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Research Article

Iterative Arrangements of the MSCFP for Strictly Pseudocontractive Mappings

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In this paper, we consider the multiple-set split common fixed point problem in Hilbert spaces. We first study a couple of critical properties of strictly pseudocontractive mappings and particularly the property under mix activity. By utilizing these properties, we propose new iterative strategies for settling this problem as well as several connected issues. Under delicate conditions, we state weak convergence of the proposed strategies that expands the current works from the case of two subsets to the instance of multiple subsets. As an application, we give an exhibit of the theoretical results to the multiple-set split equality problem and the elastic net regularization.

1. Introduction

Let t and s be the two positive integers, and H_1 and H_2 stand for two Hilbert spaces. The well-known split feasibility problem (SFP) [1] is formulated as follows: find a point $x \in H_1$ satisfying the property

$$\begin{aligned} &x \in C, \\ &Ax \in Q, \end{aligned} \tag{1}$$

where C and Q are nonempty closed convex subset of H_1 and H_2, respectively, and A is a bounded linear mapping from H_1 into H_2. There are many generalizations of the SFP, one of which is from two groups to multiple groups, that is, multiple-set split feasibility problem (MSFP) [2]. Actually, it can be formulated as the problem of finding $x \in H_1$ such that

$$\begin{aligned} &x \in \bigcap_{i=1}^{t} C_i, \\ &Ax \in \bigcap_{j=1}^{s} Q_j, \end{aligned} \tag{2}$$

where $A : H_1 \longrightarrow H_2$ is as above and $\{C_i\}_{i=1}^{t} \subset H_1$ and $\{Q_j\}_{j=1}^{s} \subset H_2$ are two classes of nonempty convex closed subsets.

The split common fixed point problem (SCFP) [3] is another generalization of the SFP, which requires to find an element in a fixed point set such that its image under a linear transformation belongs to another fixed point set. Formally, it consists in finding $x \in H_1$ such that

$$\begin{aligned} &x \in F(U), \\ &Ax \in F(T), \end{aligned} \tag{3}$$

where $A : H_1 \longrightarrow H_2$ is as above and $F(U)$ and $F(T)$ are, respectively, the fixed point sets of nonlinear mappings $U : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$. Specially, if U and T are both metric projections, then problem (3) is reduced to the SFP. As a further extension of the SFP, we recall the multiple-set split common fixed point problem (MSCFP). Indeed, the MSCFP extends the SCFP from two groups to the case of multiple groups. Formally, it consists in finding $x \in H_1$ such that

$$x \in \bigcap_{i=1}^{t} F(U_i),$$
$$Ax \in \bigcap_{j=1}^{s} F(T_j), \tag{4}$$

where $A : H_1 \longrightarrow H_2$ is as above and $F(U_i)$ and $F(T_j)$ are, respectively, the fixed point sets of nonlinear mappings $U_i : H_1 \longrightarrow H_1, i = 1, 2, \cdots, t$ and $T_j : H_2 \longrightarrow H_2, j = 1, 2, \cdots, s$. Recently, we [4] considered problem (4) whenever the involved mappings are demicontractive. These issues have been concentrated on broadly in different regions like image reconstruction and signal processing [5–9].

There are many algorithms in the literature that can solve the SCFP problem (see, e.g., [10–16]). However, in most of these algorithms, the choice of the stepsize is related to $\|A\|$. Thus, to implement these algorithms, one has to compute (or at least estimate) the norm $\|A\|$, which is generally not easy in practice. A way avoiding this is to adopt variable stepsize which ultimately has no relation with $\|A\|$ [11, 12, 17]. In this connection, Wang [18] recently proposed the following method:

$$x_{n+1} = x_n - \tau_n[(I - U)x_n + A^*(I - T)Ax_n], \tag{5}$$

where A^* is the conjugate of A, I stands for the identity mapping, and $\{\tau_n\} \subset (0,\infty)$ is chosen such that

$$\sum_{n=0}^{\infty} \tau_n = \infty,$$
$$\sum_{n=0}^{\infty} \tau_n^2 < \infty. \tag{6}$$

It is shown that if mappings U and T are firmly nonexpansive, then the sequence $\{x_n\}$ generated by (5) converges weakly to a solution of problem (3). It is clear that such a choice of the stepsize does not rely on the norm $\|A\|$. Kraikaew and Saejung [16] weakened condition (6) as follows:

$$\sum_{n=0}^{\infty} \tau_n = \infty,$$
$$\lim_{n \longrightarrow \infty} \tau_n = 0. \tag{7}$$

Furthermore, we [19] extended the above results from the class of firmly nonexpansive mappings to the class of strictly pseudocontractive mappings.

Inspired by the above work, we will continue to present and investigate strategies for addressing the MSCFP in Hilbert spaces. We initially explore a few properties of strictly pseudocontractive mappings and track down its soundness under arched combinatorial operation. Exploiting these properties, we propose another iterative algorithm to address the MSCFP, as well as the MSFP. Under gentle conditions, we acquire weak convergence of the proposed algorithm. Our outcomes broaden related work from the instance of two groups to the case of multiple groups.

2. Preliminary

Throughout the paper, assume that H, H_1, H_2, and H_3 are real Hilbert spaces, and $F(T)$ denotes its fixed point set of a mapping T. For any $\alpha, \beta \in \mathbb{R}$ and $x, y \in H$, it is well known that [20]

$$\|\beta x + \alpha y\|^2 = \beta(\beta + \alpha)\|x\|^2 + \alpha(\beta + \alpha)\|y\|^2 - \beta\alpha\|x - y\|^2. \tag{8}$$

Recall that the mapping $T : H \longrightarrow H$ is called nonexpansive if

$$\|x - y\| \le \|x - y\|, \forall x, y \in H. \tag{9}$$

It is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in H. \tag{10}$$

It is called k-strictly pseudocontractive $(k < 1)$ if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in H. \tag{11}$$

It is clear that the class of strictly pseudocontractive mappings includes the class of nonexpansive mappings, while the latter includes the class of firmly nonexpansive mappings. Indeed, a firmly nonexpansive mapping is -1-strictly pseudocontractive, while a nonexpansive mapping is 0-strictly pseudocontractive. In general, these inclusion are proper (cf. [20, 21]). The following properties of strictly pseudocontractive mappings play an import role in the subsequent analysis. It was shown [21] that if $T : H \longrightarrow H$ is k-strictly pseudocontractive, then it follows that

$$\langle Tx - z, (I - T)x\rangle \ge 0, \forall z \in F(T), x \in H,$$
$$\langle x - z, (I - T)x\rangle \ge \|(I - T)x\|^2, \forall z \in F(T), x \in H. \tag{12}$$

Moreover, the fixed point set of T is convex and closed. We now collect further properties of strictly pseudocontractive mappings.

Lemma 1. *A mapping $T : H \longrightarrow H$ is k-strictly pseudocontractive with $k < 1$ if and only if there is a nonexpansive mapping R such that*

$$T = \frac{1}{1-k}R - \frac{k}{1-k}I. \tag{13}$$

Proof. "⇒" Assume T is k-strictly pseudocontractive. Let $R = kI + (1 - k)T$. It is easy to verify that R fulfils (13). It remains to show that R is nonexpansive. To this end, fix any $x, z \in H$. It then follows from (8) and the property of strictly pseudocontractive mappings that

$$\begin{aligned} \|Rx - Rz\|^2 &= \|(kx + (1-k)Tx) - (kz + (1-k)Tz)\|^2 \\ &= \|k(x - z) + (1-k)(Tx - Tz)\|^2 \\ &= k\|x - z\|^2 + (1-k)\|Tx - Tz\|^2 \\ &\quad - k(1-k)\|(I-T)x - (I-T)z\|^2 \\ &\leq k\|x - z\|^2 + (1-k)\big(\|x - z\|^2 \\ &\quad + k\|(I-T)x - (I-T)z\|^2\big) \\ &\quad - k(1-k)\|(I-T)x - (I-T)z\|^2 \\ &= \|x - z\|^2. \end{aligned} \tag{14}$$

Hence, we have $\|Rx - Rz\| \leq \|x - z\|$; that is, R is nonexpansive.

"$\Leftarrow$" Assume that there is a nonexpansive mapping R such that (13) follows. Choose any $x, z \in H$. It then follows from (8) and the property of nonexpansive mappings that

$$\begin{aligned} \|Tx - Tz\|^2 &= \left\| \left(\frac{Rx}{1-k} - \frac{kx}{1-k} \right) - \left(\frac{Rz}{1-k} - \frac{kz}{1-k} \right) \right\|^2 \\ &= \left\| \frac{1}{1-k}(Rx - Rz) - \frac{k}{1-k}(x - z) \right\|^2 \\ &= \frac{1}{1-k}\|Rx - Rz\|^2 - \frac{k}{1-k}\|x - z\|^2 \\ &\quad + \frac{k}{(1-k)^2}\|(I-R)x - (I-R)z\|^2 \\ &\leq \frac{1}{1-k}\|x - z\|^2 - \frac{k}{1-k}\|x - z\|^2 \\ &\quad + \frac{k}{(1-k)^2}\|(I-R)x - (I-R)z\|^2 \\ &\quad \cdot \|x - z\|^2 + k\|(I-T)x - (I-T)z\|^2. \end{aligned} \tag{15}$$

Hence, T is strictly pseudocontractive, and thus, the proof is complete. □

Remark 2. Note that a firmly nonexpansive mapping is -1 -strictly pseudocontractive. It is well known that a mapping T is firmly nonexpansive if and only if there is a nonexpansive mapping R such that $T = (I + R)/2$. The following lemma can be regarded as an extension of this assertion.

Lemma 3. *Assume that $T_i : H \longrightarrow H$ is strictly pseudocontractive for each $i = 1, 2 \cdots t$. Let $T = \sum_{i=1}^t w_i T_i$, where $0 < w_i < 1$, $\sum_{i=1}^t w_i = 1$. If $\bigcap_{i=1}^t F(T_i)$ is nonempty, then*

$$F(T) = \bigcap_{i=1}^{t} F(T_i). \tag{16}$$

Proof. It suffices to show that $F(T) \subseteq \bigcap_{i=1}^t F(T_i)$. Fix $z \in \bigcap_{i=1}^t F(T_i)$ and choose any $x \in F(T)$. By our hypothesis, there exists $k_i < 1$ such that

$$\frac{1 - k_i}{2}\|x - T_i x\|^2 \leq \langle x - T_i x, x - z \rangle, \tag{17}$$

for every $i = 1, 2 \cdots t$. Adding up these inqualities, we have

$$\begin{aligned} &\sum_{i=1}^t w_i(1 - k_i)\|x - T_i x\|^2 \\ &\quad \leq 2\sum_{i=1}^t w_i \langle x - T_i x, x - z \rangle \\ &\quad = 2\left\langle x - \sum_{i=1}^t w_i T_i x, x - z \right\rangle \\ &\quad = 2\langle x - Tx, x - z \rangle = 0. \end{aligned} \tag{18}$$

Thus, $\sum_{i=1}^t w_i(1 - k_i)\|x - T_i x\|^2 = 0$. Since $w_i(1 - k_i) > 0$, we have $\|x - T_i x\| = 0$ for all $i = 1, 2 \cdots t$. Moreover, since x is chosen arbitrarily, we get $F(T) \subseteq \bigcap_{i=1}^t F(T_i)$. Hence, the proof is complete. □

Lemma 4. *For each $i = 1, 2 \cdots t$, let $0 < w_i < 1$ and $\sum_{i=1}^t w_i = 1$, and $T_i : H \longrightarrow H$ is strictly pseudocontractive with $k_i < 1$. Then, $T = \sum_{i=1}^t w_i T_i$ is strictly pseudocontractive with*

$$k = 1 - \frac{1}{\sum_{i=1}^t w_i(1 - k_i)^{-1}}. \tag{19}$$

Proof. By our hypothesis, for each $i = 1, 2 \cdots, t$, there exists a nonexpansive mapping R_i such that $T_i = (1 - k_i)^{-1} R_i - k_i (1 - k_i)^{-1} I$. Now, let us define a mapping R as

$$R = \sum_{i=1}^t \frac{(1-k)w_i}{1 - k_i} R_i, \tag{20}$$

where k is defined as in (19). It is readily seen that

$$\begin{aligned} \sum_{i=1}^t w_i T_i &= \sum_{i=1}^t \frac{w_i}{1 - k_i} R_i - \sum_{i=1}^t \frac{w_i k_i}{1 - k_i} I \\ &= \frac{1}{1-k}\sum_{i=1}^t \frac{(1-k)w_i}{1 - k_i} R_i - \frac{k}{1-k} I \\ &= \frac{1}{1-k} R - \frac{k}{1-k} I. \end{aligned} \tag{21}$$

From Lemma 1, it remains to show that R is nonexpansive. To this end, choose any $x, z \in H$. By $1 - k = \left(\sum_{i=1}^t w_i (1 - k_i)^{-1}\right)^{-1}$, we have

$$\begin{aligned}\|Rx-Rz\| &= (1-k)\left\|\sum_{i=1}^{t}\frac{w_i}{1-k_i}R_ix-\sum_{i=1}^{t}\frac{w_i}{1-k_i}R_iz\right\| \\ &= \left\|\sum_{i=1}^{t}\frac{w_i(1-k_i)^{-1}}{\sum_{i=1}^{t}w_i(1-k_i)^{-1}}(R_ix-R_iz)\right\| \\ &\le \sum_{i=1}^{t}\frac{w_i(1-k_i)^{-1}}{\sum_{i=1}^{t}w_i(1-k_i)^{-1}} \\ &\le \sum_{i=1}^{t}\frac{w_i(1-k_i)^{-1}}{\sum_{i=1}^{t}w_i(1-k_i)^{-1}}\|x-z\| = \|x-z\|.\end{aligned} \tag{22}$$

Hence, R is nonexpansive, and thus, the proof is complete. □

3. The Case for Strictly Pseudocontractive Mappings

First, let us recall a weak convergence theorem of iterative method (5) for approximating a solution of the two-set split common fixed point problem.

Theorem 5 ([19], Theorem 3.1). *Let $k, l \in (-\infty, 1)$. Assume that U and T are, respectively, k - and l -strictly pseudocontractive mappings, and $\sum_{n=1}^{\infty}\tau_n(\bar{k}-\tau_n)=\infty$, $0<\tau_n<\bar{k}$, where*

$$\bar{k}=\frac{(1-k)(1-l)}{1-l+\|A\|^2(1-k)}. \tag{23}$$

Then, the sequence $\{x_n\}$, generated by (5), converges weakly to a solution of problem (3).

We next consider the MSCFP under the following basic assumption.

(i) MSCFP is consistent; that is, it admits at least one solution

(ii) $U_i : H_1 \longrightarrow H_1, i=1,2,\cdots,t$ is k_i-strictly pseudocontractive with $k_i<1$

(iii) $T_j : H_2 \longrightarrow H_2, j=1,2,\cdots,s$ is l_j-strictly pseudocontractive with $l_j<1$

Algorithm 1. Let x_0 be arbitrary. Given x_n, update the next iteration via

$$x_{n+1}=x_n-\tau_n\left[\sum_{i=1}^{t}\alpha_i(I-U_i)x_n+\sum_{j=1}^{s}\beta_jA^*\left(I-T_j\right)Ax_n\right], \tag{24}$$

where $\{\alpha_i\}_{i=1}^{t}\subset(0,1)$ with $\sum_{i=1}^{t}\alpha_i=1$, $\{\beta_j\}_{j=1}^{s}\subset(0,1)$ with $\sum_{j=1}^{s}\beta_j=1$, and $\{\tau_n\}\subset(0,\infty)$ are properly chosen stepsizes.

Theorem 6. *Assume that conditions (A1)-(A3) hold and $\{\tau_n\}$ is chosen so that*

$$\sum_{n=1}^{\infty}\tau_n\left(\bar{k}-\tau_n\right)=\infty, 0<\tau_n<\bar{k}, \tag{25}$$

where

$$\bar{k}=\frac{1}{\sum_{i=1}^{t}\alpha_i(1-k_i)^{-1}+\|A\|^2\sum_{j=1}^{s}\beta_j\left(1-l_j\right)^{-1}}. \tag{26}$$

Then, the sequence $\{x_n\}$, generated by Algorithm 1, converges weakly to a solution of MSCFP.

Proof. Let $U=\sum_{i=1}^{t}\alpha_iU_i$ and $T=\sum_{j=1}^{s}\beta_jT_j$. By Lemma 4, we conclude that U is k-strictly pseudocontractive with $k=1-\left(\sum_{i=1}^{t}\alpha_i(1-k_i)^{-1}\right)^{-1}$, and T is l-strictly pseudocontractive with $l=1-\left(\sum_{j=1}^{s}\beta_j(1-l_j)^{-1}\right)^{-1}$. Hence, by formula (23), we have

$$\begin{aligned}&\frac{(1-k)(1-l)}{1-l+\|A\|^2(1-k)} \\ &=\frac{\left(\sum_{i=1}^{t}\alpha_i(1-k_i)^{-1}\right)^{-1}\left(\sum_{i=1}^{s}\beta_j\left(1-l_j\right)^{-1}\right)^{-1}}{\left(\left(\sum_{j=1}^{s}\beta_j\left(1-l_j\right)^{-1}\right)^{-1}+\|A\|^2\sum_{i=1}^{t}\alpha_i(1-k_i)^{-1}\right)^{-1}} \\ &=\frac{1}{\sum_{i=1}^{t}\alpha_i(1-k_i)^{-1}+\|A\|^2\sum_{j=1}^{s}\beta_j\left(1-l_j\right)^{-1}}.\end{aligned} \tag{27}$$

Moreover, by Lemma 3, $F(U)=\bigcap_{i=1}^{t}F(U_i)$ and $F(T)=\bigcap_{j=1}^{s}F(T_j)$. Therefore, by applying Theorem 5, we at once get the assertion as desired.

It seems that the choice of the stepsize above requires the prior information of k_i, l_j and the norm $\|A\|$. However, as shown below, there is a special case in which the selection of stepsizes ultimately has no relation with k_i, l_j and the norm $\|A\|$. □

Corollary 7. *Assume that conditions (A1)-(A3) hold, and the stepsize is chosen so that*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\tau_n=0, \\ &\sum_{n=1}^{\infty}\tau_n=\infty.\end{aligned} \tag{28}$$

Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution of MSFP.

Significantly, if the nonlinear mappings in (4) are all metric projections, then the MSCFP is reduced to the MSFP. Consequently, we can apply our outcome to solve the MSFP. As an application of Algorithm 1, we get the following algorithm for solving problem (2).

Algorithm 2. Let x_0 be arbitrary. Given x_n, update the next iteration via

$$x_{n+1} = x_n - \tau_n \left[\sum_{i=1}^{t} \alpha_i \left(I - P_{C_i} \right) x_n + \sum_{j=1}^{s} \beta_j A^* \left(I - P_{Q_j} \right) A x_n \right], \quad (29)$$

where $\{\alpha_i\}_{i=1}^t \subset (0,1)$ with $\sum_{i=1}^t \alpha_i = 1$, $\{\beta_j\}_{j=1}^s \subset (0,1)$ with $\sum_{j=1}^s \beta_j = 1$, and $\{\tau_n\} \subset (0,\infty)$ are properly chosen stepsize.

Corollary 8. *Assume that MSFP is consistent. If the stepsize is chosen so that*

$$\sum_{n=1}^{\infty} \tau_n \left(\frac{2}{1 + \|A\|^2} - \tau_n \right) = \infty, \tau_n < \frac{2}{1 + \|A\|^2}, \quad (30)$$

then the sequence $\{x_n\}$, generated by Algorithm 2, converges weakly to a solution of MSFP.

Proof. Let $U = \sum_{i=1}^t \alpha_i P_{C_i}$ and $T = \sum_{j=1}^s \beta_j P_{Q_j}$. By Lemma 4, we conclude that U and T are both -1-strictly pseudocontractive, that is, firmly nonexpansive. In this situation, we have $\bar{k} = 2/(1 + \|A\|^2)$. By applying Theorem 6, we at once get the assertion as desired. □

Corollary 9. *Assume MSFP is consistent. If the stepsize is chosen so that*

$$\begin{aligned} &\sum_{n=1}^{\infty} \tau_n = \infty, \\ &\lim_{n \longrightarrow \infty} \tau_n = 0, \end{aligned} \quad (31)$$

then the sequence $\{x_n\}$, generated by Algorithm 2, converges weakly to a solution of MSFP.

4. Applications

In this part, we first give an application of our theoretical results to the multiple-set split equality problem (MSEP), which is more general than the original split equality problem [22].

Example 1. The multiple-set split equality problem (MSEP) expects to find $(x_1, x_2) \in H_1 \times H_2$ such that

$$(x_1, x_2) \in \bigcap_{i=1}^{t} F(U_i) \times \bigcap_{j=1}^{s} F(T_j), A_1 x_1 = A_2 x_2, \quad (32)$$

where t and s are two positive integers, $A_1 : H_1 \longrightarrow H_3$ and $A_2 : H_2 \longrightarrow H_3$ are two bounded linear mappings, and $U_i : H_1 \longrightarrow H_1, i = 1, 2, \cdots, t$ and $T_j : H_2 \longrightarrow H_2, j = 1, 2, \cdots, s$ are two classes of nonlinear mappings.

We next consider the MSFP under the following basic assumption.

(i) MSEP is consistent; that is, it admits at least one solution

(ii) $U_i : H_1 \longrightarrow H_1, i = 1, 2, \cdots, t$ is k_i-strictly pseudocontractive with $k_i < 1$

(iii) $T_j : H_2 \longrightarrow H_2, j = 1, 2, \cdots, s$ is l_j-strictly pseudocontractive with $l_j < 1$

Under this situation, we propose a new method for solving problem (32).

Algorithm 3. For an arbitrary initial guess (x_0, y_0), define (x_n, y_n) recursively by

$$\begin{cases} x_{n+1} = x_n - \tau_n \left[\left(I - \sum_{i=1}^{t} \alpha_i U_i \right) x_n + A_1^* (A_1 x_n - A_2 y_n) \right], \\ y_{n+1} = y_n - \tau_n \left[\left(I - \sum_{j=1}^{s} \beta_j T_j \right) y_n - A_2^* (A_1 x_n - A_2 y_n) \right], \end{cases} \quad (33)$$

where $\{\tau_n\} \subset (0,\infty)$ is a sequence of positive numbers.

To proceed the convergence analysis, we consider the product space $H \coloneqq H_1 \times H_2$, in which the inner product and the norm are, respectively, defined by

$$\begin{aligned} &\langle x, y \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle, \\ &\|x\|^2 = \left(\|x_1\|^2 + \|x_2\|^2 \right)^{1/2}, \end{aligned} \quad (34)$$

where $x = (x_1, x_2), y = (y_1, y_2)$ with $x_1, y_1 \in H_1, x_2, y_2 \in H_2$. Define a linear mapping $A : H \longrightarrow H_3$ by

$$Ax = A_1 x_1 - A_2 x_2, \forall x = (x_1, x_2). \quad (35)$$

Let T be the the metric projection onto the set $\{0\} \subseteq H$, and define a nonlinear mapping $U : H \longrightarrow H$ as

$$U(x) = \left(\sum_{i=1}^{t} \alpha_i U_i x_1, \sum_{j=1}^{s} \beta_j T_j x_2 \right), \forall x = (x_1, x_2), \quad (36)$$

where α_i and β_j are as above.

Lemma 10 ([23], Lemma 12). *Let the mapping A be defined as in (35). Then A is linear bounded. Moreover, for $x = (x_1, x_2)$, it follows*

$$A^* A x = (A_1^* (A_1 x_1 - A_2 x_2), -A_2^* (A_1 x_1 - A_2 x_2)). \quad (37)$$

Lemma 11. *Let the mapping U be defined as in (36). Then, $F(U) = \bigcap_i F(U_i) \times \bigcap_j F(T_j)$. Moreover, if conditions (B1)-(B3) are met, then U is k-strictly pseudocontractive with*

$$\kappa = 1 - \frac{1}{\max \left(\sum_{i=1}^t \alpha_i (1 - k_i)^{-1}, \sum_{j=1}^s \beta_j \left(1 - l_j \right)^{-1} \right)}. \quad (38)$$

Proof. By Lemma 3, it is easy to verify the first assertion. To show the second assertion, fix any $x, y \in H$. By our hypothesis, $\sum_{i=1}^{t}\alpha_i U_i$ is k-strictly pseudocontractive with

$$k = 1 - \frac{1}{\sum_{i=1}^{t}\alpha_i (1-k_i)^{-1}}, \tag{39}$$

$\sum_{j=1}^{s}\beta_j T_j$ is l-strictly pseudocontractive with

$$l = 1 - \frac{1}{\sum_{j=1}^{s}\beta_j (1-l_j)^{-1}}. \tag{40}$$

It then follows that

$$\begin{aligned}
&\|Ux - Uy\|^2 \\
&= \left\| \sum_{i=1}^{t}\alpha_i U_i x_1 - \sum_{i=1}^{t}\alpha_i U_i y_1 \right\|^2 + \left\| \sum_{j=1}^{s}\beta_j T_j x_2 - \sum_{j=1}^{s}\beta_j T_j y_2 \right\|^2 \\
&\le \|x_1 - y_1\|^2 + k \left\| \left(I - \sum_{i=1}^{t}\alpha_i U_i \right) x_1 - \left(I - \sum_{i=1}^{t}\alpha_i U_i \right) y_1 \right\|^2 \\
&\quad + \|x_2 - y_2\|^2 + l \left\| \left(I - \sum_{j=1}^{s}\beta_j T_j \right) x_1 - \left(I - \sum_{j=1}^{s}\beta_j T_j \right) y_1 \right\|^2 \\
&\le \|x - y\|^2 + \max(k, l) \|(I - U)x - (I - U)y\|^2.
\end{aligned} \tag{41}$$

From (38), we obtain the result as desired. □

Theorem 12. *Assume that conditions (B1)-(B3) hold. If $\{\tau_n\}$ is chosen so that $\sum_{n=1}^{\infty}\tau_n(\bar{\kappa} - \tau_n) = \infty$, $0 < \tau_n < \bar{\kappa}$, where*

$$\bar{\kappa} = \frac{2(1-\kappa)}{2 + (1-\kappa)\left(\|A_1\|^2 + \|A_2\|^2\right)}, \tag{42}$$

with κ defined as in (38), then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3 converges weakly to a solution of problem (32).

Proof. Let $z_n = (x_n, y_n)$ and let A, U, T be defined as above. Thus, problem (32) is equivalently changed into finding $z \in H$ such that

$$\begin{aligned} &z \in F(U), \\ &Az \in F(T). \end{aligned} \tag{43}$$

Moreover, Algorithm 3 can be rewritten as

$$z_{n+1} = z_n - \tau_n [(I - U)z_n + A^*(I - T)Az_n]. \tag{44}$$

Note that by Lemma 10, U is κ-strictly pseudocontractive and T is -1-strictly pseudocontractive. Hence, by Theorem 5, we conclude that $\{z_n\}$ converges weakly to some $z = (x, y)$ such that

$$\begin{aligned} &z \in F(U), \\ &Az \in \{0\}. \end{aligned} \tag{45}$$

By Lemma 11, it is readily seen that $x \in \bigcap_i F(U_i)$, $y \in \bigcap_j F(T_j)$ and $A_1 x = A_2 y$. □

We next give an application of our theoretical results to a problem derived from the real world. In statistics and machine learning, least absolute shrinkage and selection operator (LASSO for short) is a regression analysis method that performs both variable selection and regularization in order to enhance the prediction accuracy and interpretability of the statistical model it produces. It was originally introduced by Tibshirani in [24] who coined out the term and provided further insights into the observed performance.

Subsequently, a number of LASSO variants have been created in order to remedy certain limitations of the original technique and to make the method more useful for particular problems. Among them, elastic net regularization adds an additional ridge regression-like penalty which improves performance when the number of predictors is larger than the sample size, allows the method to select strongly correlated variables together, and improves overall prediction accuracy. More specifically, the LASSO is a regularized regression method with the L_1 penalty, while the elastic net is a regularized regression method that linearly combines the L_1 and L_2 penalties of the LASSO and ridge methods. Here, the L_1 penalty is defined as $\|x\|_1 = \sum_{i=1}^{n}|x_i|$, and the L_2 penalty is defined as $\|x\|_2 = \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}$.

Example 2 (see [25]). The elastic net requires to solve the problem

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \frac{1}{2}\left(\|Ax - y_1\|_2^2 + \|Ax - y_2\|_2^2\right) \\ &\text{s.t.}\ \|x\|_1 \le t_1,\ \|x\|_2^2 \le t_2, \end{aligned} \tag{46}$$

where $A \in \mathbb{R}^{m\times n}$, $y_1, y_2 \in \mathbb{R}^m$, and $t_1, t_2 > 0$ are given parameters. This problem is a specific SCFP with $T_1 x = y_1$, $T_2 x = y_2$, $\forall x \in \mathbb{R}^m$ and

$$U_1 y = \begin{cases} y, & \|y\|_1 \le t_1, \\ y - \dfrac{\|y\|_1 - t_1}{\|\eta(y)\|^2}, & \|y\|_1 > t_1, \end{cases} \tag{47}$$

where $\eta(y) \in \partial(\|y\|_1)$ and

$$U_2 y = \begin{cases} y, & \|y\|_2^2 \le t_2, \\ y - \dfrac{\|y\|_2^2 - t_2}{4\|y\|^2}, & \|y\|_2^2 > t_2. \end{cases} \tag{48}$$

Algorithm 4. Let x_0 be arbitrary. Given x_n, update the next iteration via

$$x_{n+1} = x_n - \tau \sum_{i=1}^{2} \alpha_i [(I - U_i)x_n + A^* (Ax_n - y_i)], \qquad (49)$$

where $\{\alpha_i\}_{i=1}^2 \subset (0, 1)$ with $\sum_{i=1}^t \alpha_i = 1$ and τ is a properly chosen stepsize.

It is clear that the above mappings are, respectively, firmly nonexpansive and firmly quasi-nonexpansive, which implies that they are, respectively, -1-strictly pseudocontractive and -1-demicontractive mappings. As an application of Theorem 6, we can deduce that the sequence $\{x_n\}$ generated by Algorithm 4 converges to a solution to problem (46) provided that the stepsize is chosen so that

$$0 < \tau < \frac{2}{1 + \|A\|^2}. \qquad (50)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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