \mu_0\frac{\partial\mu_1}{\partial\chi}+\mu_1\frac{\partial\mu_0}{\partial\chi}, \mathscr{B}_0 = \nu_0\frac{\partial\mu_0}{\partial\beta}, \mathscr{B}_1 = \nu_0\frac{\partial\mu_1}{\partial\beta}+\nu_1\frac{\partial\mu_0}{\partial\beta},$$
$$\mathscr{C}_0 = \mu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{C}_1 = \mu_0\frac{\partial\nu_1}{\partial\chi}+\mu_1\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_0 = \nu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_1 = \nu_0\frac{\partial\nu_1}{\partial\chi}+\nu_1\frac{\partial\nu_0}{\partial\chi}. \tag{64}$$

Thus, we can quickly achieve the recursive relationship

Equation (63)

$$\mu_0(\chi,\xi,\mathfrak{T}) = -e^{\chi+\xi} + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_0(\chi,\xi,\mathfrak{T}) = e^{\chi+\xi} - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right]. \quad (65)$$

For $m=0$,

$$\mu_1(\chi,\xi,\mathfrak{T}) = e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_1(\chi,\xi,\mathfrak{T}) = -e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right]. \quad (66)$$

For $m=1$,

$$\mu_2(\chi,\xi,\mathfrak{T}) = -e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right],$$
$$\nu_2(\chi,\xi,\mathfrak{T}) = e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right]. \quad (67)$$

For $m=2$,

$$\mu_3(\chi,\xi,\mathfrak{T}) = e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{T}^{3\delta}}{\Gamma(3\delta+1)}, \nu_3(\chi,\xi,\mathfrak{T}) = -e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{T}^{3\delta}}{\Gamma(3\delta+1)}.$$
$$\vdots \quad (68)$$

In same method, the remaining μ_m and $\nu_m (m \geq 3)$ components of the SDM solution can be obtained seamlessly. Consequently, we describe the series of alternative solutions as

$$\mu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi) = \mu_0(\chi,\xi)+\mu_1(\chi,\xi)+\mu_2(\chi,\xi)+\mu_3(\chi,\xi)+\cdots,$$

$$\nu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi) = \nu_0(\chi,\xi)+\nu_1(\chi,\xi)+\nu_2(\chi,\xi)+\nu_3(\chi,\xi)+\cdots,$$

$$\begin{aligned}\mu(\chi,\xi,\mathfrak{T}) &= -e^{\chi+\xi} + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ &+ e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ &- e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right] \\ &+ e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{T}^{3\delta}}{\Gamma(3\delta+1)}-\cdots,\end{aligned}$$

$$\begin{aligned}\nu(\chi,\xi,\mathfrak{T}) &= e^{\chi+\xi} - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ &- e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ &+ e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right] \\ &- e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{T}^{3\delta}}{\Gamma(3\delta+1)}+\cdots.\end{aligned} \quad (69)$$

The exact result of Equation (56) at $\delta=1$ and $q=0$ is as follows:

$$\mu(\chi,\xi,\mathfrak{T}) = -e^{\chi+\xi+2\rho\mathfrak{T}},$$
$$\nu(\chi,\xi,\mathfrak{T}) = e^{\chi+\xi+2\rho\mathfrak{T}}. \quad (70)$$

6. The Methodology of q-HATM

Consider a nonlinear nonhomogeneous fractional partial differential equation:

$$D_{\mathfrak{T}}^{\delta}\mu(\chi,\xi,\mathfrak{T}) + R\mu(\chi,\xi,\mathfrak{T}) + N\mu(\chi,\xi,\mathfrak{T}) = f(\chi,\xi,\mathfrak{T}), \quad n-1<\delta\leq n. \quad (71)$$

Here, $D_{\mathfrak{T}}^{\delta}\mu$ is the Caputo derivative, and R and N are linear and nonlinear functions, respectively. $f(\chi,\xi,\mathfrak{T})$ is the source operator.

Now, using the Shehu transformation on Equation (71), we get

$$\begin{aligned}&S[\mu(\chi,\xi,\mathfrak{T})] - \frac{u}{s}\mu(\chi,\xi,0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \\ &\cdot \{S[R\mu(\chi,\xi,\mathfrak{T})] + S[NL(\chi,\xi,\mathfrak{T})] - S[f(\chi,\xi,\mathfrak{T})]\} = 0.\end{aligned} \quad (72)$$

The nonlinear function is

$$\begin{aligned}N[\phi(\chi,\xi,\mathfrak{T};\bar{q})] &= S[\phi(\chi,\xi,\mathfrak{T};\bar{q})] - \frac{u}{s}\phi(\chi,\xi,\mathfrak{T};\bar{q})(0^{+}) \\ &+ \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\{S[R\phi(\chi,\xi,\mathfrak{T};\bar{q})] \\ &+ S[N\phi(\chi,\xi,\mathfrak{T};\bar{q})] - S[f(\chi,\xi,\mathfrak{T})]\}.\end{aligned} \quad (73)$$

Here, $\phi(\chi,\xi,\mathfrak{T};\bar{q})$ is an unknown term, and $\bar{q} \in [0, 1/4]$ is the embedding parameter, $n \geq 1$. Construct a homotopy as

$$\begin{aligned}(1-n\bar{q})S[\phi(\chi,\xi,\mathfrak{T};\bar{q}) - \mu_0(\chi,\xi,\mathfrak{T})] \\ = \hbar q H(\chi,\xi,\mathfrak{T})N[\phi(\chi,\xi,\mathfrak{T};\bar{q})],\end{aligned} \quad (74)$$

where μ_0 is an initial condition and $\hbar \neq 0$ is an auxiliary parameter. The following solutions hold for $= 0, 1/n =$

$$\begin{aligned}\phi(\chi, \xi, \mathfrak{I}; 0) &= \mu_0(\chi, \xi, \mathfrak{I}),\\ \phi\left(\chi, \xi, \mathfrak{I}; \frac{1}{n}\right) &= \mu(\chi, \xi, \mathfrak{I}).\end{aligned} \tag{75}$$

Elevating q, ϕ converges from U_0 to U. Intensifying ϕ about q by Taylor's theorem, we get

$$\phi(\chi, \xi, \mathfrak{I}; \bar{q}) = U_0 + \sum_{m=1}^{\infty} \mu_m(\chi, \xi, \mathfrak{I})\bar{q}^m, \tag{76}$$

where

$$\mu_m = \frac{1}{m!}\frac{\partial^m \phi(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \bar{q}^m}\bigg|_{\bar{q}=0}. \tag{77}$$

By an appropriate selection of auxiliary linear operator, $U_0, n, \hbar$ and H, series (76) converges at $\bar{q} = 1/n$, thereby providing a result

$$\mu(\chi, \xi, \mathfrak{I}) = \mu_0 + \sum_{m=1}^{\infty} \mu_m(\chi, \xi, \mathfrak{I})\left(\frac{1}{n}\right)^m. \tag{78}$$

Now, differential Equation (74) m times, divide by $m!$ and taking $\bar{q} = 0$,

$$S[\mu_m(\chi, \xi, \mathfrak{I}) - k_m\mu_{m-1}(\chi, \xi, \mathfrak{I})], \tag{79}$$

where the vector is described as

$$\vec{\mu}_m = \{\mu_0(\chi, \xi, \mathfrak{I}), \mu_1(\chi, \xi, \mathfrak{I}), \cdots, \mu_m(\chi, \xi, \mathfrak{I})\}. \tag{80}$$

Applying the inverse transform on Equation (80),

$$\mu_m(\chi, \xi, I) = \quad k_m\mu_{m-1}(\chi, \xi, I) + \hbar S^{-1}\left[H(\chi, \xi, I)R_m\left(\vec{\mu}_{m-1}\right)\right]. \tag{81}$$

Here,

$$\mathfrak{R}_m\left(\vec{\mu}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\chi, \xi, \mathfrak{I}; \bar{q})]}{\partial \bar{q}^{m-1}}\bigg|_{\bar{q}=0},$$

$$k_r = \begin{cases} 0, & r \le 1, \\ n, & r > 1. \end{cases} \tag{82}$$

Lastly, by solving Equation (81), the elements of the q-HATM result are readily available.

Example 3. Consider the two dimensional fractional-order Navier-Stokes equation

$$\begin{cases} D_{\mathfrak{I}}^{\delta}\mu + \mu\dfrac{\partial \mu}{\partial \chi} + \nu\dfrac{\partial \mu}{\partial \xi} = \rho_0\left(\dfrac{\partial^2 \mu}{\partial \chi^2} + \dfrac{\partial^2 \mu}{\partial \xi^2}\right) + g, \\ D_{\mathfrak{I}}^{\delta}\nu + \mu\dfrac{\partial \nu}{\partial \chi} + \nu\dfrac{\partial \nu}{\partial \xi} = \rho_0\left(\dfrac{\partial^2 \nu}{\partial \chi^2} + \dfrac{\partial^2 \nu}{\partial \xi^2}\right) - g, \end{cases} \quad 0 < \delta \le 1, \tag{83}$$

with initial conditions

$$\begin{aligned}\nu(\chi, \xi, 0) &= \sin(\chi + \xi),\\ \mu(\chi, \xi, 0) &= -\sin(\chi + \xi).\end{aligned} \tag{84}$$

Using the Shehu transformation on Equation (83) and applying Equation (84), we have

$$\begin{aligned} &S[\mu(\chi, \xi, \mathfrak{I})] + \frac{u}{s}\sin(\chi + \xi) + \frac{1 - \delta + \delta(u/s)^{\delta}}{B(\delta)} \\ &\cdot S\left\{\mu\frac{\partial \mu}{\partial \chi} + \nu\frac{\partial \mu}{\partial \xi} - \rho_0\left(\frac{\partial^2 \mu}{\partial \chi^2} + \frac{\partial^2 \mu}{\partial \xi^2}\right) - g\right\} = 0, \\ &S[\nu(\chi, \xi, \mathfrak{I})] - \frac{u}{s}\sin(\chi + \xi) + \frac{1 - \delta + \delta(u/s)^{\delta}}{B(\delta)} \\ &\cdot S\left\{\mu\frac{\partial \nu}{\partial \chi} + \nu\frac{\partial \nu}{\partial \xi} - \rho_0\left(\frac{\partial^2 \nu}{\partial \chi^2} + \frac{\partial^2 \nu}{\partial \xi^2}\right) + g\right\} = 0. \end{aligned} \tag{85}$$

Define the nonlinear operators

$$\begin{aligned} &N^1[\phi_1(\chi, \xi, \mathfrak{I}; \bar{q}), \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})] \\ &\quad = S[\phi_1(\chi, \xi, \mathfrak{I}; \bar{q})] + \frac{u}{s}\sin(\chi + \xi) + \frac{1 - \delta + \delta(u/s)^{\delta}}{B(\delta)} \\ &\quad \cdot S\left\{\phi_1(\chi, \xi, \mathfrak{I}; \bar{q})\frac{\partial \phi_1(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \chi} + \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})\right. \\ &\quad \left.\cdot \frac{\partial \phi_1(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \xi} - \rho_0\left(\frac{\partial^2 \phi_1(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \chi^2}\frac{\partial^2 \phi_1(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \xi^2}\right) - g\right\}, \\ &N^2[\phi_1(\chi, \xi, \mathfrak{I}; \bar{q}), \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})] = S[\phi_2(\chi, \xi, \mathfrak{I}; \bar{q})] \\ &\quad - \frac{u}{s}\sin(\chi + \xi) + \frac{1 - \delta + \delta(u/s)^{\delta}}{B(\delta)} \\ &\quad \cdot S\left\{\phi_1(\chi, \xi, \mathfrak{I}; \bar{q})\frac{\partial \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \chi} + \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})\right. \\ &\quad \left.\cdot \frac{\partial \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \xi} - \rho_0\left(\frac{\partial^2 \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \chi^2} + \frac{\partial^2 \phi_2(\chi, \xi, \mathfrak{I}; \bar{q})}{\partial \xi^2}\right) + g\right\}, \end{aligned} \tag{86}$$

and the Shehu operators as

$$S[\mu_m(\chi,\xi,\mathfrak{I})-k_m\mu_{m-1}(\chi,\xi,\mathfrak{I})]=\hbar R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right],$$
$$S[\nu_m(\chi,\xi,\mathfrak{I})-k_m\nu_{m-1}(\chi,\xi,\mathfrak{I})]=\hbar R_{2,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right], \tag{87}$$

$$R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]=S[\mu_{m-1}(\chi,\xi,\mathfrak{I})]+\left(1-\frac{k_m}{n}\right)\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)}S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\mu_m-1-i}{\partial\chi}+\sum_{i=0}^{m-1}\nu\frac{\partial\mu_{m-1}-i}{\partial\xi}-\rho_0\left(\frac{\partial^2\mu_{m-1}}{\partial\chi^2}+\frac{\partial^2\mu_{m-1}}{\partial\xi^2}\right)-g\right\}, \tag{88}$$

$$R_{2,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]=S[\nu_{m-1}(\chi,\xi,\mathfrak{I})]-\left(1-\frac{k_m}{n}\right)\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^\delta}{B(\delta)}S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\nu_{m-1}-i}{\partial\chi}+\sum_{i=0}^{m-1}\nu_i\frac{\partial\nu_{m-1}-i}{\partial\xi}-\rho_0\left(\frac{\partial^2\nu_{m-1}}{\partial\chi^2}+\frac{\partial^2\nu_{m-1}}{\partial\xi^2}\right)+g\right\}. \tag{89}$$

Using the inverse Shehu transformation on Equation (87), we have

$$\mu_m(\chi,\xi,I)=k_m\mu_{m-1}+\hbar S^{-1}\left\{R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]\right\},$$
$$\nu_m(\chi,\xi,I)=k_m\nu_{m-1}+\hbar S^{-1}\left\{R_{2,m}\left[\vec{\mu}_{m-1},\bar{\nu}_{m-1}\right]\right\}. \tag{90}$$

Using μ_0 and ν_0 in Equation (90), we get

$$\mu_1=-\frac{2\rho_0\hbar\sin(\chi+\xi)}{B(\delta)}\left[1-\delta+\frac{\delta I^\delta}{\Gamma(\delta+1)}\right],$$

$$\nu_1=\frac{2\rho_0\hbar\sin(\chi+\xi)}{B(\delta)}\left[1-\delta+\frac{\delta I^\delta}{\Gamma(\delta+1)}\right],$$

$$\mu_2=-\frac{2(n+n)\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]}-\frac{4\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right],$$

$$\nu_2=\frac{2(n+h)\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]}+\frac{4\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right],$$

$$\mu_3=-\frac{2(n+\hbar)^2\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]}-\frac{8(n+\hbar)\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right]-\frac{8\rho_0^3\hbar^3\sin(\chi+\xi)I^{3\delta}}{\Gamma[3\delta+1]},$$

$$\nu_3=\frac{2(n+\hbar)^2\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]}+\frac{8(n+\hbar)\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right]+\frac{8\rho_0^3\hbar^3\sin(\chi+\xi)I^{3\delta}}{\Gamma[3\delta+1]}, \tag{91}$$

and so forth. The rest of the components are discovered in the same way. The q-HATM result of Equation (83) is then determined:

$$\mu(\chi,\xi,\mathfrak{I})=\mu_0+\sum_{m=1}^{\infty}\mu_m\left(\frac{1}{n}\right)^m,$$
$$\nu(\chi,\xi,\mathfrak{I})=2_0+\sum_{w=1}^{\infty}\nu_m\left(\frac{1}{n}\right)^m. \tag{92}$$

For $\delta=1$, $n=-1$, $n=1$ and $g=0$, solutions $\sum_{m=1}^{N}\mu_m(\chi,\xi,\mathfrak{I})(1/n)^m$ and $\sum_{m=1}^{N}\nu_m(\chi,\xi,\mathfrak{I})(1/n)^m$ are convergent to exact solutions as $N\longrightarrow\infty$:

$$\mu(\chi,\xi,\mathfrak{I})=-\sin(\chi+\xi)\left[1-\frac{2\rho_0\mathfrak{I}}{1!}+\frac{(2\rho_0\mathfrak{I})^2}{2!}-\frac{(2\rho_0\mathfrak{I})^3}{3!}+\cdots\right]=-e^{-2\rho0\mathfrak{I}}\sin(\chi+\xi),$$

$$\nu(\chi,\xi,\mathfrak{I})=\sin(\chi+\xi)\left[1-\frac{2\rho_0\mathfrak{I}}{1!}+\frac{(2\rho_0\mathfrak{I})^2}{2!}-\frac{(2\rho_0\mathfrak{I})^3}{3!}+\cdots\right]=e^{-2\rho_0\mathfrak{I}}\sin(\chi+\xi). \tag{93}$$

Example 4. In Equation (83), we take

$$\nu(\chi,\xi,0)=e^{\chi+\xi},\ \mu(\chi,\xi,0)=-e^{\chi+\xi}. \tag{94}$$

Using the Shehu transformation on Equation (83) and applying Equation (94), we have

$$S[\mu]+\frac{u}{s}e^{\chi+\xi}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}-\rho_0\left(\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right)-g\right\}=0,$$
$$S[\nu]-\frac{u}{s}e^{\chi+\xi}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho_0\left(\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right)+g\right\}=0. \tag{95}$$

Define nonlinear operators as

$$N^1[\phi_1,\phi_2]=S[\phi_1]+\frac{u}{s}e^{\chi+\xi}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\phi_1\frac{\partial\phi_1}{\partial\chi}+\phi_2\frac{\partial\phi_1}{\partial\xi}-\rho_0\left(\frac{\partial^2\phi_1}{\partial\chi^2}+\frac{\partial^2\phi_1}{\partial\xi^2}\right)-g\right\},$$

$$N^2[\phi_1,\phi_2]=S[\phi_2]-\frac{u}{s}e^{\chi+\xi}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\phi_1\frac{\partial\phi_2}{\partial\chi}+\phi_2\frac{\partial\phi_2}{\partial\xi}-\rho_0\left(\frac{\partial^2\phi_2}{\partial\chi^2}+\frac{\partial^2\phi_2}{\partial\xi^2}\right)+g\right\}, \tag{96}$$

and Shehu operators as

$$S[\mu_m(\chi,\xi,I)-k_m\mu_{m-1}(\chi,\xi,I)]=\hbar R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right],$$
$$S[\nu_m(\chi,\xi,I)-k_m\nu_{m-1}(\chi,\xi,I)]=\hbar R_{2,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right], \tag{97}$$

where

$$R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]=S[\mu_{m-1}]+\left(1-\frac{k_m}{n}\right)\frac{e^{\chi+\xi}}{s}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\mu_m-1-i}{\partial\chi}+\sum_{i=0}^{m-1}\nu_i\frac{\partial\mu_{m-1}-i}{\partial\xi}-\rho_0\left(\frac{\partial^2\mu_{m-1}}{\partial\chi^2}+\frac{\partial^2\mu_{m-1}}{\partial\xi^2}\right)-g\right\},$$

$$R_{2,\mu}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]=S[\nu_{m-1}]-\left(1-\frac{k_m}{n}\right)\frac{u}{s}e^{\chi+\xi}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\nu_{m-1-i}}{\partial\chi}+\sum_{i=0}^{m-1}\nu_i\frac{\partial\nu_{m-1-i}}{\partial\xi}-\rho_0\left(\frac{\partial^2\nu_m-1}{\partial\chi^2}+\frac{\partial^2\nu_m-1}{\partial\xi^2}\right)+g\right\}. \tag{98}$$

By the inverse Shehu transformation on Equation (97), we get

$$\mu_m(\chi,\xi,I)=k_m\mu_{m-1}+\hbar S^{-1}\left\{R_{1,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]\right\},$$
$$\nu_m(\chi,\xi,I)=k_m\nu_{m-1}+\hbar S^{-1}\left\{R_{2,m}\left[\vec{\mu}_{m-1},\vec{\nu}_{m-1}\right]\right\}. \tag{99}$$

Using μ_0 and ν_0, we get from Equation (99),

$$\mu_1=\frac{2\rho_0\hbar e^{\chi+\xi}\mathfrak{J}^{\delta}}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{J}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_1=-\frac{2\rho_0\hbar e^{\chi+\xi}\mathfrak{J}^{\delta}}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{J}^{\delta}}{\Gamma(\delta+1)}\right],$$

$$\begin{aligned}\mu_2&=\frac{2(n+\hbar)\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1-\delta+\frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right]-\frac{4\rho_0^2\hbar^2e^{\chi+\xi}}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right],\nu_2\\&=-\frac{2(n+\hbar)\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1-\delta+\frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right]+\frac{4\rho_0^2\hbar^2e^{\chi+\xi}}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right],\mu_3\\&=\frac{2(n+\hbar)^2\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1-\delta+\frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right]-\frac{8(n+\hbar)\rho_0^2\hbar^2e^{\chi+\xi}}{(B(\delta))^2}\cdot\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right]+\frac{8\rho_0^3\hbar^3e^{\chi+\xi}I^{3\delta}}{\Gamma[3\delta+1]},\\\nu_3&=-\frac{2(n+\hbar)^2\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1-\delta+\frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right]+\frac{8(n+\hbar)\rho_0^2\hbar^2e^{\chi+\xi}}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2I^{2\delta}}{\Gamma(2\delta+1)}\right]-\frac{8\rho_0^3\hbar^3e^{\chi+\xi}I^{3\delta}}{\Gamma[3\delta+1]},\end{aligned} \tag{100}$$

and so on. Accordingly, rest of the components are identified. The q-HATM result of Equation (42) is

$$\mu(\chi,\xi,\mathfrak{J})=\mu_0+\sum_{m=1}^{\infty}\mu_m\left(\frac{1}{n}\right)^m,$$
$$\nu(\chi,\xi,\mathfrak{J})=\nu_0+\sum_{i=1}^{\infty}\nu_m\left(\frac{1}{n}\right)^m. \tag{101}$$

For $\delta=1=n,\hbar=-1$ and $g=0$, solutions $\sum_{m=1}^{N}\mu_m(1/n)^m$ and $\sum_{m=1}^{N}\nu_m(1/n)^m$ are convergent to exact results as $N\longrightarrow\infty$

$$\mu(\chi,\xi,\mathfrak{J})=-e^{\chi+\xi}\left[1+\frac{2\rho_0\mathfrak{J}}{1!}+\frac{(2\rho_0\mathfrak{J})^2}{2!}+\frac{(2\rho_0\mathfrak{J})^3}{3!}+\cdots\right]=-e^{\chi+\xi+2\rho_0\mathfrak{J}},$$
$$\nu(\chi,\xi,\mathfrak{J})=e^{\chi+\xi}\left[1+\frac{2\rho_0\mathfrak{J}}{1!}+\frac{(2\rho_0\mathfrak{J})^2}{2!}+\frac{(2\rho_0\mathfrak{J})^3}{3!}+\cdots\right]=e^{\chi+\xi+2\rho_0\mathfrak{J}}. \tag{102}$$

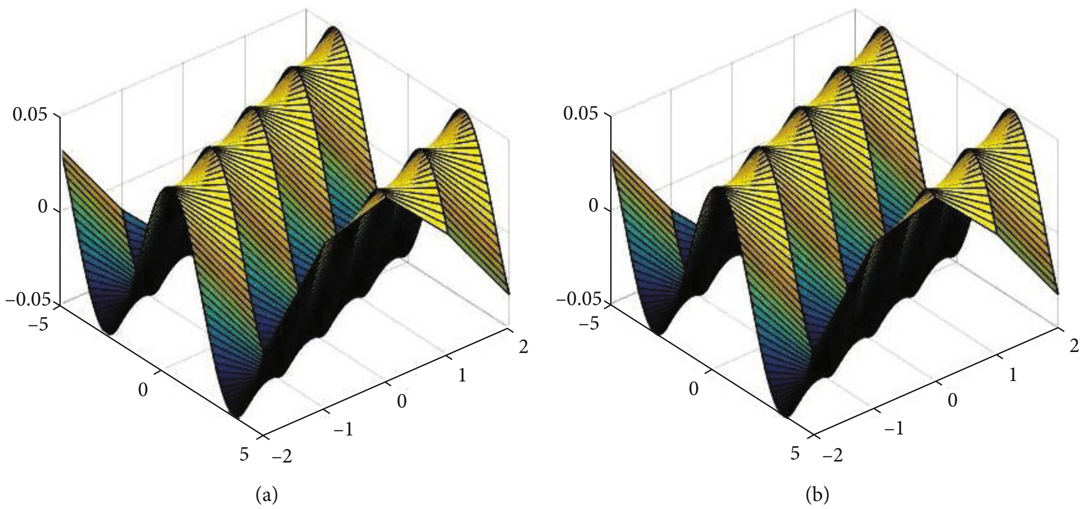


Figure 1: (a) Actual and (b) SDM/q-HATM result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.

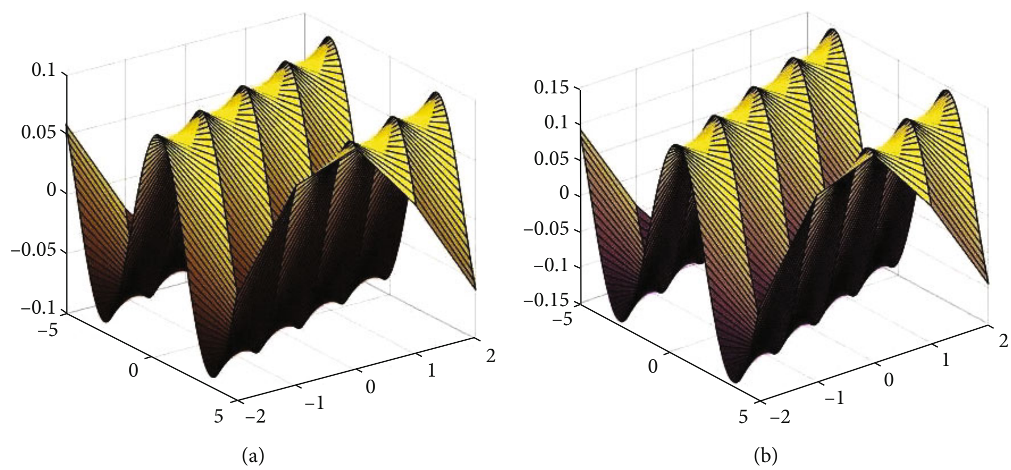


Figure 2: The different fractional-order result of $\mu(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

7. Results and Discussion

In this section, we analyze the solution-figures of the problem, which have been investigated by applying the q-homotopy analysis transform method and Adomian decomposition transform method in the sense of the Atangana-Baleanu operator. Figure 1 represents the three-dimensional solution-figures for variable μ of Example 1 at fractional-order $\delta = 1$, respectively, Figure 2 shows different fractional order of $\delta = 0.8$ and 0.6, and Figure 3 shows that $\delta = 0.4$. It is observed that the q-homotopy analysis transform method and Adomian decomposition transform method solution-figures are identical and in close contact with each other. In the same way, Figures 4–6 show the different fractional-order graphs of δ at ν of Example 1. In similar way, Figure 7 represents the three-dimensional solution-figures for variable μ of Example 2 at fractional-order $\delta = 1$, respectively, Figure 8 shows different fractional-order of $\delta = 0.8$ and 0.6, and Figure 9 shows that $\delta = 0.4$. In the same way, Figures 4–6 show the different fractional-order graphs of δ at ν of Example 2. The same graphs of the suggested methods are attained and confirm

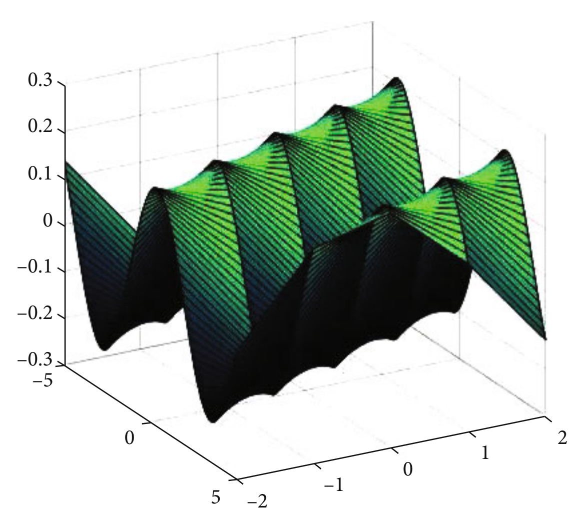


Figure 3: Analytical result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.

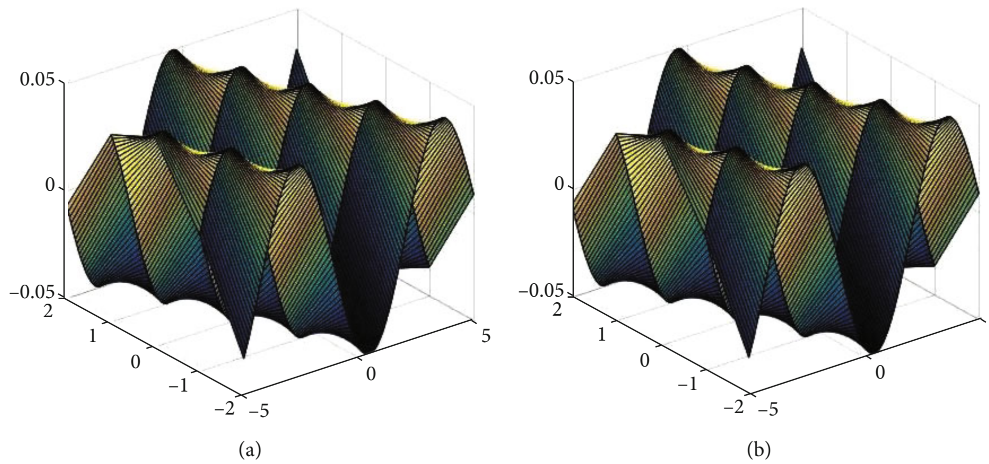


Figure 4: (a) Actual and (b) SDM/q-HATM result of $\nu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.

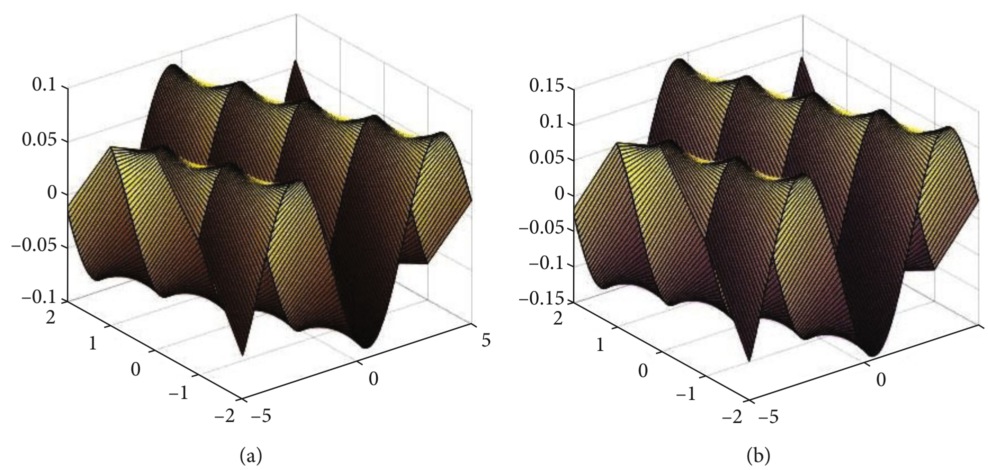


Figure 5: The different fractional-order result of $\nu(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

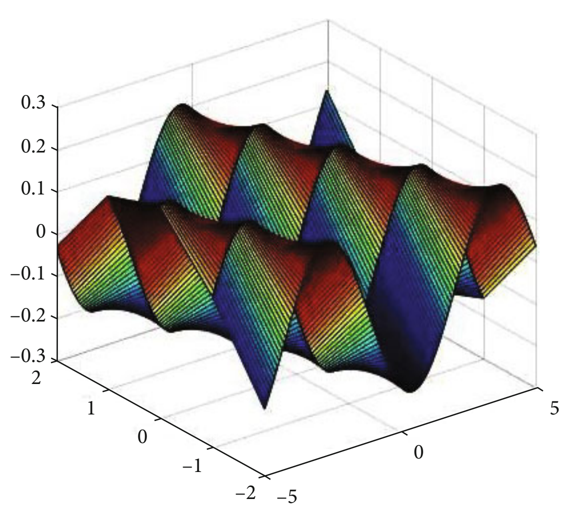
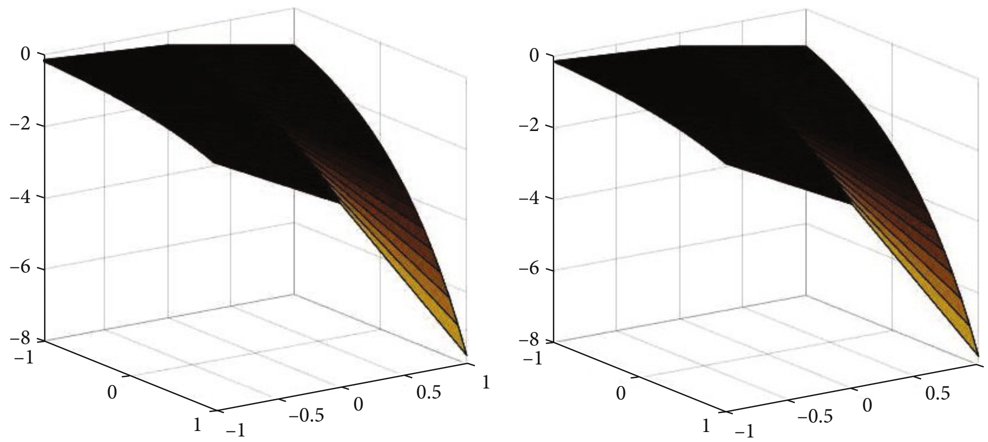


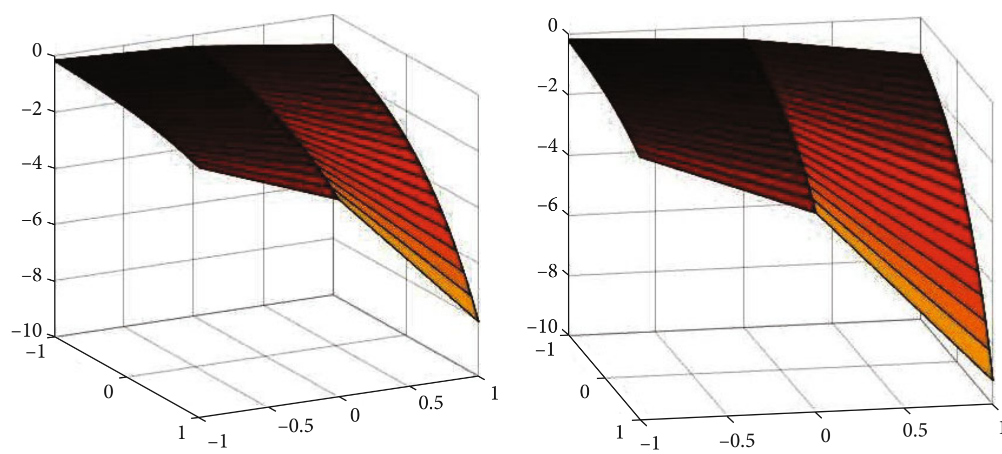
Figure 6: Analytical result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.

the applicability of the present techniques. In Figures 7–9, the q-homotopy analysis transform method and Adomian decomposition transform method solutions are plotted in three dimensional at fractional-order $\delta = 1$, 0.8, 0.6, and 0.4 of Example 2. Similarly Figures 10–12 show the exact and different fractional-order behavior of analytical solutions. The



(a) (b)

Figure 7: (a) Actual and (b) SDM/q-HATM result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.



(a) (b)

Figure 8: The different fractional-order result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta =$ (a) 0.8 and (b) 0.6 of example.

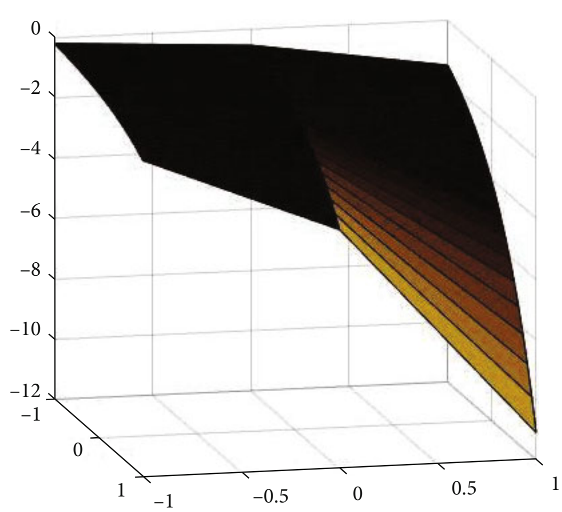


Figure 9: (a) Actual and (b) SDM/q-HATM result of $\nu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.

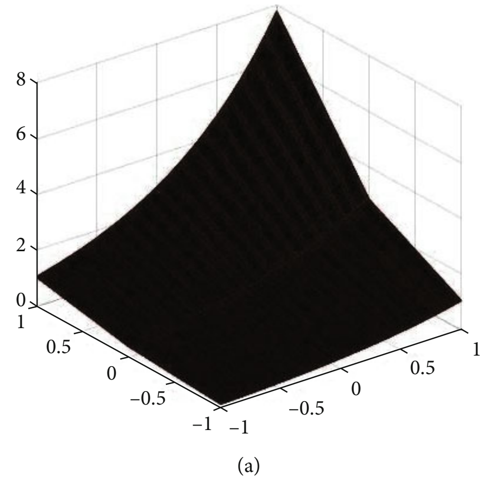


FIGURE 10: The different fractional-order result of $v(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

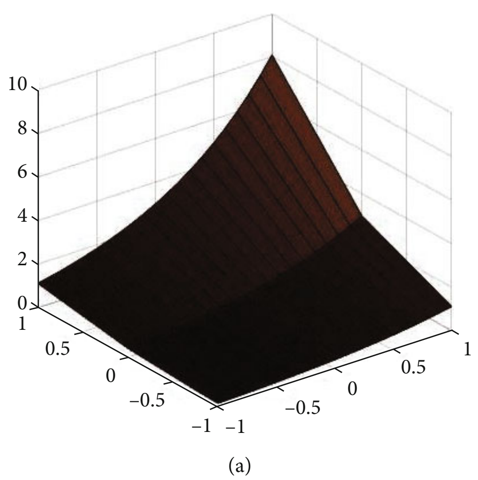


FIGURE 11: The graph of different approximate solution of $\Psi(\chi, \varphi, \eta)$ at Υ = (a) 0.8 and (b) 0.6 of example 2.

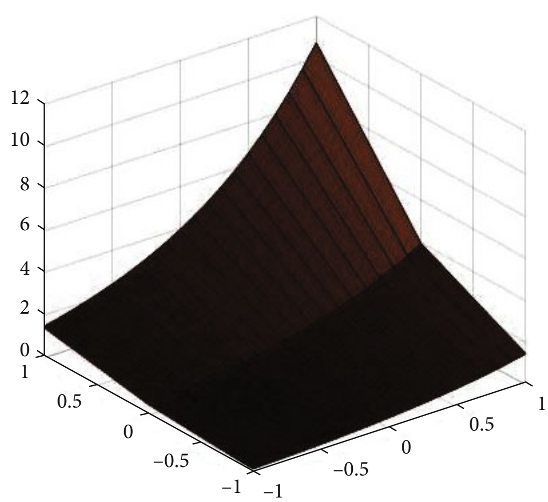


FIGURE 12: Analytical result of $v(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.

convergence phenomenon of the fractional solutions towards integer-solution is observed. The same accuracy is achieved by using the present techniques.

8. Conclusion

This article calculates a result of the fractional system of Navier-Stokes equations determined numerical solution using the suggested q-homotopy analysis transform method and Shehu decomposition method. The result is obtained in quick convergent series. The test samples provided demonstrate the approach's strength and efficacy. The proposed algorithm includes a parameter $\hbar$ that allows us to control the series solution's convergence region. As q-homotopy analysis transform method and Shehu decomposition method do not necessarily require small perturbation linearization or discretisation, it decreases computations significantly. In comparison with other techniques, q-homotopy analysis transform method and Shehu decomposition method are competent tools to obtain mathematical result of system nonlinear fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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