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Research Article

Atomic Decompositions and John-Nirenberg Theorem of Grand Martingale Hardy Spaces with Variable Exponents

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Let $\theta \geq 0$ and $p(\cdot)$ be a variable exponent, and we introduce a new class of function spaces $L_{p(\cdot),\theta}$ in a probabilistic setting which unifies and generalizes the variable Lebesgue spaces with $\theta = 0$ and grand Lebesgue spaces with $p(\cdot) \equiv p$ and $\theta = 1$. Based on the new spaces, we introduce a kind of Hardy-type spaces, grand martingale Hardy spaces with variable exponents, via the martingale operators. The atomic decompositions and John-Nirenberg theorem shall be discussed in these new Hardy spaces.

1. Introduction

The martingale theory is widely studied in the field of mathematical physics, stochastic analysis, and probability. Weisz [1] presented the atomic decomposition theorem for martingale Hardy spaces. Herz [2] established the John-Nirenberg theorem for martingales. Since then, the study of martingale Hardy spaces associated with various functional spaces has attracted a steadily increasing interest. For instance, martingale Orlicz-type Hardy spaces were investigated in [3–6], martingale Lorentz Hardy spaces were studied in [7–9], and variable martingale Hardy spaces were developed in [10–14].

Let $1 < p < \infty$, and the grand Lebesgue space $L_{p)}(E)$ introduced by Iwaniec and Sbordone [15] is defined as the Banach function space of the measurable functions f on finite E such that

$$\|f\|_{L_{p)}} = \sup_{0<\eta<p-1} \left(\eta \frac{1}{|E|} \int_E |f(x)|^{p-\eta} dx \right)^{1/(p-\eta)} < \infty. \quad (1)$$

Such spaces can be used to integrate the Jacobian under minimal hypotheses [15]. The grand Lebesgue spaces as a kind of Banach function space were investigated in the papers of Capone et al. [16, 17], Fiorenza et al. [18–21], Kokilashvili et al. [22, 23], and so forth. In particular, grand Lebesgue spaces with variable exponents were studied in [24, 25].

We find that the framework of grand Lebesgue spaces with variable exponents has not yet been studied in martingale theory. This paper is aimed at discussing the variable grand Hardy spaces defined on the probabilistic setting and showing the atomic decompositions and John-Nirenberg theorem in these new Hardy spaces. More precisely, we first present the atomic characterization of grand Hardy martingale spaces with variable exponents. To do so, we introduce the following new notations of atom.

Definition 1. Let $p(\cdot)$ be a variable exponent and $\theta \geq 0$. A measurable function a is called a $(1, p(\cdot), \theta)$ -atom (resp. $(2, p(\cdot), \theta)$ -atom, $(3, p(\cdot), \theta)$ -atom) if there exists a stopping time τ such that

$(a1) \mathbb{E}_n a = 0, \forall n \leq \tau$

$(a2) \|s(a)\|_{L_\infty} (\text{resp.} \|S(a)\|_{L_\infty}, \|Ma\|_{L_\infty}) \leq \|\chi_{\{\tau<\infty\}}\|^{-1}_{L_{p(\cdot),\theta}}$.

See Section 2 for the notation $L_{p(\cdot),\theta}$. Denote by $\mathbb{A}_s(p(\cdot), \infty)$ (resp. $\mathbb{A}_S(p(\cdot), \infty)$, $\mathbb{A}_M(p(\cdot), \infty)$) the collection of all sequences of triplet (a^k, τ_k, μ_k), where a^k are $(1, p(\cdot), \theta)$ -atoms (resp. $(2, p(\cdot), \theta)$-atoms, $(3, p(\cdot), \theta)$-atoms), τ_k are stopping times satisfying $(a1)$ and $(a2)$ in Definition 1, and μ_k are nonnegative numbers and also

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} < \infty. \tag{2}$$

Under these definitions, we show the atomic decompositions of the grand Hardy martingale spaces with variable exponents (see Section 3). To be precise, we prove that for any $f=(f_n)_{n\ge0}$, $f\in H^s_{p(\cdot),\theta}$ (resp. $Q_{p(\cdot),\theta}$, $D_{p(\cdot),\theta}$) iff there exists a sequence of triplet $(a^k,\tau_k,\mu_k)\in\mathbb{A}_s(p(\cdot),\infty)$ (resp. $\mathbb{A}_S(p(\cdot),\infty)$, $\mathbb{A}_M(p(\cdot),\infty)$) so that for each $n\ge0$,

$$\begin{gathered} f_n=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n a^k \ a.e., \\ \left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} \\ \approx \|f\|_{H^s_{p(\cdot),\theta}} \left(\text{resp.} \|f\|_{Q_{p(\cdot),\theta}}, \|f\|_{D_{p(\cdot),\theta}}\right). \end{gathered} \tag{3}$$

Moreover, we extend the classical John-Nirenberg theorem to the grand variable Hardy martingale spaces. To be precise, under suitable conditions, we present the following one:

$$\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} \approx \|f\|_{\mathrm{BMO}_1}. \tag{4}$$

See Theorem 11 for the details. This conclusion improves the recent results [12, 26], respectively.

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ denote the integer set, nonnegative integer set, and complex numbers set, respectively. We denote by C the absolute positive constant, which can vary from line to line. The symbol $A\lesssim B$ stands for the inequality $A\le CB$. If we write $A\approx B$, then it stands for $A\lesssim B\lesssim A$.

2. Preliminaries

2.1. Grand Lebesgue Spaces with Variable Exponents. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. An $\mathcal{F}$-measurable function $p(\cdot)$: $\Omega\longrightarrow(0,\infty)$ which is called a variable exponent. For convenience, we denote

$$\begin{gathered} p_- := \operatorname{essinf}\{p(\omega):\omega\in\Omega\}, p_+ := \operatorname{esssup}\{p(\omega):\omega\in\Omega\}, \\ p_-(B) = \operatorname{essinf}\{p(\omega):\omega\in B\} \text{ and} \\ p_+(B) = \operatorname{esssup}\{p(\omega):\omega\in B\}. \end{gathered} \tag{5}$$

Denote by $\mathcal{P}=\mathcal{P}(\Omega)$ the collection of all variable exponents $p(\cdot)$ satisfying with $1<p_-\le p_+<\infty$. The variable Lebesgue space $L_{p(\cdot)}=L_{p(\cdot)}(\Omega)$ consists of all $\mathcal{F}$-measurable functions f such that for some $\lambda>0$,

$$\rho\left(\frac{f}{\lambda}\right)=\int_\Omega\left(\frac{|f(\omega)|}{\lambda}\right)^{p(\omega)}d\mathbb{P}<\infty. \tag{6}$$

This leads to a Banach function space under the Luxemburg-Nakano norm

$$\|f\|_{L_{p(\cdot)}} \equiv \inf\left\{\lambda>0:\rho\left(\frac{f}{\lambda}\right)\le1\right\}. \tag{7}$$

Based on this, we begin with the definition of the grand Lebesgue space with variable exponent.

Definition 2. Suppose that $p(\cdot)\in\mathcal{P}$ and $\theta\ge0$. We define the grand Lebesgue space with variable exponent $L_{p(\cdot),\theta}=L_{p(\cdot),\theta}(\Omega)$ as the set of all $\mathcal{F}$-measurable functions f satisfying

$$\|f\|_{L_{p(\cdot),\theta}} := \sup_{0<\eta<p_--1}\eta^{\theta/(p_--\eta)}\|f\|_{L_{p(\cdot)-\eta}}<\infty. \tag{8}$$

The Grand Lebesgue space with variable exponent can unify and generalize the various function spaces. To be precise, if $\theta=0$, $L_{p(\cdot),\theta}$ degenerates to the variable Lebesgue space $L_{p(\cdot)}$. If $\theta=1$ and $p(\cdot)\equiv p$, $L_{p(\cdot),\theta}$ becomes the grand Lebesgue space $L_{p)}$.

2.2. Martingale Grand Hardy Spaces via Variable Exponents. Let $\{\mathcal{F}_n\}_{n\ge0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ sets with $\mathcal{F}=\sigma(\bigcup_{n\ge0}\mathcal{F}_n)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f=(f_n)_{n\ge0}$ of random variables is said to be a martingale, if f_n is $\mathcal{F}_n$-measurable, $\mathbb{E}(|f_n|)<\infty$, and $\mathbb{E}_n(f_{n+1})=f_n$ for every $n\ge0$. Denote $\mathcal{M}$ to be the set of all martingales $f=(f_n)_{n\ge0}$ with respect to $\{\mathcal{F}_n\}_{n\ge0}$ such that $f_0=0$. For $f\in\mathcal{M}$, write its martingale difference by $d_nf=f_n-f_{n-1}(n\ge0, f_{-1}=0)$. Define the maximal function, the square function, and the conditional square function of f, respectively, as follows:

$$\begin{gathered} M_mf=\sup_{n\le m}|f_n|, Mf=\sup_{n\ge0}|f_n|, \\ S_m(f)=\left(\sum_{n=0}^m|df_n|^2\right)^{1/2}, S(f)=\left(\sum_{n=0}^\infty|df_n|^2\right)^{1/2}, \\ s_m(f)=\left(\sum_{n=0}^m\mathbb{E}_{n-1}|df_n|^2\right)^{1/2}, s(f)=\left(\sum_{n=0}^\infty\mathbb{E}_{n-1}|df_n|^2\right)^{1/2}. \end{gathered} \tag{9}$$

Let Γ be the set of all sequences $(\lambda_n)_{n\ge0}$ of nondecreasing, nonnegative, and adapted functions, and $\lambda_\infty:=\lim_{n\longrightarrow\infty}\lambda_n$. For $f\in\mathcal{M}$, $p(\cdot)\in\mathcal{P}$, and $\theta\ge0$, denote

$$\begin{gathered} \Gamma\left[Q_{p(\cdot),\theta}\right](f)=\left\{(\lambda_n)_{n\ge0}\in\Gamma:S_n(f)\le\lambda_{n-1}(n\ge1),\lambda_\infty\in L_{p(\cdot),\theta}\right\}, \\ \Gamma\left[D_{p(\cdot),\theta}\right](f)=\left\{(\lambda_n)_{n\ge0}\in\Gamma:|f_n|\le\lambda_{n-1}(n\ge1),\lambda_\infty\in L_{p(\cdot),\theta}\right\}. \end{gathered} \tag{10}$$

Now we introduce the grand martingale Hardy spaces associated with variable exponents as follows:

$$\begin{aligned}
H^*_{p(\cdot),\theta} &= \left\{ f \in \mathcal{M} : Mf \in L_{p(\cdot),\theta} \right\}, \|f\|_{H^*_{p(\cdot),\theta}} = \|Mf\|_{L_{p(\cdot),\theta}},\\
H^S_{p(\cdot),\theta} &= \left\{ f \in \mathcal{M} : S(f) \in L_{p(\cdot),\theta)} \right\}, \|f\|_{H^S_{p(\cdot),\theta}} = \|S(f)\|_{L_{p(\cdot),\theta}},\\
H^s_{p(\cdot),\theta} &= \left\{ f \in \mathcal{M} : s(f) \in L_{p(\cdot),\theta} \right\}, \|f\|_{H^s_{p(\cdot),\theta}} = \|s(f)\|_{L_{p(\cdot),\theta}},\\
Q_{p(\cdot),\theta} &= \left\{ f \in \mathcal{M} : \|f\|_{Q_{p(\cdot),\theta}} < \infty \right\}, \|f\|_{Q_{p(\cdot),\theta}} = \inf_{(\lambda_n)_{n\geq 0} \in \Gamma\left[Q_{p(\cdot),\theta}\right](f)} \|\lambda_\infty\|_{L_{p(\cdot),\theta}},\\
D_{p(\cdot),\theta} &= \left\{ f \in \mathcal{M} : \|f\|_{D_{p(\cdot),\theta}} < \infty \right\}, \|f\|_{D_{p(\cdot),\theta}} = \inf_{(\lambda_n)_{n\geq 0} \in \Gamma\left[D_{p(\cdot),\theta}\right](f)} \|\lambda_\infty\|_{L_{p(\cdot),\theta}}.
\end{aligned} \tag{11}$$

The bounded $L_{p(\cdot),\theta}$-martingale spaces

$$L_{p(\cdot),\theta} = \left\{ f = (f_n)_{n\geq 0} : \sup_{n\geq 0} \|f_n\|_{L_{p(\cdot),\theta}} < \infty \right\}, \tag{12}$$

where

$$\|f\|_{L_{p(\cdot),\theta}} = \sup_{n\geq 0} \|f_n\|_{L_{p(\cdot),\theta}}. \tag{13}$$

Remark 3. If $\theta = 0$, then we obtain the definitions of $H^*_{p(\cdot)}$, $H^S_{p(\cdot)}$, $H^s_{p(\cdot)}$, $Q_{p(\cdot)}$, and $D_{p(\cdot)}$, respectively (see [10, 12, 27]). If we consider the special case $\theta = 1$ and $p(\cdot) \equiv p$ with the notations above, we obtain the definitions of $H^*_{p)}$, $H^S_{p)}$, $H^s_{p)}$, $Q_{p)}$, and $D_{p)}$, respectively (see [26]). In addition, if $p(\cdot) \equiv p$ and $\theta = 0$, we obtain the martingale Hardy spaces H^*_q, H^S_q, H^s_q, Q_q, and D_q, respectively (see [28]).

Refer to [29, 30] for more information on martingale theory.

3. Atomic Characterization

The method of atomic characterization plays an useful tool in martingale theory (see for instance [1, 4, 6, 31–33]). We shall construct the atomic characterizations for grand Hardy martingale spaces with variable exponents in this section.

Theorem 4. *Let $p(\cdot) \in \mathcal{P}$ and $\theta \geq 0$. If the martingale $f \in H^s_{p(\cdot),\theta}$, then there exists a sequence of triplet $(a^k, \tau_k, \mu_k) \in \mathbb{A}_s(p(\cdot),\infty)$ so that for each $n \geq 0$,*

$$\sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k = f_n, \text{ a.e.}, \tag{14}$$

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} \lesssim \|f\|_{H^s_{p(\cdot),\theta}}. \tag{15}$$

Conversely, if the martingale f has a decomposition of (14), then

$$\|f\|_{H^s_{p(\cdot),\theta}} \lesssim \inf \left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}}, \tag{16}$$

where the infimum is taken over all the admissible representations of (14).

Proof. Let $f \in H^s_{p(\cdot),\theta}$. Now consider the stopping time for each $k \in \mathbb{Z}$:

$$\tau_k = \inf \left\{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \right\}. \tag{17}$$

It is easy to see that the sequence of these stopping times is nondecreasing. For each stopping time τ, denote $f^\tau_n = f_{n\wedge\tau}$. It is easy to write that

$$f_n = \sum_{k\in\mathbb{Z}} \left(f^{\tau_{k+1}}_n - f^{\tau_k}_n \right). \tag{18}$$

For each $k \in \mathbb{Z}$, let $\mu_k = 3 \cdot 2^k \left\| \chi_{\{\tau_k<\infty\}} \right\|_{L_{p(\cdot),\theta}}$. If $\mu_k \neq 0$, we set

$$a^k_n = \frac{f^{\tau_{k+1}}_n - f^{\tau_k}_n}{\mu_k}, n \in \mathbb{N}. \tag{19}$$

If $\mu_k = 0$, we set $a^k_n = 0$ for each $n \in \mathbb{N}$. For each fixed $k \in \mathbb{Z}$, $(a^k_n)_{n\geq 0}$ is a martingale. Since $s(f^{\tau_k}) = s_{\tau_k}(f) \leq 2^k$, we get

$$s\left(a^k_n\right) \leq \frac{s(f^{\tau_{k+1}}) + s(f^{\tau_k})}{\mu_k} \leq \left\| \chi_{\{\tau_k<\infty\}} \right\|^{-1}_{L_{p(\cdot),\theta}}. \tag{20}$$

We can easily check that $(a^k_n)_{n\geq 0}$ is a bounded L_2-martingale. Hence, there exists an element $a^k \in L_2$ such that $\mathbb{E}_n a^k = a^k_n$. If $n \leq \tau_k$, then $a^k_n = 0$, and $s(a^k) \leq \|\chi_{\{\tau_k<\infty\}}\|^{-1}_{L_{p(\cdot),\theta}}$. Consequently, it implies that a^k is really a $(1, p(\cdot), \theta)$-atom.

Denote $\Lambda_k := \{\tau_k<\infty\}$. Knowing that $\{\tau_k<\infty\} = \{s(f) > 2^k\}$ and τ_k is nondecreasing for each $k \in \mathbb{Z}$, we obtain $\Lambda_{k+1} \subseteq \Lambda_k$. Now, we point out that

$$\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x) = 2 \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k\setminus\Lambda_{k+1}}(x). \tag{21}$$

Indeed, for a fixed $x_0 \in \Omega$, there is $k_0 \in \mathbb{Z}$ so that $x_0 \in \Lambda_{k_0}$ and $x_0 \in \Lambda_{k_0+1}$, then we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) &= \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) = \sum_{k=-\infty}^{k_0} 3 \cdot 2^k = 3 \cdot 2^{k_0+1},\\
\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k\setminus\Lambda_{k+1}}(x_0) &= \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\Lambda_k\setminus\Lambda_{k+1}}(x_0) = 3 \cdot 2^{k_0}.
\end{aligned} \tag{22}$$

This means

$$\begin{aligned}
&\left\|\sum_{k\in\mathbb{Z}}\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}=\left\|\sum_{k\in\mathbb{Z}}3\cdot 2^k\chi_{\{\tau_k<\infty\}}\right\|_{L_{p(\cdot),\theta}}=6\left\|\sum_{k\in\mathbb{Z}}2^k\chi_{\Lambda_k\backslash\Lambda_{k+1}}\right\|_{L_{p(\cdot),\theta}}\\
&=6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\int_\Omega\left(\sum_{k\in\mathbb{Z}}\frac{2^k\chi_{\Lambda_k\backslash\Lambda_{k+1}}(x)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\
&=6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\sum_{k\in\mathbb{Z}}\int_{\Lambda_k\backslash\Lambda_{k+1}}\left(\frac{2^k}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\
&=6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\sum_{k\in\mathbb{Z}}\int_{\Lambda_k\backslash\Lambda_{k+1}}\left(\frac{s(f)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\
&\le 6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\int_\Omega\left(\frac{s(f)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\
&=6\sup_{0<\eta<p_--1}\eta^{\theta/(p_--\eta)}\|s(f)\|_{L_{p(\cdot)-\eta}}=6\|f\|_{H^s_{p(\cdot),\theta}}.
\end{aligned}\tag{23}$$

For the converse part, according to the definition of $(1,p(\cdot),\theta)$-atom, we easily conclude

$$s\left(a^k\right)=s\left(a^k\right)\chi_{\{\tau_k<\infty\}}\le\left\|s\left(a^k\right)\right\|_{L_\infty}\chi_{\{\tau_k<\infty\}}\le\left\|\chi_{\{\tau_k<\infty\}}\right\|^{-1}_{L_{p(\cdot),\theta}}\chi_{\{\tau_k<\infty\}},\tag{24}$$

where a^k is the $(1,p(\cdot),\theta)$-atom and τ_k is the stopping time associated with a^k which, when combined with the subadditivity of the operator s, yields

$$s(f)\le\sum_{k\in\mathbb{Z}}\mu_k s\left(a^k\right)\le\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}.\tag{25}$$

This implies

$$\|f\|_{H^s_{p(\cdot),\theta}}=\|s(f)\|_{L_{p(\cdot),\theta}}\le\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}.\tag{26}$$

Taking over all the admissible representations of (14) for f, we obtain the desired result.

Next, we will characterize $Q_{p(\cdot),\theta}$ and $D_{p(\cdot),\theta}$ by atoms, respectively. The proof is similar to the proof of Theorem 4. For the completeness of this paper, we provide some details. □

Theorem 5. *Suppose $p(\cdot)\in\mathscr{P}$ and $\theta\ge 0$. If the martingale $f=(f_n)_{n\ge 0}\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$), then there exists a sequence of triplet $(a^k,\tau_k,\mu_k)\in\mathbb{A}_S(p(\cdot),\infty)$ (resp. $\mathbb{A}_M(p(\cdot),\infty)$) so that for each $n\in\mathbb{N}$,*

$$f_n=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n a^k,\tag{27}$$

$$\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}\lesssim\|f\|_{Q_{p(\cdot),\theta}}\left(resp.\|f\|_{D_{p(\cdot),\theta}}\right).\tag{28}$$

Conversely, if the martingale $f=(f_n)_{n\ge 0}$ has admissible representation (27), then $f\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$) and

$$\|f\|_{Q_{p(\cdot),\theta}}\left(resp.\|f\|_{D_{p(\cdot),\theta}}\right)\lesssim\inf\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}},\tag{29}$$

where the infimum is taken over all the admissible representations of (27).

Proof. The proof follows the ideas in Theorem 4, so we omit some details. Suppose $f=(f_n)_{n\ge 0}\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$). We define stopping times as follows:

$$\tau_k=\inf\left\{n\in\mathbb{N}:\lambda_n>2^k\right\},\inf\varnothing=\infty,\tag{30}$$

where $(\lambda_n)_{n\ge 0}$ is an adapted, nondecreasing sequence such that almost everywhere $|S_n(f)|\le\lambda_{n-1}$ (resp.$|f_n|\le\lambda_{n-1}$) and $\lambda_\infty\in L_{p(\cdot),\theta}$.

Let $(a^k)_{k\in\mathbb{Z}}$ and $(\mu_k)_{k\in\mathbb{Z}}$ be defined as in the proof of Theorem 4. And replace $\Lambda_k=\{\tau_k<\infty\}=\{s(f)>2^k\}$ by the $\Lambda_k=\{\tau_k<\infty\}=\{\lambda_\infty>2^k\}$. Then, we obtain that $f_n=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n a^k$ and (28) still hold.

For the converse part, write

$$\lambda_n=\sum_{k\in\mathbb{Z}}\mu_k\|S\left(a^k\right)\|_{L_\infty}\chi_{\{\tau_k\le n\}}\left(resp.\lambda_n=\sum_{k\in\mathbb{Z}}\mu_k\|M\left(a^k\right)\|_{L_\infty}\chi_{\{\tau_k\le n\}}\right).\tag{31}$$

Clearly, $(\lambda_n)_{n\ge 0}$ is a nonnegative, nondecreasing, and adapted sequence with $S_{n+1}(f)\le\lambda_n$ (resp.$|f_n|\le\lambda_n$). Thus, we get

$$\|f\|_{Q_{p(\cdot),\theta}}\left(\text{resp.}\|f\|_{D_{p(\cdot),\theta}}\right)=\|\lambda_\infty\|_{L_{p(\cdot),\theta}}\le\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau<\infty\}}}{\|\chi_{\{\tau<\infty\}}\|_{p(\cdot),\theta}}\right\|_{p(\cdot),\theta}.\tag{32}$$

□

Taking over all the admissible representations of (27) for f, we obtain the desired result.

Remark 6. Suppose $p(\cdot)\in\mathscr{P}$ and $\theta\ge 0$. We conclude that the sum $\sum_{k=M}^N\mu_k a^k$ in Theorem 4 converges to f in $H^s_{p(\cdot),\theta}$ as $M\longrightarrow-\infty$, $N\longrightarrow\infty$. Indeed, it follows by the subadditive of the operator s, we get, for any $M,N\in\mathbb{Z}$ with $M<N$,

$$s\left(f-\sum_{k=M}^{N}\mu_k a^k\right)\leq s\left(f-f^{\tau_{N+1}}\right)+s\left(f^{\tau_M}\right). \tag{33}$$

Moreover, $s(f-f^{\tau_{N+1}})$ is decreasing and convergent to 0 (a.e.) as $N\longrightarrow\infty$, and $s(f^{\tau_M})$ is decreasing and convergent to 0 (a.e.) as $M\longrightarrow-\infty$. From this and the dominated convergence theorem in $L_{p(\cdot)-\varepsilon}$ for $0<\varepsilon<p_--1$ (see [34], Theorem 2.62), it follows that

$$\begin{aligned}\left\|f-\sum_{k=M}^{N}\mu_k a^k\right\|_{H^s_{p(\cdot),\theta}}&\leq\left\|s\left(f-f^{\tau_{N+1}}\right)+s\left(f^{\tau_M}\right)\right\|_{L_{p(\cdot),\theta}}\\&\leq\sup_{0<\eta<p_--1}\eta^{\theta/(p_--\eta)}\left\|s\left(f-f^{\tau_{N+1}}\right)\right\|_{L_{p(\cdot)-\eta}}+\sup_{0<\eta<p_--1}\\&\quad\cdot\,\eta^{\theta/(p_--\eta)}\left\|s\left(f^{\tau_M}\right)\right\|_{L_{p(\cdot)-\eta}}\longrightarrow 0\text{ as }M\longrightarrow-\infty,N\longrightarrow\infty.\end{aligned} \tag{34}$$

Furthermore, we can also show the norm convergence of the summation $\sum_{k=M}^{N}\mu_k a^k$ in Theorems 5 as $M\longrightarrow-\infty$, $N\longrightarrow\infty$.

4. The Generalized John-Nirenberg Theorem

In the sequel of this section, we will often suppose that every $\mathscr{F}_n$ is generated by countably many atoms. Recall that $B\in\mathscr{F}_n$ is called an atom, and if for any $A\subseteq B$ with $A\in\mathscr{F}_n$ satisfying $\mathbb{P}(A)<\mathbb{P}(B)$, we have $\mathbb{P}(A)=0$. We denote by $A(\mathscr{F}_n)$ the set of all atoms in $\mathscr{F}_n$. We shall present the generalized John-Nirenberg theorem on grand Lebesgue spaces with variable exponents. For each $1\leq p<\infty$, the Banach space BMO_p (bounded mean oscillation [35]) is defined as

$$\mathrm{BMO}_p=\left\{f\in L_p:\|f\|_{\mathrm{BMO}_p}=\sup_{n\geq1}\left\|\mathbb{E}_n\left(|f-\mathbb{E}_{n-1}f|^p\right)\right\|_{L_\infty}^{1/p}<\infty\right\}. \tag{35}$$

It can be easily shown that the norm of BMO_p is equivalent to

$$\|f\|_{\mathrm{BMO}_p}=\sup_{\tau\in\mathscr{T}}\frac{\|f-f^{\tau-1}\|_{L_p}}{\|\chi_{\{\tau<\infty\}}\|_{L_p}}, \tag{36}$$

where $\mathscr{T}$ consists of all stopping times relative to $\{\mathscr{F}_n\}_{n\geq0}$. It follows immediately from the John-Nirenberg theorem [2, 30] that

$$\mathrm{BMO}_p=\mathrm{BMO}_1,\ 1<p<\infty. \tag{37}$$

What is more, in [2], there has

$$C\cdot p\|f\|_{\mathrm{BMO}_1}\geq\|f\|_{\mathrm{BMO}_p}\geq\|f\|_{\mathrm{BMO}_1}. \tag{38}$$

Definition 7. For $p(\cdot)\in\mathscr{P}$ and $\theta\geq0$, the generalized BMO martingale space is defined by

$$\mathrm{BMO}_{p(\cdot),\theta}=\left\{f\in L_{p(\cdot),\theta}:\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}<\infty\right\}, \tag{39}$$

where

$$\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}=\sup_{\tau\in\mathscr{T}}\frac{\|f-f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot),\theta}}}. \tag{40}$$

Remark 8. If $\theta=0$, $\mathrm{BMO}_{p(\cdot),\theta}$ degenerates to the variable exponent BMO martingale space $\mathrm{BMO}_{p(\cdot)}$ introduced and studied in [12]. If $\theta=1$ and $p(\cdot)\equiv p$, $\mathrm{BMO}_{p(\cdot),\theta}$ becomes the grand BMO martingale space $\mathrm{BMO}_{p)}$ studied in [26].

In order to establish the generalized John-Nirenberg theorem in the framework of $\mathrm{BMO}_{p(\cdot),\theta}$, we need the following lemmas and notations.

Lemma 9 (Hölder's inequality, see [34]). *Let $p(\cdot),q(\cdot),r(\cdot)\in\mathscr{P}$ satisfy*

$$\frac{1}{p(\omega)}+\frac{1}{q(\omega)}=\frac{1}{r(\omega)},\ a.e.\,\omega\in\Omega. \tag{41}$$

Then, there exists a constant C such that for all $f\in L_{p(\cdot)}$ and $g\in L_{q(\cdot)}$, we have $fg\in L_{r(\cdot)}$ and

$$\|fg\|_{L_{r(\cdot)}}\leq C\|f\|_{L_{p(\cdot)}}\|g\|_{L_{q(\cdot)}}. \tag{42}$$

We mention that if the variable exponent $p(x)$ meets the log-Hölder continuity condition in Euclidean spaces, the inverse Hölder's inequality holds for characteristic functions in $L_{p(\cdot)}(\mathbb{R}^n)$ (see [36]). Compared with Euclidean space $\mathbb{R}^n$, the probability space $(\Omega,\mathbb{P})$ has no natural metric structure. Fortunately, Jiao et al. [11, 27] put forward the following condition: there exists an absolute constant $\kappa\geq1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(B)^{p_-(B)-p_+(B)}\leq\kappa,\forall B\in\bigcup_{n\geq0}A(\mathscr{F}_n). \tag{43}$$

Lemma 10 (see [27]). *Suppose $p(\cdot)\in\mathscr{P}$ satisfying (43).*

(1) For each $B\in\bigcup_{n\geq0}A(\mathscr{F}_n)$, we get

$$\mathbb{P}(B)\approx\|\chi_B\|_{L_{p(\cdot)}}\|\chi_B\|_{L_{p'(\cdot)}}. \tag{44}$$

(2) Let $q(\cdot)\in\mathscr{P}$ satisfy (43). If $r(\cdot)$ satisfies

$$\frac{1}{r(\omega)}=\frac{1}{p(\omega)}+\frac{1}{q(\omega)},\ a.e.\,\omega\in\Omega, \tag{45}$$

then $r(\cdot)$ also satisfies condition (43). Moreover, for each $B\in\bigcup_{n\geq0}A(\mathscr{F}_n)$, we deduce

$$\|\chi_B\|_{L_{r(\cdot)}} \approx \|\chi_B\|_{L_{p(\cdot)}}\|\chi_B\|_{L_{q(\cdot)}}. \tag{46}$$

Theorem 11. *Suppose that $p(\cdot) \in \mathscr{P}$ satisfies (43) and $\theta \geq 0$. Then, for every $f \in BMO_1$, there has*

$$\|f\|_{BMO_1} \lesssim \|f\|_{BMO_{p(\cdot),\theta}} \lesssim \|f\|_{BMO_1}. \tag{47}$$

Proof. If $p(\cdot) \in \mathscr{P}$ satisfies (43), then we clearly get that $p(\cdot) - \eta$ also satisfies (43) for $0 < \eta < p_- - 1$. It follows from Lemmas 9 and 10 that

$$\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \leq \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}\|\chi_{\{\tau<\infty\}}\|_{L_{(p(\cdot)-\eta)'}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \approx \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot)-\eta}}}, \tag{48}$$

for any $0 < \eta < p_- - 1$. Here, the variable exponent $(p(\cdot) - \eta)'$ is defined by

$$\frac{1}{(p(\omega) - \eta)'} + \frac{1}{p(\omega) - \eta} = 1, \text{ a.e.} \omega \in \Omega. \tag{49}$$

This is equivalent to the following inequality:

$$\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \cdot \left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot)-\eta}} \lesssim \|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}. \tag{50}$$

Hence, we have

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} &= \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\left(\|f - f^{\tau-1}\|_{L_1}/\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}\right) \cdot \left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \\ &\lesssim \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} = \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}}.\end{aligned} \tag{51}$$

Taking the supremum over all stopping times, we deduce

$$\|f\|_{\mathrm{BMO}_1} \leq \|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}. \tag{52}$$

Conversely, from the definition of $L_{p(\cdot),\theta}$, we get

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}} &= \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)}\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \\ &\leq \sup_{0<\eta<p_--1}\left\{\frac{\eta^{\theta/(p_--\eta)}\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\eta^{\theta/(p_--\eta)}\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}}\right\} \\ &= \sup_{0<\eta<p_--1}\left\{\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}}\right\}.\end{aligned} \tag{53}$$

It follows from Lemma 9 that

$$\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \leq \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}\|\chi_{\{\tau<\infty\}}\|_{L_{q(\cdot)}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \approx \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}}, \tag{54}$$

where $q(\cdot)$ satisfies

$$\frac{1}{p(\omega) - \eta} = \frac{1}{2p_+} + \frac{1}{q(\omega)}, \text{ a.e.} \omega \in \Omega. \tag{55}$$

Hence, by (38), we deduce that

$$\begin{aligned}\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} &= \sup_{\tau\in\mathcal{T}} \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}} \lesssim \sup_{\tau\in\mathcal{T}} \sup_{0<\eta<p_--1} \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}} \\ &= \sup_{\tau\in\mathcal{T}} \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}} = \|f\|_{\mathrm{BMO}_{2p_+}} \leq C \cdot 2p_+\|f\|_{\mathrm{BMO}_1}.\end{aligned} \tag{56}$$

From what has been discussed above, we draw the conclusion that

$$\|f\|_{\mathrm{BMO}_1} \lesssim \|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} \lesssim \|f\|_{\mathrm{BMO}_1}. \tag{57}$$

Theorem 11 improves the recent results [12, 26], respectively. More precisely, if we consider the case $\theta = 0$, then the following result holds: □

Corollary 12. *If $p(\cdot)$ satisfies (43) with $1 < p_- \leq p_+ < \infty$, then for $f \in BMO_1$,*

$$\|f\|_{BMO_{p(\cdot)}} \approx \|f\|_{BMO_1}. \tag{58}$$

And especially for $\theta = 1$ and $p(\cdot) \equiv p$, we get the conclusion as follows.

Corollary 13 (see [26]). *Suppose $1 < p < \infty$, then for $f \in BMO_1$,*

$$\|f\|_{BMO_{p)}} \approx \|f\|_{BMO_1}. \tag{59}$$

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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