

Research Article

General Decay of the Moore–Gibson–Thompson Equation with Viscoelastic Memory of Type II

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This study deals with the general decay of solutions of a new class of Moore–Gibson–Thompson equation with respect to the memory kernel of type II. By using the energy method in the Fourier space, we establish the main results.

1. Introduction and Preliminaries

In this work, we are interested in the study of the general decay of solutions of the following problem:

$$\begin{cases} au_{ttt} + \beta u_{tt} - \gamma^2 \Delta u - b \Delta u_t + \int_0^t h(t - \sigma) \Delta u_t(\sigma) d\sigma = 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), u_{tt}(x, 0) = u_2(x), \quad x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $a, b, \beta > 0$ are physical parameters and γ is the speed of sound. The convolution term $\int_0^t h(t - s) \Delta u_t(s) ds$ reflects the memory type-II effect of materials due to viscoelasticity, where h is the relaxation function.

Moore–Gibson–Thompson (MGT) equation is based on the modeling of high amplitude sound waves. There has been quite a bit of work in this area of research due to a wide range of applications such as medical and industrial use of high intensity ultrasound in lithotripsy, heat therapy, ultrasonic cleaning, etc. Classical models of nonlinear acoustics include the Kuznetsov equation, the Westervelt equation, and the Kokhlov–Zabulotskaya–Kuznetsov

equation. An in-depth study of linear models is a good starting point for a better understanding of the approximate behaviors of nonlinear models. Indeed, even in the linear case, the works [1] have shown a rich dynamic. In [2], Marchand et al. presented a detailed analysis of this equation. Using a quasi-abstract group approach and refined spectral analysis, they establish the well posedness of the problem and define the accumulation point of eigenvalues. Kaltenbacher et al. [3] also studied the fully nonlinear version of the MGT equation and established the global well posedness and the exponential decay for the nonlinear equation under consideration.

In the case of a third-order equation, the modeling of memory effects is quite complex. The memory term may affect u , or u_t , or even a combination of both. Accordingly, the corresponding models are called as memory of type I, type II, and type III. This classification raises a fundamental question on the impact of the memory terms on the stability properties of the corresponding evolutions. We mention that the memory has some stabilizing effects (see, for instance, [4, 5]). On the contrary, a quantification of the stability results via decay rates shows that the memory may destroy the stability properties of a stable system.

In this paper, we are interested in studying the dynamics that results from the memory kernel of type II (see [6]) in our problem (1).

Natural questions can be asked based on the study of the viscoelastic memory and integral condition system (see [7–10]): could the addition of the memory kernel of type II harm the stability of this kind of problem in any way? If the answer to the question in the wave condition with friction damping are relatively simple, it is not easy to answer in the case of memory kernel of type II that we present below, especially in Fourier space. This work is part of the effort to understand this the MGT problem and memory kernel of type II.

The coupling between the Fourier law of heat condition and different systems has been considered by many authors and there are many results. For example, Bresse system (Bresse–Fourier) has been studied in [11], the Bresse system coupled with the Cattaneo law of heat condition (Bresse–Cattaneo) has been studied in [12], and Timoshenko system with past history has been considered in [13]. For further details, we refer the reader to the following papers [14, 15].

We mention also that several results related to viscoelasticity have been obtained by using the theory of Lie symmetries. Precisely, thanks to the symmetry reduction obtained by means of the classical Lie symmetries, it was possible to obtain exact solutions of considerable interest. For further details in this direction, we refer the reader to [16].

Based on all last mentioned works, especially [6, 10], we would like to prove the general decay result in the Fourier space to problem (1). To the best of our knowledge, this is one of the first works that deals with the MGT problem with viscoelastic memory kernel of type II in the Fourier space.

The paper has the following structure. In Section 1, we put our assumptions and preliminaries that will be employed in our main decay result. In Section 2, by using the energy method in the Fourier space, we construct the Lyapunov functional and find the estimate for the Fourier image. Section 3 is devoted to the conclusion.

To prove our main result, we need the following hypotheses and lemmas.

(H1) $h \in C^1(\mathbb{R}_+, \mathbb{R}_+) \cap C(\mathbb{R}_+, \mathbb{R}_+)$ is a nonincreasing function; there exists a positive constant κ satisfying

$$h'(t) \leq -\kappa h(t), \quad t \geq 0. \quad (2)$$

(H2) There exist positive numbers θ and $\lambda \in (\gamma^2/b, \beta/a)$ such that $\lambda/\theta < \kappa$ and

$$H(\infty) \leq \left(b - \frac{\gamma^2}{\lambda}\right) \min\left\{\frac{2}{\lambda(2+\theta)}, 1\right\}, \quad (3)$$

where $H(\infty) = \int_0^\infty h(s)ds > 0$ and $H(t) = \int_0^t h(s)ds$.

(H3)

$$\beta - \frac{\gamma^2 a}{b} > 0. \quad (4)$$

Throughout this paper, we use $c, C, C_i, i = 1, 2$, to denote a generic positive constant.

Lemma 1. *Suppose that (3) holds. Then, for all $\omega \in L^2_{loc}(\mathbb{R}_+, \mathbb{C})$, we have*

$$\left| \int_0^t h(t-\sigma)(\omega(t) - \omega(\sigma))d\sigma \right|^2 \leq c(h \circ \omega)(t), \quad \forall t \geq 0, \quad (5)$$

and

$$\left| \int_0^t h'(t-\sigma)(\omega(t) - \omega(\sigma))d\sigma \right|^2 \leq -h(0)(h' \circ \omega)(t), \quad \forall t \geq 0, \quad (6)$$

where

$$(h \circ \omega)(t) := \int_0^t h(t-\sigma)|\omega(t) - \omega(\sigma)|^2 d\sigma. \quad (7)$$

Lemma 2. *For any $k \geq 0$ and $c > 0$, there exist a constant $C > 0$ such that, for all $t \geq 0$, the following estimate holds:*

$$\int_{|\xi|^2 \leq 1} |\xi|^k e^{-c|\xi|^2 t} d\xi \leq C(1+t)^{-(k+n)/2}, \quad \xi \in \mathbb{R}^n. \quad (8)$$

2. The Energy Method in the Fourier Space

In this section, we get the decay estimate of the Fourier image of the solution for system (1). By using Plancherel's theorem together with some integral estimates such as (2), this method will allow us to give the decay rate of the solution in the energy space. To this goal, we use the energy method in the Fourier space and construct appropriate Lyapunov functionals. Finally, we prove our main result.

Applying the Fourier transform to (1), we get the following problem:

$$\begin{cases} a\widehat{u}_{ttt} + \beta\widehat{u}_{tt} + \gamma^2|\xi|^2\widehat{u} + b|\xi|^2\widehat{u}_t - |\xi|^2 \int_0^t h(t-\sigma)\widehat{u}_t(\sigma)d\sigma = 0, & \xi \in \mathbb{R}^n, t > 0 \\ \widehat{u}(\xi, 0) = \widehat{u}_0(\xi), \widehat{u}_t(\xi, 0) = \widehat{u}_1(\xi), \widehat{u}_{tt}(\xi, 0) = \widehat{u}_2(\xi). \end{cases} \quad (9)$$

Lemma 3. Suppose that (3)–(4) hold. Let $\widehat{u}(\xi, t)$ be the solution of (9). Then, the energy functional $\widehat{E}(t)$, given by

$$\widehat{E}(\xi, t) = \widehat{E}(t) = \frac{a}{2}|\widehat{u}_{tt} + \lambda\widehat{u}_t|^2 + |\xi|^2 \frac{\gamma^2}{2\lambda} |\widehat{u}_t + \lambda\widehat{u}|^2 + |\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \right) |\widehat{u}_t|^2 + \frac{\lambda}{2} (\beta - a\lambda) |\widehat{u}_t|^2 + \frac{1}{2} |\xi|^2 (h \circ \widehat{u}_t)(t), \quad (10)$$

satisfies

$$\widehat{E}'(t) \leq -|\xi|^2 \lambda C_1 |\widehat{u}_t|^2 + C_2 |\xi|^2 (h' \circ \widehat{u}_t)(t), \quad (11)$$

where $C_1 = ((b - \gamma^2/\lambda) - H(t)(1 + \theta/2)) > 0$, $C_2 = \kappa_0/2\kappa$, and $\kappa_0 = (\kappa - \lambda/\theta) > 0$.

Proof. Firstly, multiplying (9) by (\widehat{u}_{tt}) and taking the real part, we obtain

$$\operatorname{Re} \left\{ a\widehat{u}_{ttt}\widehat{u}_{tt} + \beta\widehat{u}_{tt}\widehat{u}_{tt} + \gamma^2|\xi|^2\widehat{u}\widehat{u}_{tt} + b|\xi|^2\widehat{u}_t\widehat{u}_{tt} - |\xi|^2 \int_0^t h(t-\sigma)\widehat{u}_t(\sigma)\widehat{u}_{tt}d\sigma \right\} = \frac{d}{dt}\widehat{E}_1(t) + \widehat{R}_1(t) = 0. \quad (12)$$

Then, we have

$$\begin{aligned} a\operatorname{Re}\{\widehat{u}_{ttt}\widehat{u}_{tt}\} &= \frac{a}{2} \frac{d}{dt} |\widehat{u}_{tt}|^2, \\ \beta\operatorname{Re}\{\widehat{u}_{tt}\widehat{u}_{tt}\} &= \beta |\widehat{u}_{tt}|^2, \\ b|\xi|^2 \operatorname{Re}\{\widehat{u}_t\widehat{u}_{tt}\} &= \frac{b}{2} |\xi|^2 \frac{d}{dt} |\widehat{u}_t|^2 \\ \gamma^2|\xi|^2 \operatorname{Re}\{\widehat{u}\widehat{u}_{tt}\} &= \gamma^2|\xi|^2 \operatorname{Re} \left\{ \frac{d}{dt} \widehat{u}\widehat{u}_t \right\} - \gamma^2|\xi|^2 |\widehat{u}_t|^2, \end{aligned} \quad (13)$$

and

$$\begin{aligned}
& -|\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) \widehat{u}_t(\sigma) \overline{\widehat{u}_{tt}} d\sigma \right\} \\
& = |\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) (\widehat{u}_t(t) - \widehat{u}_t(\sigma)) \overline{\widehat{u}_{tt}} d\sigma \right\} - |\xi|^2 H(t) \operatorname{Re} \left\{ \widehat{u}_t(t) \overline{\widehat{u}_{tt}} \right\} \\
& = |\xi|^2 \frac{1}{2} \frac{d}{dt} \int_0^t h(t-\sigma) |\widehat{u}_t(t) - \widehat{u}_t(\sigma)|^2 d\sigma - \frac{H(t)}{2} |\xi|^2 \frac{d}{dt} |\widehat{u}_t|^2 \\
& \quad - |\xi|^2 \frac{1}{2} \int_0^t h'(t-\sigma) |\widehat{u}_t(t) - \widehat{u}_t(\sigma)|^2 d\sigma \tag{14} \\
& = |\xi|^2 \frac{1}{2} \frac{d}{dt} (h \circ \widehat{u}_t)(t) - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) - \frac{H(t)}{2} |\xi|^2 \frac{d}{dt} |\widehat{u}_t|^2 \\
& = |\xi|^2 \frac{1}{2} \frac{d}{dt} (h \circ \widehat{u}_t)(t) - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) - \frac{1}{2} |\xi|^2 \frac{d}{dt} (H(t) |\widehat{u}_t|^2) \\
& \quad + \frac{1}{2} |\xi|^2 h(t) |\widehat{u}_t|^2.
\end{aligned}$$

Substituting (13) and (14) into (12), we obtain

$$\begin{aligned}
\widehat{E}_1(t) & = \frac{a}{2} |\widehat{u}_{tt}|^2 + \frac{b}{2} |\xi|^2 |\widehat{u}_t|^2 + \gamma^2 |\xi|^2 \operatorname{Re} \left\{ \widehat{u} \overline{\widehat{u}_t} \right\} - \frac{H(t)}{2} |\xi|^2 |\widehat{u}_t|^2 + |\xi|^2 \frac{1}{2} (h \circ \widehat{u}_t)(t) \\
\widehat{R}_1(t) & = \beta |\widehat{u}_{tt}|^2 - \gamma^2 |\xi|^2 |\widehat{u}_t|^2 - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) + \frac{1}{2} |\xi|^2 h(t) |\widehat{u}_t|^2.
\end{aligned} \tag{15}$$

Secondly, multiplying (9) by $(\overline{\widehat{u}_t})$ and taking the real part, we obtain

$$\operatorname{Re} \left\{ a \widehat{u}_{ttt} \overline{\widehat{u}_t} + \beta \widehat{u}_{tt} \overline{\widehat{u}_t} + \gamma^2 |\xi|^2 \widehat{u} \overline{\widehat{u}_t} + b |\xi|^2 \widehat{u}_t \overline{\widehat{u}_t} - |\xi|^2 \int_0^t h(t-\sigma) \widehat{u}_t(\sigma) \overline{\widehat{u}_t} d\sigma \right\} = \frac{d}{dt} \widehat{E}_2(t) + \widehat{R}_2(t) = 0. \tag{16}$$

We have

$$\begin{aligned}
a \operatorname{Re} \left\{ \widehat{u}_{ttt} \overline{\widehat{u}_t} \right\} & = a \operatorname{Re} \left\{ \frac{d}{dt} (\widehat{u}_{tt} \overline{\widehat{u}_t}) \right\} - a |\widehat{u}_{tt}|^2, \\
\beta \operatorname{Re} \left\{ \widehat{u}_{tt} \overline{\widehat{u}_t} \right\} & = \frac{\beta}{2} \frac{d}{dt} |\widehat{u}_t|^2, \\
b |\xi|^2 \operatorname{Re} \left\{ \widehat{u}_t \overline{\widehat{u}_t} \right\} & = b |\xi|^2 |\widehat{u}_t|^2, \\
\gamma^2 |\xi|^2 \operatorname{Re} \left\{ \widehat{u} \overline{\widehat{u}_t} \right\} & = \frac{\gamma^2}{2} |\xi|^2 \frac{d}{dt} |\widehat{u}|^2,
\end{aligned} \tag{17}$$

and

$$-|\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) \widehat{u}_t(\sigma) \overline{\widehat{u}_t} d\sigma \right\} = |\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) (\widehat{u}_t(t) - \widehat{u}_t(\sigma)) \overline{\widehat{u}_t} d\sigma \right\} - |\xi|^2 H(t) |\widehat{u}_t|^2. \tag{18}$$

Substituting (17) and (18) into (16), we obtain

$$\begin{aligned} \widehat{E}_2(t) &= a \operatorname{Re} \{ \widehat{u}_{tt} \overline{\widehat{u}_t} \} + \frac{\beta}{2} |\widehat{u}_t|^2 + \frac{\gamma^2}{2} |\xi|^2 |\widehat{u}|^2, \\ \widehat{R}_2(t) &= -a |\widehat{u}_{tt}|^2 + b |\xi|^2 |\widehat{u}_t|^2 + |\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) (\widehat{u}_t(t) - \widehat{u}_t(\sigma)) \overline{\widehat{u}_t} d\sigma \right\} \\ &\quad - |\xi|^2 H(t) |\widehat{u}_t|^2. \end{aligned} \tag{19}$$

At this point, by (4), we have $\beta - \gamma^2 a/b > 0$, and by (3), we select λ such that $\gamma^2/b < \lambda < \beta/a$.

By (15) and (19), we have

Let $\widehat{E}(t)$ be the energy functional. Then, we have

$$\begin{aligned} \frac{d}{dt} \widehat{E}(t) + \widehat{R}(t) &= 0, \\ \widehat{E}(t) &= \widehat{E}_1(t) + \lambda \widehat{E}_2(t), \\ \widehat{R}(t) &= \widehat{R}_1(t) + \lambda \widehat{R}_2(t). \end{aligned} \tag{20}$$

$$\begin{aligned} \widehat{E}(t) &= \widehat{E}_1(t) + \lambda \widehat{E}_2(t) \\ &= \frac{a}{2} |\widehat{u}_{tt}|^2 + \frac{b}{2} |\xi|^2 |\widehat{u}_t|^2 + \gamma^2 |\xi|^2 \operatorname{Re} \{ \widehat{u} \overline{\widehat{u}_t} \} - \frac{H(t)}{2} |\xi|^2 |\widehat{u}_t|^2 + |\xi|^2 \frac{1}{2} (h \circ \widehat{u}_t)(t) \\ &\quad + a \lambda \operatorname{Re} \{ \widehat{u}_{tt} \overline{\widehat{u}_t} \} + \frac{\beta}{2} \lambda |\widehat{u}_t|^2 + \frac{\gamma^2 \lambda}{2} |\xi|^2 |\widehat{u}|^2 \end{aligned} \tag{21}$$

$$\begin{aligned} &= \frac{a}{2} |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + \frac{1}{2} |\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \right) |\widehat{u}_t|^2 + \frac{\gamma^2}{2\lambda} |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 \\ &\quad + \frac{\lambda}{2} (\beta - a\lambda) |\widehat{u}_t|^2 + |\xi|^2 \frac{1}{2} (h \circ \widehat{u}_t)(t), \\ \widehat{R}(t) &= \widehat{R}_1(t) + \lambda \widehat{R}_2(t) \\ &= (\beta - a\lambda) |\widehat{u}_{tt}|^2 - \gamma^2 |\xi|^2 |\widehat{u}_t|^2 - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) + \frac{1}{2} |\xi|^2 h(t) |\widehat{u}_t|^2 \\ &\quad + b \lambda |\xi|^2 |\widehat{u}_t|^2 + \lambda |\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) (\widehat{u}_t(t) - \widehat{u}_t(\sigma)) \overline{\widehat{u}_t} d\sigma \right\} \\ &\quad - \lambda |\xi|^2 H(t) |\widehat{u}_t|^2. \end{aligned} \tag{22}$$

By (3), we have $(\beta - a\lambda > 0)$, and using Young's inequality, we obtain

$$\begin{aligned}
\widehat{R}(t) &= \lambda|\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \right) |\widehat{u}_t|^2 - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) - \frac{1}{2} |\xi|^2 h(t) |\widehat{u}_t|^2 \\
&\quad + \lambda|\xi|^2 \operatorname{Re} \left\{ \int_0^t h(t-\sigma) (\widehat{u}_t(t) - \widehat{u}_t(\sigma)) \overline{\widehat{u}_t} d\sigma \right\} \\
&\geq \lambda|\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \right) |\widehat{u}_t|^2 - |\xi|^2 \frac{1}{2} (h' \circ \widehat{u}_t)(t) \\
&\quad - \lambda|\xi|^2 H(t) \frac{\theta}{2} |\widehat{u}_t|^2 - |\xi|^2 \frac{\lambda}{2\theta} (h' \circ \widehat{u}_t)(t) \\
&= \lambda|\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \left(1 + \frac{\theta}{2} \right) \right) |\widehat{u}_t|^2 - |\xi|^2 \frac{1}{2} (h \circ \widehat{u}_t)(t) \\
&\quad - |\xi|^2 \frac{\lambda}{2\theta} (h \circ \widehat{u}_t)(t).
\end{aligned} \tag{23}$$

By using the fact that $\lambda/\theta < \kappa$ and under supposition (2), we have

$$h'(t) \leq -\kappa h(t) \Rightarrow \kappa h(t) \leq -h'(t). \tag{24}$$

Thus, (23) becomes

$$\widehat{R}(t) \geq \lambda|\xi|^2 \left(b - \frac{\gamma^2}{\lambda} - H(t) \left(1 + \frac{\theta}{2} \right) \right) |\widehat{u}_t|^2 - |\xi|^2 \frac{\kappa_0}{2} (h \circ \widehat{u}_t)(t), \tag{25}$$

where $\kappa_0 = (\kappa - \lambda/\theta) > 0$,

Finally, by setting $C_1 = (b - \gamma^2/\lambda - H(t)(1 + \theta/2)) > 0$ and $C_2 = \kappa_0/2\kappa$, we find (11). \square

Now, to achieve our goal, we need the following lemmas.

Lemma 4. Suppose that (3)-(4) hold. Then, the functional

$$D_1(t) := a \operatorname{Re} \{ (\widehat{u}_{tt} + \lambda \widehat{u}_t)(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \} + (\beta - a\lambda) \operatorname{Re} \{ \widehat{u}_t(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \}, \tag{26}$$

satisfies

$$\begin{aligned}
D_1'(t) &\leq -\frac{\gamma^2}{2\lambda} |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 + c(1 + |\xi|^2) |\widehat{u}_t|^2 + c |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 \\
&\quad + c |\xi|^2 (h' \circ \widehat{u}_t)(t).
\end{aligned} \tag{27}$$

Proof. Differentiating D_1 and by using (9), we obtain

$$\begin{aligned}
D_1'(t) &= a \operatorname{Re} \{ (\widehat{u}_{ttt} + \lambda \widehat{u}_{tt})(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \} + a |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 \\
&\quad + (\beta - a\lambda) \operatorname{Re} \{ \widehat{u}_{tt}(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \} + (\beta - a\lambda) \operatorname{Re} \{ \widehat{u}_t(\overline{\widehat{u}_{tt}} + \lambda \overline{\widehat{u}_t}) \} \\
&= \underbrace{(\beta - a\lambda) \operatorname{Re} \{ \widehat{u}_t(\overline{\widehat{u}_{tt}} + \lambda \overline{\widehat{u}_t}) \}}_{I_1} + a |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 \\
&\quad + \underbrace{|\xi|^2 \operatorname{Re} \{ (-b \widehat{u}_t)(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \}}_{I_2} + \underbrace{|\xi|^2 \operatorname{Re} \{ (-\gamma^2 \widehat{u})(\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \}}_{I_3} \\
&\quad + \underbrace{|\xi|^2 \operatorname{Re} \left\{ \left(\int_0^t h(t-\sigma) \widehat{u}_t(\sigma) d\sigma \right) (\overline{\widehat{u}_t} + \lambda \overline{\widehat{u}}) \right\}}_{I_4}.
\end{aligned} \tag{28}$$

In what follows, we estimate the terms I_i , $i = 1, \dots, 4$, that appear in the right-hand side of (28); using Young's inequality, we obtain

$$I_1 \leq \frac{(\beta - a\lambda)}{2} |\widehat{u}_t|^2 + \frac{(\beta - a\lambda)}{2} |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2, \tag{29}$$

$$\begin{aligned} I_2 + I_3 &= -b|\xi|^2 \operatorname{Re}\{\widehat{u}_t(\widehat{u}_t + \lambda \widehat{u})\} - \frac{\gamma^2}{\lambda} |\xi|^2 \operatorname{Re}\{\lambda \widehat{u}(\widehat{u}_t + \lambda \widehat{u})\} \\ &= -\frac{\gamma^2}{\lambda} |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 - \left(b - \frac{\gamma^2}{\lambda}\right) |\xi|^2 \operatorname{Re}\{\widehat{u}_t(\widehat{u}_t + \lambda \widehat{u})\} \\ &\leq -\frac{\gamma^2}{\lambda} |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 + \left(b - \frac{\gamma^2}{\lambda}\right)^2 |\xi|^2 \frac{1}{4\varepsilon_1} |\widehat{u}_t|^2 + \varepsilon_1 |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2, \end{aligned} \tag{30}$$

$$\begin{aligned} I_4 &= -|\xi|^2 \operatorname{Re}\left\{\left(\int_0^t h(t - \sigma)(\widehat{u}_t(\sigma) - \widehat{u}_t(\sigma))d\sigma\right)(\widehat{u}_t + \lambda \widehat{u})\right\} \\ &\quad + |\xi|^2 H(t) \operatorname{Re}\{\widehat{u}_t(\widehat{u}_t + \lambda \widehat{u})\} \\ &\leq \frac{c}{4\varepsilon_1} |\xi|^2 (h \circ \widehat{u}_t)(t) + \frac{H^2(t)}{4\varepsilon_1} |\xi|^2 |\widehat{u}_t|^2 + (2\varepsilon_1) |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2. \end{aligned} \tag{31}$$

By letting $\varepsilon_1 = \gamma^2/6\lambda$, we obtain (27). \square

Lemma 5. *The function,*

$$D_2(t) := -a \operatorname{Re}\left\{(\widehat{u}_{tt} + \lambda \widehat{u}_t)\left(\int_0^t h(t - \sigma)(\widehat{u}_t(t) - \widehat{u}_t(\sigma))d\sigma\right)\right\}, \tag{32}$$

satisfies, for any $\varepsilon, \varepsilon_2 > 0$,

$$\begin{aligned} D_2'(t) &\leq -(aH(t) - 3\varepsilon_2) |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + c(1 + |\xi|^2) |\widehat{u}_t|^2 + \varepsilon |\widehat{u}_t + \lambda \widehat{u}|^2 \\ &\quad + c\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon}\right) (1 + |\xi|^2) (h \circ \widehat{u}_t)(t) - \frac{c}{\varepsilon_2} (h' \circ \widehat{u}_t)(t). \end{aligned} \tag{33}$$

Proof. Differentiating D_2 and by using (9), we obtain

$$\begin{aligned}
D_2'(t) &= -a\operatorname{Re}\left\{\left(\widehat{u}_{ttt} + \lambda\widehat{u}_{tt}\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\quad - a\operatorname{Re}\left\{\left(\widehat{u}_{tt} + \lambda\widehat{u}_t\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\quad - aH(t)\operatorname{Re}\left\{\left(\widehat{u}_{tt} + \lambda\widehat{u}_t\right)\overline{\widehat{u}}_{tt}(t)\right\} \\
&= \underbrace{-(\beta - a\lambda)\operatorname{Re}\left\{\widehat{u}_{tt}\int_0^t h'(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right\}}_{J_1} \\
&\quad + \underbrace{|\xi|^2\operatorname{Re}\left\{-b\widehat{u}_t\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right\}}_{J_2} \\
&\quad + \underbrace{|\xi|^2\operatorname{Re}\left\{-\gamma^2\widehat{u}\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\}}_{J_3} \\
&\quad + \underbrace{|\xi|^2\operatorname{Re}\left\{\left(\int_0^t h(t-\sigma)\widehat{u}_t(\sigma)d\sigma\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\}}_{J_4} \\
&\quad - \underbrace{a\operatorname{Re}\left\{\left(\widehat{u}_{tt} + \lambda\widehat{u}_t\right)\int_0^t h'(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right\}}_{J_5} \\
&\quad - aH(t)|\widehat{u}_{tt} + \lambda\widehat{u}_t|^2 + \underbrace{aH(t)\lambda\operatorname{Re}\left\{\left(\widehat{u}_{tt} + \lambda\widehat{u}_t\right)\overline{\widehat{u}}_t\right\}}_{J_6}.
\end{aligned} \tag{34}$$

Similarly, we estimate the terms J_i , $i = 1, \dots, 6$, that appear in the right-hand side of (34). Using Young's inequality, we obtain

$$\begin{aligned}
J_1 &= -(\beta - a\lambda)\operatorname{Re}\left\{\left(\widehat{u}_{tt} + \lambda\widehat{u}_t\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\quad + (\beta - a\lambda)\lambda\operatorname{Re}\left\{\widehat{u}_t\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\leq \left(\frac{(\beta - a\lambda)^2c}{4\varepsilon_2} + \frac{(\beta - a\lambda)c}{2}\right)(h \circ \widehat{u}_t)(t) + \varepsilon_2|\widehat{u}_{tt} + \lambda\widehat{u}_t|^2 + \frac{(\beta - a\lambda)}{2}|\widehat{u}_t|^2 \\
&\leq c\left(1 + \frac{1}{\varepsilon_2}\right)(h \circ \widehat{u}_t)(t) + \varepsilon_2|\widehat{u}_{tt} + \lambda\widehat{u}_t|^2 + c|\widehat{u}_t|^2, \\
J_2 + J_3 &= -\frac{\gamma^2}{\lambda}|\xi|^2\operatorname{Re}\left\{\left(\widehat{u}_t + \lambda\widehat{u}\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\quad - \left(b - \frac{\gamma^2}{\lambda}\right)|\xi|^2\operatorname{Re}\left\{\widehat{u}_t\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\leq \varepsilon|\xi|^2|\widehat{u}_t + \lambda\widehat{u}|^2 + c\left(1 + \frac{1}{\varepsilon}\right)|\xi|^2(h \circ \widehat{u}_t)(t) + \left(b - \frac{\gamma^2}{\lambda}\right)^2|\xi|^2\frac{1}{2}|\widehat{u}_t|^2,
\end{aligned} \tag{35}$$

$$\begin{aligned}
J_2 + J_3 &= -\frac{\gamma^2}{\lambda}|\xi|^2\operatorname{Re}\left\{\left(\widehat{u}_t + \lambda\widehat{u}\right)\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\quad - \left(b - \frac{\gamma^2}{\lambda}\right)|\xi|^2\operatorname{Re}\left\{\widehat{u}_t\left(\int_0^t h(t-\sigma)\left(\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)\right)d\sigma\right)\right\} \\
&\leq \varepsilon|\xi|^2|\widehat{u}_t + \lambda\widehat{u}|^2 + c\left(1 + \frac{1}{\varepsilon}\right)|\xi|^2(h \circ \widehat{u}_t)(t) + \left(b - \frac{\gamma^2}{\lambda}\right)^2|\xi|^2\frac{1}{2}|\widehat{u}_t|^2,
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
 J_4 &= -|\xi|^2 \operatorname{Re} \left\{ \left(\int_0^t h(t-\sigma) (\widehat{u}_t(\sigma) - \widehat{u}_t(\sigma)) d\sigma \right) \left(\int_0^t h(t-\sigma) (\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)) d\sigma \right) \right\} \\
 &\quad + |\xi|^2 H(t) \operatorname{Re} \left\{ \widehat{u}_t \left(\int_0^t h(t-\sigma) (\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)) d\sigma \right) \right\} \\
 &\leq c|\xi|^2 (h \circ \widehat{u}_t)(t) + \frac{H^2(t)}{2} |\xi|^2 |\widehat{u}_t|^2.
 \end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}
 J_5 &= -a \operatorname{Re} \left\{ (\widehat{u}_{tt} + \lambda \widehat{u}_t) \left(\int_0^t h'(t-\sigma) (\overline{\widehat{u}}_t(t) - \overline{\widehat{u}}_t(\sigma)) d\sigma \right) \right\} \\
 &\leq \varepsilon_2 |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 - \frac{c}{4\varepsilon_2} (h' \circ \widehat{u}_t)(t),
 \end{aligned} \tag{38}$$

and

$$\begin{aligned}
 J_6 &= aH(t) \lambda \operatorname{Re} \left\{ (\widehat{u}_{tt} + \lambda \widehat{u}_t) \overline{\widehat{u}}_t \right\} \\
 &\leq \varepsilon_2 |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + \frac{(aH(t)\lambda)^2}{4\varepsilon_2} |\widehat{u}_t|^2.
 \end{aligned} \tag{39}$$

By substituting (35)–(39) into (34), we obtain (33). \square

At this stage, we define the functional

$$\mathcal{K}(t) := N(1 + |\xi|^2) \widehat{E}(t) + N_1 |\xi|^2 D_1(t) + N_2 |\xi|^2 D_2(t), \tag{40}$$

where N, N_1 , and N_2 are positive constants to be properly chosen later.

Lemma 6. *There exist $\mu_i, t_0 > 0, i = 1, \dots, 4$, such that the functional $\mathcal{K}(t)$ given by (40) satisfies*

$$\mathcal{K}'(t) \leq -\mu_1 |\xi|^2 \widehat{E}(t) + \mu_2 (1 + |\xi|^2)^2 (h' \circ \nabla u)(t), t > t_0, \tag{41}$$

and

$$\mu_3 (1 + |\xi|^2) \widehat{E}(t) \leq \mathcal{K}(t) \leq \mu_4 (1 + |\xi|^2) \widehat{E}(t). \tag{42}$$

Proof. Since the function h is a positive and continuous, for all $t_0 > 0$, we have

$$H(t) = \int_0^t h(\sigma) d\sigma \geq \int_0^{t_0} h(\sigma) d\sigma =: h_0, \forall t \geq t_0. \tag{43}$$

Firstly, by differentiating (40) and using (11), (27), and (33), we have

$$\begin{aligned}
 \mathcal{K}'(t) &\leq -|\xi|^2 [N_2 (aH(t) - 3\varepsilon_2) - cN_1] |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 \\
 &\quad - |\xi|^2 \left[\frac{\gamma^2}{2\lambda} |\xi|^2 N_1 - \varepsilon N_2 \right] |\widehat{u}_t + \lambda \widehat{u}|^2 \\
 &\quad - |\xi|^2 (1 + |\xi|^2) [C_1 N - cN_1 - cN_2] |\widehat{u}_t|^2 \\
 &\quad + |\xi|^2 (1 + |\xi|^2) \left[cN_1 + c \left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon} \right) N_2 \right] (h \circ \widehat{u}_t)(t) \\
 &\quad + |\xi|^2 (1 + |\xi|^2) \left[C_2 N - \frac{c}{\varepsilon_2} N_2 \right] (h' \circ \widehat{u}_t)(t).
 \end{aligned} \tag{44}$$

By setting

$$\begin{aligned}
 \varepsilon &= \frac{\gamma^2 |\xi|^2 N_1}{4\lambda N_2}, \\
 \varepsilon_2 &= \frac{ah_0}{6},
 \end{aligned} \tag{45}$$

we obtain

$$\begin{aligned}
 \mathcal{K}'(t) &\leq -|\xi|^2 \left[N_2 \frac{ah_0}{2} - cN_1 \right] |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 \\
 &\quad - |\xi|^2 \left[\frac{\gamma^2}{4\lambda} |\xi|^2 N_1 \right] |\widehat{u}_t + \lambda \widehat{u}|^2 \\
 &\quad - |\xi|^2 (1 + |\xi|^2) [C_1 N - cN_1 - cN_2] |\widehat{u}_t|^2 \\
 &\quad + (1 + |\xi|^2)^2 [cN_1 + cN_2] (h \circ \widehat{u}_t)(t) \\
 &\quad + |\xi|^2 (1 + |\xi|^2) [C_2 N - cN_2] (h' \circ \widehat{u}_t)(t).
 \end{aligned} \tag{46}$$

Next, we fixed N_1 and chose N_2 large enough such that

$$N_2 \frac{ah_0}{2} - cN_1 > 0. \tag{47}$$

Hence, we arrive at

$$\begin{aligned} \mathcal{K}'(t) \leq & -|\xi|^2 \alpha_0 |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 - |\xi|^4 \alpha_1 |\widehat{u}_t + \lambda \widehat{u}|^2 - |\xi|^2 (1 + |\xi|^2) [C_1 N - c] |\widehat{u}_t|^2 \\ & + c(1 + |\xi|^2)^2 (h \circ \widehat{u}_t)(t) + |\xi|^2 (1 + |\xi|^2) [C_2 N - c] (h' \circ \widehat{u}_t)(t). \end{aligned} \quad (48)$$

Secondly, we have

$$\begin{aligned} |\mathcal{K}(t) - N(1 + |\xi|^2) \widehat{E}(t)| &= N_1 |\xi|^2 D_1(t) + N_2 |\xi|^2 D_2(t) \\ &\leq a N_1 |\xi|^2 \left| \operatorname{Re} \{ (\widehat{u}_{tt} + \lambda \widehat{u}_t) (\overline{\widehat{u}_t + \lambda \widehat{u}}) \} \right| \\ &\quad + N_1 |\xi|^2 (\beta - a\lambda) \left| \operatorname{Re} \{ \widehat{u}_t (\overline{\widehat{u}_t + \lambda \widehat{u}}) \} \right| \\ &\quad + a N_2 |\xi|^2 \left| \operatorname{Re} \left\{ (\widehat{u}_{tt} + \lambda \widehat{u}_t) \left(\int_0^t h(t - \sigma) (\overline{\widehat{u}_t(t) - \widehat{u}_t(\sigma)}) d\sigma \right) \right\} \right|. \end{aligned} \quad (49)$$

Using Young's inequality and Lemma 1, we find

$$\begin{aligned} |\mathcal{K}(t) - N(1 + |\xi|^2) \widehat{E}(t)| &\leq \frac{1}{2} N_1 |\xi|^2 |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + \frac{1}{2} N_1 |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 \\ &\quad + \frac{1}{2} N_1 |\xi|^2 |\widehat{u}_t|^2 + \frac{1}{2} N_1 |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 \\ &\quad + \frac{a}{2} N_2 |\xi|^2 |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + \frac{c}{2} N_2 (h \circ \widehat{u}_t)(t) \\ &\leq c |\xi|^2 \{ |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + |\widehat{u}_t + \lambda \widehat{u}|^2 + |\widehat{u}_t|^2 + (h \circ \widehat{u}_t)(t) \} \\ &\leq c(1 + |\xi|^2) \{ |\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + |\xi|^2 |\widehat{u}_t + \lambda \widehat{u}|^2 \\ &\quad + (1 + |\xi|^2) |\widehat{u}_t|^2 + |\xi|^2 (h \circ \widehat{u}_t)(t) \} \\ &\leq c(1 + |\xi|^2) \widehat{E}(t). \end{aligned} \quad (50)$$

Hence, we obtain

$$(N - c)(1 + |\xi|^2) \widehat{E}(t) \leq \mathcal{K}(t) \leq (N + c)(1 + |\xi|^2) \widehat{E}(t). \quad (51)$$

Now, we choose N large enough such that

$$\begin{aligned} N - c &> 0, \\ C_1 N - c &> 0, \\ C_2 N - c &> 0, \end{aligned} \quad (52)$$

and exploiting (10), estimates (48) and (51), respectively, give

$$\mathcal{K}'(t) \leq -\mu_1 |\xi|^2 \widehat{E}(t) + \mu_2 (1 + |\xi|^2)^2 (h \circ \widehat{u}_t)(t), \quad \forall t \geq t_0. \quad (53)$$

and

$$\mu_3 (1 + |\xi|^2) \widehat{E}(t) \leq \mathcal{K}(t) \leq \mu_4 (1 + |\xi|^2) \widehat{E}(t). \quad (54)$$

for some $\mu_i > 0, i = 1, \dots, 4$. \square

Theorem 1. Suppose (3)-(4) hold. Then, there exist positive constants d_1 and d_2 such that the energy functional given by (10) satisfies

$$\widehat{E}(t) \leq d_1 \widehat{E}(0) e^{-d_2 \rho(\xi)t}, \quad \forall t \geq 0, \quad (55)$$

where $\rho(\xi) = |\xi|^2 / (1 + |\xi|^2)$.

Proof. From (53) and (2), we have

$$\begin{aligned} \mathcal{K}'(t) &\leq -\mu_1|\xi|^2\widehat{E}(t) - \frac{\mu_2}{\kappa}(1+|\xi|^2)^2(h' \circ \widehat{u}_t)(t) \\ &\leq -\mu_1|\xi|^2\widehat{E}(t) - c(1+|\xi|^2)^2\widehat{E}'(t) \\ &\leq -\mu_1|\xi|^2\widehat{E}(t) - c(1+|\xi|^2)\widehat{E}'(t), \forall t \geq t_0. \end{aligned} \tag{56}$$

Hence,

$$(\mathcal{K}(t) + c(1+|\xi|^2)\widehat{E}(t))' \leq -\mu_1|\xi|^2\widehat{E}(t), \quad \forall t \geq t_0. \tag{57}$$

By exploiting (54), it can easily be shown that

$$\mathcal{K}_1(t) := \mathcal{K}(t) + c(1+|\xi|^2)\widehat{E}(t) \sim (1+|\xi|^2)\widehat{E}(t). \tag{58}$$

Consequently, for some $d_2 > 0$, we find

$$\mathcal{K}'_1(t) \leq -d_2 \frac{|\xi|^2}{(1+|\xi|^2)} \mathcal{K}_1(t), \quad \forall t \geq t_0. \tag{59}$$

By integrating of (59) over (t_0, t) , we obtain

$$\mathcal{K}_1(t) \leq \mathcal{K}_1(t_0)e^{-(d_2|\xi|^2/(1+|\xi|^2))(t-t_0)}, \quad \forall t \geq t_0. \tag{60}$$

Hence, by invoking (54), (58), and (60), the continuity, and the boundedness of E , we establish (55). \square

Now, we state and prove the following result.

Theorem 2. *Let k be a nonnegative integer, and suppose that $|\widehat{U}(\xi, 0)|^2$ is bounded. Then, $U = (u_{tt} + \lambda u_t, \nabla(u_t + \lambda u), \nabla u_t)^T$ satisfies; for all $t \geq 0$, the decay estimate is*

$$\|\nabla^k U(\xi, t)\|_2 \leq C\|U_0\|_1(1+t)^{-n/4-k/2} + Ce^{-ct}\|\nabla^k U_0\|_2. \tag{61}$$

Proof. From (10), let

$$\widehat{E}_*(t) = \widehat{E}_*(\xi, t) := \frac{1}{2} [|\widehat{u}_{tt} + \lambda \widehat{u}_t|^2 + |\xi|^2|\widehat{u}_t + \lambda \widehat{u}|^2 + (1+|\xi|^2)|\widehat{u}_t|^2], \tag{62}$$

and we have $|\widehat{U}(\xi, t)|^2 \sim \widehat{E}_*(t)$ and $\widehat{E}_*(t) \leq c\widehat{E}(t), \forall t \geq 0$; then, by applying the Plancherel theorem and exploiting (55), we find

$$\begin{aligned} \|\nabla^k U(\xi, t)\|_2^2 &= \int_{\mathbb{R}^n} |\xi|^{2k} |\widehat{U}(\xi, t)|^2 d\xi \leq \int_{\mathbb{R}^n} |\xi|^{2k} \widehat{E}_*(\xi, t) d\xi \\ &\leq c \int_{\mathbb{R}^n} |\xi|^{2k} e^{-d_2\rho(\xi)t} |\widehat{U}(\xi, 0)|^2 d\xi \\ &\leq c \underbrace{\int_{|\xi|^2 \leq 1} |\xi|^{2ke^{-d_2\rho(\xi)t}} |\widehat{U}(\xi, 0)|^2 d\xi}_{M_1} \\ &\quad + c \underbrace{\int_{|\xi|^2 \geq 1} |\xi|^{2k} e^{-d_2\rho(\xi)t} |\widehat{U}(\xi, 0)|^2 d\xi}_{M_2}. \end{aligned} \tag{63}$$

Now, we estimate M_1, M_2 , the low-frequency part $|\xi|^2 \leq 1$, and the high-frequency part $|\xi|^2 \geq 1$.

Firstly, we have $\rho(\xi) \geq 1/2|\xi|^2$, for $|\xi|^2 \leq 1$. Then,

$$M_1 \leq c \int_{|\xi|^2 \leq 1} |\xi|^{2k} e^{-(d_2/2)|\xi|^2 t} |\widehat{U}(\xi, 0)|^2 d\xi \leq c \sup_{|\xi|^2 \leq 1} \{|\widehat{U}(\xi, 0)|^2\} \int_{|\xi|^2 \leq 1} |\xi|^{2k} e^{-(d_2/2)|\xi|^2 t} d\xi, \tag{64}$$

and, by using Lemma 2, we obtain

$$M_1 \leq c \sup_{|\xi|^2 \leq 1} \{|\widehat{U}(\xi, 0)|^2\} (1+t)^{-k-n/2} \leq c\|U_0\|_1^2 (1+t)^{-k-n/2}. \tag{65}$$

Secondly, we have $\rho(\xi) \geq 1/2$, for $|\xi|^2 \geq 1$. Then,

$$\begin{aligned} M_2 &\leq c \int_{|\xi|^2 \geq 1} |\xi|^{2k} e^{-(d_2/2)t} |\widehat{U}(\xi, 0)|^2 d\xi \\ &\leq ce^{-(d_2/2)t} \|\nabla^k U(x, 0)\|_2^2, \forall t \geq 0. \end{aligned} \tag{66}$$

Substituting (65) and (66) into (63), we find (61). \square

3. Conclusion

The aim of this work is the study of the general decay estimate of solutions of a new class of Moore–Gibson–Thompson (MGT) equation with respect to the memory kernel of type II by using the energy method in Fourier space. MGT equation is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. In this paper, we examined the different mechanism resulting from the memory kernel of type II (see [6]), which dictates the emergence of the memory term in the system in the framework of Fourier space.

In the next work, we will try to use the same method with MGT equations, but in light of a generalization of assumption (3). Precisely, we will assume that κ is a positive number and ϑ is a function that fulfills the following conditions:

$$\begin{aligned} \exists \kappa > 0, 0 < \kappa \leq \vartheta(t) \leq \vartheta(0), \\ h'(t) \leq -\vartheta(t)h(t), \forall t \geq 0. \end{aligned} \quad (67)$$

Also, we will study the same problem, but with the memory of the type III. In the future, we will try to use the same method with Boussinesq and Hall-MHD equations which are nonlinear partial differential equations that arises in hydrodynamics and some physical applications (see, for example, [17–19]).

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

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