Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9004239, 7 pages
https://doi.org/10.1155/2022/9004239

Research Article

Applications of (α, β)-Symmetrical Functions on a Certain Class of Spirallike Functions

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Received 21 April 2022; Revised 18 May 2022; Accepted 23 May 2022; Published 30 June 2022

Academic Editor: Teodor Bulboaca

In this note, we use the notions of (α, β)-symmetrical, generalized Janowski-type and spirallike functions to define a new class $\mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$ defined in the open unit disk. In particular, we obtain a structural formula, a representation theorem, Marx-Strohacker inequality. Our results continue to hold the covering and distortion properties.

1. Introduction

Let $\mathscr{H}(\Delta)$ denote the class of analytic functions in the open unit disk $\Delta = \{\omega \in \mathbb{C} : |\omega| < 1\}$. Also, $\mathscr{H}$ indicates the subclass of $\mathscr{H}(\Delta)$ which has the form

$$h(\omega) = \omega + \sum_{\nu=2}^{\infty} a_\nu \omega^\nu. \quad (1)$$

The family of functions $h \in \mathscr{H}$ that are univalent in Δ is represented by $\tilde{\mathscr{H}}$.

The family of *Schwarz functions* is denoted by Ω where

$$\Omega := \{s \in \mathscr{H}, s(0) = 0, |s(\omega)| < 1, \omega \in \Delta\}. \quad (2)$$

Let $h, \tilde{h}$ be analytic, the function h is said to be *subordinate* to $\tilde{h}$ in Δ if there exists $s \in \Omega$ and $h(\omega) = \tilde{h}(s(\omega))$, and we denote this by $h(\omega) \prec \tilde{h}(\omega)$. Whereover $\tilde{h}$ is univalent in Δ, then the subordination is equivalent to $h(0) = \tilde{h}(0)$ and $h(\Delta) \subset \tilde{h}(\Delta)$.

By using the concept of the subordination, let define the well-known Carathĕodory class $\mathscr{P} = \{p \in \mathscr{H}(\Delta)$ satisfying $p(0) = 1, \mathrm{Re}p(\omega) > 0;\ \omega \in \Delta$, and for any function $p(0) = 1$ is said to belong to the class $\mathscr{P}$ has the representation $p(\omega) = (1 + s(\omega))/(1 - s(\omega))$, for some $s \in \Omega$.

In [1], Janowski introduced the class $\mathscr{P}[N, M]$ with $-1 \le M < N \le 1$, a function p analytic in Δ with $p(0) = 1$ is said to belong to the class $\mathscr{P}[N, M]$ which has the representation $p(\omega) = (1 + N\omega)/(1 - M\omega)$.

The class $\mathscr{P}[N, M, \mu]$ of generalized Janowski functions was introduced in [2]. For arbitrary numbers N, M and μ with $-1 \le M < N \le 1$, $0 \le \mu < 1$, a function p analytic in Δ with $p(0) = 1$ is said to belong to the class $\mathscr{P}[N, M, \mu]$ if and only if

$$p(\omega) \prec \frac{1 + [(1-\mu)N + \mu M]\omega}{1 + M\omega} \Leftrightarrow p(\omega) = \frac{1 + [(1-\mu)N + \mu M]s(\omega)}{1 + Ms(\omega)}, \quad s \in \Omega. \quad (3)$$

In order to define new classes of symmetrical functions defined in Δ, we first recall the notion of β-fold symmetric functions defined in β-fold symmetric domain, where β is any positive integer. A domain $\mathscr{G}$ is said to be β-fold symmetric if a rotation of $\mathscr{G}$ about the origin through an angle $2\pi/\beta$ carries $\mathscr{G}$ onto itself. A function h is said to be β-fold symmetric in $\mathscr{G}$ if for every ω in $\mathscr{G}$ we have

$$h(\varepsilon\omega) = \varepsilon h(\omega), \left(\varepsilon = e^{2\pi i/\beta}\right), \quad \omega \in \mathscr{G}. \quad (4)$$

The family of all β-fold symmetric functions is denoted by $\tilde{\mathcal{S}}^{\beta}$; we get the class of odd univalent functions for $\beta = 2$. In 1995, Liczberski and Polubinski [3] constructed the theory of (α, β)-symmetrical functions for $(\alpha = 0, 1, 2, \cdots, \beta - 1)$ and $(\beta = 2, 3, \cdots)$. If $\mathcal{G}$ is β-fold symmetric domain and α any integer, then a function $h : \mathcal{G} \longrightarrow \mathbb{C}$ is called (α, β) -symmetrical if for each $\omega \in \mathcal{G}$, $h(\varepsilon\omega) = \varepsilon^{\alpha} h(\omega)$. We note that the (α, β)-symmetrical functions are a generalization of the notions of even, odd, and β-symmetrical functions.

In [3], we observe that the theory of (α, β)-symmetrical functions has many interesting applications; we now investigate some results in the classes of analytic functions.

Denote the family of all (α, β)-symmetrical functions by $\tilde{\mathcal{S}}^{(\alpha,\beta)}$. We observe that, $\tilde{\mathcal{S}}^{(0,2)}$, $\tilde{\mathcal{S}}^{(1,2)}$, and $\tilde{\mathcal{S}}^{(1,\beta)}$ are the classes of even, odd, and β-symmetric functions, respectively.

Theorem 1 (see [3], page 16). *For every mapping $h : \Delta \mapsto \mathbb{C}$ and a β-fold symmetric set Δ, there exists exactly one sequence of (α, β)-symmetrical functions $h_{\alpha,\beta}$ such that*

$$h(\omega) = \sum_{\alpha=0}^{\beta-1} h_{\alpha,\beta}(\omega), h_{\alpha,\beta}(\omega) = \frac{1}{\beta} \sum_{m=0}^{\beta-1} \varepsilon^{-m\alpha} h(\varepsilon^m \omega), \quad \omega \in \Delta. \tag{5}$$

Furthermore, we say that h is λ-spirallike if and only if Re $(e^{i\lambda}\omega h'(\omega)/h(\omega)) > 0, \lambda$ is real and $|\lambda| < \pi/2$.

Recently, see [4–6] obtained many interesting results for various subclasses of Janowski-type functions by using the concept of (α, β)-symmetrical functions.

By taking motivation from the above-cited work and using the generalized Janowski functions, (α, β)-symmetrical functions, and λ-spirallike, we introduce a new subclass of analytic functions.

Definition 2. A function h in $\mathcal{H}$ is said to belong to the class $\mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, $(-1 \le M < N \le 1), 0 \le \mu < 1$ if

$$\frac{1}{\cos(\lambda)} \left[\frac{e^{i\lambda}\omega h'(\omega)}{h_{\alpha,\beta}(\omega)} - i \sin(\lambda) \right] \in \mathcal{P}[N, M, \mu], |\lambda| < \frac{\pi}{2}, \quad \omega \in \Delta, \tag{6}$$

where $h_{\alpha,\beta}$ is defined in (5).

Our defined class generalizes many classes by choosing particular values of the parameters for various choices of α , β, λ, N, M, and μ; Definition 2 yields several known and new subclasses of $\mathcal{H}$, as $\mathcal{S}^{0}_{\alpha,\beta}(N, M, \mu) \coloneqq \mathcal{S}_{\alpha,\beta}(N, M, \mu)$ introduced by the authors in [5]; $\mathcal{S}^{\lambda}_{1,1}(N, M, 0) \coloneqq \mathcal{S}(N, M,$ $\mu)$ and $\mathcal{S}^{\lambda}_{1,1}(N, M, 0) \coloneqq \mathcal{S}(N, M, \mu)$ motivated by Polatoḡlu et al. [2, 7]; $\mathcal{S}^{0}_{1,\beta}(N, M, \mu) \coloneqq \mathcal{S}_{\beta}(N, M, \mu)$ introduced by Latha and Darus [8]; $\mathcal{S}^{0}_{1,\beta}(1,-1, 0) \coloneqq \mathcal{S}_{\beta}$ defined by Sakaguchi [9]; $\mathcal{S}^{0}_{1,1}(N, M, 0) \coloneqq \mathcal{S}[N, M]$ these class reduce to well-known class defined by Janowski [1].

Lemma 3 (see [5]). *If h belongs to the class $\mathcal{S}^{(\alpha,\beta)}(N, M, \mu)$, then*

$$h_{\alpha,\beta}(\omega) = \begin{cases} \omega(1 + Ms(\omega))^{(1-\mu)(N-M)/M}, & \text{if } M \ne 0, \\ \omega \exp[(1-\mu)Ns(\omega)], & \text{if } M = 0, \end{cases} \tag{7}$$

and $h_{\alpha,\beta} \in \mathcal{S}^{(\alpha,\beta)}(N, M, \mu)$ for some $s \in \Omega$, where $h_{\alpha,\beta}$ are defined by ((5)).

Lemma 4 (see [2]). *For τ is an arbitrary fixed point of Δ and $H(p) = p(\tau), p \in \mathcal{P}[N, M, \mu]$, then the set of the values of H is in the closed disc with center at $C(r)$ and having the radius $\rho(r)$, where*

$$\begin{cases} C(r) = \left(\dfrac{1 - M[(1-\mu)N + \mu M]r^2}{1 - M^2 r^2}, 0 \right), \rho(r) = \dfrac{(1-\mu)(N-M)r}{1 - M^2 r^2}, & \text{if } M \ne 0, \\ C(r) = (1, 0), \rho(r) = (1-\mu)|N|r, & \text{if } M = 0. \end{cases} \tag{8}$$

Lemma 5 (see [10]). *Let $p \in \mathcal{P}[N, M, \mu]$, then*

$$\frac{1 - [(1-\mu)N + \mu M]r}{1 - Mr} \le |p(\omega)| \le \frac{1 + [(1-\mu)N + \mu M]r}{1 + Mr}, \quad |\omega| \le r < 1. \tag{9}$$

2. Main Results

Theorem 6. *A function h belongs to the class $\mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$ if and only if*

$$h(\omega) = \int_0^{\omega} \mathcal{B}(t) q(t) dt, \tag{10}$$

where

$$q(t) = \exp \left\{ \int_0^t \frac{1}{\beta u} \left(\sum_{m=0}^{\beta-1} \mathcal{B}(\varepsilon^m u) - \beta \right) du \right\}, \tag{11}$$

$$\mathcal{B}(\omega) = [\cos(\lambda) p(\omega) + i \sin(\lambda)] e^{-i\lambda}, p \in \mathcal{P}[N, M, \mu]. \tag{12}$$

Proof. Suppose that $h \in \mathcal{S}_{\alpha,\beta}^{\lambda}(N, M, \mu)$, we have

$$\frac{\omega h'(\omega)}{h_{\alpha,\beta}(\omega)} = [\cos(\lambda)p(\omega) + i\sin(\lambda)]e^{-i\lambda}, p \in \mathcal{P}[N, M, \mu]. \tag{13}$$

Replacing ω by $\varepsilon^m\omega$ in (13), we obtain

$$\frac{\varepsilon^{m(1-\alpha)}\omega h'(\varepsilon^m\omega)}{h_{\alpha,\beta}(\omega)} = \mathcal{B}(\varepsilon^m\omega). \tag{14}$$

From (13) and (14), we get

$$h'(\varepsilon^m\omega) = \mathcal{B}(\varepsilon^m\omega)\frac{\varepsilon^{m(\alpha-1)}h'(\omega)}{\mathcal{B}(\omega)}. \tag{15}$$

By differentiation (13), we have

$$h_{\alpha,\beta}'(\omega) = \frac{\omega h''(\omega) + h'(\omega)}{\mathcal{B}(\omega)} - \omega h'(\omega)\frac{\mathcal{B}'(\omega)}{\mathcal{B}^2(\omega)}. \tag{16}$$

From (5) and (15), we get

$$h_{\alpha,\beta}'(\omega) = \frac{1}{\beta}\frac{h'(\omega)}{\mathcal{B}(\omega)}\sum_{m=0}^{\beta-1}\mathcal{B}(\varepsilon^m\omega). \tag{17}$$

From (5) and (17), we get

$$\frac{h''(\omega)}{h'(\omega)} = \frac{\mathcal{B}'(\omega)}{\mathcal{B}(\omega)} + \frac{1}{\beta\omega}\left(\sum_{m=0}^{\beta-1}\mathcal{B}(\varepsilon^m\omega) - \beta\right). \tag{18}$$

Integrating repeatedly, we get the required structural formula

$$h(\omega) = \int_0^{\omega}\mathcal{B}(t)q(t)dt, \tag{19}$$

which proves the necessity. To prove the sufficiency of (10), suppose that (10) holds with $p \in \mathcal{P}[N, M, \mu]$. The function h defined by (10) is obviously in $\mathcal{H}$ with $h(0) = 0$ and $h'(0) = 1$. The following identity can be verified by differentiation

$$\omega q(\omega) = \int_0^{\varepsilon^m\omega}\left[\frac{1}{\beta}\sum_{m=0}^{\beta-1}\varepsilon^{-\alpha m}\mathcal{B}(t).q(t)\right]dt, \tag{20}$$

where q and $\mathcal{B}$ are given by (10) and (11), respectively. Also, using (10), we have

$$h'(\omega) = \mathcal{B}(\omega).q(\omega), \tag{21}$$

which shows that $h' \neq 0$ in Δ.

From (10), since ε is the root of unity, we conclude that

$$h_{\alpha,\beta}(\omega) = \int_0^{\varepsilon^m\omega}\left[\frac{1}{\beta}\sum_{m=0}^{\beta-1}\varepsilon^{-\alpha m}\mathcal{B}(t).q(t)\right]dt. \tag{22}$$

Using (20), (21), and (22), we arrive at the result

$$h_{\alpha,\beta}(\omega) = \frac{\omega h'(\omega)}{\mathcal{B}(\omega)}, \tag{23}$$

thus proving the sufficiency of (10). □

Lemma 7. *If $h \in \mathcal{S}_{\alpha,\beta}^{\lambda}(N, M, \mu)$, then*

$$h_{\alpha,\beta}(\omega) = \begin{cases} \omega(1 + Ms(\omega))^{e^{-i\lambda}(1-\mu)(N-M)\cos(\lambda)/M}, & M \neq 0, \\ \omega e^{(1-\mu)N\cos(\lambda)e^{-i\lambda}s(\omega)}, & M = 0, \end{cases} \tag{24}$$

for some $s \in \Omega$, where $h_{\alpha,\beta}$ are defined by (5).

Proof. By using Lemma 3 and similar technique proof in Theorem 6 in [5]. □

Corollary 8. *Marx-Strohacker inequality for the class $\mathcal{S}_{\alpha,\beta}^{\lambda}(N, M, \mu)$ is*

$$\begin{cases} \left|\log\left(\frac{h_{\alpha,\beta}(\omega)}{\omega}\right)^{e^{i\lambda}/(1-\mu)N\cos(\lambda)}\right| < 1, & \text{if } M = 0, \\ \left|\left(\frac{h_{\alpha,\beta}(\omega)}{\omega}\right)^{Me^{i\lambda}/(1-\mu)(N-M)\cos(\lambda)} - 1\right| < 1, & \text{if } M \neq 0. \end{cases} \tag{25}$$

Proof. The proof of this corollary is a simple consequence of Lemma 7. Indeed,

$$\begin{aligned} \frac{h_{\alpha,\beta}(\omega)}{\omega} &= (1 + Ms(\omega))^{e^{-i\lambda}(1-\mu)(N-M)\cos(\lambda)/M} \\ &\Rightarrow \left|\left(\frac{h_{\alpha,\beta}(\omega)}{\omega}\right)^{Me^{i\lambda}/(1-\mu)(N-M)\cos(\lambda)} - 1\right| < 1, \\ \frac{h_{\alpha,\beta}(\omega)}{\omega} &= e^{(1-\mu)N\cos(\lambda)e^{-i\lambda}s(\omega)} \\ &\Rightarrow \left|\log\left(\frac{h_{\alpha,\beta}(\omega)}{\omega}\right)^{e^{i\lambda}/(1-\mu)N\cos(\lambda)}\right| < 1. \end{aligned} \tag{26}$$

□

The next covering, starlikeness and distortion theorems for the class $\mathcal{S}_{\alpha,\beta}^{\lambda}(N, M, \mu)$ hold.

Theorem 9. *Let $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, with $-1 \le M < N \le 1$ and $0 \le \mu < 1$. Then,*

$$h(\omega) = \begin{cases} \int_0^{\omega} \frac{1+\gamma\tilde{s}(\zeta)}{1+M\tilde{s}(\zeta)} (1+Ms(\zeta))^{e^{-i\lambda}(1-\mu)(N-M)\cos(\lambda)/M}, d\zeta, & \text{if } M \neq 0, \\ \int_0^{\omega} \left[1+e^{-i\lambda}(1-\mu)N\cos(\lambda)\tilde{s}(\zeta)\right] e^{e^{-i\lambda}(1-\mu)N\cos(\lambda)s(\zeta)}\, d\zeta, & \text{if } M = 0, \end{cases} \tag{27}$$

for some $s, \tilde{s} \in \Omega$ and

$$\gamma = [[(1-\mu)N + \mu M]\cos(\lambda) + iM\sin(\lambda)]e^{-i\lambda}. \tag{28}$$

Proof. Supposing that $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, it follows that there exists a function $\tilde{s} \in \Omega$ such that

$$\begin{aligned} &\frac{e^{i\lambda}\left(\omega h'(\omega)/h_{\alpha,\beta}(\omega)\right) - i\sin(\lambda)}{\cos(\lambda)} \\ &\quad = \frac{1+[(1-\mu)N+\mu M]\tilde{s}(\omega)}{1+M\tilde{s}(\omega)}, \quad \omega \in \Delta. \end{aligned} \tag{29}$$

Combining the above relation with Lemma 7, we have

$$h'(\omega) = \begin{cases} \frac{1+[(1-\mu)N+\mu M]\tilde{s}(\omega)}{1+\tilde{s}(\omega)} (1+Ms(\omega))^{e^{-i\lambda}(1-\mu)(N-M)\cos(\lambda)/M}, & \text{if } M \neq 0, \\ \left[1+e^{-i\lambda}(1-\mu)N\cos(\lambda)\tilde{s}(\omega)\right] e^{e^{-i\lambda}(1-\mu)N\cos(\lambda)s(\omega)}, & \text{if } M = 0. \end{cases} \tag{30}$$

Integrating the above equation along the line connecting the origin with $\omega \in \Delta$, we obtain our result. □

Theorem 10. *The radius of starlikeness of the class $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$ is*

$$r = \begin{cases} \frac{2}{(1-\mu)(N-M)\cos(\lambda) + \sqrt{(1-\mu)^2(N-M)^2\cos^2(\lambda) + 4\{[(1-\mu)N+\mu M]M\cos^2(\lambda) + M^2\sin^2(\lambda)\}}}, & \text{if } M \neq 0, \\ \frac{1}{(1-\mu)N\cos(\lambda)}, & \text{if } M = 0. \end{cases} \tag{31}$$

This radius is sharp because the extremal function is

$$h_{\alpha,\beta}(\omega) = \begin{pmatrix} \omega(1+M\omega)^{e^{-i\lambda}(1-\mu)(N-M)\cos(\lambda)/M}, M \neq 0, \\ \omega e^{(1-\mu)N\cos(\lambda)e^{-i\lambda}\omega}, M = 0. \end{pmatrix} \tag{32}$$

Proof. Since

$$\frac{\left(e^{i\lambda}\omega h'(\omega)/h_{\alpha,\beta}(\omega)\right) - i\sin(\lambda)}{\cos(\lambda)} = p(\omega), p \in \mathcal{P}[N, M, \mu], \tag{33}$$

using Lemma 4, that is

$$\left| p(\omega) - \frac{1 - M[(1-\mu)N+\mu M]r^2}{1-M^2r^2} \right| \le \frac{(1-\mu)(N-M)r}{1-M^2r^2}. \tag{34}$$

Using (33) in (34) and after straightforward calculations, we get

$$\begin{aligned} &\left.\begin{matrix} \frac{1-(1-\mu)(N-M)\cos(\lambda)r - \left\{[(1-\mu)N+\mu M]M\cos^2(\lambda) + M^2\sin^2(\lambda)\right\}r^2}{1-M^2r^2}, & \text{if } M \neq 0, \\ 1-(1-\mu)N\cos(\lambda)r, & \text{if } M = 0, \end{matrix}\right\} \\ &\le \Re\left\{\omega\frac{h'(\omega)}{h_{\alpha,\beta}(\omega)}\right\} \le \begin{cases} \frac{1+(1-\mu)(N-M)\cos(\lambda)r - \left\{[(1-\mu)N+\mu M]M\cos^2(\lambda) + M^2\sin^2(\lambda)\right\}r^2}{1-M^2r^2}, & \text{if } M \neq 0, \\ 1+(1-\mu)N\cos(\lambda)r, & \text{if } M = 0, \end{cases} \end{aligned} \tag{35}$$

where $|\omega| \le r < 1$. The above inequalities shows that this theorem is true. □

Remark 11.

(i) For $N = -M = 1, \lambda = \mu = 0$, we obtain $r = 1$

(ii) For $N = -M = 1, \mu = 0$, we obtain $r = 1/(\cos(\lambda) + |\sin(\lambda)|)$

We also note that if we give another special values to N, M, α, β, and μ, we obtain radius of starlikeness of the subclass of λ-spirallike functions.

Corollary 12. *If $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, then*

$$\left.\begin{array}{ll} \dfrac{(1-[(1-\mu)N+\mu M]r)\cos(\lambda)-(1-Mr)|\sin(\lambda)|}{1-Mr}, & \text{if } M \neq 0, \\ (1-(1-\mu)Nr)\cos(\lambda)-|\sin(\lambda)|, & \text{if } M = 0, \end{array}\right\} \le \left|\frac{\omega h'(\omega)}{h_{\alpha,\beta}(\omega)}\right| \le \begin{cases} \dfrac{(1+[(1-\mu)N+\mu M]r)\cos(\lambda)+(1+Mr)|\sin(\lambda)|}{1+Mr}, & \text{if } M \neq 0, \\ (1+(1-\mu)Nr)\cos(\lambda)+|\sin(\lambda)|, & \text{if } M = 0, \end{cases} \tag{36}$$

where $|\omega| \le r < 1$.

Proof. For an arbitrary function $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, we have

$$\frac{1}{\cos(\lambda)}\left[\frac{e^{i\lambda}\omega h'(\omega)}{h_{\alpha,\beta}(\omega)} - i\sin(\lambda)\right] = p(\omega), p \in \mathcal{P}[N, M, \mu]. \tag{37}$$

Using Lemma 5 and after the straightforward calculations, we get the result. □

Theorem 13. *For $M \ge 0$, if $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, then*

$$\left.\begin{array}{ll} \left[\cos(\lambda)\dfrac{1-[(1-\mu)N+\mu M]r}{1-Mr} - |\sin(\lambda)|\right](1-Ms(\omega))^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}, & \text{if } M \neq 0, \\ \left[\cos(\lambda)\{1-(1-\mu)Nr\} - |\sin(\lambda)|\right]\exp\left[-\cos(\lambda)(1-\mu)Nr\right], & \text{if } M = 0, \end{array}\right\} \le |h'(\omega)| \le \begin{cases} \left[\cos(\lambda)\dfrac{1+[(1-\mu)N+\mu M]r}{1+Mr} + |\sin(\lambda)|\right](1+Ms(\omega))^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}, & \text{if } M \neq 0, \\ \left[\cos(\lambda)\{1+(1-\mu)Nr\} + |\sin(\lambda)|\right]\exp\left[\cos(\lambda)(1-\mu)Nr\right], & \text{if } M = 0, \end{cases} \tag{38}$$

where $|\omega| \le r < 1$.

Proof. For function $h \in \mathcal{S}^{\lambda}_{\alpha,\beta}(N, M, \mu)$, according to Lemma 7, we have to distinguish the next two cases.

(i) For $M \neq 0$, then there exists a function $s \in \Omega$ such that $h_{\alpha,\beta}(\omega) = \omega(1+Ms(\omega))^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}$, and by Lemma 5, for, $|\omega| \le r < 1$, we get

$$\left[\cos(\lambda)\frac{1-[(1-\mu)N+\mu M]r}{1-Mr} - |\sin(\lambda)|\right]|1+Ms(\omega)|^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M} \le |h'(\omega)| \le \left[\cos(\lambda)\frac{1+[(1-\mu)N+\mu M]r}{1+Mr} + |\sin(\lambda)|\right]|1+Ms(\omega)|^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}. \tag{39}$$

Since $s \in \Omega$, we have

$$1 - |M|r \leq |1 + Ms(\omega)| \leq 1 + |M|r, \quad |\omega| \leq r < 1. \tag{40}$$

For $M > 0$, we can easily use the fact that $-1 \leq M < N \leq 1, 0 \leq \mu < 1$ and $|\omega| \leq r < 1$; we have

$$\begin{aligned} &(1 - |M|r)^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M} \\ &\leq |1 + Ms(\omega)|^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M} \\ &\leq (1 + |M|r)^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}, \end{aligned} \tag{41}$$

and from (39), we obtain

$$\begin{aligned} &\left[\cos(\lambda)\frac{1 - [(1-\mu)N + \mu M]r}{1 - Mr} - |\sin(\lambda)|\right](1 - |M|s(\omega))^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M} \\ &\leq |h'(\omega)| \leq \left[\cos(\lambda)\frac{1 + [(1-\mu)N + \mu M]r}{1 + Mr} + |\sin(\lambda)|\right](1 + |M|s(\omega))^{\cos(\lambda)e^{-i\lambda}(1-\mu)(N-M)/M}. \end{aligned} \tag{42}$$

(ii) If $M = 0$, there exists a function $s \in \Omega$ such that $h_{\alpha,\beta}(\omega) = \omega \exp[\cos(\lambda)e^{-i\lambda}(1-\mu)Ns(\omega)], \quad , |\omega| \leq r < 1$ and therefore

$$\begin{aligned} &[\cos(\lambda)\{1 - (1-\mu)Ns(\omega)\} - |\sin(\lambda)|]\left|\exp\left[\cos(\lambda)e^{-i\lambda}(1-\nu)Ns(\omega)\right]\right| \\ &\leq |h'(\omega)| \leq [\cos(\lambda)\{1 + (1-\mu)Ns(\omega)\} + |\sin(\lambda)|]|\exp \\ &\cdot \left[\cos(\lambda)e^{-i\lambda}(1-\mu)Ns(\omega)\right]|. \end{aligned} \tag{43}$$

Since

$$\begin{aligned} &\left|\exp\left[\cos(\lambda)e^{-i\lambda}(1-\mu)Ns(\omega)\right]\right| \\ &= \exp\left[\cos(\lambda)(1-\mu)NRe\left\{e^{-i\lambda}s(\omega)\right\}\right], \quad \omega \in \Delta, \end{aligned} \tag{44}$$

using a similar computation as in the previous case, we deduce

$$\begin{aligned} &\exp[-\cos(\lambda)(1-\mu)Nr] \\ &\leq \left|\exp\left[\cos(\lambda)e^{-i\lambda}(1-\mu)Ns(\omega)\right]\right| \\ &\leq \exp[\cos(\lambda)(1-\mu)Nr], \quad |\omega| \leq r < 1. \end{aligned} \tag{45}$$

Thus, (43) yields to

$$\begin{aligned} &[\cos(\lambda)\{1 - (1-\mu)Nr\} - |\sin(\lambda)|]\exp[-\cos(\lambda)(1-\mu)Nr] \\ &\leq |h'(\omega)| \leq [\cos(\lambda)\{1 + (1-\mu)Nr\} \\ &+ |\sin(\lambda)|]\exp[\cos(\lambda)(1-\mu)Nr], \end{aligned} \tag{46}$$

for $|\omega| \leq r < 1$, which completes the proof of our theorem. □

Data Availability

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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