Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9011506, 24 pages
https://doi.org/10.1155/2022/9011506

Research Article

Decision-Making on the Solution of a Stochastic Nonlinear Dynamical System of Kannan-Type in New Sequence Space of Soft Functions

Meshayil M. Alsolmi [1] **and Awad A. Bakery** [2]

[1] *University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo 11566, Abbassia, Egypt*

Correspondence should be addressed to Awad A. Bakery; awad_bakery@yahoo.com

Received 30 June 2022; Accepted 1 August 2022; Published 28 August 2022

Academic Editor: Santosh Kumar

In this paper, we construct and investigate the space of weighted Gamma matrix of order r in Nakano sequence space of soft functions. The idealization of the mappings has been achieved through the use of extended s-soft functions and this sequence space of soft functions. This new space's topological and geometric properties, the multiplication mappings that stand in on it, and the mappings' ideal that correspond to them are discussed. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated prequasi ideal. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of soft functions are introduced.

1. Introduction

Probability theory, fuzzy set theory, soft sets, and rough sets have all contributed substantially to the study of uncertainty. But there are drawbacks to these theories that must be considered. For more information and real-world examples, please refer to [1–10]. Numerous mathematicians have investigated potential expansions to the theorem and its applications in various contexts since the publication of the book [11] on the Banach fixed point theorem. The Banach contraction principle is an important part of nonlinear analysis, which uses it as a powerful tool [12–15]. Kannan [16] presented a collection of mappings with the same actions at fixed places as contractions. However, this collection is discontinuous. In Reference [17], an explanation of Kannan operators in modular vector spaces was once tried. Only this one try was ever made as [18–23] show that much attention has been paid to the s-number mapping ideal and the multiplication operator hypothesis in functional analysis. Bakery and Mohamed [24] offered the idea of a prequasi norm on the Nakano sequence space with a variable exponent that fell somewhere in the range $(0, 1]$. They talked about the conditions that must be met to generate prequasi Banach and closed space when it is endowed with a specified prequasi norm and the Fatou property of various prequasi norms on it. They also determined a fixed point for Kannan prequasi norm contraction mappings on it, in addition to the ideal of prequasi Banach mappings derived from s-numbers in this sequence space. Both of these ideals were established. In addition, several fixed point findings of Kannan nonexpansive mappings on generalized Cesàro backward difference sequence space of a nonabsolute type were discovered in [25]. Assume that $\mathfrak{R}$ is the set of real numbers and $\mathcal{N}$ is the set of nonnegative integers. We denote the collection of all nonempty bounded subsets of $\mathscr{R}$ by $\mathfrak{B}(\mathscr{R})$, and E is the set of parameters. By $\mathscr{R}(A)^*$ and $\mathscr{R}(A)$, we indicate the set of nonnegative and all soft real numbers (corresponding to A), where $A \subset E$. The additive identity and multiplicative identity in $\mathscr{R}(A)$ are denoted by $\tilde{0}$ and $\tilde{1}$, respectively. For more details on the arithmetic operations on $\mathscr{R}(A)$, see [26]. Let $\mu : \mathscr{R}(A) \times \mathscr{R}(A) \longrightarrow \mathscr{R}(A)^*$, where $\mu(\tilde{f},\tilde{g})=|\tilde{f}-\tilde{g}|$, for all $\tilde{f},\tilde{g}\in\mathscr{R}(A)$. Assume $\tilde{\rho}:\mathscr{R}(A)\times\mathscr{R}(A)\longrightarrow\mathfrak{R}^+$ is defined by

$$\tilde{\rho}\left(\tilde{f},\tilde{g}\right)=\max_{\lambda\in A}\mu\left(\tilde{f},\tilde{g}\right)(\lambda). \tag{1}$$

Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both of these options are viable; we have constructed the space, $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, which is the domain of weighted Gamma matrix of order r in Nakano soft sequence space since it is constructed by the domain of weighted Gamma matrix of order r defined in $\ell^{\mathfrak{S}}_{((v_l))}$, where the weighted Gamma matrix of order r, $W\Gamma_r=(\lambda^r_{lz}(q))$, is defined as

$$\lambda^r_{lz}(q)=\begin{cases}\dfrac{\begin{bmatrix}r+z-1\\ z\end{bmatrix}q_z}{\begin{bmatrix}r+l\\ l\end{bmatrix}}, & 0\le z\le l,\\ 0, & z>l,\end{cases} \tag{2}$$

where r is a positive integer, $q_z\in(0,\infty)$, for all $z\in\mathscr{N}$ and

$$\begin{bmatrix}r+z-1\\ z\end{bmatrix}=\frac{(r+z-1)!}{z!(r-1)!}. \tag{3}$$

In [27], Roopaei and Basar studied the Gamma spaces, including the spaces of absolutely p-summable, null, convergent, and bounded sequences.

In this article, we have introduced a new general space called $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and the mappings' ideal space of solutions for many stochastic nonlinear and matrix systems of Kannan contraction type. We have offered some geometric and topological structures of the soft function space, $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, multiplication operator acting on it, and its operators' ideal. A fixed point of the Kannan contraction operator exists in this space, and its prequasi operator ideal is confirmed. Finally, we discuss many applications of solutions to nonlinear stochastic dynamical systems and illustrative examples of our findings.

2. Properties of $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and Its Operators' Ideal

Some geometric and topological structures of the soft function space, $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, and its operators' ideals are presented in this section.

By c_0, ℓ_∞, and ℓ_r, we denote the space of null, bounded, and r-absolutely summable sequences of reals. We indicate the space of all bounded, finite rank linear mappings from an infinite-dimensional Banach space $\mathscr{G}$ into an infinite-dimensional Banach space $\mathscr{V}$ by $\mathbb{D}(\mathscr{G},\mathscr{V})$ and $\mathbb{F}(\mathscr{G},\mathscr{V})$, and if $\mathscr{G}=\mathscr{V}$, we write $\mathbb{D}(\mathscr{G})$ and $\mathbb{F}(\mathscr{G})$. The space of approximable and compact bounded linear operators from $\mathscr{G}$ into $\mathscr{V}$ will be marked by $\mathscr{A}(\mathscr{G},\mathscr{V})$ and $\mathscr{K}(\mathscr{G},\mathscr{V})$, respectively. The ideal of bounded, approximable, and compact mappings between every two infinite-dimensional Banach spaces will be denoted by $\mathbb{D}$, $\mathscr{A}$, and $\mathscr{K}$, respectively. Suppose $\omega^{\mathfrak{S}}$ is the class of all sequence spaces of soft reals.

Definition 1. If $(v_l)\in\mathfrak{R}^{+\mathscr{N}}$, $\mathfrak{R}^{+\mathscr{N}}$ is the space of all sequences of positive reals. The sequence space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ with the function τ is defined by

$$\left(\Gamma_r^{\mathfrak{S}}(q,v)\right)_\tau=\left\{\tilde{h}=\left(\widetilde{h_m}\right)\in\omega^{\mathfrak{S}}:\tau\left(\delta\tilde{h}\right)<\infty,\text{for some }\varepsilon>0\right\},$$
$$\text{where }\tau\left(\tilde{h}\right)=\sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\widetilde{h_z},\bar{0}\right)}{\begin{bmatrix}r+m\\ m\end{bmatrix}}\right)^{v_m}. \tag{4}$$

Lemma 2 (see [28]). *If $v_b>0$ and $x_b,z_b\in\mathfrak{R}$, for all $b\in\mathscr{N}$, and $\hbar=\max\{1,\sup_b v_b\}$, then*

$$|x_b+z_b|^{v_b}\le 2^{\hbar-1}\left(|x_b|^{v_b}+|z_b|^{v_b}\right). \tag{5}$$

Theorem 3. *Suppose $(v_l)\in\ell_\infty\cap\mathfrak{R}^{+\mathscr{N}}$, then*

$$\left(\Gamma_r^{\mathfrak{S}}(q,v)\right)_\tau=\left\{\tilde{h}=\left(\widetilde{h_b}\right)\in\omega^{\mathfrak{S}}:\tau\left(\delta\tilde{h}\right)<\infty,\text{for all }\delta>0\right\}. \tag{6}$$

Proof. Obviously, (v_l) is a bounded sequence. □

Theorem 4. *The space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a nonabsolute type, whenever $(v_l)\in[1,\infty)^{\mathscr{N}}\cap\ell_\infty$.*

Proof. Clearly, since

$$\begin{aligned}\tau\left(\tilde{1},-\tilde{1},\tilde{0},\tilde{0},\tilde{0},\cdots\right)&=(q_0)^{v_0}+\left(\frac{|q_0-rq_1|}{1+r}\right)^{v_1}+\left(\frac{|q_0-rq_1|}{\begin{bmatrix}r+2\\ 2\end{bmatrix}}\right)^{v_2}\\&\quad+\cdots\neq(q_0)^{v_0}+\left(\frac{q_0+rq_1}{1+r}\right)^{v_1}+\left(\frac{q_0+rq_1}{\begin{bmatrix}r+2\\ 2\end{bmatrix}}\right)^{v_2}\\&\quad+\cdots=\tau\left(\tilde{1},\tilde{1},\tilde{0},\tilde{0},\tilde{0},\cdots\right).\end{aligned} \tag{7}$$

□

Definition 5. Assume $(v_b) \in \mathfrak{R}^{+\mathcal{N}}$ and $v_b \geq 1$, for all $b \in \mathcal{N}$:

$$\left(\left|\Gamma_r^{\mathfrak{S}}\right|(q,v)\right)_\varphi := \left\{\tilde{h} = \left(\tilde{h_b}\right) \in \omega^{\mathfrak{S}} : \varphi(\delta f) < \infty, \text{for some } \delta > 0\right\}, \quad (8)$$

where

$$\varphi\left(\tilde{h}\right) = \sum_{b=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{h_z}\right|, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}} \right)^{v_b}. \quad (9)$$

Theorem 6. *Suppose $(v_l) \in (1,\infty)^{\mathcal{N}} \cap \ell_\infty$ with*

$$\left(\frac{l+1}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right) \notin \ell_{(v_l)}, \quad (10)$$

hence $(|\Gamma_r^{\mathfrak{S}}|(q,v))_\varphi \subsetneq (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$.

Proof. Assume $\tilde{f} \in (|\Gamma_r^{\mathfrak{S}}|(q,v))_\varphi$, as

$$\begin{aligned} &\sum_{b=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \tilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}} \right)^{v_b} \\ &\leq \sum_{b=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{f_z}\right|, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}} \right)^{v_b} < \infty. \end{aligned} \quad (11)$$

Then $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. If we choose

$$\tilde{g} = \left(\frac{\left(-\tilde{1}\right)^z}{\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z} \right)_{z \in \mathcal{N}}, \quad (12)$$

one gets $\tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and $\tilde{g} \notin (|\Gamma_r^{\mathfrak{S}}|(q,v))_\varphi$.

Suppose $\mathscr{E}^{\mathfrak{S}}$ is a linear space of sequences of soft functions, and $[p]$ describes an integral part of the real number p. □

Definition 7. The space $\mathscr{E}^{\mathfrak{S}}$ is said to be a private sequence space of soft functions ($\mathfrak{pssf}$) if it satisfies the next setups:

(a1) For all $b \in \mathcal{N}$, then $\tilde{e_b} \in \mathscr{E}^{\mathfrak{S}}$, where $\tilde{e_b} = (\tilde{0}, \tilde{0}, \cdots, \tilde{1}, \tilde{0}, \tilde{0}, \cdots)$, while $\tilde{1}$ displays at the b^{th} place

(a2) If $\tilde{f} = (\tilde{f_b}) \in \omega^{\mathfrak{S}}$, $|\tilde{g}| = (|\tilde{g_b}|) \in \mathscr{E}^{\mathfrak{S}}$ and $|\tilde{f_b}| \leq |\tilde{g_b}|$, with $b \in \mathcal{N}$, then $|\tilde{f}| \in \mathscr{E}^{\mathfrak{S}}$

(a3) $(|\widetilde{h_{[b/2]}}|)_{b=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, whenever $(|\tilde{h_b}|)_{b=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$

Definition 8 (see [29]). An s-number is a function $s : \mathbb{D}(\mathscr{G}, \mathscr{V}) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ that gives all $V \in \mathbb{D}(\mathscr{G}, \mathscr{V})$ a $(s_d(V))_{d=0}^{\infty}$ holds the following conditions:

(1) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \cdots \geq 0$, for all $V \in \mathbb{D}(\mathscr{G}, \mathscr{V})$

(2) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for every $W \in \mathbb{D}(\mathscr{G}_0, \mathscr{G})$, $Y \in \mathbb{D}(\mathscr{G}, \mathscr{V})$ and $V \in \mathbb{D}(\mathscr{V}, \mathscr{V}_0)$, where $\mathscr{G}_0$ and $\mathscr{V}_0$ are arbitrary Banach spaces

(3) $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, for every $V_1, V_2 \in \mathbb{D}(\mathscr{G}, \mathscr{V})$ and $l, d \in \mathcal{N}$

(4) Assume $V \in \mathbb{D}(\mathscr{G}, \mathscr{V})$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$

(5) If rank $(V) \leq d$, then $s_d(V) = 0$, for all $V \in \mathbb{D}(\mathscr{G}, \mathscr{V})$

(6) $s_{l \geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a indicates the unit mapping on the a-dimensional Hilbert space ℓ_2^a

Some examples of s-numbers:

(a) The bth approximation number is defined as $\alpha_b(X) = \inf \{\|X - Y\| : Y \in \mathbb{D}(\mathscr{G}, \mathscr{V}) \text{ and } \operatorname{rank}(Y) \leq b\}$

(b) The bth Kolmogorov number is defined as $d_b(X) = \inf_{\dim J \leq b} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|$

Notation 9 (see [30]).

$$\begin{aligned} \widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}} &:= \left\{\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V})\right\}, \text{where } \widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V}) \\ &:= \left\{V \in D(\mathscr{G}, \mathscr{V}) : \left(\left(\widetilde{s_j(V)}\right)_{j=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}\right\}, \end{aligned}$$

$$\begin{aligned} \widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}} &:= \left\{\widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V})\right\}, \text{where } \widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V}) \\ &:= \left\{V \in D(\mathscr{G}, \mathscr{V}) : \left(\left(\widetilde{\alpha_j(V)}\right)_{j=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}\right\}, \end{aligned}$$

$$\begin{aligned} \widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}} &:= \left\{\widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V})\right\}, \text{where } \widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}, \mathscr{V}) \\ &:= \left\{V \in D(\mathscr{G}, \mathscr{V}) : \left(\left(\widetilde{d_j(V)}\right)_{j=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}\right\}, \end{aligned}$$

$$\left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma} := \left\{\left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma}(\mathscr{G},\mathscr{V})\right\}, \text{ where } \left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma}(\mathscr{G},\mathscr{V}) := \left\{V \in D(\mathscr{G},\mathscr{V}) : \left(\left(\gamma_b\widetilde{(V)}\right)_{b=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}} \text{ and } \cdot \left\|V - \tilde{\rho}\left(\gamma_b\widetilde{(V)}, \tilde{0}\right)I\right\| = 0, \text{for all } b \in \mathscr{N}\right\}. \tag{13}$$

Theorem 10. *Assume the linear sequence space $\mathscr{E}^{\mathfrak{S}}$ is a $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, then $\widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}$ is an operator ideal.*

Proof.

(i) Assume $V \in \mathbb{F}(\mathscr{G},\mathscr{V})$ and rank $(V) = n$ with $n \in \mathscr{N}$, as $\widetilde{e_i} \in \mathscr{E}^{\mathfrak{S}}$ for all $i \in \mathscr{N}$ and $\mathscr{E}^{\mathfrak{S}}$ is a linear space, one has $(s_i\widetilde{(V)})_{i=0}^{\infty} = (s_0\widetilde{(V)}, s_1\widetilde{(V)}, \cdots, s_{n-1}\widetilde{(V)}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) = \sum_{i=0}^{n-1} s_i\widetilde{(V)}\widetilde{e_i} \in \mathscr{E}^{\mathfrak{S}}$, for that $V \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ then $\mathbb{F}(\mathscr{G},\mathscr{V}) \subseteq \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$

(ii) Suppose $V_1, V_2 \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ and $\beta_1, \beta_2 \in \mathfrak{R}$ then by Definition 7 condition (iii), one has $(s_{[i/2]}\widetilde{(V_1)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$ and $(s_{[i/2]}\widetilde{(V_1)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, as $i \geq 2[i/2]$, by the definition of $\tilde{s}$-numbers and $s_i\widetilde{(P)}$ is a decreasing sequence, we have

$$s_i\left(\beta_1\widetilde{V_1 + \beta_2}V_2\right) \leq s_{2[i/2]}\left(\beta_1\widetilde{V_1 + \beta_2}V_2\right) \leq s_{[i/2]}\widetilde{(\beta_1 V_1) + s_{[i/2]}}(\beta_2 V_2) = |\beta_1| s_{[i/2]}\widetilde{(V_1)} + |\beta_2| s_{[i/2]}\widetilde{(V_2)}, \tag{14}$$

for each $i \in \mathscr{N}$. In view of Definition 7 condition (ii) and $\mathscr{E}^{\mathfrak{S}}$ is a linear space, one obtains $(s_i(\beta_1\widetilde{V_1 + \beta_2}V_2))_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, then $\beta_1 V_1 + \beta_2 V_2 \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$

(iii) If $P \in \mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, and $R \in \mathbb{D}(\mathscr{V},\mathscr{V}_0)$, one has $(s_i\widetilde{(T)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$ and as $s_i\widetilde{(RTP)} \leq \|R\| \|s_i\widetilde{(T)}\| \|P\|$, by Definition 7 conditions (i) and (ii), one gets $(s_i\widetilde{(RTP)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, hence $RTP \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}_0,\mathscr{V}_0)$

Assume $\widetilde{\theta} = (\tilde{0}, \tilde{0}, \tilde{0}, \cdots)$ and $\mathscr{F}$ is the space of finite sequences of soft numbers. □

Definition 11. A subspace of the $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$ is called a premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if there is a function $\tau : \mathscr{E}^{\mathfrak{S}} \longrightarrow [0,\infty)$ satisfies the next setups:

(i) If $\tilde{h} \in \mathscr{E}^{\mathfrak{S}}$, $\tilde{h} = \widetilde{\theta} \Leftrightarrow \tau(|\tilde{h}|) = 0$, and $\tau(\tilde{h}) \geq 0$

(ii) Assume $\tilde{h} \in \mathscr{E}^{\mathfrak{S}}$ and $\varepsilon \in \mathfrak{R}$, one has $E_0 \geq 1$ so that $\tau(\varepsilon\tilde{h}) \leq |\varepsilon| E_0 \tau(\tilde{h})$

(iii) There are $G_0 \geq 1$ so that $\tau(\tilde{f} + \tilde{g}) \leq G_0(\tau(\tilde{f}) + \tau(\tilde{g}))$, for all $\tilde{f}, \tilde{g} \in \mathscr{E}^{\mathfrak{S}}$

(iv) Assume $|\widetilde{f_b}| \leq |\widetilde{g_b}|$, for all $b \in \mathscr{N}$, then $\tau(|\widetilde{f_b}|) \leq \tau(|\widetilde{g_b}|)$

(v) One gets $D_0 \geq 1$ such that $\tau(|\tilde{f}|) \leq \tau(|\widetilde{f_{[\cdot]}}|) \leq D_0 \tau(|\tilde{f}|)$

(vi) The closure of $\mathscr{F} = \mathscr{E}_{\tau}^{\mathfrak{S}}$

(vii) There are $\varepsilon > 0$ with $\tau(\tilde{v}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) \geq \varepsilon |v| \tau(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots)$

Definition 12. The $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}\mathscr{E}_{\tau}^{\mathfrak{S}}$ is said to be a prequasi normed $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if τ confirms the setups (i)-(iii) of Definition 11. The space $\mathscr{E}_{\tau}^{\mathfrak{S}}$ is called a prequasi Banach $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, whenever $\mathscr{E}^{\mathfrak{S}}$ is complete under τ.

Theorem 13. *The space $\mathscr{E}_{\tau}^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, whenever it is premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$. By $\uparrow$ and $\downarrow$, we denote the space of all monotonic increasing and decreasing sequences of positive reals, respectively.*

Theorem 14. *$(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$ is a prequasi Banach $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if the next setups are confirmed:*

(f1) $(v_l) \in \uparrow \cap \ell_{\infty}$ with $v_0 > 1/r$

(f2) $\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)_{b=0}^{\infty} \in \downarrow$ or $\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)_{b=0}^{\infty} \in \uparrow \cap \ell_{\infty}$ and there exists $C \geq 1$ such that

$$\begin{bmatrix} 2b+r \\ 2b+1 \end{bmatrix} q_{2b+1} \leq C \begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b \tag{15}$$

Proof. First, we have to show that $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$ is a premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}$.

(i) Obviously, $\tau(|\tilde{h}|) = 0 \Leftrightarrow \tilde{h} = \widetilde{\theta}$ and $\tau(\tilde{h}) \geq 0$

(a1) and (iii) If $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$, then

$$\tau\left(\tilde{f}+\tilde{g}\right) = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\tilde{f}_z+\tilde{g}_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \leq 2^{h-1}\left(\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\tilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\tilde{g}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right) = 2^{h-1}\left(\tau\left(\tilde{f}\right) + \tau(\tilde{g})\right) < \infty, \tag{16}$$

hence $\tilde{f} + \tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$

(ii) Next, suppose $\lambda \in \mathfrak{R}$, $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and as $(v_l) \in \uparrow \cap \ell_\infty$, we get

$$\begin{aligned}\tau\left(\lambda \tilde{f}\right) &= \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \lambda \tilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sup_m |\lambda|^{v_m} \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \tilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le E_0 |\lambda| \tau\left(\tilde{f}\right) < \infty,\end{aligned} \tag{17}$$

where $E_0 = \max\{1, \sup_l |\lambda|^{v_l - 1}\} \ge 1$. So, $\lambda \tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$.

As $(v_l) \in \uparrow \cap \ell_\infty$ and $v_0 > 1/r$, one obtains

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{(e_b)}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} &= \sum_{m=b}^{\infty}\left(\frac{\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sup_{m=b}^{\infty}\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)^{v_m} \sum_{m=b}^{\infty}\left(\frac{1}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} < \infty.\end{aligned} \tag{18}$$

Therefore, $\tilde{e_b} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for every $b \in \mathcal{N}$.

(a2) and (iv) If $|\widetilde{f_m}| \le |\widetilde{g_m}|$, for all $m \in \mathcal{N}$ and $|\tilde{g}| \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, then

$$\begin{aligned}\tau\left(\left|\tilde{f}\right|\right) &= \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z |\tilde{g}_z|, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &= \tau(|\tilde{g}|) < \infty,\end{aligned} \tag{19}$$

hence $|\tilde{f}| \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$

(a3) and (v) Assume $(|\widetilde{f_z}|) \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, with $(v_l) \in \uparrow \cap \ell_\infty$ and

$$\left(\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right)_{z=0}^{\infty} \in \downarrow, \tag{20}$$

we get

$$\begin{aligned}\tau\left(\left|\widetilde{f_{[z/2]}}\right|\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+2l \\ 2l \end{bmatrix}}\right)^{v_{2l}} \\ &\quad + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l+1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+2l+1 \\ 2l+1 \end{bmatrix}}\right)^{v_{2l+1}} \le \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\quad + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l+1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\le \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\begin{bmatrix} 2l+r-1 \\ 2l \end{bmatrix} q_{2l} \left|\tilde{f}_l\right| + \sum_{z=0}^{l}\left(\begin{bmatrix} 2z+r-1 \\ z \end{bmatrix} q_{2z} + \begin{bmatrix} 2z+r \\ 2z+1 \end{bmatrix} q_{2z+1}\right)\left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\quad + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\begin{bmatrix} 2z+r-1 \\ z \end{bmatrix} q_{2z} + \begin{bmatrix} 2z+r \\ 2z+1 \end{bmatrix} q_{2z+1}\right)\left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + \sum_{l=0}^{\infty}\left(\frac{2\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right) \\ &\quad + \sum_{l=0}^{\infty}\left(\frac{2\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\tilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \le D_0 \tau\left(\left|\tilde{f}\right|\right) < \infty,\end{aligned} \tag{21}$$

where $D_0 \ge (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}) \ge 1$. Hence, $(|\widetilde{f_{[z/2]}}|) \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$

(vi) It is clear that the closure of $\mathcal{F} = \Gamma_r^{\mathfrak{S}}(q,v)$

(vii)There are $0 < \delta \le \sup_l |\lambda|^{v_l - 1}$ so that $\tau(\tilde{\lambda}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) \ge \delta |\lambda| \tau(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots)$, for all $\lambda \ne 0$ and $\delta > 0$, if $\lambda = 0$

By Theorem 13, the space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a prequasi normed $\mathfrak{pSSSf}$. Second, to prove that $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a Banach space, suppose $\widetilde{h^i} = (\widetilde{h^i_k})_{k=0}^{\infty}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, hence for every $\gamma \in (0,1)$, one has $i_0 \in \mathcal{N}$ with $i, j \ge i_0$, we have

$$\tau\left(\widetilde{h^i} - \widetilde{h^j}\right) = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left(\widetilde{h^i_z} - \widetilde{f^j_z}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \gamma^{\hbar}. \tag{22}$$

That implies

$$\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{h_z^i}-\widetilde{h_z^j}\right),\tilde{0}\right)<\gamma. \tag{23}$$

As $(\mathscr{R}(A),\tilde{\rho})$ is a complete metric space. Therefore, $(\widetilde{h_k^j})$ is a Cauchy sequence in $\mathscr{R}(A)$, for constant $k\in\mathscr{N}$. So, it is convergent to $\widetilde{h_k^0}\in\mathscr{R}(A)$. This implies $\tau(\widetilde{h^i}-\widetilde{h^0})<\gamma^{\hbar}$, for every $i\geq i_0$. Clearly, from condition (iii) that $\widetilde{h^0}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$.

In view of Theorems 10 and 14, we have the next theorem. □

Theorem 15. *The space $\widetilde{\mathbb{D}^s}_{\Gamma_r^{\mathfrak{S}}(q,v)}$ is an operator ideal, if the conditions of Theorem 14 are verified.*

Theorem 16. *If s-type $\mathscr{E}_\tau^{\mathfrak{S}}:=\{\tilde{h}=(\widetilde{s_j(H)})\in\mathfrak{R}^{\mathscr{N}}: H\in D(\mathscr{G},\mathscr{V})$ and $\tau(\tilde{h})<\infty\}$. Assume $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau}$ is an operator ideal, one has the next setups:*

- *(a) s-type $\mathscr{E}_\tau^{\mathfrak{S}}\supset\mathscr{F}$*
- *(b) Suppose $(\widetilde{s_j(H_1)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ and $(\widetilde{s_j(H_2)})_{j=0}^{\infty}\in$ s -type $\mathscr{E}_\tau^{\mathfrak{S}}$, then $(\widetilde{s_j(H_1+H_2)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$*
- *(c) If $\varepsilon\in\mathfrak{R}$ and $(\widetilde{s_j(H)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, one has $|\varepsilon|$ $(\widetilde{s_j(H)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$*
- *(d) Suppose $(\widetilde{s_j(U)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ and $\widetilde{s_j(T)}\leq\widetilde{s_j(U)}$, for all $j\in\mathscr{N}$ and $T,U\in\mathbb{D}(\mathscr{G},\mathscr{V})$, one gets $(\widetilde{s_j(T)})_{j=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, i.e., $\mathscr{E}_\tau^{\mathfrak{S}}$ is a solid space*

Proof. If $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau}$ is a mappings' ideal.

(a)We have $\mathbb{F}(\mathscr{G},\mathscr{V})\subset\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$. Hence, for all $X\in\mathbb{F}(\mathscr{G},\mathscr{V})$, we have $(\widetilde{s_r(X)})_{r=0}^{\infty}\in\mathscr{F}$. This gives $(\widetilde{s_r(X)})_{r=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. Hence, $\mathscr{F}\subset$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$

(b) and (c)The space $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ is linear over $\mathfrak{R}$. Hence, for each $\lambda\in\mathfrak{R}$ and $X_1,X_2\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, we have $X_1+X_2\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ and $\lambda X_1\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$. That implies

$$\begin{gathered}\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty}\in s\text{-type }\mathscr{E}_\tau^{\mathfrak{S}}\ \text{ and }\ \left(\widetilde{s_r(X_2)}\right)_{r=0}^{\infty}\in s\text{-type }\mathscr{E}_\tau^{\mathfrak{S}}\Rightarrow\left(\widetilde{s_r(X_1+X_2)}\right)_{r=0}^{\infty}\in s\text{-type }\mathscr{E}_\tau^{\mathfrak{S}},\\ \lambda\in\mathfrak{R}\ \text{ and }\ \left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty}\in s\text{-type }\mathscr{E}_\tau^{\mathfrak{S}}\Rightarrow|\lambda|\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty}\in s\text{-type }\mathscr{E}_\tau^{\mathfrak{S}}\end{gathered} \tag{24}$$

(d)If $A\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $B\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, and $D\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $DBA\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G}_0,\mathscr{V}_0)$. Therefore, since $(\widetilde{s_r(B)})_{r=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, then $(\widetilde{s_r(DBA)})_{r=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. Since $\widetilde{s_r(DBA)}\leq\|D\|\widetilde{s_r(B)}\|A\|$. By using condition (c), if $(\|D\|\|A\|\widetilde{s_r(B)})_{r=0}^{\infty}\in\mathscr{E}_\tau^{\mathfrak{S}}$, we have $(\widetilde{s_r(DBA)})_{r=0}^{\infty}\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. This means s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ is solid

Some properties of s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ are presented in the next theorem according to Theorems 16 and 15. □

Theorem 17.

- *(a) s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau\supset\mathscr{F}$*
- *(b) If $(\widetilde{s_n(X_1)})_{n=0}^{\infty}\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and $(\widetilde{s_n(X_2)})_{n=0}^{\infty}\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, then $(s_n(X_1+X_2))_{n=0}^{\infty}\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$*
- *(c) Assume $\lambda\in\mathfrak{R}$ and $(\widetilde{s_n(X)})_{n=0}^{\infty}\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, hence $|\lambda|(\widetilde{s_n(X)})_{n=0}^{\infty}\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$*
- *(d) s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a solid space*

Definition 18 (see [31]). A subclass $\mathscr{U}$ of $\mathbb{D}$ is said to be a mappings' ideal, if every $\mathscr{U}(\mathscr{G},\mathscr{V})=\mathscr{U}\cap\mathbb{D}(\mathscr{G},\mathscr{V})$ satisfies the following setups:

- (i) $I_\Gamma\in\mathscr{U}$, where Γ indicates Banach space of one dimension
- (ii) The space $\mathscr{U}(\mathscr{G},\mathscr{V})$ is linear over $\mathfrak{R}$
- (iii) If $W\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $X\in\mathscr{U}(\mathscr{G},\mathscr{V})$, and $Y\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $YXW\in\mathscr{U}(\mathscr{G},\mathscr{V}_0)$

Definition 19 (see [32]). A function $H\in[0,\infty)^{\mathscr{U}}$ is said to be a prequasi norm on the ideal $\mathscr{U}$ if the following conditions hold:

1. Assume $V\in\mathscr{U}(\mathscr{G},\mathscr{V})$, $H(V)\geq0$ and $H(V)=0$, if and only if, $V=0$
2. One has $Q\geq1$ with $H(\alpha V)\leq D|\alpha|H(V)$, for all $V\in\mathscr{U}(\mathscr{G},\mathscr{V})$ and $\alpha\in\mathfrak{R}$
3. There are $P\geq1$ such that $H(V_1+V_2)\leq P[H(V_1)+H(V_2)]$, for all $V_1,V_2\in\mathscr{U}(\mathscr{G},\mathscr{V})$
4. There are $\sigma\geq1$ so that if $V\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $X\in\mathscr{U}(\mathscr{G},\mathscr{V})$, and $Y\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $H(YXV)\leq\sigma\|Y\|H(X)\|V\|$

Theorem 20 (see [32]). *Every quasi norm on the ideal $\mathscr{U}$ is a prequasi norm.*

We have discussed some properties of the ideal constructed by our soft space and extended s-numbers, supposing that the conditions of Theorem 14 are verified.

Theorem 21. *The conditions of Theorem 14 are sufficient only for $\mathbb{D}^{\alpha}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})=$ the closure of $\mathbb{F}(\mathscr{G},\mathscr{V})$.*

Proof. Clearly, the closure of $\mathbb{F}(\mathscr{G},\mathscr{V})\subseteq\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$ from the linearity of the space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and $\widetilde{e_m}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for all $m\in\mathscr{N}$. Next, to show that $\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})\subseteq$ the closure of $\mathbb{F}(\mathscr{G},\mathscr{V})$. If $H\in\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$, one has $(\widetilde{\alpha_l(H)})_{m=0}^{\infty}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. As $\tau(\widetilde{\alpha_m(H)})_{m=0}^{\infty}<\infty$, assume $\gamma\in(0,1)$, we have $l_0\in\mathscr{N}-\{0\}$ so that $\tau((\widetilde{\alpha_m(H)})_{m=l_0}^{\infty})<\gamma/2^{\hbar+3}\delta j$, for some $j\geq 1$ and

$$\delta=\max\left\{1,\sum_{l=l_0}^{\infty}\left(\frac{1}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\right\}. \tag{25}$$

Since $\widetilde{\alpha_l(H)}\in\mathfrak{I}_{\searrow}^{\mathfrak{S}}$, we get

$$\begin{aligned}
&\sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{\alpha_{2l_0}(H)},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\leq\sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{\alpha_z(H)},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\leq\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{\alpha_z(H)},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}<\frac{\gamma}{2^{\hbar+3}\delta j}.
\end{aligned} \tag{26}$$

We get $U\in\mathbb{F}_{2l_0}(\mathscr{G},\mathscr{V})$ with rank $(U)\leq 2l_0$ and

$$\begin{aligned}
&\sum_{l=2l_0+1}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\leq\sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&<\frac{\gamma}{2^{\hbar+3}\delta j},
\end{aligned} \tag{27}$$

since $(v_l)\in\uparrow\cap\ell_\infty$, we have

$$\sup_{l=l_0}^{\infty}\tilde{\rho}^{v_l}\left(\sum_{z=0}^{l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)<\frac{\gamma}{2^{2\hbar+2}\delta}. \tag{28}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}<\frac{\gamma}{2^{\hbar+3}\delta j}. \tag{29}$$

Because of inequalities (5), (26), (27), (28), and (29), one gets

$$\begin{aligned}
d(H,U)&=\tau\left(\alpha_l(\widetilde{H-U})\right)_{l=0}^{\infty}=\sum_{l=0}^{3l_0-1}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+\sum_{l=3l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)\leq\sum_{l=0}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l+2l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l+2l_0\\ l+2l_0\end{bmatrix}}\right)^{v_{l+2l_0}}\leq\sum_{l=0}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l+2l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U})+\sum_{z=2l_0}^{l+2l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=2l_0}^{l+2l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\alpha_z(\widetilde{Z-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\quad+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+2l_0+r-1\\ z+2l_0\end{bmatrix}q_{z+2l_0}\alpha_{z+2l_0}(\widetilde{H-U}),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}+2^{2\hbar-1}\sup_{l=l_0}^{\infty}\tilde{\rho}^{v_l}\left(\sum_{z=0}^{l_0}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\|\widetilde{H-U}\|,\tilde{0}\right)\\
&\quad\cdot\sum_{l=l_0}^{\infty}\left(\frac{1}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{\alpha_z(H)},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}<\gamma.
\end{aligned} \tag{30}$$

On the other hand, one has a negative example as $I_2\in\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$, where $z+r-1zq_z=1$, for all $z\in\mathscr{N}$ and

$v = (0,-1, 2, 2, 2)$, but $(v_l) \notin \uparrow$. This gives a negative answer to the Rhoades [33] open problem about the linearity of s-type $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ spaces. □

Theorem 22. *The class $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}, \Xi)$ is a prequasi Banach ideal, where $\Xi(H) = \tau((\widetilde{s_b(H)})_{b=0}^{\infty})$.*

Proof. Evidently, Ξ is a prequasi norm on $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}$ since τ is a prequasi norm on $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$. Assume $(H_m)_{m\in\mathcal{N}}$ is a Cauchy sequence in $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$. Since $\mathbb{D}(\mathcal{G},\mathcal{V}) \supseteq \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$, we have

$$\Xi(H_j - H_m) = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_z\left(\widetilde{H_j - H_m}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \geq \left(\|H_j - H_m\|\right)^{v_0}, \tag{31}$$

then $(H_m)_{m\in\mathcal{N}}$ is a Cauchy sequence in $\mathbb{D}(\mathcal{G},\mathcal{V})$. As $\mathbb{D}(\mathcal{G},\mathcal{V})$ is a Banach space, one has $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ so that $\lim_{m\longrightarrow\infty}\|H_m - H\| = 0$. As $(\widetilde{s_l(H_m)})_{l=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q, v))_\tau$, for all $m \in \mathcal{N}$. By Definition 11 conditions (ii), (iii), and (v), we have

$$\begin{aligned}
\Xi(H) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_{[z/2]}\left(\widetilde{H - H_m}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\quad + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_{[z/2]}(H_m)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H - H_m}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\quad + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_z(H_m)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \infty.
\end{aligned} \tag{32}$$

Hence, $(\widetilde{s_b(H)})_{b=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q, v))_\tau$, so $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$. □

Theorem 23. *If $1 < v_b^{(1)} < v_b^{(2)}$, and $0 < q_b^{(2)} \leq q_b^{(1)}$, for every $b \in \mathcal{N}$, then*

$$\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_b^{(1)}\right),\left(v_b^{(1)}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_b^{(2)}\right),\left(v_b^{(2)}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \mathbb{D}(\mathcal{G},\mathcal{V}). \tag{33}$$

Proof. Let $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_b^{(1)}),(v_b^{(1)})))_\tau}(\mathcal{G},\mathcal{V})$, then $(\widetilde{s_b(H)}) \in (\Gamma_r^{\mathfrak{S}}((q_b^{(1)}), (v_b^{(1)})))_\tau$. One obtains

$$\begin{aligned}
&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(2)} \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b^{(2)}} \\
&\quad < \sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(1)} \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b^{(1)}} < \infty,
\end{aligned} \tag{34}$$

then $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathcal{G},\mathcal{V})$. Take $(\widetilde{s_b(H)})_{b=0}^{\infty}$ with

$$\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(1)} \widetilde{s_z(H)}, \tilde{0}\right) = \frac{\begin{bmatrix} r+b \\ b \end{bmatrix}}{\sqrt[vb1]{b+1}}, \tag{35}$$

we have $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ with

$$\begin{aligned}
&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(1)} \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b^{(1)}} = \sum_{b=0}^{\infty}\frac{1}{b+1} = \infty, \\
&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(2)} \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b^{(2)}} \\
&\quad \leq \sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(1)} \widetilde{s_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b^{(2)}} = \sum_{b=0}^{\infty}\left(\frac{1}{b+1}\right)^{v_b^{(2)}/v_b^{(1)}} < \infty.
\end{aligned} \tag{36}$$

Hence, $H \notin \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(1)}),(v_b^{(1)})))_\tau}(\mathscr{G},\mathscr{V})$ and $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V})$.

Clearly, $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V}) \subset \mathbb{D}(\mathscr{G},\mathscr{V})$. Take $(\widetilde{s_b(H)})_{b=0}^{\infty}$ with

$$\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(2)}\widetilde{s_z(H)}, \tilde{0}\right) = \frac{\begin{bmatrix} r+b \\ b \end{bmatrix}}{\sqrt[vb2]{b+1}}. \tag{37}$$

Then $H \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and $H \notin \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V})$.

Recall that if $\mathscr{G}$ and $\mathscr{V}$ are infinite-dimensional, by Dvoretzky's theorem [34], there are $\mathscr{G}/Y_j$ and $M_j \subseteq \mathscr{V}$ operated onto ℓ_2^j through isomorphisms V_j and X_j such that $\|V_j\|\|V_j^{-1}\| \le 2$ and $\|X_j\|\|X_j^{-1}\| \le 2$, for all $j \in \mathscr{N}$. Assume T_j is the quotient mapping from $\mathscr{G}$ onto $\mathscr{G}/Y_j$, I_j is the identity operator on ℓ_2^j and J_j is the natural embedding operator from M_j into $\mathscr{V}$. Assume m_j is the Bernstein numbers [18]. □

Theorem 24. *The class $\widetilde{\mathbb{D}^\alpha}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}$ is minimum, whenever*

$$\left(\frac{\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)_{l=0}^{\infty} \notin \ell_{((v_l))}. \tag{38}$$

Proof. Assume $\widetilde{\mathbb{D}^\alpha}_{\Gamma_r^{\mathfrak{C}}(q,v)}(\mathscr{G},\mathscr{V}) = \mathbb{D}(\mathscr{G},\mathscr{V})$, one has $\gamma > 0$ so that $\Xi(H) \le \gamma\|H\|$, for all $H \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and

$$\Xi(H) = \sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\widetilde{\alpha_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b}. \tag{39}$$

We have

$$\begin{aligned} 1 = m_z(I_j) &= m_z\left(X_jX_j^{-1}I_jV_jV_j^{-1}\right) \\ &\le \|X_j\| m_z\left(X_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| = \|X_j\| m_z\left(J_jX_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| \\ &\le \|X_j\| d_z\left(J_jX_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| = \|X_j\| d_z\left(J_jX_j^{-1}I_jV_jT_j\right) \\ &\quad\cdot\left\|V_j^{-1}\right\| \le \|X_j\|\alpha_z\left(J_jX_j^{-1}I_jV_jT_j\right)\left\|V_j^{-1}\right\|. \end{aligned} \tag{40}$$

Take $0 \le m \le j$, one has

$$\begin{aligned} &\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \le \tilde{\rho}\left(\sum_{z=0}^{m}\|X_j\|\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right)\left\|V_j^{-1}\right\|, \tilde{0}\right) \Rightarrow \\ &\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \left(\|X_j\|\left\|V_j^{-1}\right\|\right)^{v_m} \\ &\cdot\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right), \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m}. \end{aligned} \tag{41}$$

Therefore, for some $\lambda \ge 1$, we obtain

$$\begin{aligned} &\sum_{m=0}^{j}\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \lambda\|X_j\|\left\|V_j^{-1}\right\|\sum_{m=0}^{j} \\ &\cdot\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right), \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \Rightarrow \sum_{m=0}^{j} \\ &\cdot\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \lambda\|X_j\|\left\|V_j^{-1}\right\|\Xi\left(J_jX_j^{-1}I_jV_jT_j\right) \\ &\le \lambda\gamma\|X_j\|\left\|V_j^{-1}\right\|\left\|J_jX_j^{-1}I_jV_jT_j\right\| \le 4\lambda\gamma. \end{aligned} \tag{42}$$

When $j \longrightarrow \infty$, one has a contradiction. So, $\mathscr{G}$ and $\mathscr{V}$ both cannot be infinite-dimensional when $\widetilde{\mathbb{D}^\alpha}_{\Gamma_r^{\mathfrak{C}}(q,v)}(\mathscr{G},\mathscr{V}) = \mathbb{D}(\mathscr{G},\mathscr{V})$. □

Theorem 25. *The class $\widetilde{\mathbb{D}^d}_{\Gamma_r^{\mathfrak{C}}(q,v)}$ is minimum, whenever*

$$\left(\frac{\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)_{l=0}^{\infty} \notin \ell_{((v_l))}. \tag{43}$$

Lemma 26 (see [19]). *Suppose $W \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and $W \notin \mathscr{A}(\mathscr{G},\mathscr{V})$, one has $P \in \mathbb{D}(\mathscr{G})$ and $A \in \mathbb{D}(\mathscr{V})$ with $AWPe_j = e_j$, for every $j \in \mathscr{N}$.*

Theorem 27 (see [19]). *If $\mathscr{E}^{\mathfrak{S}}$ is an infinite-dimensional Banach space, then*

$$\mathbb{F}\left(\mathscr{E}^{\mathfrak{S}}\right) \subsetneqq \mathscr{A}\left(\mathscr{E}^{\mathfrak{S}}\right) \subsetneqq \mathscr{K}\left(\mathscr{E}^{\mathfrak{S}}\right) \subsetneqq \mathbb{D}\left(\mathscr{E}^{\mathfrak{S}}\right). \quad (44)$$

Theorem 28. *If $1 < v_l^{(1)} < v_l^{(2)}$ and $0 < q_l^{(2)} \le q_l^{(1)}$, for every $l \in \mathcal{N}$, then*

$$\begin{aligned}&\mathbb{D}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right)\\&\quad=\mathscr{A}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right).\end{aligned} \quad (45)$$

Proof. Let $X \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$ and $X \notin \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$. In view of Lemma 26, there are $Y \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}))$ and $Z \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathcal{N}$, one has

$$\begin{aligned}&\|I_b\|_{\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})}\\&\quad=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(1)}\widetilde{s_z(I_b)},\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l^{(1)}}\\&\quad\le\|ZXY\|\|I_b\|_{\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})}\\&\quad\le\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(2)}\widetilde{s_z(I_b)},\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l^{(2)}}.\end{aligned} \quad (46)$$

This contradicts Theorem 23; hence, $X \in \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$. □

Corollary 29. *Suppose $1 < v_l^{(1)} < v_l^{(2)}$, and $0 < q_l^{(2)} \le q_l^{(1)}$, for every $l \in \mathcal{N}$, then*

$$\begin{aligned}&\mathbb{D}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right)\\&\quad=\mathscr{K}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}),\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right).\end{aligned} \quad (47)$$

Proof. Evidently, as $\mathscr{A} \subset \mathscr{K}$. □

Definition 30 (see [19]). A Banach space $\mathscr{E}^{\mathfrak{S}}$ *is said to be simple when $\mathbb{D}(\mathscr{E}^{\mathfrak{S}})$ has a unique nontrivial closed ideal.*

Theorem 31. *The class $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}$ is simple.*

Proof. Let the closed ideal $\mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$ contain a mapping $H \notin \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. By Lemma 26, there are $P, A \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$ so that $AHPI_j = I_j$. Therefore, $I_{\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})} \in \mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. Hence, $\mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})) = \mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. Therefore, $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}$ is a simple Banach space. □

Theorem 32. *Assume*

$$\inf_l\left(\frac{\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l} > 0, \quad (48)$$

then $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau})^{\gamma}(\mathscr{G},\mathscr{V}) = \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$.

Proof. Let $H \in (\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau})^{\gamma}(\mathscr{G},\mathscr{V})$, then $(\widetilde{\gamma_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and $\|H - \tilde{\rho}(\widetilde{\gamma_m(H)},\tilde{0})I\| = 0$, for every $m \in \mathcal{N}$. One has $H = \tilde{\rho}(\widetilde{\gamma_m(H)},\tilde{0})I$, for all $m \in \mathcal{N}$, then

$$\tilde{\rho}\left(\widetilde{s_m(H)},\tilde{0}\right) = \tilde{\rho}\left(\widetilde{s_m(\tilde{\rho}\left(\widetilde{\gamma_m(H)},\tilde{0}\right)I)},\tilde{0}\right) = \tilde{\rho}\left(\widetilde{\gamma_m(H)},\tilde{0}\right), \quad (49)$$

for all $m \in \mathcal{N}$. Hence $(\widetilde{s_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, so $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$.

Next, assume $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$. Hence, $(\widetilde{s_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. Therefore, one has

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+m\\m\end{bmatrix}}\right)^{v_m}\\&\quad\ge\inf_m\left(\frac{\sum_{z=0}^{m}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z}{\begin{bmatrix}r+m\\m\end{bmatrix}}\right)^{v_m}\sum_{m=0}^{\infty}\left[\tilde{\rho}\left(\widetilde{s_m(H)},\tilde{0}\right)\right]^{v_m}.\end{aligned} \quad (50)$$

Hence, $\lim_{m\longrightarrow\infty}\widetilde{s_m(H)} = \tilde{0}$. If $\|H - \tilde{\rho}(\widetilde{s_m(H)},\tilde{0})I\|^{-1}$

exists, for all $m \in \mathcal{N}$. Then $\|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\|^{-1}$ exists and bounded, for all $m \in \mathcal{N}$. So, $\lim_{m\longrightarrow\infty}\|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\|^{-1} = \|H\|^{-1}$ exists and bounded. Since $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}, \Xi)$ is a pre-quasi ideal, one obtains

$$I = HH^{-1} \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}(\mathcal{G}, \mathcal{V}) \Rightarrow \left(\widetilde{s_m(I)}\right)_{m=0}^{\infty} \in \Gamma_r^{\mathfrak{S}}(q, \nu) \Rightarrow \lim_{m\longrightarrow\infty} \widetilde{s_m(I)} = \tilde{0}. \tag{51}$$

One has a contradiction, as $\lim_{m\longrightarrow\infty}\widetilde{s_m(I)} = \tilde{1}$. Then, $\|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\| = 0$, for all $m \in \mathcal{N}$. So, $\|H - \tilde{\rho}(\widetilde{\gamma_m(H)}, \tilde{0})I\| = 0$, for all $m \in \mathcal{N}$. Therefore, $H \in (\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau})^{\gamma}(\mathcal{G}, \mathcal{V})$. □

3. Multiplication Mappings on $(\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$

Under the conditions of Theorem 14, we have presented in this section some properties of the multiplication mapping acting on $(\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$.

Let $(\mathrm{Range}(V))^c$ indicate the complement of $\mathrm{Range}(V)$. Let $\mathfrak{I}$ be the space of all sets with a finite number of elements. Assume $\ell_\infty^{\mathfrak{S}}$ is the space of bounded sequences of soft functions.

Definition 33. Suppose $\mathcal{E}_\tau^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{pssf}$ and $\lambda = (\lambda_k) \in \mathfrak{R}^{\mathcal{N}}$ *. The mapping $H_\lambda : \mathcal{E}_\tau^{\mathfrak{S}} \longrightarrow \mathcal{E}_\tau^{\mathfrak{S}}$ is said to be a multiplication mapping on $\mathcal{E}_\tau^{\mathfrak{S}}$, if $H_\lambda\tilde{f} = (\lambda_b\widetilde{f_b}) \in \mathcal{E}_\tau^{\mathfrak{S}}$,* for all $f \in \mathcal{E}_\tau^{\mathfrak{S}}$. The multiplication mapping is called constructed by λ, if $H_\lambda \in \mathbb{D}(\mathcal{E}_\tau^{\mathfrak{S}})$.

Definition 34 (see [35]). A mapping $V \in \mathbb{D}(\mathcal{E})$ is said to be Fredholm if $\dim(\mathrm{Range}(V))^c < \infty$, $\mathrm{Range}(V)$ is closed and $\dim(\ker(V)) < \infty$.

Theorem 35.

(1) $\lambda \in \ell_\infty \Leftrightarrow H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$

(2) $|\lambda_a| = 1$, for every $a \in \mathcal{N}$, if and only if, H_λ is an isometry

(3) $H_\lambda \in \mathcal{A}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \Leftrightarrow (\lambda_a)_{a=0}^{\infty} \in c_0$

(4) $H_\lambda \in \mathcal{K}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \Leftrightarrow (\lambda_b)_{b=0}^{\infty} \in c_0$

(5) $\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \subsetneqq \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$

(6) $0 < \alpha < |\lambda_a| < \eta$, for every $a \in (\ker(\lambda))^c$, if and only if, $\mathrm{Range}(H_\lambda)$ is closed

(7) $0 < \alpha < |\lambda_a| < \eta$, for all $a \in \mathcal{N}$, if and only if, $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$ is invertible

(8) H_λ is Fredholm operator, if and only if (g1) $\ker(\lambda) \subsetneqq \mathcal{N} \cap \mathfrak{I}$ and (g2) $|\lambda_a| \geq \rho$, for all $a \in (\ker(\lambda))^c$

Proof.

(1) Suppose $\lambda \in \ell_\infty$, one has $\nu > 0$ with $|\lambda_a| \leq \nu$, for all $a \in \mathcal{N}$. If $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$, we have

$$\begin{aligned}\tau\left(H_\lambda\tilde{f}\right) = \tau\left(\lambda\tilde{f}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\lambda_z\begin{bmatrix} z+r-1 \\ z\end{bmatrix}q_z\widetilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\ &\leq \sup_l \nu^{\nu_l}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix}q_z\widetilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\ &= \sup_l \nu^{\nu_l}\tau\left(\tilde{f}\right).\end{aligned} \tag{52}$$

Therefore, $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$.

Next, if $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$ and $\lambda \notin \ell_\infty$. One has $x_b \in \mathcal{N}$, for every $b \in \mathcal{N}$ with $\lambda_{x_b} > b$. Then,

$$\begin{aligned}\tau\left(H_\lambda\widetilde{e_{x_b}}\right) = \tau\left(\lambda\widetilde{e_{x_b}}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\lambda_z\begin{bmatrix} z+r-1 \\ z\end{bmatrix}q_z\left(\widetilde{e_{x_b}}\right)_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\ &= \sum_{l=x_b}^{\infty}\left(\frac{\lambda_{(x_b)}\begin{bmatrix} x_b+r-1 \\ x_b\end{bmatrix}q_{x_b}}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} > \sum_{l=x_b}^{\infty}\left(\frac{b\begin{bmatrix} x_b+r-1 \\ x_b\end{bmatrix}q_{x_b}}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} > b^{\nu_0}\tau\left(\widetilde{e_{x_b}}\right).\end{aligned} \tag{53}$$

Hence, $H_\lambda \notin \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$. So, $\lambda \in \ell_\infty$.

(2) Let $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$ and $|\lambda_b| = 1$, for every $b \in \mathcal{N}$. One obtains

$$\begin{aligned}\tau\left(H_\lambda\tilde{f}\right) = \tau\left(\lambda\tilde{f}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix}q_z\lambda_z\widetilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\ &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix}q_z\widetilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} = \tau\left(\tilde{f}\right),\end{aligned} \tag{54}$$

then H_λ is an isometry.

Next, if for some $b=b_0$ that $|\lambda_b|<1$, one has

$$\tau(H_\lambda\widetilde{e_{b_0}})=\tau(\lambda\widetilde{e_{b_0}})=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\lambda_z(\widetilde{e_{b_0}})_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=b_0}^{\infty}\left(\frac{|\lambda_{b_0}|\begin{bmatrix}b_0+r-1\\ b_0\end{bmatrix}q_{b_0}}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}<\sum_{l=b_0}^{\infty}\left(\frac{\begin{bmatrix}b_0+r-1\\ b_0\end{bmatrix}q_{b_0}}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}=\tau(\widetilde{e_{b_0}}). \tag{55}$$

When $|\lambda_{b_0}|>1$, so $\tau(H_\lambda\widetilde{e_{b_0}})>\tau(\widetilde{e_{b_0}})$. Hence, $|\lambda_a|=1$, for every $a\in\mathcal{N}$.

(3) Assume $H_\lambda\in\mathcal{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$, so $H_\lambda\in\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$. If $\lim_{b\longrightarrow\infty}\lambda_b\neq 0$. One has $\rho>0$ with $K_\varrho=\{a\in\mathcal{N}:|\lambda_a|\geq\rho\}\subsetneqq\mathfrak{I}$. Let $\{\alpha_a\}_{a\in\mathcal{N}}\subset K_\rho$. We have $\{\widetilde{e_{\alpha_a}}:\alpha_a\in K_\varrho\}\in\ell_\infty^{\mathfrak{S}}$ be an infinite set in $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. For all $\alpha_a,\alpha_b\in K_\rho$, one gets

$$\tau(H_\lambda\widetilde{e_{\alpha_a}}-H_\lambda\widetilde{e_{\alpha_b}})=\tau(\lambda\widetilde{e_{\alpha_a}}-\lambda\widetilde{e_{\alpha_b}})$$
$$=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\lambda_z\left((\widetilde{e_{\alpha_a}})_z-(\widetilde{e_{\alpha_b}})_z\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\geq\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\rho\left((\widetilde{e_{\alpha_a}})_z-(\widetilde{e_{\alpha_b}})_z\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\geq\inf_l\rho^{v_l}\tau(\widetilde{e_{\alpha_a}}-\widetilde{e_{\alpha_b}}). \tag{56}$$

Hence, $\{\widetilde{e_{\alpha_b}}:\alpha_b\in K_\rho\}\in\ell_\infty^{\mathfrak{S}}$ has not a convergent subsequence under H_λ. So, $H_\lambda\notin\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$. Therefore, $H_\lambda\notin\mathcal{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$; this is a contradiction. So, $\lim_{b\longrightarrow\infty}\lambda_b=0$. Next, let $\lim_{a\longrightarrow\infty}\lambda_a=0$. Hence, for every $\rho>0$, we have $K_\rho=\{b\in\mathcal{N}:|\lambda_b|\geq\rho\}\subset\mathfrak{I}$. Therefore, for all $\rho>0$, one gets $\dim(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_\rho})=\dim(\mathfrak{R}^{K_\rho})<\infty$. So, $H_\lambda\in\mathbb{F}(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_\rho})$. If $\lambda_a\in\mathfrak{R}^{\mathcal{N}}$, for all $a\in\mathcal{N}$, where

$$(\lambda_a)_b=\begin{cases}\lambda_b, & b\in K_{1/a+1},\\ 0, & \text{otherwise.}\end{cases} \tag{57}$$

Obviously, $H_{\lambda_a}\in\mathbb{F}(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_{1/a+1}})$, since $\dim(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_{1/a+1}})<\infty$, for all $a\in\mathcal{N}$. According to $(v_l)\in\uparrow\cap\ell_\infty$ with $v_0>1/r$, we have

$$\tau\left((H_\lambda-H_{\lambda_a})\tilde{f}\right)=\tau\left(\left((\lambda_b-(\lambda_a)_b)\tilde{f}_b\right)_{b=0}^{\infty}\right)$$
$$=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z(\lambda_z-(\lambda_a)_z)\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=0,l\in K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z(\lambda_z-(\lambda_a)_z)\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$+\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z(\lambda_z-(\lambda_a)_z)\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\lambda_z\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\leq\frac{1}{(a+1)^{v_0}}\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$<\frac{1}{(a+1)^{v_0}}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\tilde{f}_z,\tilde{0}\right)}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\frac{1}{(a+1)^{v_0}}\tau(\tilde{f}). \tag{58}$$

Therefore, $\|H_\lambda-H_{\lambda_a}\|\leq 1/(a+1)^{v_0}$. This implies H_λ is a limit of finite rank mappings.

(4) As $\mathcal{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)\subsetneqq\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$, the proof follows

(5) Since $I=I_\lambda$, where $\lambda=(1,1,)$, one has $I\notin\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$ and $I\in\mathbb{D}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$

(6) Let the sufficient setups be verified. One has $\rho>0$ with $|\lambda_a|\geq\rho$, for every $a\in(\ker(\lambda))^c$. We have to show that Range(H_λ) is closed; let $\tilde{g}$ be a limit point of Range(H_λ). One has $H_\lambda\tilde{f}_b\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for all b

$\in \mathcal{N}$ with $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{f_b} = \tilde{g}$. Clearly, $H_\lambda \widetilde{f_b}$ is a Cauchy sequence. Since $(v_l) \in \uparrow \cap \ell_\infty$, we have

$$\begin{aligned}\tau\left(H_\lambda \widetilde{f_a} - H_\lambda \widetilde{f_b}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\lambda_z(\widetilde{f_a})_z - \lambda_z(\widetilde{f_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &= \sum_{l=0, l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\lambda_z(\widetilde{f_a})_z - \lambda_z(\widetilde{f_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &+ \sum_{l=0, l\notin(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\lambda_z(\widetilde{f_a})_z - \lambda_z(\widetilde{f_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &\geq \sum_{l=0, l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\lambda_z(\widetilde{f_a})_z - \lambda_z(\widetilde{f_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\lambda_z(\widetilde{u_a})_z - \lambda_z(\widetilde{u_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &> \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\rho\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left((\widetilde{u_a})_z - (\widetilde{u_b})_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &\geq \inf_l \rho^{v_l}\tau(\widetilde{u_a} - \widetilde{u_b}),\end{aligned} \tag{59}$$

where

$$(\widetilde{u_a})_k = \begin{cases} (\widetilde{f_a})_k, & k \in (\ker(\lambda))^c, \\ 0, & k \notin (\ker(\lambda))^c. \end{cases} \tag{60}$$

Therefore, $\{\widetilde{u_a}\}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. Since $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is complete. One has $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ with $\lim_{b\longrightarrow\infty}\widetilde{u_b} = \tilde{f}$. As $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$, we have $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{u_b} = H_\lambda \tilde{f}$. As $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{u_b} = \lim_{b\longrightarrow\infty} H_\lambda \widetilde{f_b} = \tilde{g}$. So, $H_\lambda \tilde{f} = \tilde{g}$. Then, $\tilde{g} \in \text{Range}(H_\lambda)$, i.e., $\text{Range}(H_\lambda)$ is closed. Next, suppose the necessary condition is satisfied. One has $\rho > 0$ with $\tau(H_\lambda \tilde{f}) \geq \rho\tau(\tilde{f})$ and $\tilde{f} \in ((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{(\ker(\lambda))^c}$. Let $K = \{b \in (\ker(\lambda))^c : |\lambda_b| < \rho\} \neq \varnothing$, then for $a_0 \in K$, we have

$$\begin{aligned}\tau(H_\lambda \widetilde{e_{a_0}}) &= \tau\left(\left(\lambda_b(\widetilde{e_{a_0}})_b\right)_{b=0}^{\infty}\right) \\ &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\lambda_z(\widetilde{e_{a_0}})_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &< \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\rho\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z(\widetilde{e_{a_0}})_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l} \\ &\leq \sup_l \rho^{v_l}\tau(\widetilde{e_{a_0}}),\end{aligned} \tag{61}$$

which introduces a contradiction. So $K = \phi$, we have $|\lambda_a| \geq \rho$, for all $a \in (\ker(\lambda))^c$.

(7) First, assume $\kappa \in \mathfrak{R}^{\mathcal{N}}$ so that $\kappa_a = 1/\lambda_a$. By Theorem 35 part (1), we have $H_\lambda, H_\kappa \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$. One has $H_\lambda \cdot H_\kappa = H_\kappa \cdot H_\lambda = I$. So, $H_\kappa = H_\lambda^{-1}$. Second, if H_λ is invertible. Then $\text{Range}(H_\lambda) = ((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{\mathcal{N}}$. Therefore, $\text{Range}(H_\lambda)$ is closed. From Theorem 35 part (5), one has $\alpha > 0$ with $|\lambda_a| \geq \alpha$, for all $a \in (\ker(\lambda))^c$. Then, $\ker(\lambda) = \varnothing$, when $\lambda_{a_0} = 0$, where $a_0 \in \mathcal{N}$; this implies $e_{a_0} \in \ker(H_\lambda)$, which is a contradiction, since $\ker(H_\lambda)$ is trivial. Then, $|\lambda_a| \geq \alpha$, for all $a \in \mathcal{N}$. As $H_\lambda \in \ell_\infty$. From Theorem 35 part (1), one has $\eta > 0$ with $|\lambda_a| \leq \eta$, for all $a \in \mathcal{N}$. So $\alpha \leq |\lambda_a| \leq \eta$, for all $a \in \mathcal{N}$

(8) First, if $\ker(\lambda) \subsetneqq \mathcal{N}$ and $\ker(\lambda) \notin \mathfrak{I}$, one has $\widetilde{e_a} \in \ker(H_\lambda)$, for all $a \in \ker(\lambda)$. As $\widetilde{e_a}$'s are linearly independent, we have $\dim(\ker(H_\lambda)) = \infty$; this is a contradiction. Therefore, $\ker(\lambda) \subsetneqq \mathcal{N} \in \mathfrak{I}$. The condition (g2) comes from Theorem 35 part (6). Next, assume the setups (g1) and (g2) are satisfied. According to Theorem 35 part (6), the setup (g2) gives that $\text{Range}(H_\lambda)$ is closed. The condition (g1) implies that $\dim((\text{Range}(H_\lambda))^c) < \infty$ and $\dim(\ker(H_\lambda)) < \infty$. Therefore, H_λ is Fredholm

□

4. Fixed Points of Kannan Contraction Type

In this section, we offer the existence of a fixed point of Kannan contraction mapping acting on this new space under the conditions of Theorem 14 and its associated prequasi ideal. Interestingly, several numerical experiments are presented to illustrate our results.

Definition 36. A prequasi normed $\mathfrak{pssft}\tau$ on $\mathscr{E}^{\mathfrak{S}}$ confirms the Fatou property, if for every sequence $\{\widetilde{h^b}\} \subseteq \mathscr{E}_\tau^{\mathfrak{S}}$ so that $\lim_{b\longrightarrow\infty}\tau(\widetilde{h^b} - \tilde{h}) = \tilde{0}$ and every $\tilde{g} \in \mathscr{E}_\tau^{\mathfrak{S}}$, one has $\tau(\tilde{g} - \tilde{h}) \leq \sup_p \inf_{b\geq p}\tau(\tilde{g} - \widetilde{h^b})$.

Throughout the next part of this article, we will use the two functions τ_1 and τ_2 as

$$\begin{aligned}\tau_1(\tilde{f}) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\tilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}, \\ \tau_2(\tilde{f}) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\tilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l},\end{aligned} \tag{62}$$

for all $\tilde{f} \in \Gamma_r^{\mathfrak{S}}(q,v)$.

Theorem 37. *The function τ_1 satisfies the Fatou property.*

Proof. Assume $\{\widetilde{g^b}\} \subseteq (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ so that $\lim_{b\longrightarrow\infty}\tau_1(\widetilde{g^b} - \tilde{g}) = 0$. Clearly, $\tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$. For every $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$, one has

$$\begin{aligned}\tau_1\left(\tilde{f}-\tilde{g}\right) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &\le \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\widetilde{g_z^b}\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &\quad + \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\widetilde{g_z^b}-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &\le \sup_j \inf_{b\ge j}\tau_1\left(\tilde{f}-\widetilde{g^b}\right).\end{aligned} \tag{63}$$

□

Theorem 38. *Suppose $\nu_0 > 1$, then τ_2 does not verify the Fatou property.*

Proof. If $\{\widetilde{g^b}\} \subseteq (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$ so that $\lim_{b\longrightarrow\infty}\tau_2(\widetilde{g^b} - \tilde{g}) = 0$. Clearly, $\tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$. For every $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$, one has

$$\begin{aligned}\tau_2\left(\tilde{f}-\tilde{g}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\ &\le 2^{\hbar-1}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\widetilde{g_z^b}\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right. \\ &\quad \left. + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\widetilde{g_z^b}-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right] \\ &\le 2^{\hbar-1}\sup_j \inf_{b\ge j}\tau_2\left(\tilde{f}-\widetilde{g^b}\right).\end{aligned} \tag{64}$$

□

Hence, τ_2 does not satisfy the Fatou property.

Definition 39 (see [30]). A mapping $G : \mathscr{E}_\tau^{\mathfrak{S}} \longrightarrow \mathscr{E}_\tau^{\mathfrak{S}}$ is called a Kannan τ-contraction, if one has $\zeta \in [0, 1/2)$, with $\tau(G\tilde{g} - G\tilde{h}) \le \zeta(\tau(G\tilde{g}-\tilde{g}) + \tau(G\tilde{h}-\tilde{h}))$, for all $\tilde{g},\tilde{h} \in \mathscr{E}_\tau^{\mathfrak{S}}$. When $G(\tilde{g}) = \tilde{g}$, then $\tilde{g} \in \mathscr{E}_\tau^{\mathfrak{S}}$ is called a fixed point of G.

Theorem 40. *Suppose $G : (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is Kannan τ_1-contraction operator, then G has a unique fixed point.*

Proof. If $\tilde{h} \in \Gamma_r^{\mathfrak{S}}(q,\nu)$, one has $G^m\tilde{h} \in \Gamma_r^{\mathfrak{S}}(q,\nu)$. As G is a Kannan τ_1-contraction, one has

$$\begin{aligned}&\tau_1\left(G^{m+1}\tilde{h}-G^m\tilde{h}\right) \le \zeta\left(\tau_1\left(G^{m+1}\tilde{h}-G^m\tilde{h}\right)+\tau_1\left(G^m\tilde{h}-G^{m-1}\tilde{h}\right)\right) \\ &\quad \Rightarrow \tau_1\left(G^{m+1}\tilde{h}-G^m\tilde{h}\right) \le \frac{\zeta}{1-\zeta}\tau_1\left(G^m\tilde{h}-G^{m-1}\tilde{h}\right) \\ &\quad \le \left(\frac{\zeta}{1-\zeta}\right)^2\tau_1\left(G^{m-1}\tilde{h}-G^{m-2}\tilde{h}\right) \\ &\quad \le \le \left(\frac{\zeta}{1-\zeta}\right)^m\tau_1\left(G\tilde{h}-\tilde{h}\right).\end{aligned} \tag{65}$$

We get for all $m, n \in \mathscr{N}$ so that $n > m$ that

$$\begin{aligned}\tau_1\left(G^m\tilde{h}-G^n\tilde{h}\right) &\le \zeta\left(\tau_1\left(G^m\tilde{h}-G^{m-1}\tilde{h}\right)+\tau_1\left(G^n\tilde{h}-G^{n-1}\tilde{h}\right)\right) \\ &\le \zeta\left(\left(\frac{\zeta}{1-\zeta}\right)^{m-1}+\left(\frac{\zeta}{1-\zeta}\right)^{n-1}\right)\tau_1\left(G\tilde{h}-\tilde{h}\right).\end{aligned} \tag{66}$$

□

Therefore, $\{G^m\tilde{h}\}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$. As $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is prequasi Banach space. One has $\tilde{J} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ with $\lim_{m\longrightarrow\infty}G^m\tilde{h} = \tilde{J}$. To show that $G(\tilde{J}) = \tilde{J}$. Since τ_1 satisfies the Fatou property, one can see

$$\begin{aligned}\tau_1\left(G\tilde{J}-\tilde{J}\right) &\le \sup_i \inf_{m\ge i}\tau_1\left(G^{m+1}\tilde{h}-G^m\tilde{h}\right) \\ &\le \sup_i \inf_{m\ge i}\left(\frac{\zeta}{1-\zeta}\right)^m\tau_1\left(G\tilde{h}-\tilde{h}\right) = 0,\end{aligned} \tag{67}$$

then $G(\tilde{J}) = \tilde{J}$. Therefore, $\tilde{J}$ is a fixed point of G. To indicate the uniqueness of the fixed point. Let us have two different fixed points $\tilde{f}, \tilde{J} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ of G. We have

$$\tau_1\left(\tilde{f}-\tilde{J}\right) \le \tau_1\left(G\tilde{f}-G\tilde{J}\right) \le \zeta\left(\tau_1\left(G\tilde{f}-\tilde{f}\right)+\tau_1\left(G\tilde{J}-\tilde{J}\right)\right) = 0. \tag{68}$$

Therefore, $\tilde{f} = \tilde{J}$.

□

Corollary 41. *If $G : (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$ is Kannan τ_1-contraction, then G has a unique fixed point $\tilde{J}$ so that $\tau_1(G^m\tilde{h} - \tilde{J}) \le \zeta(\zeta/1-\zeta)^{m-1}\tau_1(G\tilde{h} - \tilde{h})$.*

Proof. By Theorem 40, one has a unique fixed point $\tilde{J}$ of G. Hence,

$$\begin{aligned}\tau_1\left(G^m\tilde{h} - \tilde{J}\right) &= \tau_1\left(G^m\tilde{h} - G\tilde{J}\right) \\ &\le \zeta\left(\tau_1\left(G^m\tilde{h} - G^{m-1}\tilde{h}\right) + \tau_1\left(G\tilde{J} - \tilde{J}\right)\right) \\ &= \zeta\left(\frac{\zeta}{1-\zeta}\right)^{m-1}\tau_1\left(G\tilde{h} - \tilde{h}\right).\end{aligned} \tag{69}$$

□

Definition 42. If $\mathscr{E}_\tau^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{psssf}$, $G : \mathscr{E}_\tau^{\mathfrak{S}} \longrightarrow \mathscr{E}_\tau^{\mathfrak{S}}$ and $\tilde{j} \in \mathscr{E}_\tau^{\mathfrak{S}}$. The mapping G is called τ-sequentially continuous at $\tilde{j}$, if and only if, when $\lim_{i\longrightarrow\infty}\tau(\tilde{g}_i - \tilde{j}) = 0$, then $\lim_{i\longrightarrow\infty}\tau(G\tilde{g}_i - G\tilde{j}) = 0$.

Theorem 43. *If $v_0 > 1$, and $G : (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$. The element $\tilde{h} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ is the unique fixed point of G, when the following conditions are confirmed:*

(i) *G is Kannan τ_2-contraction*

(ii) *G is τ_2-sequentially continuous at $\tilde{h} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$*

(iii) *One has $\tilde{j} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ with $\{G^m\tilde{j}\}$ has $\{G^{m_i}\tilde{j}\}$ converges to $\tilde{h}$*

Proof. Assume $\tilde{h}$ is not a fixed point of G, one has $G\tilde{h} \ne \tilde{h}$. According to conditions (ii) and (iii), we have

$$\begin{aligned}&\lim_{m_i\longrightarrow\infty}\tau_2\left(G^{m_i}\tilde{j} - \tilde{h}\right) = 0, \\ &\lim_{m_i\longrightarrow\infty}\tau_2\left(G^{m_i+1}\tilde{j} - G\tilde{h}\right) = 0.\end{aligned} \tag{70}$$

As G is Kannan τ_2-contraction, one has

$$\begin{aligned}0 < \tau_2\left(G\tilde{h} - \tilde{h}\right) &= \tau_2\left(\left(G\tilde{h} - G^{m_i+1}\tilde{j}\right) + \left(G^{m_i}\tilde{j} - \tilde{h}\right)\right. \\ &\quad \left.+ \left(G^{m_i+1}\tilde{j} - G^{m_i}\tilde{j}\right)\right) \le 2^{2\hbar-2}\tau_2 \\ &\quad \cdot \left(G^{m_i+1}\tilde{j} - G\tilde{h}\right) + 2^{2\hbar-2}\tau_2\left(G^{m_i}\tilde{j} - \tilde{h}\right) \\ &\quad + 2^{\hbar-1}\zeta\left(\frac{\zeta}{1-\zeta}\right)^{m_i-1}\tau_2\left(G\tilde{j} - \tilde{j}\right).\end{aligned} \tag{71}$$

Take $m_i \longrightarrow \infty$, one obtains a contradiction. Therefore, $\tilde{h}$ is a fixed point of G. To explain the uniqueness of $\tilde{h}$. Suppose we have two different fixed points $\tilde{h}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ of G. Then

$$\tau_2\left(\tilde{h} - \tilde{g}\right) \le \tau_2\left(G\tilde{h} - G\tilde{g}\right) \le \zeta\left(\tau_2\left(G\tilde{h} - \tilde{h}\right) + \tau_2\left(G\tilde{g} - \tilde{g}\right)\right) = 0. \tag{72}$$

So $\tilde{h} = \tilde{g}$. □

Example 44. If $T : (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^\infty, (2l+3/l+2)_{l=0}^\infty))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^\infty, (2l+3/l+2)_{l=0}^\infty))_{\tau_1}$ and

$$T\left(\tilde{f}\right) = \begin{cases}\dfrac{\tilde{f}}{4}, & \tau_1\left(\tilde{f}\right) \in [0,1), \\ \dfrac{\tilde{f}}{5}, & \tau_1\left(\tilde{f}\right) \in [1,\infty).\end{cases} \tag{73}$$

For all $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^\infty, (2l+3/l+2)_{l=0}^\infty))_{\tau_1}$. If $\tau_1(\tilde{f}), \tau_1(\tilde{g}) \in [0,1)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right) + \tau_1\left(\frac{3\tilde{g}}{4}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1\left(T\tilde{g} - \tilde{g}\right)\right).\end{aligned} \tag{74}$$

For every $\tau_1(\tilde{f}), \tau_1(\tilde{g}) \in [1,\infty)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{\tilde{f}}{5} - \frac{\tilde{g}}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(\tau_1\left(\frac{4\tilde{f}}{5}\right) + \tau_1\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{64}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1\left(T\tilde{g} - \tilde{g}\right)\right).\end{aligned} \tag{75}$$

For every $\tau_1(\tilde{f}) \in [0,1)$ and $\tau_1(\tilde{g}) \in [1,\infty)$, one has

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{5}\right) \le \frac{1}{\sqrt[4]{27}}\tau_1\left(\frac{3\tilde{f}}{4}\right) + \frac{1}{\sqrt[4]{64}}\tau_1 \\ &\quad \cdot \left(\frac{4\tilde{g}}{5}\right) \le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right) + \tau_1\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1\left(T\tilde{g} - \tilde{g}\right)\right).\end{aligned} \tag{76}$$

Hence, T is Kannan τ_1-contraction, as τ_1 satisfies the Fatou property. By Theorem 40, T has a unique fixed point $\tilde{\theta}$. Assume

$$\left\{\widetilde{h^{(k)}}\right\} \subseteq \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+5)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}, \tag{77}$$

so that $\lim_{k\longrightarrow\infty}\tau_1(\widetilde{h^{(k)}} - \widetilde{h^{(0)}}) = 0$, where

$$\widetilde{h^{(0)}} \in \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+5)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}, \tag{78}$$

such that $\tau_1(\widetilde{h^{(0)}}) = 1$. As τ_1 is continuous, one has

$$\begin{aligned} \lim_{k\longrightarrow\infty}\tau_1\left(T\widetilde{h^{(k)}} - T\widetilde{h^{(0)}}\right) &= \lim_{k\longrightarrow\infty}\tau_1\left(\frac{\widetilde{h^{(k)}}}{4} - \frac{\widetilde{h^{(0)}}}{5}\right) \\ &= \tau_1\left(\frac{\widetilde{h^{(0)}}}{20}\right) > 0. \end{aligned} \tag{79}$$

So T is not τ_1-sequentially continuous at $\widetilde{h^{(0)}}$. This implies T is not continuous at $\widetilde{h^{(0)}}$.

For every $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2}$. If $\tau_2(\tilde{f}), \tau_2(\tilde{g}) \in [0,1)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{4}\right) \le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{3\tilde{g}}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{80}$$

Let $\tau_2(\tilde{f}), \tau_2(\tilde{g}) \in [1,\infty)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{5} - \frac{\tilde{g}}{5}\right) \le \frac{1}{4}\left(\tau_2\left(\frac{4\tilde{f}}{5}\right) + \tau_2\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{1}{4}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{81}$$

For every $\tau_2(\tilde{f}) \in [0,1)$ and $\tau_2(\tilde{g}) \in [1,\infty)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{5}\right) \le \frac{2}{\sqrt{27}}\tau_2\left(\frac{3\tilde{f}}{4}\right) + \frac{1}{4}\tau_2\left(\frac{4\tilde{g}}{5}\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{82}$$

Hence, T is Kannan τ_2-contraction and

$$T^m\left(\tilde{f}\right) = \begin{cases} \dfrac{\tilde{f}}{4^m}, & \tau_2\left(\tilde{f}\right) \in [0,1), \\ \dfrac{\tilde{f}}{5^m}, & \tau_2\left(\tilde{f}\right) \in [1,\infty). \end{cases} \tag{83}$$

Evidently, T is τ_2-sequentially continuous at $\widetilde{\theta}$ and $\{T^m\tilde{f}\}$ has a subsequence $\{T^{m_j}\tilde{f}\}$ converges to $\widetilde{\theta}$. According to Theorem 43, the element $\widetilde{\theta}$ is the only fixed point of T.

Example 45. Let $T : (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2} \longrightarrow (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2}$ and

$$T\left(\tilde{f}\right) = \begin{cases} \dfrac{1}{4}\left(\tilde{e_1} + \tilde{f}\right), & \tilde{f_0}(t) \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{3}\tilde{e_1}, & \tilde{f_0}(t) = \dfrac{1}{3}, \\ \dfrac{1}{4}\tilde{e_1}, & \tilde{f_0}(t) \in \left(\dfrac{1}{3}, 1\right]. \end{cases} \tag{84}$$

As $\tilde{f_0}(t), \tilde{g_0}(t) \in [0,1/3)$, we get

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{1}{4}\left(\tilde{f_0} - \tilde{g_0}, \tilde{f_1} - \tilde{g_1}, \tilde{f_2} - \tilde{g_2}, \cdots\right)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{3\tilde{g}}{4}\right)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{85}$$

For all $\tilde{f_0}(t), \tilde{g_0}(t) \in (1/3, 1]$, hence for all $\varepsilon > 0$, we have

$$\tau_2\left(T\tilde{f} - T\tilde{g}\right) = 0 \le \varepsilon\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \tag{86}$$

For all $\widetilde{f_0}(t) \in [0, 1/3)$ and $\widetilde{g_0}(t) \in (1/3, 1]$, one has

$$\begin{aligned}\tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{4}\right) \le \frac{1}{\sqrt{27}}\tau_2\left(\frac{3\tilde{f}}{4}\right) = \frac{1}{\sqrt{27}}\tau_2\left(T\tilde{f} - \tilde{f}\right) \\ &\le \frac{1}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right).\end{aligned} \tag{87}$$

Hence, T is Kannan τ_2-contraction. Obviously, T is τ_2-sequentially continuous at $1/3\widetilde{e_1}$, and there is $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^{\infty}, (2l + 3/l + 2)_{l=0}^{\infty}))_{\tau_2}$ with $\widetilde{f_0}(t) \in [0, 1/3)$ such that the sequence of iterates $\{T^m \tilde{f}\} = \{\sum_{a=1}^{m} 1/4^a \widetilde{e_1} + 1/4^m \tilde{f}\}$ includes a subsequence $\{T^{m_j}\tilde{f}\} = \{\sum_{a=1}^{m_j} 1/4^a \widetilde{e_1} + 1/4^{m_j}\tilde{f}\}$ converges to $1/3\widetilde{e_1}$. In view of Theorem 43, the operator T has one fixed point $1/3\widetilde{e_1}$. Note that T is not continuous at $1/3\widetilde{e_1}$.

For all $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^{\infty}, (2l + 3/l + 2)_{l=0}^{\infty}))_{\tau_1}$. If $\widetilde{f_0}(t), \widetilde{g_0}(t) \in [0, 1/3)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{1}{4}\left(\widetilde{f_0} - \widetilde{g_0}, \widetilde{f_1} - \widetilde{g_1}, \widetilde{f_2} - \widetilde{g_2}, \cdots\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right) + \tau_1\left(\frac{3\tilde{g}}{4}\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right).\end{aligned} \tag{88}$$

For all $\widetilde{f_0}(t), \widetilde{g_0}(t) \in (1/3, 1]$, hence for all $\varepsilon > 0$, one has

$$\tau_1\left(T\tilde{f} - T\tilde{g}\right) = 0 \le \varepsilon\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right). \tag{89}$$

For all $\widetilde{f_0}(t) \in [0, 1/3)$ and $\widetilde{g_0}(t) \in (1/3, 1]$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{\tilde{f}}{4}\right) \le \frac{1}{\sqrt[4]{27}}\tau_1\left(\frac{3\tilde{f}}{4}\right) = \frac{1}{\sqrt[4]{27}}\tau_1\left(T\tilde{f} - \tilde{f}\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right).\end{aligned} \tag{90}$$

Therefore, the operator T is Kannan τ_1-contraction. Since τ_1 confirms the Fatou property. By Theorem 40, the operator T has a unique fixed point $1/3\widetilde{e_1}$.

In this part, we will use

$$\Xi(V) = \tau\left(\left(s_b\widetilde{(V)}\right)_{b=0}^{\infty}\right) = \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_z\widetilde{(V)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}, \tag{91}$$

for every $V \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$.

Definition 46. A function Ξ on $\widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}$ satisfies the Fatou property if for all $\{V_b\}_{b\in\mathcal{N}} \subseteq \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ so that $\lim_{b\longrightarrow\infty}\Xi(V_b - V) = 0$ and all $T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$, one has $\Xi(T - V) \le \sup_b \inf_{j\ge b}\Xi(T - V_j)$.

Theorem 47. *The function Ξ does not verify the Fatou property.*

Proof. Assume $\{W_m\}_{m\in\mathcal{N}} \subseteq \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$ so that $\lim_{m\longrightarrow\infty}\Xi(W_m - W) = 0$. Clearly, $W \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$. Hence, for every $V \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$, we have

$$\begin{aligned}\Xi(V - W) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_z(\widetilde{V - W}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\le \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_{[z/2]}(\widetilde{V - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\quad + \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_{[z/2]}(\widetilde{W - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\le \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{1/\hbar}\sup_m \inf_{i\ge m} \\ &\quad \cdot \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z s_z(\widetilde{V - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}.\end{aligned} \tag{92}$$

□

Therefore, Ξ does not satisfy the Fatou property.

Definition 48 (see [30]). A mapping $W : \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M) \longrightarrow \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ is said to be a Kannan Ξ-contraction, assume there is $\zeta \in [0, 1/2)$ with $\Xi(WV - WT) \le \zeta(\Xi(WV - V) + \Xi(WT - T))$, for all $V, T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$.

Definition 49. Assume $G : \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M) \longrightarrow \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ and $B \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$. The mapping G is called Ξ-sequentially continuous at B, if and only if, when $\lim_{m\longrightarrow\infty}\Xi(W_m - B) = 0$, one has $\lim_{m\longrightarrow\infty}\Xi(GW_m - GB) = 0$.

Theorem 50. *If $G : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$. The operator $A \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$ is the only fixed point of G, when the following conditions are confirmed:*

(i) G is Kannan Ξ-contraction

(ii) G is Ξ-sequentially continuous at $A \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$

(iii) One has $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$ *with* $\{G^m B\}$ *has* $\{G^{m_i} B\}$ *converges to A*

Proof. Suppose A is not a fixed point of G, then $GA \neq A$. By conditions (ii) and (iii), one has

$$\lim_{m_i \longrightarrow \infty} \Xi(G^{m_i}B - A) = 0, \quad \lim_{m_i \longrightarrow \infty} \Xi\left(G^{m_i+1}B - GA\right) = 0. \tag{93}$$

As G is Kannan Ξ-contraction operator, we get

$$\begin{aligned} 0 < \Xi(GA - A) &= \Xi\left(\left(GA - G^{m_i+1}B\right) + \left(G^{m_i}B - A\right) + \left(G^{m_i+1}B - G^{m_i}B\right)\right) \\ &\leq \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{1/\hbar} \Xi\left(G^{m_i+1}B - GA\right) \\ &\quad + \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{2/\hbar} \Xi\left(G^{m_i}B - A\right) \\ &\quad + \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{2/\hbar} \zeta \left(\frac{\zeta}{1-\zeta}\right)^{m_i-1} \Xi(GB - B). \end{aligned} \tag{94}$$

By $m_i \longrightarrow \infty$, we have a contradiction. Then, A is a fixed point of G. To show the uniqueness of the fixed point A. If one has two different fixed points $A, D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$ of G. So

$$\Xi(A - D) \leq \Xi(GA - GD) \leq \zeta(\Xi(GA - A) + \Xi(GD - D)) = 0. \tag{95}$$

Therefore, $A = D$. □

Example 51. Assume

$$M : S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}) \longrightarrow S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}), \tag{96}$$

$$M(H) = \begin{cases} \dfrac{H}{6}, & \Xi(H) \in [0,1), \\ \dfrac{H}{7}, & \Xi(H) \in [1,\infty). \end{cases} \tag{97}$$

For all

$$H_1, H_2 \in S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_\tau}. \tag{98}$$

If $\Xi(H_1), \Xi(H_2) \in [0,1)$, we have

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{6} - \frac{H_2}{6}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Xi\left(\frac{5H_1}{6}\right) + \Xi\left(\frac{5H_2}{6}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{99}$$

Suppose $\Xi(H_1), \Xi(H_2) \in [1,\infty)$, one has

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{7} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Xi\left(\frac{6H_1}{7}\right) + \Xi\left(\frac{6H_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{100}$$

Assume $\Xi(H_1) \in [0,1)$ and $\Xi(H_2) \in [1,\infty)$, one gets

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{6} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}} \Xi\left(\frac{5H_1}{6}\right) \\ &\quad + \frac{\sqrt{2}}{\sqrt[4]{216}} \Xi\left(\frac{6H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}} \\ &\quad \cdot (\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{101}$$

Hence, M is Kannan Ξ-contraction and

$$M^m(H) = \begin{cases} \dfrac{H}{6^m}, & \Xi(H) \in [0,1), \\ \dfrac{H}{7^m}, & \Xi(H) \in [1,\infty). \end{cases} \tag{102}$$

Evidently, M is Ξ-sequentially continuous at the zero operator Θ and $\{M^m H\}$ has a subsequence $\{M^{m_j} H\}$ converges to Θ. According to Theorem 50, the zero operator is the only fixed point of M.

If

$$\left\{H^{(a)}\right\} \subseteq S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_\tau}, \tag{103}$$

with $\lim_{a\longrightarrow\infty}\Xi(H^{(a)} - H^{(0)}) = 0$, where

$$H^{(0)} \in S\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_{\tau}, \tag{104}$$

so that $\Xi(H^{(0)}) = 1$. As Ξ is continuous, one has

$$\begin{aligned}\lim_{a\longrightarrow\infty} \Xi\left(MH^{(a)} - MH^{(0)}\right) &= \lim_{a\longrightarrow\infty} \Xi\left(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\right) \\ &= \Xi\left(\frac{H^{(0)}}{42}\right) > 0.\end{aligned} \tag{105}$$

Therefore, M is not Ξ-sequentially continuous at $H^{(0)}$. This implies M is not continuous at $H^{(0)}$.

5. Applications on Stochastic Nonlinear Dynamical System

We investigate in this section a solution in $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ to stochastic nonlinear dynamical system (106) under the conditions of Theorem 14. For every $\tilde{f} \in \Gamma_r^{\mathfrak{S}}(q,\nu)$.

Consider the stochastic nonlinear dynamical system [36]:

$$\widetilde{f}_z = \widetilde{y}_z + \sum_{m=0}^{\infty} \Pi(z,m) g\left(m, \widetilde{f_m}\right), \tag{106}$$

and assume $W : (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is constructed by

$$W\left(\widetilde{f}_z\right)_{z\in\mathcal{N}} = \left(\widetilde{y}_z + \sum_{m=0}^{\infty} \Pi(z,m) g\left(m, \widetilde{f_m}\right)\right)_{z\in\mathcal{N}}. \tag{107}$$

Theorem 52. *The stochastic nonlinear dynamical system (106) has one and only one solution in* $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$, *if* $\Pi : \mathcal{N}^2 \longrightarrow \mathfrak{R}, g : \mathcal{N} \times \mathscr{R}(A) \longrightarrow \mathscr{R}(A), \tilde{f} : \mathcal{N} \longrightarrow \mathscr{R}(A)$, $\tilde{y} : \mathcal{N} \longrightarrow \mathscr{R}(A), \tilde{\eta} : \mathcal{N} \longrightarrow \mathscr{R}(A)$, *one has* $\lambda \in \mathfrak{R}$ *with* $\sup_l |\lambda|^{\nu_l/\hbar} \in [0, 1/2)$ *and for every* $l \in \mathcal{N}$, *one obtains*

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathcal{N}} \Pi(z,m)\left[g\left(m,\widetilde{f_m}\right) - g(m,\widetilde{\eta_m})\right]\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\ &\quad \tilde{\le} |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y}_z - \widetilde{f}_z + \sum_{m=0}^{\infty}\Pi(z,m)g\left(m,\widetilde{f_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\ &\quad + |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y}_z - \widetilde{\eta}_z + \sum_{m=0}^{\infty}\Pi(z,m)g(m,\widetilde{\eta_m})\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right|.\end{aligned} \tag{108}$$

Proof. Let the conditions be established. Assume the mapping $W : (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is defined by equation (11). Hence,

$$\begin{aligned}\tau_1\left(W\tilde{f} - W\tilde{\eta}\right) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(W\widetilde{f}_z - W\widetilde{\eta_z}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\sum_{m\in\mathcal{N}}\Pi(z,m)\left[g\left(m,\widetilde{f_m}\right) - g(m,\widetilde{\eta_m})\right]\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &\le \sup_l |\lambda|^{\nu_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\widetilde{y}_z - \widetilde{f}_z + \sum_{m=0}^{\infty}\Pi(z,m)g\left(m,\widetilde{f_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &\quad + \sup_l |\lambda|^{\nu_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}(\widetilde{y}_z - \widetilde{\eta}_z + \sum_{m=0}^{\infty}\Pi(z,m)g(m,\widetilde{\eta_m}))\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\ &= \sup_l |\lambda|^{\nu_l/\hbar}\left(\tau_1\left(W\tilde{f} - \tilde{f}\right) + \tau_1(W\tilde{\eta} - \tilde{\eta})\right).\end{aligned} \tag{109}$$

□

From Theorem 40, one has one and only one solution of (106) in $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$.

Example 53. Consider

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}. \tag{110}$$

Suppose the stochastic nonlinear dynamical system:

$$\widetilde{f}_z = e^{-\widetilde{(3z+6)}} + \sum_{m=0}^{\infty}(-1)^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d} + m^2 + 1}}, \tag{111}$$

with $b, d, \widetilde{f_{-2}}(t), \widetilde{f_{-1}}(t) > 0$, for all $t \in A$ and suppose

$$\begin{aligned}W : &\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1} \\ &\longrightarrow \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}\end{aligned} \tag{112}$$

is defined by

$$W\left(\widetilde{f_z}\right)_{z=0}^{\infty} = \left(\widetilde{e^{-(3z+6)}} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2+1}}\right)_{z=0}^{\infty}. \tag{113}$$

Evidently, one has $\lambda \in \Re$ with $\sup_l |\lambda|^{2l+3/2l+4} \in [0, 1/2)$ and for every $l \in \mathcal{N}$, we have

$$\begin{aligned}
&\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty}(-1)^z \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2+1}}\left((-1)^m - (-1)^m\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&\tilde{\leq} |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{e^{-(3z+6)}} - f_z + \sum_{m=0}^{\infty}(-1)^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2+1}}\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&+ |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{e^{-(3z+6)}} - \widetilde{\eta_z} + \sum_{m=0}^{\infty}(-1)^{z+m} \frac{\widetilde{\eta_{z-2}^b}}{\widetilde{\eta_{z-1}^d} + \widetilde{m^2+1}}\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right|.
\end{aligned} \tag{114}$$

From Theorem 52, system (111) has one and only one solution in

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}. \tag{115}$$

Example 54. Suppose the sequence space

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}. \tag{116}$$

Assume the stochastic nonlinear dynamical system:

$$\widetilde{f_z} = \widetilde{y_z} + \sum_{m=0}^{\infty} e^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{f_{z-1}^b} + \tilde{2}}, \tag{117}$$

with $b, d, \widetilde{f_{-2}}(t), \widetilde{f_{-1}}(t) > 0$, for all $t \in A$ and suppose

$$\begin{aligned}
W : &\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1} \\
&\longrightarrow \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}
\end{aligned} \tag{118}$$

is defined by

$$W\left(\widetilde{f_z}\right)_{z=0}^{\infty} = \left(\widetilde{y_z} + \sum_{m=0}^{\infty} e^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{f_{z-1}^b} + \tilde{2}}\right)_{z=0}^{\infty}. \tag{119}$$

Evidently, there is $\lambda \in \Re$ such that $\sup_l |\lambda|^{2l+3/2l+4} \in [0, 1/2)$ and for every $l \in \mathcal{N}$, we have

$$\begin{aligned}
&\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty} e^z \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{f_{z-1}^b} + \tilde{2}}\left(e^m - e^m\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&\tilde{\leq} |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - f_z + \sum_{m=0}^{\infty} e^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{f_{z-1}^b} + \tilde{2}}\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&+ |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} e^{z+m} \frac{\widetilde{\eta_{z-2}^b}}{\widetilde{\eta_{z-1}^d} + \widetilde{\eta_{z-1}^b} + \tilde{2}}\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right|.
\end{aligned} \tag{120}$$

According to Theorem 52, the stochastic nonlinear dynamical system (14) contains a unique solution in

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}. \tag{121}$$

Theorem 55. *If $W : (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2}$ is defined by (11) and $v_0 > 1$. The stochastic nonlinear dynamical system (106) has a unique solution $\tilde{Z} \in (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2}$, when the following conditions are satisfied:*

(1) If $\Pi : \mathcal{N}^2 \longrightarrow \Re, g : \mathcal{N} \times \mathcal{R}(A) \longrightarrow \mathcal{R}(A)$, $\tilde{f} : \mathcal{N} \longrightarrow \mathcal{R}(A), \tilde{y} : \mathcal{N} \longrightarrow \mathcal{R}(A), \tilde{\eta} : \mathcal{N} \longrightarrow \mathcal{R}(A)$, assume there is $\lambda \in \Re$ so that $2^{\hbar-1} \sup_l |\lambda|^{v_l} \in [0, 1/2)$ and for every $l \in \mathcal{N}$, one has

$$\begin{aligned}
&\left|\sum_{z=0}^{l}\left(\sum_{m \in \mathcal{N}} \Pi(z, m)\left[g\left(m, \widetilde{f_m}\right) - g\left(m, \widetilde{\eta_m}\right)\right]\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&\tilde{\leq} |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{f_z} + \sum_{m=0}^{\infty} \Pi(z, m) g\left(m, \widetilde{f_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\
&+ |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} \Pi(z, m) g\left(m, \widetilde{\eta_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right|
\end{aligned} \tag{122}$$

(2) W is τ_2-sequentially continuous at $\tilde{Z} \in (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2}$

(3) *There is* $\tilde{Y} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_{\tau_2}$ *with* $\{W^m \tilde{Y}\}$ *has* $\{W^{m_j} \tilde{Y}\}$ *converging to* $\tilde{Z}$

Proof. One has

$$\begin{aligned}
\tau_2\left(W\tilde{f} - W\tilde{\eta}\right) &= \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} r+z-1 \\ z \end{bmatrix} q_z \left(W\widetilde{f_z} - W\widetilde{\eta_z} \right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} \\
&= \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \left(\sum_{m \in \mathcal{N}} \Pi(z, m) \left[g\left(m, \widetilde{f_m}\right) - g(m, \widetilde{\eta_m}) \right] \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} \\
&\le 2^{\hbar-1} \sup_l |\lambda|^{\nu_l} \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \left(\widetilde{y_z} - \widetilde{f_z} + \sum_{m=0}^{\infty} \Pi(z, m) g\left(m, \widetilde{f_m}\right) \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} \\
&\quad + 2^{\hbar-1} \sup_l |\lambda|^{\nu_l} \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} (\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} \Pi(z, m) g(m, \widetilde{\eta_m})) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} \\
&= 2^{\hbar-1} \sup_l |\lambda|^{\nu_l} \left(\tau_2\left(W\tilde{f} - \tilde{f} \right) + \tau_2(W\tilde{\eta} - \tilde{\eta}) \right).
\end{aligned} \tag{123}$$

□

By Theorem 43, one gets a unique solution $\tilde{Z} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_{\tau_2}$ of equation (106).

Example 56. Suppose the sequence space

$$\left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}. \tag{124}$$

Consider the summable equation (111):
Let

$$\begin{aligned}
W : &\left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2} \\
&\longrightarrow \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}
\end{aligned} \tag{125}$$

defined by (13). Assume W is τ_2-sequentially continuous at

$$\tilde{Z} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}, \tag{126}$$

and there is

$$\tilde{Y} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2} \tag{127}$$

with $\{W^m \tilde{Y}\}$ has $\{W^{m_j} \tilde{Y}\}$ converging to $\tilde{Z}$. Evidently, there is $\lambda \in \mathfrak{R}$ such that $2^{\hbar-1} \sup_l |\lambda|^{2l+3/l+2} \in [0, 1/2)$ and for all $l \in \mathcal{N}$, one has

$$\begin{aligned}
&\left| \sum_{z=0}^{l} \left(\sum_{m=0}^{\infty} (-1)^z \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2 + 1}} ((-1)^m - (-1)^m) \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right| \\
&\tilde{\le} |\lambda| \left| \sum_{z=0}^{l} \left(\widetilde{e^{-(3z+6)}} - f_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2 + 1}} \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right| \\
&\quad + |\lambda| \left| \sum_{z=0}^{l} \left(\widetilde{e^{-(3z+6)}} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{\eta_{z-2}^b}}{\widetilde{\eta_{z-1}^d} + \widetilde{m^2 + 1}} \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right|.
\end{aligned} \tag{128}$$

By Theorem 57, the stochastic nonlinear dynamical system (111) has one and only one solution

$$\tilde{Z} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}. \tag{129}$$

In this part, we search for a solution to nonlinear matrix equation (131) at $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}(\mathcal{G}, \mathcal{V})$, the conditions of Theorem 14 are satisfied, and

$$\Xi(G) = \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} r+z-1 \\ z \end{bmatrix} q_z \widetilde{s_z(G)}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} \right]^{1/\hbar}, \tag{130}$$

for every $G \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}(\mathcal{G}, \mathcal{V})$. Consider the stochastic nonlinear dynamical system:

$$\widetilde{s_z(G)} = \widetilde{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right), \tag{131}$$

and suppose $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$ is defined by

$$W(G) = \left(\widetilde{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right) \right) I. \tag{132}$$

Theorem 57. *The stochastic nonlinear dynamical system (131) has one and only one solution $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$, if the following conditions are satisfied:*

(1) $\Pi : \mathcal{N}^2 \longrightarrow \mathfrak{R}, f : \mathcal{N} \times \mathcal{R}(A) \longrightarrow \mathcal{R}(A)$, $P \in \mathbb{D}(\mathcal{G},\mathcal{V}), T \in \mathbb{D}(\mathcal{G},\mathcal{V})$, and for every $z \in \mathcal{N}$, there is a positive real number κ so that $\sup_z \kappa^{t_z/\hbar} \in [0, 0.5)$, with

$$\begin{aligned} &\left| \sum_{m\in\mathcal{N}} \Pi(z,m) \left(f\left(m, \widetilde{s_m(G)}\right) - f\left(m, s_m(T)^{\sim}\right)\right) \right| \\ &\quad \tilde{\le} \kappa \left[\left| \widetilde{s_z(P)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} A(a,m) f\left(m, \widetilde{s_m(G)}\right) \right| \right. \\ &\quad \left. + \left| \widetilde{s_z(P)} - \widetilde{s_z(T)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(T)}\right) \right| \right] \end{aligned} \tag{133}$$

(2) W is Ξ-sequentially continuous at a point $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$

(3) There is $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D

Proof. Suppose the settings are verified. Consider the mapping $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$ defined by (132). We have

$$\begin{aligned} \Xi(WG - WT) &= \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left(\widetilde{s_z(G)} - \widetilde{s_z(T)} \right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right]^{1/\hbar} \\ &= \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \sum_{m\in\mathcal{N}} A(a,m) \left(f\left(m, \widetilde{s_m(G)}\right) - f\left(m, \widetilde{s_m(T)}\right)\right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right]^{1/\hbar} \\ &\le \sup_z \kappa^{t_z/\hbar} \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left(\widetilde{s_z(P)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right)\right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right]^{1/\hbar} \\ &\quad + \sup_z \kappa^{t_z/\hbar} \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left(\widetilde{s_z(T)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(T)}\right)\right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right]^{1/\hbar} \\ &= \sup_z \kappa^{t_z/\hbar} \left(\Xi(WG - G) + \Xi(WT - T) \right). \end{aligned} \tag{134}$$

In view of Theorem 50, one obtains a unique solution of equation (131) at $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$. □

Example 58. Assume the class $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V})$, where

$$\Xi(G) = \sqrt{\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_z(G)}, \tilde{0} \right)}{l! \sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z} \right)^{2l+3/l+2}}, \quad \text{for all } G \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V}).. \tag{135}$$

Consider the stochastic nonlinear dynamical system:

$$\widetilde{s_z(G)} = e^{-\widetilde{(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b \left| \widetilde{s_{z-2}(G)} \right|}{\sinh^d \left| \widetilde{s_{z-1}(G)} \right| + \widetilde{\sin mz} + \tilde{1}}, \tag{136}$$

where $z \ge 2$ and $b, d > 0$ and let $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V})$ be defined as

$$W(G) = \left(e^{-\widetilde{(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b \left| \widetilde{s_{z-2}(G)} \right|}{\sinh^d \left| \widetilde{s_{z-1}(G)} \right| + \widetilde{\sin mz} + \tilde{1}} \right) I. \tag{137}$$

Suppose W is Ξ-sequentially continuous at a point $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V})$, and there is $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathcal{V})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D. It is easy to see that

$$\begin{aligned} &\left| \sum_{m=0}^{\infty} \frac{\cosh(3m-z)\cos^b \left| \widetilde{s_{z-2}(G)} \right|}{\sinh^d \left| \widetilde{s_{z-1}(G)} \right| + \widetilde{\sin mz} + \tilde{1}} (\tan(2m+1) - \tan(2m+1)) \right| \\ &\tilde{\le} \frac{1}{25} \left| e^{-\widetilde{(2z+3)}} - \widetilde{s_z(G)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b \left| \widetilde{s_{z-2}(G)} \right|}{\sinh^d \left| \widetilde{s_{z-1}(G)} \right| + \widetilde{\sin mz} + \tilde{1}} \right| \\ &\quad + \frac{1}{25} \left| e^{-\widetilde{(2z+3)}} - \widetilde{s_z(T)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b \left| \widetilde{s_{z-2}(T)} \right|}{\sinh^d \left| \widetilde{s_{z-1}(T)} \right| + \widetilde{\sin mz} + \tilde{1}} \right|. \end{aligned} \tag{138}$$

By Theorem 57, the stochastic nonlinear dynamical system (18) has one solution D.

6. Conclusion

In this article, we introduced a new general space called $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ and the mappings' ideal space of solutions for many stochastic nonlinear and matrix systems of Kannan contraction type. We have presented some topological and geometric properties of it, of the multiplication operators acting on it, of the mappings' ideal, and of the spectrum of its mappings' ideal. The existence of a fixed point in the Kannan contraction mapping on these spaces is explored. To put our findings to the test, we introduced several numerical experiments. In addition, various effective implementations of the stochastic nonlinear dynamical and matrix system are discussed. The ideal spectrum of mappings, multiplication operators, and the fixed points of any contraction mappings in this new soft functions space are investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] L. Guo and Q. Zhu, "Stability analysis for stochastic Volterra–Levin equations with Poisson jumps: fixed point approach," *Journal of Mathematical Physics*, vol. 52, no. 4, article 042702, 2011.

[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] D. Dubois and H. Prade, *Possibility theory: an approach to computerized processing of uncertainty*, Plenum, New York, 1998.

[4] H. Ahmad, M. Younis, and M. E. Koksal, "Double controlled partial metric type spaces and convergence results," *Journal of Mathematics*, vol. 2021, Article ID 7008737, 11 pages, 2021.

[5] W. Mao, Q. Zhu, and X. Mao, "Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps," *Applied Mathematics and Computation*, vol. 254, pp. 252–265, 2015.

[6] D. Molodtsov, "Soft set theory–first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[7] P. K. Maji, R. Biswas, and A. R. Roy, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[8] X. Yang and Q. Zhu, "Existence, uniqueness, and stability of stochastic neutral functional differential equations of Sobolev-type," *Journal of Mathematical Physics*, vol. 56, no. 12, p. 122701, 2015.

[9] M. Abbas, G. Murtaza, and S. Romaguera, "Soft contraction theorem," *Journal of Nonlinear and Convex Analysis*, vol. 16, pp. 423–435, 2015.

[10] C. M. Chen and I. J. Lin, "Fixed point theory of the soft Meir-Keeler type contractive mappings on a complete soft metric space," *Fixed Point Theory and Applications*, vol. 2015, no. 1, 2015.

[11] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[12] I. Beg, M. Abbas, and M. W. Asghar, "Polytopic fuzzy sets and their applications to multiple-attribute decision-making problems," *International Journal of Fuzzy Systems*, 2022.

[13] M. Younis, S. Deepak, S. Radenovic, and M. Imdad, "Convergence theorems for generalized contractions and applications," *Univerzitet u Nišu*, vol. 34, no. 3, pp. 945–964, 2020.

[14] M. Ruzicka, "Electrorheological fluids. Modeling and mathematical theory," in *In Lecture Notes in Mathematics*, p. 1748, Springer, Berlin, Germany,, 2000.

[15] I. Beg, "Ordered uniform convexity in ordered convex metric spaces with an application to fixed point theory," *Journal of Function Spaces*, vol. 2022, Article ID 8237943, 7 pages, 2022.

[16] R. Kannan, "Some results on fixed points- II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[17] S. J. H. Ghoncheh, "Some fixed point theorems for Kannan mapping in the modular spaces," *Ciĕncia eNatura*, vol. 37, no. 6-1, pp. 462–466, 2015.

[18] A. Pietsch, "s-numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

[19] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, MR582655 (81j: 47001), 1980.

[20] A. A. Bakery and A. R. Abou Elmatty, "A note on Nakano generalized difference sequence space," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[21] M. İlkhan, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, no. 3, pp. 605–614, 2020.

[22] M. Mursaleen and A. K. Noman, "On some new sequence spaces of non-absolute type related to the spaces ℓp and ℓ∞ I," *Filomat*, vol. 25, no. 2, pp. 33–51, 2011.

[23] M. Mursaleen and F. Basar, "Domain of Cesàro mean of order one in some spaces of double sequences," *Studia Scientiarum Mathematicarum Hungarica*, vol. 51, no. 3, pp. 335–356, 2014.

[24] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Function Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.

[25] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 2021, no. 1, pp. 1–132, 2021.

[26] S. Das and S. K. Samanta, "Soft metric," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, pp. 77–94, 2013.

[27] H. Roopaei and F. Basar, "On the gamma spaces including the spaces of absolutely p-summable, null, convergent and bounded sequences," *Numerical Functional Analysis and Optimization*, vol. 43, no. 6, pp. 723–754, 2022.

[28] B. Altay and F. Basar, "Generalization of the sequence space ℓ (p) derived by weighted mean," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 174–185, 2007.

[29] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, New York, NY, USA, 1986.

[30] A. A. Bakery and M. M. Mohammed, “Kannan non-expansive mappings on Nakano sequence space of soft reals with some applications,” *Journal of function spaces*, vol. 2022, Article ID 2307519, 18 pages, 2022.

[31] A. A. Bakery and O. S. K. Mohamed, “Orlicz generalized difference sequence space and the linked pre-quasi operator ideal,” *Journal of Mathematics*, vol. 2020, Article ID 6664996, 9 pages, 2020.

[32] N. Faried and A. A. Bakery, “Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces,” *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[33] B. E. Rhoades, “Operators of A-p type,” *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 59, no. 3-4, pp. 238–241, 1975.

[34] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, 1978.

[35] T. Mrowka, *A Brief Introduction to Linear Analysis: Fredholm Operators, Geometry of Manifolds*, Massachusetts Institute of Technology, MIT OpenCouseWare), Fall, 2004.

[36] P. Salimi, A. Latif, and N. Hussain, “Modified α-ψ-contractive mappings with applications,” *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.