Hindawi
Journal of Function Spaces
Volume 2023, Article ID 9818094, 1 page
https://doi.org/10.1155/2023/9818094

Retraction

Retracted: The Power XLindley Distribution: Statistical Inference, Fuzzy Reliability, and COVID-19 Application

Journal of Function Spaces

Received 3 October 2023; Accepted 3 October 2023; Published 4 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] B. Meriem, A. M. Gemeay, E. M. Almetwally et al., "The Power XLindley Distribution: Statistical Inference, Fuzzy Reliability, and COVID-19 Application," *Journal of Function Spaces*, vol. 2022, Article ID 9094078, 21 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9094078, 21 pages
https://doi.org/10.1155/2022/9094078

Research Article

The Power XLindley Distribution: Statistical Inference, Fuzzy Reliability, and COVID-19 Application

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Received 13 February 2022; Accepted 4 May 2022; Published 7 June 2022

Academic Editor: Muhammad Gulzar

The power XLindley (PXL) distribution is introduced in this study. It is a two-parameter distribution that extends the XLindley distribution established in this paper. Numerous statistical characteristics of the suggested model were determined analytically. The proposed model's fuzzy dependability was statistically assessed. Numerous estimation techniques have been devised for the purpose of estimating the proposed model parameters. The behaviour of these factors was examined using randomly generated data and developed estimation approaches. The suggested model seems to be superior to its base model and other well-known and related models when applied to the COVID-19 data set.

1. Introduction

Modern numerical approaches may be used in a wide range of fields, including medicine, finance, bioengineering, and statistics. Last but not least, statistics play an important role in our daily life. There is a lot of statistical analysis that relies on the assumption of a probability model or distribution.

Many viruses, including COVID-19, have been making headlines in the previous two years. To comprehend and analyze COVID-19 infections as statisticians, we needed a statistical model that could handle both continuous and discrete random variables, such as COVID-19 infections. We went to great lengths in this work to select a model that accurately reflects COVID-19 infections. In the end, we came to the conclusion that we needed to introduce a new distribution as a blend of other distributions to solve the shortcomings of the baseline distribution. COVID-19 mortality data were used to create a statistical model. More information may be found at [1–7].

Let Y be a random variable following one-parameter Lindley (L) distribution [8] with probability density function (PDF) defined as the following

$$f(y;\theta) = \frac{\theta^2(1+y)e^{-t\theta}}{1+\theta}, y, \theta > 0. \tag{1}$$

Recently, new generalised models have taken a shine to the Lindley model and its extensions. The Lindley model and its extensions have been shown to properly represent real-world data in a variety of scenarios. Because of this, a number of scholars are looking for new generalisations and extensions of the Lindley distribution to describe life events in diverse contexts. Due to the difficulty of applying a generalised distribution to describe real data, other generalised distributions have been devised and employed in various sectors. However, real-world data still presents a number of significant challenges that existing models cannot solve (see [9–18], for further information on the Lindley distribution and its expansion).

Recently, Chouia and Zeghdoudi [19] introduced a new statistical distribution named XLindley distribution based on a special mixture of exponential and Lindley distributions. Its PDF is defined as follows:

$$f_{XL}(t\,;\theta) = \frac{\theta^2(2+\theta+t)}{(1+\theta)^2} e^{\theta t} = p_1 f_1(t) + p_2 f_2(t), \quad (2)$$

and its cumulative distribution function (CDF) is defined as follows:

$$F_{XL}(t\,;\theta) = 1 - \left(1 + \frac{\theta t}{(1+\theta)^2}\right) e^{-\theta t}, t, \theta > 0, \quad (3)$$

where $f_1(t) \sim Exp(\theta)$, $f_2(t) \sim Lindley(\theta)$, $p_1 = \theta/(\theta+1)$, and $p_2 = 1/(\theta+1)$.

This article proposes a power XLindley distribution (PXL). We study the forms of the density and hazard rate functions, the moments and related measures, the quantile function, stochastic ordering, and order statistics' limiting distributions. Fuzzy reliability is defined as numerical values. Nine different estimation methods are discussed for our model. COVID-19 data is shown in real-time to illustrate the proposed distribution's flexibility and to compare it to the fit obtained by many well-known two-parameter distributions. The following things impacted the creation of this work:

(i) Although the PXL distribution is limited to the tail of a distribution, it is simple to implement

(ii) Explicitly defining the statistical characteristics is straightforward

(iii) This novel distribution offers a number of benefits, including the presence of numerous parameters (two) that may be used to model survival analysis, actuarial science, and so on

(iv) The PXL distribution may be utilised pretty well to analyse a large number of real-life data sets and fits them quite well

The following is the organisation of the paper. In Section 2, we present the PXL distribution formulation and discuss its immediate features. In Section 3, we discuss the PXL distribution's many statistical features. Section 4 discusses fuzzy reliability and its numerical values. In Section 5, we will examine the estimation of parameters using nine different estimate approaches. Section 6 contains simulation studies. In Section 7, we try to compare the model to alternative distributions using the real-world COVID-19 data set.

The primary objective and goal of this research are to provide a unique extension of the Lindley distribution that is capable of fitting a wide variety of data types, most notably COVID 19 data; also, we used fuzzy statistics in our suggested distribution.

2. Formulation of PXL Distribution

Lindley and XLindley distributions may be inapplicable to a large number of theoretical issues. To create a model that is adaptable, we developed the power XLindley (PXL) distribution based on the power transformation. $X = T^{1/\alpha}$. The X PDF can be obtained as follows:

$$\begin{aligned} f_{PXL}(x) &= \frac{\alpha\theta^2(2+\theta+x^\alpha)x^{\alpha-1}}{(1+\theta)^2} e^{-\theta x^\alpha} \\ &= p g_1(x) + (1-p) g_2(t), x, \alpha, \theta > 0, \end{aligned} \quad (4)$$

where $p = (\theta^2 + 2\theta)/(1+\theta)^2$, $g_1(x) = \alpha\theta x^{\alpha-1} e^{-\theta x^\alpha}$ follows the Weibull distribution with shape parameter α and scale parameter θ, and $g_2(t) = \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^\alpha}$ follows the generalized gamma distribution with shape parameters 2 and α and scale θ. So, we can say that The PDF PXL distribution (4) is a two-component mixture of the Weibull distribution and a generalized gamma distribution with mixing proportion $p = (\theta^2 + 2\theta)/(1+\theta)^2$.

The survival function (SF) and hazard rate function (HRF) of PXL distribution are, respectively, defined as follows:

$$S_{PXL}(x) = \left(1 + \frac{\theta x^\alpha}{(1+\theta)^2}\right) e^{-\theta x^\alpha}, h_{PXL}(x) = \frac{\alpha\theta^2 x^\alpha(\theta + x^\alpha + 2)}{\theta x(\theta + x^\alpha + 2) + x}. \quad (5)$$

3. Statistical Properties

Numerous statistical features were discussed in this section, including the behaviour of the PDF, HRF, and quantile function, as well as moments, incomplete moments, stochastic ordering, and order statistics limits.

3.1. Asymptotic Behavior. This subsection discusses the behaviour and probable shapes features of the PDF $f_{PXL}(x)$ and HRF $h_{PXL}(x)$ in (4) and (5), respectively, of the PXL distribution. The PDF's behaviour is as follows:

$$\lim_{x \to 0} f(x) = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ \dfrac{\theta^2(\theta+2)}{(\theta+1)^2} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1, \end{cases} \quad (6)$$

$$\lim_{x \to \infty} f(x) = 0.$$

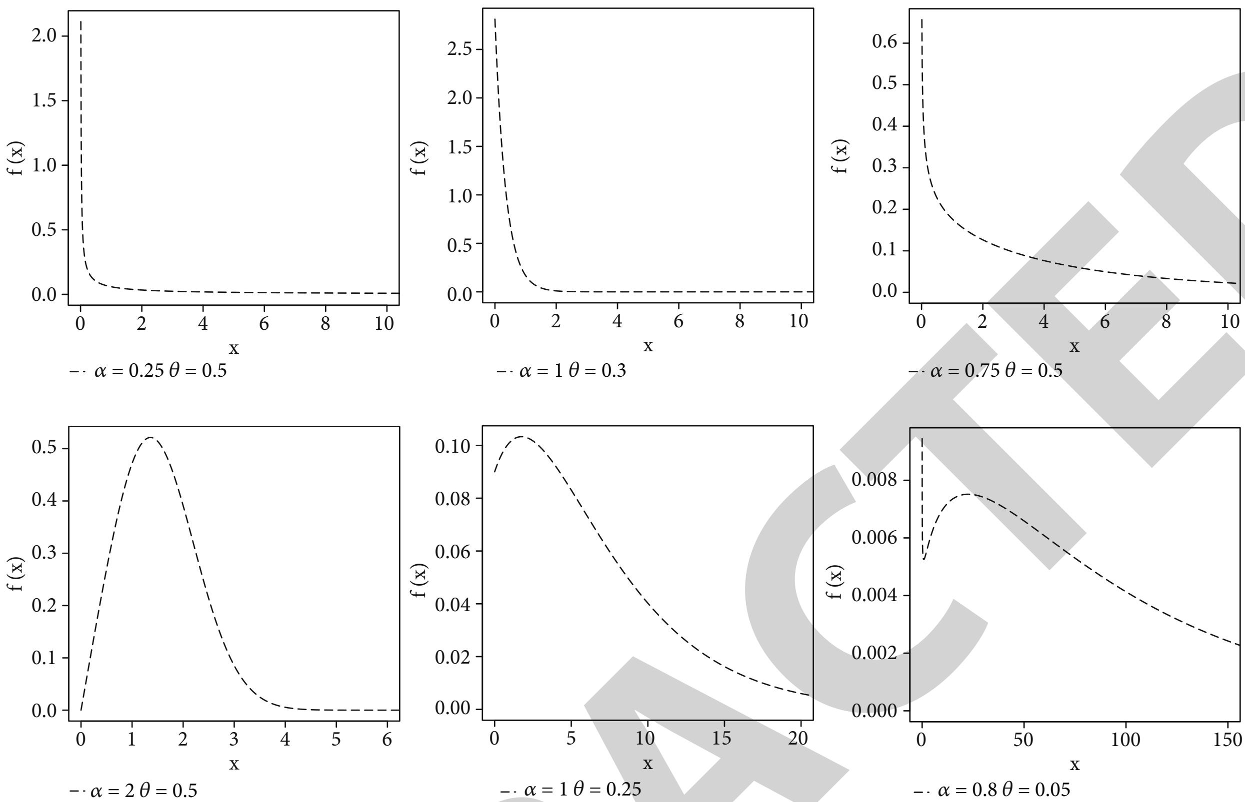

Figure 1: Plots of the PDF of PXL distribution for some parameter values.

The behavior of $h_{PXL}(x)$ at $x = 0$ and $x = \infty$, respectively, is given by

$$\lim_{x \longrightarrow 0} h(x) = \begin{cases} \text{Indeterminate} & \text{if } 0 < \alpha < 1 \\ \dfrac{\theta^2(\theta+2)}{(\theta+1)^2} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1, \end{cases} \qquad (7)$$

$$\lim_{x \longrightarrow \infty} h(x) = \begin{cases} 0 & \text{if } \alpha < 1 \\ \theta & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1. \end{cases}$$

According to the preceding proposition, there are three distinct forms for the PDF of the power XLindley distribution, depending on the parameter range, α and θ. Additionally, Figure 1 illustrates all conceivable PDF forms for the proposed model.

Proposition 1. *The PDF $f_{PXL}(x)$ in (4) of the PXLD is*

(I) *Decreasing if $0 < \alpha \leq 1/2, \theta > 0, 1/2 < \alpha < 1, \theta \geq J(\alpha)$, or $\alpha = 1, \theta \geq \sqrt{2} - 1$*

(II) *Unimodel if $\alpha > 1, \theta > 0$, or $\alpha = 1, 0 < \theta < \sqrt{2} - 1$*

(III) *Decreasing–increasing–decreasing $1/2 < \alpha < 1, 0 < \theta < J(\alpha)$, where $J(\alpha) = -1 + \sqrt{\alpha(3\alpha - 1)}/\alpha$*

Proof. The first derivative of $f_{PXL}(x)$ is

$$\frac{df_{PXL}(x)}{dx} = x^{\alpha-2}\theta^2\alpha \frac{e^{-x^\alpha\theta}}{(\theta+1)^2} L(x^\alpha), \qquad (8)$$

where $L(x^\alpha) = c + bx^\alpha + ax^{2\alpha}$, with $c = 2\alpha - \theta + \theta\alpha - 2$, $b = -\alpha\theta^2 - 2\alpha\theta + 2\alpha - 1$, and $a = -\theta\alpha$.

We can see that $df_{PXL}(x)/dx$ and $L(x^\alpha)$ have the same sign. Using the characteristics of the quadratic function $L(x^\alpha)$ under the previous conditions in (I), (II), and (III), the function $L(x^\alpha)$ is negative for $b < 0$ and $c < 0$, is unimodel with maximum value at the point $x_0^\alpha = -b/2a$ for $b > 0$ and $c > 0$, and changes sign from − to + to − for $b > 0$ and $c > 0$, respectively. When $b = 0$, the solution of θ is $\{-(1/\alpha)(\alpha + \sqrt{\alpha(3\alpha-1)}), -(1/\alpha)(\alpha - \sqrt{\alpha(3\alpha-1)})\}$, $\alpha \neq 0$, where $J(\alpha) = -1 + \sqrt{\alpha(3\alpha-1)}/\alpha$.

$b > 0$, if $\theta < J(\alpha)$, with $\alpha(3\alpha - 1) > 0$ and $J(\alpha) > 0 (\alpha > 1/2)$, $b>0$, if $\theta > J(\alpha)$, with $\alpha(3\alpha - 1) > 0$, and $J(\alpha) > 0 (\alpha > 1/2)$, if $0 < \alpha \leq 1/2$, $b < 0$. This completes the proof of Proposition 1. □

The following proposition showed the different shapes for the HRF of PXL distribution, depending on the values of the parameters α and θ. All possible shapes of HRF of the proposed model are presented in Figure 2.

Proposition 2. *The HRF $h_{PXL}(x)$ in (5) of the PXL distribution is*

(I) *Decreasing if $0 < \alpha \le 1/2, \theta > 0$, or $1/2 < \alpha < 1, \theta \ge Z(\alpha)$*

(II) *Increasing if $\alpha \ge 1, \theta > 0$*

(III) *Decreasing–increasing–decreasing if $1/2 < \alpha < 1, 0 < \theta < Z(\alpha)$, where $Z(\alpha) = -1/(2\alpha - 2)(2\alpha + \sqrt{2}\sqrt{-\alpha + 1} - 2)$*

Proof. The first derivative of $h_{PXL}(x)$ is

$$\frac{dh_{PXL}(x)}{dx} = x^{\alpha-2}\theta^2 \frac{\alpha}{\left(2\theta + \theta^2 + x^\alpha\theta + 1\right)^2} M(x^\alpha), \quad (9)$$

where $M(x^\alpha) = c' + b'x^\alpha + a'x^{2\alpha}$,

$c' = 2\alpha - 5\theta - 4\theta^2 - \theta^3 + 5\theta\alpha + \theta^3\alpha + 4\theta^2\alpha - 2$, $b' = 4\theta\alpha + 2\theta^2\alpha - 2\theta^2 - 4\theta + 2\alpha - 1$, and $a' = \theta\alpha - \theta$. We can see that $dh_{PXL}(x)/dx$ and $M(x^\alpha)$ have the same sign and $c' > 0$ if $\alpha > 1$ and $c' < 0$ if $\alpha < 1$. The rest of the proof of this proposition follows similarly to that of Proposition 1. □

3.2. The Quantile Function of PXL Distribution. According to Chouia and Zeghdoudi [19] the quantile function of the XLindley distribution is given by

$$F_T^{-1}(u) = -\frac{(1+\theta)^2}{\theta} - \frac{1}{\theta} W_{-1}\left[\frac{(1+\theta)^2}{\exp(1+\theta)^2}(u-1)\right], u\varepsilon[0,1], \quad (10)$$

where $W_{-1}()$ denotes the negative branch of the Lambert W function.

Then, The PXL distribution quantile function is given by

$$F_X^{-1}(u) == \left[-\frac{(1+\theta)^2}{\theta} - \frac{1}{\theta} W_{-1}\left[\frac{(1+\theta)^2}{\exp(1+\theta)^2}(u-1)\right]\right]^{1/\alpha}. \quad (11)$$

3.3. Moments and Associated Measures. The rth moment concerning the PXL distribution's origin may be calculated as follows:

$$\begin{aligned}\mu_r' = E(X^r) &= \int_0^\infty x^{(r)} f_{PXL}(x)dx \\ &= p\frac{\Gamma((r/\alpha)+1)}{\theta^{(r/\alpha)}} + (1-p)\frac{\Gamma((r/\alpha)+2)}{\theta^{(r/\alpha)}} \\ &= \frac{\theta^2+2\theta}{(1+\theta)^2}\frac{\Gamma((r/\alpha)+1)}{\theta^{(r/\alpha)}} + \left(1 - \frac{\theta^2+2\theta}{(1+\theta)^2}\right)\frac{\Gamma((r/\alpha)+2)}{\theta^{(r/\alpha)}} \\ &= \frac{r\left(\alpha\theta^2 + 2\alpha\theta + r + \alpha\right)\Gamma(r/\alpha)}{\alpha^2\theta^{(r/\alpha)}(\theta+1)^2},\end{aligned} \quad (12)$$

The mean and variance of the PXL distribution are given by

$$\begin{aligned}\mu &= \frac{r\left(\theta^2 + 2\theta + 1 + \alpha\right)\Gamma(1/a)}{\alpha^2\theta^{(1/\alpha)}(\theta+1)^2}, \\ \sigma^2 &= \frac{2\left(\alpha\theta^2 + 2\alpha\theta + \alpha + 2\right)\Gamma(2/\alpha)}{\alpha^2\theta^{(2/\alpha)}(\theta+1)^2} - \frac{r^2\left(\theta^2 + 2\theta + \alpha + 1\right)^2(\Gamma(1/\alpha))^2}{\alpha^4\theta^{(2/\alpha)}(\theta+1)^4},\end{aligned} \quad (13)$$

respectively, and the moment generating function of the PXL distribution takes the form

$$\begin{aligned}M(t) &= \int_0^\infty e^{tx} f(x)dx = \sum_{m=0}^\infty \frac{t^m}{m!}\int_0^\infty x^m f(x)dx \\ &= \sum_{m=0}^\infty \frac{t^m}{m!}\frac{m\left(\alpha\theta^2 + 2\alpha\theta + m + \alpha\right)\Gamma(m/\alpha)}{\alpha^2\theta^{(m/\alpha)}(\theta+1)^2}.\end{aligned} \quad (14)$$

Its characteristic function is obtained by replacing t with it in the previous equation.

The rth incomplete moments of PXL distribution is given by

$$\Psi_r(t) = \int_0^t x^r f(x)dx = \frac{\theta^{-a/\alpha}\left((\theta+2)\theta\Gamma((a+\alpha)/\alpha) + \Gamma(a/\alpha+2) + \theta^2(-\Gamma((a+\alpha)/\alpha, t^\alpha\theta)) - 2\theta\Gamma((a+\alpha)/\alpha, t^\alpha\theta) - \Gamma((a/\alpha)+2, t^\alpha\theta)\right)}{(\theta+1)^2}, \quad (15)$$

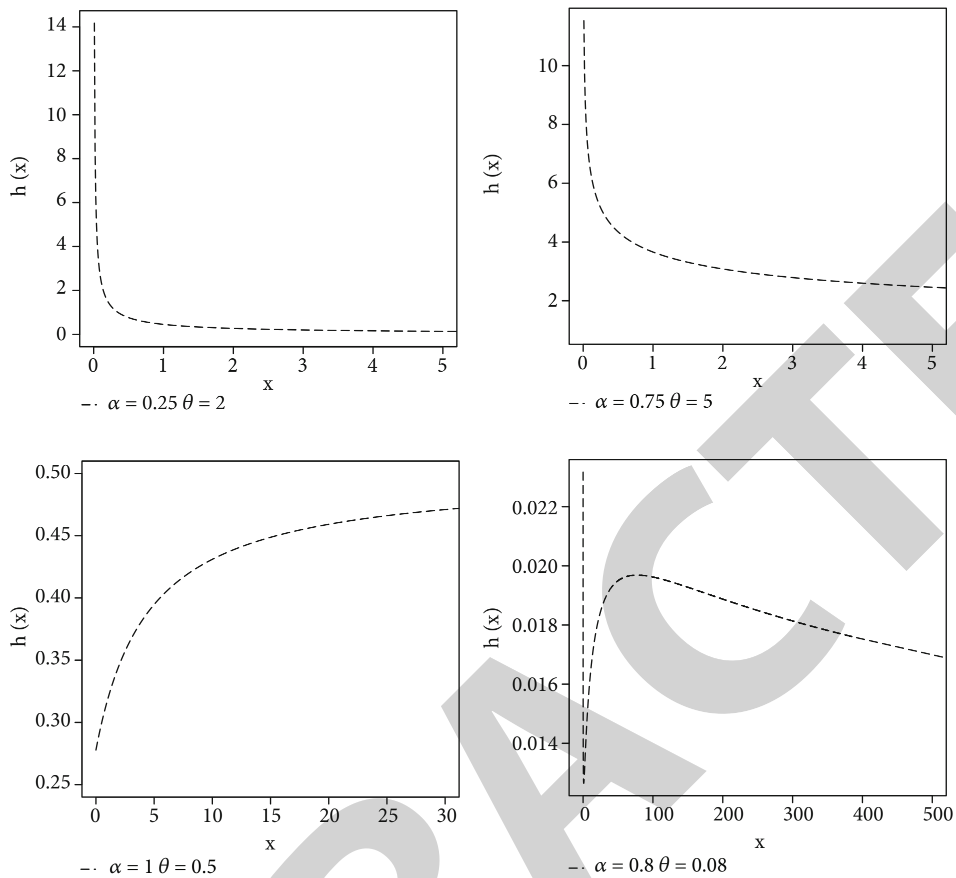

FIGURE 2: Plots of HRF of PXL distribution for some parameter values.

TABLE 1: Traditional and fuzzy reliability with different values.

θ	α	t_1	t_2	$S(t_1)$	$S(t_2)$	R_F $\gamma = 0.25$	$\gamma = 0.55$	$\gamma = 0.90$
			1		0.5880	0.2938	0.3494	0.3876
		0.000001	1.5	0.9841	0.5542	0.3215	0.3806	0.4210
	0.25		2		0.5292	0.3423	0.4039	0.4456
			1		0.5880	0.3047	0.3602	0.3984
		1E-08	1	0.9950	0.5542	0.3324	0.3915	0.4318
			2		0.5292	0.3531	0.4147	0.4565
0.75			1		0.5880	0.0924	0.2237	0.3721
		0.001	1.5	0.9999	0.4128	0.1467	0.3415	0.5389
			3		0.1161	0.3102	0.6316	0.8471
			1		0.5880	0.0863	0.1917	0.3091
		0.2	2	0.9288	0.2790	0.1961	0.4122	0.6063
			4		0.0438	0.3976	0.7099	0.8648
	1.2	1	3	0.5880	0.1161	0.1752	0.3312	0.4485
			5		0.0152	0.3091	0.4918	0.5646
			1		0.2767	0.1773	0.3586	0.5193
		0.2	1.5	0.8326	0.1212	0.2767	0.5169	0.6812
1.5			3		0.0070	0.5095	0.7531	0.8203
			2		0.0494	0.2393	0.4117	0.5098
		0.5	3	0.5749	0.0070	0.3481	0.5127	0.5633
			1		0.0001	0.4775	0.5676	0.5747

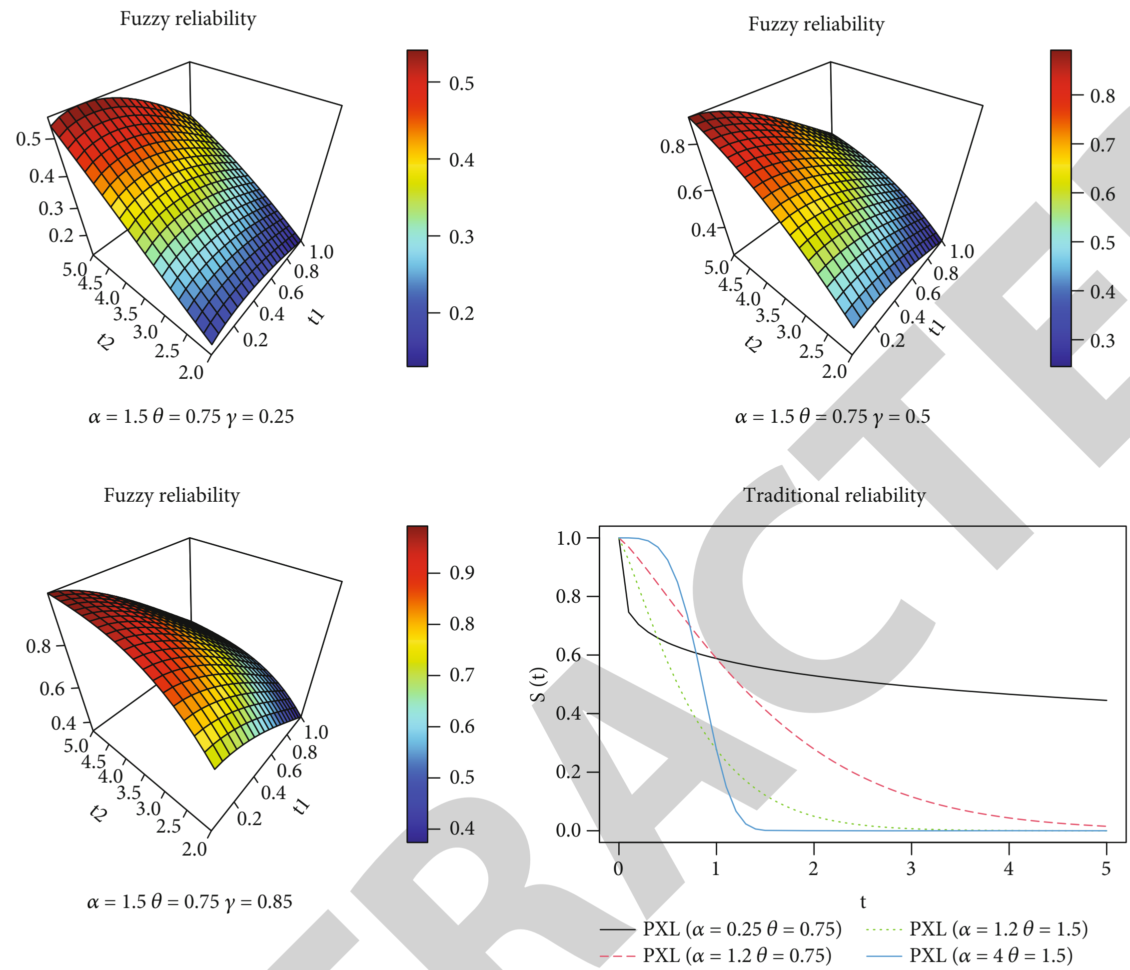

FIGURE 3: Plots of traditional and fuzzy reliability.

Input: initial values of α and θ.
Input: interval time (t_1, t_2) based on properties of sample with α and θ.
Input: γ-Cut where $0 < \gamma < 1$.
Calculate $x(\gamma) = t_1 + \gamma(t_2 - t_1)$.
for each method **do**
  Set: $i = 1$.
  Estimate parameters as $\widehat{\alpha}$, and $\widehat{\theta}$.
  Calculate $\widehat{R}_F(t)^{(i)} = (1 + (\widehat{\theta} t_1^{\widehat{\alpha}}/(1 + \widehat{\theta})^2))e^{-\widehat{\theta} t_1^{\widehat{\alpha}}} - (1 + (\widehat{\theta} x(\gamma)^{\widehat{\alpha}}/(1 + \widehat{\theta})^2))e^{-\widehat{\theta} x(\gamma)^{\widehat{\alpha}}}$
end

ALGORITHM 1: Fuzzy estimation algorithm.

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$.

3.4. Stochastic Ordering. Consider two random variables X_1 and X_2. Then X_1 is said smaller than X_2 in the following cases:

(I) Stochastic order $(X_1 <_S X_2)$, if $F_{X_1}(t) < F_{X_2}(t), \forall t$

(II) Convex order $(X_1 \leq_{cx} X_2)$, if for all convex functions ϕ and provided expectation exist, $E[\phi(X_1)] \leq E[\phi(X_2)]$

(III) Hazard rate order $(X_1 \leq_{hr} X_2)$, if $h_{X_1}(t) \geq h_{X_2}(t), \forall t$

(IV) Likelihood ratio order $(X_1 <_{lr} X_2)$, if $f_{X_1}(t)/f_{X_2}(t)$ is decreasing in t

Table 2: Simulation values of BIAS, MSE, and MRE for $\alpha = 0.25$ and $\theta = 0.75$.

n	Est.	*Est.Par.*	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
20	BIAS	$\hat{\alpha}$	$0.064023^{\{1\}}$	$0.068075^{\{2\}}$	$0.068368^{\{3\}}$	$0.07006^{\{6\}}$	$0.073795^{\{8\}}$	$0.101013^{\{9\}}$	$0.070065^{\{7\}}$	$0.069329^{\{5\}}$	$0.068664^{\{4\}}$
		$\hat{\theta}$	$0.092713^{\{1\}}$	$0.101222^{\{3\}}$	$0.119548^{\{7\}}$	$0.097822^{\{2\}}$	$0.108895^{\{5\}}$	$0.12919^{\{9\}}$	$0.106095^{\{4\}}$	$0.110435^{\{6\}}$	$0.119816^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.006631^{\{1\}}$	$0.007714^{\{4\}}$	$0.007307^{\{2\}}$	$0.008126^{\{7\}}$	$0.008629^{\{8\}}$	$0.021442^{\{9\}}$	$0.007567^{\{3\}}$	$0.007728^{\{6\}}$	$0.00772^{\{5\}}$
		$\hat{\theta}$	$0.014687^{\{1\}}$	$0.018103^{\{3\}}$	$0.026483^{\{8\}}$	$0.014963^{\{2\}}$	$0.019875^{\{4\}}$	$0.026787^{\{9\}}$	$0.020317^{\{5\}}$	$0.022185^{\{6\}}$	$0.025131^{\{7\}}$
	MRE	$\hat{\alpha}$	$0.256092^{\{1\}}$	$0.2723^{\{2\}}$	$0.273473^{\{3\}}$	$0.28024^{\{6\}}$	$0.29518^{\{8\}}$	$0.404052^{\{9\}}$	$0.280262^{\{7\}}$	$0.277317^{\{5\}}$	$0.274658^{\{4\}}$
		$\hat{\theta}$	$0.123618^{\{1\}}$	$0.134962^{\{3\}}$	$0.159397^{\{7\}}$	$0.13043^{\{2\}}$	$0.145193^{\{5\}}$	$0.172253^{\{9\}}$	$0.14146^{\{4\}}$	$0.147246^{\{6\}}$	$0.159755^{\{8\}}$
	$\sum Ranks$		$6^{\{1\}}$	$17^{\{2\}}$	$30^{\{4.5\}}$	$25^{\{3\}}$	$38^{\{8\}}$	$54^{\{9\}}$	$30^{\{4.5\}}$	$34^{\{6\}}$	$36^{\{7\}}$
60	BIAS	$\hat{\alpha}$	$0.037326^{\{1\}}$	$0.039833^{\{5\}}$	$0.040231^{\{7\}}$	$0.038925^{\{3\}}$	$0.040337^{\{8\}}$	$0.069403^{\{9\}}$	$0.039993^{\{6\}}$	$0.038216^{\{2\}}$	$0.039601^{\{4\}}$
		$\hat{\theta}$	$0.050794^{\{1\}}$	$0.05777^{\{5\}}$	$0.06605^{\{8\}}$	$0.055565^{\{3\}}$	$0.061158^{\{6\}}$	$0.086042^{\{9\}}$	$0.056687^{\{4\}}$	$0.05452^{\{2\}}$	$0.065329^{\{7\}}$
	MSE	$\hat{\alpha}$	$0.00217^{\{1\}}$	$0.002512^{\{5\}}$	$0.002603^{\{8\}}$	$0.00241^{\{3\}}$	$0.002561^{\{7\}}$	$0.009956^{\{9\}}$	$0.002533^{\{6\}}$	$0.002286^{\{2\}}$	$0.002455^{\{4\}}$
		$\hat{\theta}$	$0.00417^{\{1\}}$	$0.005282^{\{5\}}$	$0.007248^{\{8\}}$	$0.004663^{\{3\}}$	$0.005927^{\{6\}}$	$0.012073^{\{9\}}$	$0.005263^{\{4\}}$	$0.004642^{\{2\}}$	$0.006944^{\{7\}}$
	MRE	$\hat{\alpha}$	$0.149306^{\{1\}}$	$0.15933^{\{5\}}$	$0.160925^{\{7\}}$	$0.155702^{\{3\}}$	$0.161348^{\{8\}}$	$0.277612^{\{9\}}$	$0.159972^{\{6\}}$	$0.152865^{\{2\}}$	$0.158404^{\{4\}}$
		$\hat{\theta}$	$0.067725^{\{1\}}$	$0.077026^{\{5\}}$	$0.088067^{\{8\}}$	$0.074086^{\{3\}}$	$0.081544^{\{6\}}$	$0.114722^{\{9\}}$	$0.075583^{\{4\}}$	$0.072693^{\{2\}}$	$0.087105^{\{7\}}$
	$\sum Ranks$		$6^{\{1\}}$	$30^{\{4.5\}}$	$46^{\{8\}}$	$18^{\{3\}}$	$41^{\{7\}}$	$54^{\{9\}}$	$30^{\{4.5\}}$	$12^{\{2\}}$	$33^{\{6\}}$
100	BIAS	$\hat{\alpha}$	$0.02863^{\{1\}}$	$0.029407^{\{2\}}$	$0.030867^{\{5\}}$	$0.030961^{\{6\}}$	$0.031234^{\{7\}}$	$0.052187^{\{9\}}$	$0.030282^{\{4\}}$	$0.029491^{\{3\}}$	$0.031495^{\{8\}}$
		$\hat{\theta}$	$0.039342^{\{1\}}$	$0.042789^{\{3\}}$	$0.047827^{\{7\}}$	$0.04366^{\{4\}}$	$0.045723^{\{6\}}$	$0.065933^{\{9\}}$	$0.044617^{\{5\}}$	$0.041852^{\{2\}}$	$0.049134^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.001314^{\{1\}}$	$0.001384^{\{2\}}$	$0.001492^{\{5\}}$	$0.001518^{\{6\}}$	$0.00155^{\{7\}}$	$0.005117^{\{9\}}$	$0.001425^{\{4\}}$	$0.00139^{\{3\}}$	$0.001555^{\{8\}}$
		$\hat{\theta}$	$0.002494^{\{1\}}$	$0.0029^{\{4\}}$	$0.003692^{\{7\}}$	$0.002878^{\{3\}}$	$0.003361^{\{6\}}$	$0.007025^{\{9\}}$	$0.003156^{\{5\}}$	$0.002807^{\{2\}}$	$0.003987^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.114521^{\{1\}}$	$0.11763^{\{2\}}$	$0.123467^{\{5\}}$	$0.123846^{\{6\}}$	$0.124935^{\{7\}}$	$0.20875^{\{9\}}$	$0.121129^{\{4\}}$	$0.117962^{\{3\}}$	$0.125979^{\{8\}}$
		$\hat{\theta}$	$0.052456^{\{1\}}$	$0.057051^{\{3\}}$	$0.06377^{\{7\}}$	$0.058214^{\{4\}}$	$0.060965^{\{6\}}$	$0.087911^{\{9\}}$	$0.059489^{\{5\}}$	$0.055803^{\{2\}}$	$0.065512^{\{8\}}$
	$\sum Ranks$		$6^{\{1\}}$	$16^{\{3\}}$	$36^{\{6\}}$	$29^{\{5\}}$	$39^{\{7\}}$	$54^{\{9\}}$	$27^{\{4\}}$	$15^{\{2\}}$	$48^{\{8\}}$
200	BIAS	$\hat{\alpha}$	$0.019995^{\{1\}}$	$0.021001^{\{5\}}$	$0.021949^{\{6\}}$	$0.020978^{\{4\}}$	$0.022156^{\{7\}}$	$0.040799^{\{9\}}$	$0.022494^{\{8\}}$	$0.020346^{\{2\}}$	$0.020931^{\{3\}}$
		$\hat{\theta}$	$0.028384^{\{1\}}$	$0.029909^{\{4\}}$	$0.033126^{\{7\}}$	$0.029388^{\{3\}}$	$0.032708^{\{6\}}$	$0.051383^{\{9\}}$	$0.031332^{\{5\}}$	$0.02934^{\{2\}}$	$0.034695^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.000626^{\{1\}}$	$0.000679^{\{3\}}$	$0.000748^{\{6\}}$	$0.00068^{\{4\}}$	$0.00075^{\{7\}}$	$0.003^{\{9\}}$	$0.000774^{\{8\}}$	$0.000663^{\{2\}}$	$0.000726^{\{5\}}$
		$\hat{\theta}$	$0.001258^{\{1\}}$	$0.001411^{\{4\}}$	$0.001728^{\{7\}}$	$0.001269^{\{2\}}$	$0.00166^{\{6\}}$	$0.004359^{\{9\}}$	$0.001516^{\{5\}}$	$0.00141^{\{3\}}$	$0.001922^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.079982^{\{1\}}$	$0.084005^{\{5\}}$	$0.087797^{\{6\}}$	$0.083912^{\{4\}}$	$0.088622^{\{7\}}$	$0.163195^{\{9\}}$	$0.089978^{\{8\}}$	$0.081384^{\{2\}}$	$0.083724^{\{3\}}$
		$\hat{\theta}$	$0.037845^{\{1\}}$	$0.039878^{\{4\}}$	$0.044169^{\{7\}}$	$0.039183^{\{3\}}$	$0.04361^{\{6\}}$	$0.06851^{\{9\}}$	$0.041776^{\{5\}}$	$0.03912^{\{2\}}$	$0.04626^{\{8\}}$
	$\sum Ranks$		$6^{\{1\}}$	$25^{\{4\}}$	$39^{\{7\}}$	$20^{\{3\}}$	$39^{\{7\}}$	$54^{\{9\}}$	$39^{\{7\}}$	$13^{\{2\}}$	$35^{\{9\}}$

Table 2: Continued.

n	Est.	*Est.Par.*	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
350	BIAS	$\widehat{\alpha}$	$0.015255^{\{1\}}$	$0.015994^{\{4\}}$	$0.017253^{\{7\}}$	$0.015957^{\{3\}}$	$0.017389^{\{8\}}$	$0.030119^{\{9\}}$	$0.016228^{\{5\}}$	$0.015814^{\{2\}}$	$0.016784^{\{6\}}$
		$\widehat{\theta}$	$0.020888^{\{1\}}$	$0.022521^{\{4\}}$	$0.025787^{\{7\}}$	$0.021487^{\{2\}}$	$0.025267^{\{6\}}$	$0.038265^{\{9\}}$	$0.022927^{\{5\}}$	$0.022151^{\{3\}}$	$0.026993^{\{8\}}$
	MSE	$\widehat{\alpha}$	$0.000367^{\{1\}}$	$0.000397^{\{3\}}$	$0.000457^{\{7\}}$	$0.000402^{\{4\}}$	$0.000468^{\{8\}}$	$0.001574^{\{9\}}$	$0.000414^{\{5\}}$	$0.000383^{\{2\}}$	$0.000431^{\{6\}}$
		$\widehat{\theta}$	$0.000699^{\{1\}}$	$0.000789^{\{4\}}$	$0.00103^{\{7\}}$	$0.000725^{\{2\}}$	$0.001002^{\{6\}}$	$0.002374^{\{9\}}$	$0.000821^{\{5\}}$	$0.000771^{\{3\}}$	$0.001115^{\{8\}}$
	MRE	$\widehat{\alpha}$	$0.06102^{\{1\}}$	$0.063975^{\{4\}}$	$0.069014^{\{7\}}$	$0.063828^{\{3\}}$	$0.069554^{\{8\}}$	$0.120477^{\{9\}}$	$0.064911^{\{5\}}$	$0.063255^{\{2\}}$	$0.067135^{\{6\}}$
		$\widehat{\theta}$	$0.027851^{\{1\}}$	$0.030028^{\{4\}}$	$0.034383^{\{7\}}$	$0.028649^{\{2\}}$	$0.033689^{\{6\}}$	$0.051019^{\{9\}}$	$0.030569^{\{5\}}$	$0.029535^{\{3\}}$	$0.03599^{\{8\}}$
	$\sum Ranks$		$6^{\{1\}}$	$23^{\{4\}}$	$42^{\{7\}}$	$16^{\{3\}}$	$47^{\{7\}}$	$54^{\{9\}}$	$30^{\{5\}}$	$15^{\{2\}}$	$42^{\{7\}}$
500	BIAS	$\widehat{\alpha}$	$0.01275^{\{1\}}$	$0.013037^{\{4\}}$	$0.01402^{\{7\}}$	$0.012839^{\{2\}}$	0.013655	$0.026477^{\{9\}}$	$0.014055^{\{8\}}$	$0.012991^{\{3\}}$	$0.013913^{\{6\}}$
		$\widehat{\theta}$	$0.017803^{\{2\}}$	$0.01881^{\{4\}}$	$0.021562^{\{7\}}$	$0.017448^{\{1\}}$	$0.020936^{\{6\}}$	$0.034149^{\{9\}}$	$0.020136^{\{5\}}$	$0.018195^{\{3\}}$	$0.021659^{\{8\}}$
	MSE	$\widehat{\alpha}$	$0.000251^{\{1\}}$	$0.000267^{\{4\}}$	$0.000308^{\{7\}}$	$0.000266^{\{3\}}$	$3e\text{-}04^{\{5\}}$	$0.001244^{\{9\}}$	$0.000311^{\{8\}}$	$0.000263^{\{2\}}$	$0.000303^{\{6\}}$
		$\widehat{\theta}$	$0.000496^{\{2\}}$	$0.000545^{\{4\}}$	$0.000717^{\{7\}}$	$0.000487^{\{1\}}$	$0.000696^{\{6\}}$	$0.001899^{\{9\}}$	$0.000639^{\{5\}}$	$0.000527^{\{3\}}$	$0.00074^{\{8\}}$
	MRE	$\widehat{\alpha}$	$0.050999^{\{1\}}$	$0.052149^{\{4\}}$	$0.056079^{\{7\}}$	$0.051354^{\{2\}}$	$0.054598^{\{5\}}$	$0.10591^{\{9\}}$	$0.056218^{\{8\}}$	$0.051962^{\{3\}}$	$0.05565^{\{6\}}$
		$\widehat{\theta}$	$0.023738^{\{1\}}$	$0.025081^{\{4\}}$	$0.02875^{\{7\}}$	$0.023264^{\{1\}}$	$0.027914^{\{6\}}$	$0.045532^{\{9\}}$	$0.026848^{\{5\}}$	$0.02426^{\{3\}}$	$0.028879^{\{8\}}$
	$\sum Ranks$		$9^{\{1\}}$	$24^{\{4\}}$	$41^{\{7.5\}}$	$10^{\{2\}}$	$37^{\{5\}}$	$53^{\{9\}}$	$38^{\{6\}}$	$17^{\{3\}}$	$41^{\{7.5\}}$

Table 3: Simulation values of BIAS, MSE, and MRE for $\alpha = 0.75$ and $\theta = 1.5$.

n	Est.	Est. par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
20	BIAS	$\hat{\alpha}$	$0.137977^{\{7\}}$	$0.13617^{\{5\}}$	$0.142029^{\{9\}}$	$0.12337^{\{1\}}$	$0.135548^{\{4\}}$	$0.132766^{\{3\}}$	$0.136454^{\{6\}}$	$0.131292^{\{2\}}$	$0.138844^{\{8\}}$
		$\hat{\theta}$	$0.229513^{\{4\}}$	$0.227968^{\{3\}}$	$0.279356^{\{8\}}$	$0.218778^{\{1\}}$	$0.269247^{\{7\}}$	$0.221119^{\{2\}}$	$0.25409^{\{6\}}$	$0.244711^{\{5\}}$	$0.285505^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.029637^{\{5\}}$	$0.030681^{\{7\}}$	$0.032568^{\{9\}}$	$0.024965^{\{1\}}$	$0.029622^{\{4\}}$	$0.027414^{\{2\}}$	$0.030111^{\{6\}}$	$0.027964^{\{3\}}$	$0.030942^{\{8\}}$
		$\hat{\theta}$	$0.094556^{\{4\}}$	$0.088348^{\{3\}}$	$0.147204^{\{9\}}$	$0.072238^{\{1\}}$	$0.12245^{\{7\}}$	$0.079028^{\{2\}}$	$0.118112^{\{6\}}$	$0.096773^{\{5\}}$	$0.141887^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.183969^{\{7\}}$	$0.18156^{\{5\}}$	$0.189372^{\{9\}}$	$0.164493^{\{1\}}$	$0.180731^{\{4\}}$	$0.177022^{\{3\}}$	$0.181939^{\{6\}}$	$0.175056^{\{2\}}$	$0.185125^{\{8\}}$
		$\hat{\theta}$	$0.153009^{\{4\}}$	$0.151979^{\{3\}}$	$0.186237^{\{8\}}$	$0.145852^{\{1\}}$	$0.179498^{\{7\}}$	$0.147412^{\{2\}}$	$0.169393^{\{6\}}$	$0.163141^{\{5\}}$	$0.190337^{\{9\}}$
	$\sum Ranks$		$31^{\{5\}}$	$26^{\{4\}}$	$52^{\{9\}}$	$6^{\{1\}}$	$33^{\{6\}}$	$14^{\{2\}}$	$36^{\{7\}}$	$22^{\{3\}}$	$50^{\{8\}}$
60	BIAS	$\hat{\alpha}$	$0.076508^{\{3\}}$	$0.077561^{\{5\}}$	$0.07744^{\{4\}}$	$0.076171^{\{2\}}$	$0.081792^{\{9\}}$	$0.079472^{\{7\}}$	$0.080599^{\{8\}}$	$0.075043^{\{1\}}$	$0.078246^{\{6\}}$
		$\hat{\theta}$	$0.121216^{\{1\}}$	$0.123493^{\{3\}}$	$0.142752^{\{7\}}$	$0.122812^{\{2\}}$	$0.151239^{\{9\}}$	$0.138644^{\{6\}}$	$0.136138^{\{5\}}$	$0.1297^{\{4\}}$	$0.149234^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.009225^{\{3\}}$	$0.009755^{\{6\}}$	$0.0093^{\{4\}}$	$0.009041^{\{2\}}$	$0.010473^{\{9\}}$	$0.01029^{\{8\}}$	$0.010226^{\{7\}}$	$0.008758^{\{1\}}$	$0.009742^{\{5\}}$
		$\hat{\theta}$	$0.024438^{\{2\}}$	$0.024734^{\{3\}}$	$0.033754^{\{7\}}$	$0.023414^{\{1\}}$	$0.036508^{\{9\}}$	$0.029362^{\{5\}}$	$0.030011^{\{6\}}$	$0.027302^{\{4\}}$	$0.036072^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.102011^{\{3\}}$	$0.103415^{\{5\}}$	$0.103254^{\{4\}}$	$0.101561^{\{2\}}$	$0.109055^{\{9\}}$	$0.105963^{\{7\}}$	$0.107465^{\{8\}}$	$0.100057^{\{1\}}$	$0.104328^{\{6\}}$
		$\hat{\theta}$	$0.080811^{\{1\}}$	$0.082329^{\{3\}}$	$0.095168^{\{7\}}$	$0.081875^{\{2\}}$	$0.100826^{\{9\}}$	$0.09243^{\{6\}}$	$0.090758^{\{5\}}$	$0.086467^{\{4\}}$	$0.099489^{\{8\}}$
	$\sum Ranks$		$13^{\{2\}}$	$25^{\{4\}}$	$33^{\{5\}}$	$11^{\{1\}}$	$54^{\{9\}}$	$39^{\{6.5\}}$	$39^{\{6.5\}}$	$15^{\{3\}}$	$41^{\{8\}}$
100	BIAS	$\hat{\alpha}$	$0.06041^{\{7\}}$	$0.058974^{\{4\}}$	$0.059138^{\{5\}}$	$0.055183^{\{1\}}$	$0.055893^{\{2\}}$	$0.061067^{\{9\}}$	$0.059588^{\{6\}}$	$0.058633^{\{3\}}$	$0.060994^{\{8\}}$
		$\hat{\theta}$	$0.094829^{\{1\}}$	$0.096446^{\{2\}}$	$0.112153^{\{7\}}$	$0.098434^{\{3\}}$	$0.112176^{\{8\}}$	$0.106085^{\{6\}}$	$0.10123^{\{5\}}$	$0.098785^{\{4\}}$	$0.117038^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.005624^{\{6\}}$	$0.005516^{\{4\}}$	$0.005761^{\{7\}}$	$0.004883^{\{1\}}$	$0.004924^{\{2\}}$	$0.005935^{\{9\}}$	$0.005555^{\{5\}}$	$0.005365^{\{3\}}$	$0.005922^{\{8\}}$
		$\hat{\theta}$	$0.01427^{\{1\}}$	$0.014931^{\{3\}}$	$0.020641^{\{8\}}$	$0.014577^{\{2\}}$	$0.019882^{\{7\}}$	$0.017311^{\{6\}}$	$0.01633^{\{5\}}$	$0.01555^{\{4\}}$	$0.022179^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.080547^{\{7\}}$	$0.078632^{\{4\}}$	$0.078851^{\{5\}}$	$0.073577^{\{1\}}$	$0.074524^{\{2\}}$	$0.081423^{\{9\}}$	$0.07945^{\{6\}}$	$0.078178^{\{3\}}$	$0.081326^{\{8\}}$
		$\hat{\theta}$	$0.063219^{\{1\}}$	$0.064298^{\{2\}}$	$0.074769^{\{7\}}$	$0.065623^{\{3\}}$	$0.074784^{\{8\}}$	$0.070724^{\{6\}}$	$0.067487^{\{5\}}$	$0.065857^{\{4\}}$	$0.078026^{\{9\}}$
	$\sum Ranks$		$23^{\{4\}}$	$19^{\{2\}}$	$39^{\{7\}}$	$11^{\{1\}}$	$29^{\{5\}}$	$45^{\{8\}}$	$32^{\{6\}}$	$21^{\{3\}}$	$51^{\{9\}}$
200	BIAS	$\hat{\alpha}$	$0.042184^{\{8\}}$	$0.04013^{\{2\}}$	$0.041208^{\{5\}}$	$0.040012^{\{1\}}$	$0.043211^{\{9\}}$	$0.041577^{\{7\}}$	$0.041234^{\{6\}}$	$0.040823^{\{3\}}$	$0.041173^{\{4\}}$
		$\hat{\theta}$	$0.067061^{\{2\}}$	$0.067813^{\{3\}}$	$0.081558^{\{8\}}$	$0.065002^{\{1\}}$	$0.078475^{\{7\}}$	$0.072065^{\{6\}}$	$0.071978^{\{5\}}$	$0.069975^{\{4\}}$	$0.082133^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.002753^{\{7\}}$	$0.002509^{\{1\}}$	$0.002766^{\{8\}}$	$0.002564^{\{2\}}$	$0.002927^{\{9\}}$	$0.0027^{\{6\}}$	$0.002698^{\{5\}}$	$0.002593^{\{4\}}$	$0.002581^{\{3\}}$
		$\hat{\theta}$	$0.007107^{\{2\}}$	$0.007212^{\{3\}}$	$0.010524^{\{8\}}$	$0.006594^{\{1\}}$	$0.009767^{\{7\}}$	$0.008261^{\{5\}}$	$0.008311^{\{6\}}$	$0.007747^{\{4\}}$	$0.010726^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.056246^{\{8\}}$	$0.053507^{\{2\}}$	$0.054945^{\{5\}}$	$0.05335^{\{1\}}$	$0.057615^{\{9\}}$	$0.055436^{\{7\}}$	$0.054978^{\{6\}}$	$0.054431^{\{3\}}$	$0.054898^{\{4\}}$
		$\hat{\theta}$	$0.044707^{\{2\}}$	$0.045209^{\{3\}}$	$0.054372^{\{8\}}$	$0.043335^{\{1\}}$	$0.052316^{\{7\}}$	$0.048043^{\{6\}}$	$0.047985^{\{5\}}$	$0.04665^{\{4\}}$	$0.054755^{\{9\}}$
	$\sum Ranks$		$29^{\{4\}}$	$14^{\{2\}}$	$42^{\{8\}}$	$7^{\{1\}}$	$48^{\{9\}}$	$37^{\{6\}}$	$33^{\{5\}}$	$22^{\{3\}}$	$38^{\{7\}}$
350	BIAS	$\hat{\alpha}$	$0.030915^{\{3\}}$	$0.03161^{\{6\}}$	$0.032622^{\{9\}}$	$0.029985^{\{1\}}$	$0.031154^{\{4\}}$	$0.032585^{\{8\}}$	$0.030298^{\{2\}}$	$0.031687^{\{7\}}$	$0.031251^{\{5\}}$
		$\hat{\theta}$	$0.048118^{\{1\}}$	$0.050489^{\{3\}}$	$0.060015^{\{8\}}$	$0.050242^{\{2\}}$	$0.059371^{\{7\}}$	$0.056874^{\{6\}}$	$0.051969^{\{4\}}$	$0.052368^{\{5\}}$	$0.064127^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.001521^{\{3\}}$	$0.001531^{\{5\}}$	$0.001647^{\{8\}}$	$0.001434^{\{1\}}$	$0.001536^{\{6\}}$	$0.00166^{\{9\}}$	$0.001459^{\{2\}}$	$0.001591^{\{7\}}$	$0.001523^{\{4\}}$
		$\hat{\theta}$	$0.003598^{\{1\}}$	$0.004076^{\{3\}}$	$0.00567^{\{8\}}$	$0.003797^{\{2\}}$	$0.005585^{\{7\}}$	$0.005042^{\{6\}}$	$0.004238^{\{4\}}$	$0.004254^{\{5\}}$	$0.00639^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.041221^{\{3\}}$	$0.042147^{\{6\}}$	$0.043496^{\{9\}}$	$0.03998^{\{1\}}$	$0.041539^{\{4\}}$	$0.043447^{\{8\}}$	$0.040398^{\{2\}}$	$0.04225^{\{7\}}$	$0.041668^{\{5\}}$
		$\hat{\theta}$	$0.032078^{\{1\}}$	$0.03366^{\{3\}}$	$0.04001^{\{8\}}$	$0.033495^{\{2\}}$	$0.039581^{\{7\}}$	$0.037916^{\{6\}}$	$0.034646^{\{4\}}$	$0.034912^{\{5\}}$	$0.042751^{\{9\}}$
	$\sum Ranks$		$12^{\{2\}}$	$26^{\{4\}}$	$50^{\{9\}}$	$9^{\{1\}}$	$35^{\{5\}}$	$43^{\{8\}}$	$18^{\{3\}}$	$36^{\{6\}}$	$41^{\{7\}}$

Table 3: Continued.

n	Est.	Est. par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
600	BIAS	$\hat{\alpha}$	$0.026684^{\{7\}}$	$0.026033^{\{3\}}$	$0.02713^{\{8\}}$	$0.024621^{\{1\}}$	$0.026566^{\{6\}}$	$0.02625^{\{4\}}$	$0.027299^{\{9\}}$	$0.025838^{\{2\}}$	$0.026553^{\{5\}}$
		$\hat{\theta}$	$0.040432^{\{2\}}$	$0.041825^{\{3\}}$	$0.048019^{\{7\}}$	$0.039776^{\{1\}}$	$0.048301^{\{8\}}$	$0.045466^{\{5\}}$	$0.046326^{\{6\}}$	$0.043181^{\{4\}}$	$0.051627^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.001141^{\{8\}}$	$0.001066^{\{3\}}$	$0.001117^{\{7\}}$	$0.00097^{\{1\}}$	$0.001109^{\{5\}}$	$0.001072^{\{4\}}$	$0.001152^{\{9\}}$	$0.001058^{\{2\}}$	$0.001114^{\{6\}}$
		$\hat{\theta}$	$0.002594^{\{2\}}$	$0.002772^{\{3\}}$	$0.003593^{\{7\}}$	$0.00249^{\{1\}}$	$0.003644^{\{8\}}$	$0.00323^{\{5\}}$	$0.003314^{\{6\}}$	$0.002989^{\{4\}}$	$0.004147^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.035579^{\{7\}}$	$0.03471^{\{3\}}$	$0.036174^{\{8\}}$	$0.032827^{\{1\}}$	$0.035422^{\{6\}}$	$0.035^{\{4\}}$	$0.036398^{\{9\}}$	$0.034451^{\{2\}}$	$0.035404^{\{5\}}$
		$\hat{\theta}$	$0.026955^{\{2\}}$	$0.027884^{\{3\}}$	$0.032013^{\{7\}}$	$0.026517^{\{1\}}$	$0.032201^{\{8\}}$	$0.03031^{\{5\}}$	$0.030884^{\{6\}}$	$0.028787^{\{4\}}$	$0.034418^{\{9\}}$
	$\sum$ *Ranks*		$28^{\{5\}}$	$18^{\{2.5\}}$	$44^{\{8\}}$	$6^\wedge{}^{\{1\}}$	$41^{\{6\}}$	$27^{\{4\}}$	$45^{\{9\}}$	$18^{\{2.5\}}$	$43^{\{7\}}$

Remark 3. Likelihood ratio order ⇒ hazard rate order ⇒ stochastic order. If $E[X_1] = E[X_2]$, then convex order ⇔ stochastic order.

Theorem 4. *Let $X_i \sim PXLD(\theta_i\,;\alpha_i)\,; i = 1, 2$ be two random variables. If $\theta_1 \geq \theta_2$ and $\alpha_1 \leq \alpha_2$, then $X_1 <_{lr} X_2$, $X_1 <_{hr} X_2$, $X_1 <_S X_2$, and $X_1 \leq_{cx} X_2$.*

Proof. We have

$$\frac{f_{X_1}(t)}{f_{X_2}(t)} = \frac{\alpha_1\theta_1^2(2+\theta_1+t^{\alpha_1})t^{\alpha_1-1}(1+\theta_2)^2}{\alpha_2\theta_2^2(2+\theta_2+t^{\alpha})t^{\alpha_2-1}(1+\theta_1)^2}e^{-\theta_1 t^{\alpha}+\theta_2 t^{\alpha_2}}. \quad (16)$$

Using the ln $(f_X(t)/f_Y(t))$ for simplification, we can find

$$\begin{aligned}\frac{d}{dt}\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right) &= \frac{\alpha_1 t^{-1+\alpha_1}}{2+\theta_1+t^{\alpha_1}} - \frac{\alpha_2 t^{-1+\alpha_2}}{2+\theta_2+t^{\alpha_2}} + \frac{\alpha_1-\alpha_2}{t} \\ &\quad - \theta_1\alpha_1 t^{-1+\alpha_1} + \theta_2\alpha_2 t^{-1+\alpha_2}, \\ &= -\frac{(2+\theta_2+t^{\alpha_2})\alpha_1 t^{\alpha_1}(\theta_1(2+\theta_1+t^{\alpha_1})-1)}{s} \\ &\quad - \frac{(2+\theta_1+t^{\alpha_1})\alpha_2 t^{\alpha_2}(1-\theta_2(2+\theta_2+t^{\alpha_2}))}{s} \\ &\quad - \frac{(\alpha_2-\alpha_1)(2+\theta_1+t^{\alpha_1})(2+\theta_2+t^{\alpha_2})}{s},\end{aligned} \quad (17)$$

where $s = t(2+\theta_1+t^{\alpha_1})(2+\theta_2+t^{\alpha_2})$.

To this end, if $\theta_1 \geq \theta_2$ and $\alpha_1 \leq \alpha_2$, we have $(d/dt)\ln(f_{X_1}(t)/f_{X_2}(t)) \leq 0$. This means that $X_1 <_{lr} X_2$. Also, according to Remark 3 the theorem is proved. □

4. Fuzzy Reliability

Let T be a continuous random variable reflecting the time required for a system to fail (component). The fuzzy reliability of the formula (20) may then be computed using the fuzzy probability.

$$R_F(t) = P(T > t) = \int_t^{\infty}\mu(x)f_{PXL}(x)dx, 0 \leq t \leq x < \infty, \quad (18)$$

where $\mu(x)$ is a membership function that describes the degree to which each element of a given universe belongs to a fuzzy set. For more examples, see [21] and [22]. Now, assume that $\mu(x)$ is

$$\mu(x) = \begin{cases} 0 & , x \leq t_1 \\ \dfrac{x-t_1}{t_2-t_1} & , t_1 < x < t_{2,} t_1 \geq 0 \\ 1 & , x \geq t_2. \end{cases} \quad (19)$$

For $\mu(x)$, by the computational analysis of the function of fuzzy numbers, the lifetime $x(\gamma)$ can be obtained corresponds to a certain value of $\gamma - Cut, \gamma \in 0, 1]$ and can by obtained as: $\mu(x) = \gamma \longrightarrow (x-t_1)/(t_2-t_1) = \gamma$; then

$$\begin{cases} x(\gamma) \leq t_1 & , \gamma = 0 \\ x(\gamma) = t_1 + \gamma(t_2 - t_1) & , 0 < \gamma < 1 \\ x(\gamma) \geq t_2 & , \gamma = 1. \end{cases} \quad (20)$$

As a consequence, the fuzzy reliability values for all γ values may be computed. The fuzzy reliability definition is used to calculate the PXL distribution's fuzzy dependability. The PXL distribution's fuzzy dependability may be described as

$$R_F(t) = \left(1 + \frac{\theta t_1^{\alpha}}{(1+\theta)^2}\right)e^{-\theta t_1^{\alpha}} - \left(1 + \frac{\theta x(\gamma)^{\alpha}}{(1+\theta)^2}\right)e^{-\theta x(\gamma)^{\alpha}}. \quad (21)$$

Then $R_F(t)_{\gamma=0} = 0$.

4.1. Numerical Values of Fuzzy Reliability. In this part, we compare conventional and fuzzy reliability, where traditional reliability is defined as a survival function $S(x) = (1 + (\theta x^{\alpha})/(1+\theta)^2)e^{-\theta x^{\alpha}}$. Table 1 discusses this comparison. Also, plots for both of them are presented in Figure 3. The following observations are made based on the findings:

(i) When the γ-Cut is increased, the fuzzy reliability increases

Table 4: Simulation values of BIAS, MSE, and MRE for $\alpha = 2.5$ and $\theta = 0.5$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
20	BIAS	$\hat{\alpha}$	0.521184$^{\{5\}}$	0.467495$^{\{2\}}$	0.753993$^{\{9\}}$	0.426985$^{\{1\}}$	0.581814$^{\{7\}}$	0.491352$^{\{3\}}$	0.526135$^{\{6\}}$	0.503814$^{\{4\}}$	0.701128$^{\{8\}}$
		$\hat{\theta}$	0.076772$^{\{2\}}$	0.076104$^{\{1\}}$	0.102718$^{\{8\}}$	0.078158$^{\{3\}}$	0.086544$^{\{5\}}$	0.148815$^{\{9\}}$	0.090632$^{\{6\}}$	0.084575$^{\{4\}}$	0.096793$^{\{7\}}$
	MSE	$\hat{\alpha}$	0.604624$^{\{6\}}$	0.411346$^{\{2\}}$	3.032123$^{\{9\}}$	0.331002$^{\{1\}}$	0.801514$^{\{7\}}$	0.575714$^{\{4\}}$	0.596513$^{\{5\}}$	0.540103$^{\{3\}}$	1.553027$^{\{8\}}$
		$\hat{\theta}$	0.010574$^{\{3\}}$	0.009815$^{\{2\}}$	0.02066$^{\{8\}}$	0.009109$^{\{1\}}$	0.01216$^{\{5\}}$	0.03465$^{\{9\}}$	0.014927$^{\{6\}}$	0.012094$^{\{4\}}$	0.016837$^{\{7\}}$
	MRE	$\hat{\alpha}$	0.208474$^{\{5\}}$	0.186998$^{\{2\}}$	0.301597$^{\{9\}}$	0.170794$^{\{1\}}$	0.232726$^{\{7\}}$	0.196541$^{\{3\}}$	0.210454$^{\{6\}}$	0.201525$^{\{4\}}$	0.280451$^{\{8\}}$
		$\hat{\theta}$	0.153544$^{\{2\}}$	0.152208$^{\{1\}}$	0.205435$^{\{8\}}$	0.156317$^{\{3\}}$	0.173088$^{\{5\}}$	0.297631$^{\{9\}}$	0.181264$^{\{6\}}$	0.16915$^{\{4\}}$	0.193587$^{\{7\}}$
	$\sum Ranks$		23$^{\{3.5\}}$	10$^{\{1.5\}}$	51$^{\{9\}}$	10$^{\{1.5\}}$	36$^{\{6\}}$	37$^{\{7\}}$	35$^{\{5\}}$	23$^{\{3.5\}}$	45$^{\{8\}}$
60	BIAS	$\hat{\alpha}$	0.265319$^{\{3\}}$	0.273776$^{\{6\}}$	0.306174$^{\{8\}}$	0.238352$^{\{1\}}$	0.290567$^{\{7\}}$	0.257912$^{\{2\}}$	0.271254$^{\{4\}}$	0.273431$^{\{5\}}$	0.339426$^{\{9\}}$
		$\hat{\theta}$	0.041686$^{\{2\}}$	0.044312^({3})	0.052367$^{\{8\}}$	0.041296$^{\{1\}}$	0.050037$^{\{6\}}$	0.112478$^{\{9\}}$	0.048004$^{\{5\}}$	0.045441$^{\{4\}}$	0.05171$^{\{7\}}$
	MSE	$\hat{\alpha}$	0.125759$^{\{4\}}$	0.132248$^{\{6\}}$	0.181615$^{\{8\}}$	0.090083$^{\{1\}}$	0.140764$^{\{7\}}$	0.110871$^{\{2\}}$	0.126727$^{\{5\}}$	0.125333$^{\{3\}}$	0.215019$^{\{9\}}$
		$\hat{\theta}$	0.002811$^{\{2\}}$	0.003134$^{\{3\}}$	0.004529$^{\{8\}}$	0.002637$^{\{1\}}$	0.004102$^{\{6\}}$	0.020149$^{\{9\}}$	0.003732$^{\{5\}}$	0.00343$^{\{4\}}$	0.004432$^{\{7\}}$
	MRE	$\hat{\alpha}$	0.106128$^{\{3\}}$	0.10951$^{\{6\}}$	0.12247$^{\{8\}}$	0.095341$^{\{1\}}$	0.116227$^{\{7\}}$	0.103165$^{\{2\}}$	0.108502$^{\{4\}}$	0.109372$^{\{5\}}$	0.135771$^{\{8\}}$
		$\hat{\theta}$	0.083371$^{\{2\}}$	0.088625$^{\{3\}}$	0.104733$^{\{8\}}$	0.082593$^{\{1\}}$	0.100074$^{\{6\}}$	0.224957$^{\{9\}}$	0.096009$^{\{5\}}$	0.090882$^{\{4\}}$	0.103419$^{\{7\}}$
	$\sum Ranks$		16$^{\{2\}}$	27$^{\{4\}}$	48^({8.5})	6$^{\{1\}}$	39$^{\{7\}}$	33$^{\{6\}}$	28$^{\{5\}}$	25$^{\{3\}}$	48$^{\{8.5\}}$
100	BIAS	$\hat{\alpha}$	0.199122$^{\{2\}}$	0.206032$^{\{4\}}$	0.22673$^{\{7\}}$	0.19245$^{\{1\}}$	0.229466$^{\{8\}}$	0.220894$^{\{6\}}$	0.199846$^{\{3\}}$	0.208453$^{\{5\}}$	0.25666$^{\{9\}}$
		$\hat{\theta}$	0.032435$^{\{1\}}$	0.034336$^{\{3\}}$	0.038934$^{\{6\}}$	0.033479$^{\{2\}}$	0.039013$^{\{7\}}$	0.092339$^{\{9\}}$	0.036641$^{\{5\}}$	0.035227$^{\{4\}}$	0.039599$^{\{8\}}$
	MSE	$\hat{\alpha}$	0.066608$^{\{3\}}$	0.071395$^{\{4\}}$	0.089779$^{\{7\}}$	0.057914$^{\{1\}}$	0.090105$^{\{8\}}$	0.08029$^{\{6\}}$	0.066601$^{\{2\}}$	0.073165$^{\{5\}}$	0.116057$^{\{9\}}$
		$\hat{\theta}$	0.001705$^{\{2\}}$	0.001877$^{\{3\}}$	0.002458$^{\{7\}}$	0.001685$^{\{1\}}$	0.002398$^{\{6\}}$	0.014214$^{\{9\}}$	0.002186$^{\{5\}}$	0.001999$^{\{4\}}$	0.002515$^{\{8\}}$
	MRE	$\hat{\alpha}$	0.079649$^{\{2\}}$	0.082413$^{\{4\}}$	0.090692$^{\{7\}}$	0.07698$^{\{1\}}$	0.091786$^{\{8\}}$	0.088358$^{\{6\}}$	0.079938$^{\{3\}}$	0.083381$^{\{5\}}$	0.102664$^{\{9\}}$
		$\hat{\theta}$	0.064871$^{\{1\}}$	0.068673$^{\{3\}}$	0.077867$^{\{6\}}$	0.066958$^{\{1\}}$	0.078027$^{\{7\}}$	0.184678$^{\{9\}}$	0.073282$^{\{5\}}$	0.070454$^{\{4\}}$	0.079197^({8})
	$\sum Ranks$		11$^{\{2\}}$	21$^{\{3\}}$	40$^{\{6\}}$	8$^{\{1\}}$	44$^{\{7\}}$	45$^{\{8\}}$	23$^{\{4\}}$	27$^{\{5\}}$	51$^{\{9\}}$
200	BIAS	$\hat{\alpha}$	0.139654$^{\{3\}}$	0.145635$^{\{5\}}$	0.162861$^{\{8\}}$	0.132133$^{\{1\}}$	0.161872$^{\{7\}}$	0.157093$^{\{6\}}$	0.136732$^{\{2\}}$	0.144871$^{\{4\}}$	0.178635$^{\{9\}}$
		$\hat{\theta}$	0.022005$^{\{1\}}$	0.023786$^{\{3\}}$	0.028828$^{\{8\}}$	0.023118$^{\{2\}}$	0.026891$^{\{6\}}$	0.075261$^{\{9\}}$	0.025796$^{\{5\}}$	0.024335$^{\{4\}}$	0.027892$^{\{7\}}$
	MSE	$\hat{\alpha}$	0.030723$^{\{3\}}$	0.033699$^{\{5\}}$	0.043523$^{\{8\}}$	0.026881$^{\{1\}}$	0.041419$^{\{6\}}$	0.042857$^{\{7\}}$	0.030332$^{\{2\}}$	0.032729$^{\{4\}}$	0.053613$^{\{9\}}$
		$\hat{\theta}$	0.000777$^{\{1\}}$	0.000882$^{\{3\}}$	0.001319$^{\{8\}}$	0.000815$^{\{2\}}$	0.001169$^{\{6\}}$	0.009177$^{\{9\}}$	0.001054$^{\{5\}}$	0.00094$^{\{4\}}$	0.001226$^{\{7\}}$
	MRE	$\hat{\alpha}$	0.055862$^{\{3\}}$	0.058254$^{\{5\}}$	0.065144$^{\{8\}}$	0.052853$^{\{1\}}$	0.064749$^{\{7\}}$	0.062837$^{\{6\}}$	0.054693$^{\{2\}}$	0.057948$^{\{4\}}$	0.071454$^{\{9\}}$
		$\hat{\theta}$	0.044011$^{\{1\}}$	0.047571$^{\{3\}}$	0.057656$^{\{8\}}$	0.046236$^{\{2\}}$	0.053782$^{\{6\}}$	0.150522$^{\{9\}}$	0.051593$^{\{5\}}$	0.04867$^{\{4\}}$	0.055783$^{\{7\}}$
	$\sum Ranks$		12$^{\{2\}}$	24$^{\{4.5\}}$	48$^{\{8\}}$	9$^{\{1\}}$	38$^{\{6\}}$	46$^{\{7\}}$	21$^{\{3\}}$	24$^{\{4.5\}}$	48$^{\{8.5\}}$
350	BIAS	$\hat{\alpha}$	0.104313$^{\{3\}}$	0.109432$^{\{5\}}$	0.118824$^{\{8\}}$	0.09866$^{\{1\}}$	0.117734$^{\{6\}}$	0.125123$^{\{8\}}$	0.102642$^{\{2\}}$	0.106265$^{\{4\}}$	0.144369$^{\{9\}}$

Table 4: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
		$\hat{\theta}$	0.016325$^{\{1\}}$	0.018767$^{\{5\}}$	0.020675$^{\{8\}}$	0.016489$^{\{2\}}$	0.021033$^{\{8\}}$	0.061037$^{\{9\}}$	0.019663$^{\{5\}}$	0.018562$^{\{3\}}$	0.020636$^{\{6\}}$
	MSE	$\hat{\alpha}$	0.017754$^{\{3\}}$	0.018707$^{\{5\}}$	0.023275$^{\{8\}}$	0.015368$^{\{1\}}$	0.022592$^{\{6\}}$	0.027329$^{\{8\}}$	0.016928$^{\{2\}}$	0.018148$^{\{4\}}$	0.033717$^{\{9\}}$
		$\hat{\theta}$	0.000429$^{\{2\}}$	0.00054$^{\{4\}}$	0.00069$^{\{8\}}$	0.000426$^{\{1\}}$	0.000691$^{\{8\}}$	0.006229$^{\{9\}}$	0.000617$^{\{5\}}$	0.000534$^{\{3\}}$	0.000671$^{\{6\}}$
	MRE	$\hat{\alpha}$	0.041725$^{\{3\}}$	0.043773$^{\{5\}}$	0.047529$^{\{8\}}$	0.039464$^{\{1\}}$	0.047094$^{\{6\}}$	0.050049$^{\{8\}}$	0.041057$^{\{2\}}$	0.042506$^{\{4\}}$	0.057747$^{\{9\}}$
		$\hat{\theta}$	0.03265$^{\{1\}}$	0.037534$^{\{4\}}$	0.04135$^{\{8\}}$	0.032979$^{\{2\}}$	0.042065$^{\{8\}}$	0.122074$^{\{9\}}$	0.039325$^{\{5\}}$	0.037124$^{\{3\}}$	0.041273$^{\{6\}}$
	$\sum Ranks$		13$^{\{2\}}$	27$^{\{5\}}$	42$^{\{6.5\}}$	8$^{\{1\}}$	42$^{\{6.5\}}$	51$^{\{9\}}$	21$^{\{3.5\}}$	21$^{\{3.5\}}$	45$^{\{8\}}$
	BIAS	$\hat{\alpha}$	0.080085$^{\{1\}}$	0.092965$^{\{5\}}$	0.098139$^{\{6\}}$	0.082415$^{\{2\}}$	0.101594$^{\{7\}}$	0.114768$^{\{9\}}$	0.088289$^{\{3\}}$	0.091908$^{\{4\}}$	0.109989$^{\{8\}}$
		$\hat{\theta}$	0.013154$^{\{1\}}$	0.015722$^{\{4\}}$	0.017762$^{\{7\}}$	0.014324$^{\{3\}}$	0.017534$^{\{6\}}$	0.055976$^{\{9\}}$	0.016353$^{\{5\}}$	0.014203$^{\{2\}}$	0.018341$^{\{8\}}$
	MSE	$\hat{\alpha}$	0.010171$^{\{1\}}$	0.013879$^{\{5\}}$	0.015258$^{\{6\}}$	0.011104$^{\{2\}}$	0.016673$^{\{7\}}$	0.026277$^{\{9\}}$	0.012824$^{\{3\}}$	0.012978$^{\{4\}}$	0.019589$^{\{8\}}$
500		$\hat{\theta}$	0.000286$^{\{1\}}$	0.000386$^{\{4\}}$	0.000488$^{\{7\}}$	0.000316$^{\{2\}}$	0.000486$^{\{6\}}$	0.005328$^{\{9\}}$	0.000412$^{\{5\}}$	0.000322$^{\{3\}}$	0.000527$^{\{8\}}$
	MRE	$\hat{\alpha}$	0.032034$^{\{1\}}$	0.037186$^{\{5\}}$	0.039256$^{\{6\}}$	0.032966$^{\{2\}}$	0.040637$^{\{7\}}$	0.045907$^{\{9\}}$	0.035316$^{\{3\}}$	0.036763$^{\{4\}}$	0.043995$^{\{8\}}$
		$\hat{\theta}$	0.026307$^{\{1\}}$	0.031444$^{\{4\}}$	0.035524$^{\{7\}}$	0.028648$^{\{3\}}$	0.035067$^{\{6\}}$	0.111952$^{\{9\}}$	0.032707$^{\{5\}}$	0.028405$^{\{2\}}$	0.036682$^{\{8\}}$
	$\sum Ranks$		6$^{\{1\}}$	27$^{\{5\}}$	39$^{\{6.5\}}$	14$^{\{2\}}$	39$^{\{6.5\}}$	54$^{\{9\}}$	24$^{\{4\}}$	19$^{\{3\}}$	48$^{\{8\}}$

Table 5: Simulation values of BIAS, MSE, and MRE for $\alpha = 1.5$ and $\theta = 2.5$.

n	Est.	Est. par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
20	BIAS	$\hat{\alpha}$	$0.260322^{\{7\}}$	$0.252966^{\{4\}}$	$0.286052^{\{8\}}$	$0.212118^{\{1\}}$	$0.259951^{\{6\}}$	$0.230689^{\{2\}}$	$0.239306^{\{3\}}$	$0.253787^{\{5\}}$	$0.324086^{\{9\}}$
		$\hat{\theta}$	$0.390096^{\{3\}}$	$0.412361^{\{4\}}$	$0.505593^{\{8\}}$	$0.380992^{\{2\}}$	$0.458447^{\{7\}}$	$0.379715^{\{1\}}$	$0.414099^{\{5\}}$	$0.435619^{\{6\}}$	$0.510942^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.120027^{\{5\}}$	$0.10955^{\{4\}}$	$0.186034^{\{8\}}$	$0.075026^{\{1\}}$	$0.147429^{\{7\}}$	$0.094146^{\{2\}}$	$0.106249^{\{3\}}$	$0.122371^{\{6\}}$	$0.250058^{\{9\}}$
		$\hat{\theta}$	$0.274304^{\{3\}}$	$0.296691^{\{4\}}$	$0.534281^{\{9\}}$	$0.222639^{\{1\}}$	$0.346186^{\{7\}}$	$0.22481^{\{2\}}$	$0.312561^{\{5\}}$	$0.337199^{\{6\}}$	$0.494843^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.173548^{\{7\}}$	$0.168644^{\{4\}}$	$0.190701^{\{8\}}$	$0.141412^{\{1\}}$	$0.173301^{\{6\}}$	$0.153793^{\{2\}}$	$0.159537^{\{3\}}$	$0.169191^{\{5\}}$	$0.216057^{\{9\}}$
		$\hat{\theta}$	$0.156038^{\{3\}}$	$0.164944^{\{4\}}$	$0.202237^{\{8\}}$	$0.152397^{\{2\}}$	$0.183379^{\{7\}}$	$0.151886^{\{1\}}$	$0.16564^{\{5\}}$	$0.174248^{\{6\}}$	$0.204377^{\{9\}}$
	$\sum Ranks$		$28^{\{5\}}$	$24^{\{3.5\}}$	$49^{\{8\}}$	$8^{\{1\}}$	$40^{\{7\}}$	$10^{\{2\}}$	$24^{\{3.5\}}$	$34^{\{6\}}$	$53^{\{9\}}$
60	BIAS	$\hat{\alpha}$	$0.127447^{\{1\}}$	$0.140202^{\{5\}}$	$0.149724^{\{8\}}$	$0.127918^{\{2\}}$	$0.138552^{\{4\}}$	$0.128937^{\{3\}}$	$0.140356^{\{6\}}$	$0.145771^{\{7\}}$	$0.154086^{\{9\}}$
		$\hat{\theta}$	$0.206986^{\{1\}}$	$0.225093^{\{4\}}$	$0.253965^{\{7\}}$	$0.23259^{\{5\}}$	$0.254481^{\{8\}}$	$0.213088^{\{2\}}$	$0.238331^{\{6\}}$	$0.224397^{\{3\}}$	$0.256915^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.026638^{\{2\}}$	$0.031322^{\{5\}}$	$0.038656^{\{8\}}$	$0.026291^{\{1\}}$	$0.030462^{\{4\}}$	$0.027024^{\{3\}}$	$0.031407^{\{6\}}$	$0.034376^{\{7\}}$	$0.040969^{\{9\}}$
		$\hat{\theta}$	$0.071275^{\{2\}}$	$0.081426^{\{5\}}$	$0.111348^{\{9\}}$	$0.078337^{\{3\}}$	$0.101455^{\{7\}}$	$0.069738^{\{1\}}$	$0.096487^{\{6\}}$	$0.079157^{\{4\}}$	$0.10806^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.084964^{\{1\}}$	$0.093468^{\{5\}}$	$0.099816^{\{8\}}$	$0.085279^{\{2\}}$	$0.092368^{\{4\}}$	$0.085958^{\{3\}}$	$0.093571^{\{6\}}$	$0.097181^{\{7\}}$	$0.102724^{\{9\}}$
		$\hat{\theta}$	$0.082794^{\{1\}}$	$0.090037^{\{4\}}$	$0.101586^{\{7\}}$	$0.093036^{\{5\}}$	$0.101792^{87\}}$	$0.085235^{\{2\}}$	$0.095332^{\{6\}}$	$0.089759^{\{3\}}$	$0.102766^{\{9\}}$
	$\sum Ranks$		$8^{\{1\}}$	$28^{\{4\}}$	$47^{\{8\}}$	$18^{\{3\}}$	$35^{\{6\}}$	$14^{\{2\}}$	$36^{\{7\}}$	$31^{\{5\}}$	$53^{\{9\}}$
100	BIAS	$\hat{\alpha}$	$0.108672^{\{6\}}$	$0.105696^{\{5\}}$	$0.117563^{\{9\}}$	$0.09787^{\{2\}}$	$0.112172^{\{7\}}$	$0.096309^{\{1\}}$	$0.100645^{\{3\}}$	$0.102352^{\{4\}}$	$0.113071^{\{8\}}$
		$\hat{\theta}$	$0.160012^{\{2\}}$	$0.168575^{\{3\}}$	$0.198132^{\{9\}}$	$0.170146^{\{4\}}$	$0.193601^{\{7\}}$	$0.159442^{\{1\}}$	$0.181613^{\{6\}}$	$0.172704^{\{5\}}$	$0.195852^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.018883^{\{6\}}$	$0.017835^{\{5\}}$	$0.021818^{\{9\}}$	$0.015112^{\{2\}}$	$0.020421^{\{7\}}$	$0.014428^{\{1\}}$	$0.016357^{\{3\}}$	$0.016985^{\{4\}}$	$0.020676^{\{8\}}$
		$\hat{\theta}$	$0.042642^{\{2\}}$	$0.044001^{\{4\}}$	$0.065373^{\{9\}}$	$0.042984^{\{3\}}$	$0.057578^{\{7\}}$	$0.038296^{\{1\}}$	$0.053659^{\{6\}}$	$0.047041^{\{5\}}$	$0.062995^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.072448^{\{6\}}$	$0.070464^{\{5\}}$	$0.078375^{\{9\}}$	$0.065246^{\{2\}}$	$0.074781^{\{7\}}$	$0.064206^{\{1\}}$	$0.067097^{\{3\}}$	$0.068235^{\{4\}}$	$0.075381^{\{8\}}$
		$\hat{\theta}$	$0.064005^{\{2\}}$	$0.06743^{\{3\}}$	$0.079253^{\{9\}}$	$0.068059^{\{4\}}$	$0.07744^{\{7\}}$	$0.063777^{\{1\}}$	$0.072645^{\{6\}}$	$0.069082^{\{5\}}$	$0.078341^{\{8\}}$
	$\sum Ranks$		$24^{\{3\}}$	$25^{\{4\}}$	$54^{\{9\}}$	$17^{\{2\}}$	$42^{\{7\}}$	$6^{\{1\}}$	$27^{\{5.5\}}$	$27^{\{5.5\}}$	$48^{\{8\}}$
200	BIAS	$\hat{\alpha}$	$0.072573^{\{3\}}$	$0.0739^{\{4\}}$	$0.075672^{\{7\}}$	$0.071735^{\{2\}}$	$0.077291^{\{8\}}$	$0.071665^{\{1\}}$	$0.0749^{\{6\}}$	$0.074856^{\{5\}}$	$0.082323^{\{9\}}$
		$\hat{\theta}$	$0.116527^{\{3\}}$	$0.117408^{\{4\}}$	$0.135815^{\{8\}}$	$0.111618^{\{1\}}$	$0.129664^{\{7\}}$	$0.112908^{\{2\}}$	$0.12543^{\{6\}}$	$0.121674^{\{5\}}$	$0.141145^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.00847^{\{3\}}$	$0.008563^{\{4\}}$	$0.009018^{\{7\}}$	$0.008238^{\{2\}}$	$0.009339^{\{8\}}$	$0.007981^{\{1\}}$	$0.008825^{\{6\}}$	$0.008743^{\{5\}}$	$0.010496^{\{9\}}$
		$\hat{\theta}$	$0.021346^{\{3\}}$	$0.022249^{\{4\}}$	$0.030064^{\{8\}}$	$0.019931^{\{1\}}$	$0.02741^{\{7\}}$	$0.02014^{\{2\}}$	$0.024372^{\{6\}}$	$0.022852^{\{5\}}$	$0.031395^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.048382^{\{3\}}$	$0.049267^{\{4\}}$	$0.050448^{\{7\}}$	$0.047823^{\{2\}}$	$0.051527^{\{8\}}$	$0.047776^{\{1\}}$	$0.049934^{\{6\}}$	$0.049904^{\{5\}}$	$0.054882^{\{9\}}$
		$\hat{\theta}$	$0.046611^{\{3\}}$	$0.046963^{\{4\}}$	$0.054326^{\{8\}}$	$0.044647^{\{1\}}$	$0.051865^{\{7\}}$	$0.045163^{\{2\}}$	$0.050172^{\{6\}}$	$0.04867^{\{5\}}$	$0.056458^{\{9\}}$
	$\sum Ranks$		$18^{\{3\}}$	$24^{\{4\}}$	$45^{\{7.5\}}$	$9^{\{1.5\}}$	$45^{\{7.5\}}$	$9^{\{1.5\}}$	$36\wedge^{\{6\}}$	$30^{\{5\}}$	$54^{\{9\}}$
350	BIAS	$\hat{\alpha}$	$0.053528^{\{2\}}$	$0.054066^{\{3\}}$	$0.059041^{\{7\}}$	$0.054357^{\{4\}}$	$0.059289^{\{8\}}$	$0.052958^{\{1\}}$	$0.05575^{\{6\}}$	$0.055388^{\{5\}}$	$0.060781^{\{9\}}$
		$\hat{\theta}$	$0.079826^{\{1\}}$	$0.091069^{\{5\}}$	$0.103266^{\{9\}}$	$0.085171^{\{2\}}$	$0.099207^{\{7\}}$	$0.087793^{\{3\}}$	$0.09334^{\{6\}}$	$0.088486^{\{4\}}$	$0.102409^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.004605^{\{2\}}$	$0.004705^{\{4\}}$	$0.00544^{\{7\}}$	$0.004695^{\{3\}}$	$0.005646^{\{8\}}$	$0.004399^{\{1\}}$	$0.004836^{\{5\}}$	$0.00492^{\{6\}}$	$0.005795^{\{9\}}$
		$\hat{\theta}$	$0.010508^{\{1\}}$	$0.012945^{\{5\}}$	$0.017093^{\{9\}}$	$0.011567^{\{2\}}$	$0.015637^{\{7\}}$	$0.012121^{\{3\}}$	$0.014118^{\{6\}}$	$0.012408^{\{4\}}$	$0.016951^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.035685^{\{2\}}$	$0.036044^{\{3\}}$	$0.039361^{\{7\}}$	$0.036238^{\{4\}}$	$0.039526^{\{8\}}$	$0.035305^{\{1\}}$	$0.037166^{\{6\}}$	$0.036926^{\{5\}}$	$0.04052^{\{9\}}$
		$\hat{\theta}$	$0.03193^{\{1\}}$	$0.036428^{\{5\}}$	$0.041306^{\{9\}}$	$0.034068^{\{2\}}$	$0.039683^{\{7\}}$	$0.035117^{\{3\}}$	$0.037336^{\{6\}}$	$0.035394^{\{4\}}$	$0.040964^{\{8\}}$
	$\sum Ranks$		$9^{\{1\}}$	$25^{\{4\}}$	$48^{\{8\}}$	$17^{\{3\}}$	$45^{\{7\}}$	$12^{\{2\}}$	$35^{\{6\}}$	$28^{\{5\}}$	$51^{\{9\}}$

Table 5: Continued.

n	Est.	Est. par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
600	BIAS	$\hat{\alpha}$	$0.046772^{\{4\}}$	$0.046659^{\{3\}}$	$0.048574^{\{8\}}$	$0.047313^{\{6\}}$	$0.047942^{\{7\}}$	$0.045318^{\{1\}}$	$0.045952^{\{2\}}$	$0.046996^{\{5\}}$	$0.055547^{\{9\}}$
		$\hat{\theta}$	$0.070225^{\{1\}}$	$0.075111^{\{4\}}$	$0.087494^{\{8\}}$	$0.070843^{\{2\}}$	$0.082491^{\{7\}}$	$0.074186^{\{3\}}$	$0.082064^{\{6\}}$	$0.076815^{\{5\}}$	$0.090869^{\{9\}}$
	MSE	$\hat{\alpha}$	$0.003462^{\{5\}}$	$0.003455^{\{4\}}$	$0.003917^{\{8\}}$	$0.003465^{\{6\}}$	$0.003678^{\{7\}}$	$0.003229^{\{1\}}$	$0.003322^{\{2\}}$	$0.003402^{\{3\}}$	$0.004951^{\{9\}}$
		$\hat{\theta}$	$0.00806^{\{2\}}$	$0.008834^{\{4\}}$	$0.011932^{\{8\}}$	$0.007829^{\{1\}}$	$0.010944^{\{7\}}$	$0.008551^{\{3\}}$	$0.010789^{\{6\}}$	$0.009387^{\{5\}}$	$0.012818^{\{9\}}$
	MRE	$\hat{\alpha}$	$0.031181^{\{4\}}$	$0.031106^{\{3\}}$	$0.032383^{\{8\}}$	$0.031542^{\{6\}}$	$0.031961^{\{7\}}$	$0.030212^{\{1\}}$	$0.030634^{\{2\}}$	$0.031331^{\{5\}}$	$0.037031^{\{9\}}$
		$\hat{\theta}$	$0.02809^{\{1\}}$	$0.030044^{\{4\}}$	$0.034998^{\{8\}}$	$0.028337^{\{2\}}$	$0.032996^{\{7\}}$	$0.029675^{\{3\}}$	$0.032826^{\{6\}}$	$0.030726^{\{5\}}$	$0.036348^{\{9\}}$
	$\sum Ranks$		$17^{\{2\}}$	$22^{\{3\}}$	$48^{\{8\}}$	$23^{\{4\}}$	$42^{\{7\}}$	$12^{\{1\}}$	$24^{\{5\}}$	$28^{\{6\}}$	$54^{\{9\}}$

(ii) When the t_2 of the interval of the membership function is increased, the fuzzy reliability increases

(iii) When the t_1 is decreased, the fuzzy reliability increases, and vice versa

(iv) The traditional reliability with t_2 is lower than the traditional reliability with t_1

The fuzzy estimation algorithm produces a series of draws from PXL distribution as follows in Algorithm 1.

5. The Classical Methods Used for Estimating the Parameters

This section addresses several approaches for estimating the suggested model parameters, including the maximum likelihood estimation (MLE), the most well-known classical approach. The Anderson–Darling estimate (ADE) is another key technique that is employed in place of MLE. Another significant approach that is used in lieu of MLE is the Cramer-von Mises estimate (CVME). Another significant method that is used in lieu of MLE is the maximum product of spacings estimation (MPSE). Another major technique that is employed in lieu of MLE is the conventional least-squares estimate (OLSE). Another major technique that is employed in place of MLE is percentile estimation (PCE). Another significant approach used in place of MLE is the right-tailed Anderson–Darling estimation (RTADE). Another significant method used in place of MLE is the weighted least squares estimation (WLSE). Another significant technique that is employed in place of MLE is the left-tailed Anderson–Darling estimate (LTADE).

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from the PDF of the proposed model; then the log-likelihood function takes the form

$$L = -\theta \sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n} \log\left(\theta + x_i^{\alpha} + 2\right) + (\alpha - 1) \sum_{i=1}^{n} \log\left(x_i\right) + n \log\left(\frac{\alpha\theta^2}{(\theta+1)^2}\right). \quad (22)$$

To get the estimates, we must find the first derivative for Equation (22) regarding the distribution's parameters.

A sorted random sample from the suggested distribution is $x1 : n, x2 : n, \cdots, x2 : n$. As a result, by minimizing the following equation, we will get the OLSE of the suggested model parameters α and θ:

$$O = \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{i}{n+1}\right]^2. \quad (23)$$

The following expression is minimized to determine the ADEs of the suggested model parameters.

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S(x_{i:n})\right]. \quad (24)$$

The WLSE of the suggested model parameters α and θ is determined by minimizing the following formula:

$$W = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1}\right]^2. \quad (25)$$

The CVME of the given model parameters is determined by minimizing the formula given:

$$CV = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2. \quad (26)$$

As just an equivalent to the MLE approach, the MPS technique is used to estimate the parameters of continuous univariate models. The uniform spacings of a random sample of size n drawn from the suggested distribution can be described this way:

$$D_i = F(x_i) - F(x_{i-1}), \quad (27)$$

where D_i denotes to the uniform spacings, $F(x_0) = 0$, $F(x_{n+1} = 1)$, and $\sum_{i=1}^{n+1} D_i = 1$. MPS obtained by maximizing

Table 6: Simulation values of BIAS, MSE, and MRE for $\alpha = 3$ and $\theta = 5$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
20	BIAS	$\hat{\alpha}$	$0.717631^{\{5\}}$	$0.660787^{\{3\}}$	$1.001041^{\{8\}}$	$0.560465^{\{1\}}$	$0.791574^{\{7\}}$	$0.586262^{\{2\}}$	$0.679892^{\{4\}}$	$0.737606^{\{6\}}$	$1.071329^{\{9\}}$
		$\hat{\theta}$	$0.8064^{\{4\}}$	$0.780501^{\{2\}}$	$1.017863^{\{9\}}$	$0.775191^{\{1\}}$	$0.877717^{\{7\}}$	$0.796661^{\{3\}}$	$0.877084^{\{6\}}$	$0.856637^{\{5\}}$	$0.994721^{\{8\}}$
	MSE	$\hat{\alpha}$	$1.260224^{\{5\}}$	$0.978326^{\{3\}}$	$4.39799^{\{8\}}$	$0.502231^{\{1\}}$	$2.122434^{\{7\}}$	$0.865303^{\{2\}}$	$1.04072^{\{4\}}$	$1.899592^{\{6\}}$	$4.751738^{\{9\}}$
		$\hat{\theta}$	$1.104086^{\{4\}}$	$1.046869^{\{3\}}$	$2.097186^{\{9\}}$	$0.876038^{\{1\}}$	$1.294^{\{6\}}$	$1.031908^{\{2\}}$	$1.322306^{\{7\}}$	$1.272431^{\{5\}}$	$1.887763^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.23921^{\{5\}}$	$0.220262^{\{3\}}$	$0.33368^{\{8\}}$	$0.186822^{\{1\}}$	$0.263858^{\{7\}}$	$0.195421^{\{2\}}$	$0.226631^{\{4\}}$	$0.245869^{\{6\}}$	$0.35711^{\{9\}}$
		$\hat{\theta}$	$0.16128^{\{4\}}$	$0.1561^{\{2\}}$	$0.203573^{\{9\}}$	$0.155038^{\{1\}}$	$0.175543^{\{7\}}$	$0.159332^{\{3\}}$	$0.175417^{\{6\}}$	$0.171327^{\{5\}}$	$0.198944^{\{8\}}$
	$\sum$ *Ranks*		$27^{\{4\}}$	$16^{\{3\}}$	$51^{\{8.5\}}$	$6^{\{1\}}$	$41^{\{7\}}$	$14^{\{2\}}$	$31^{\{5\}}$	$33^{\{6\}}$	$51^{\{8.5\}}$
60	BIAS	$\hat{\alpha}$	$0.336795^{\{3\}}$	$0.352473^{\{4\}}$	$0.429429^{\{8\}}$	$0.324061^{\{1\}}$	$0.401071^{\{7\}}$	$0.329515^{\{2\}}$	$0.357751^{\{5\}}$	$0.374126^{\{6\}}$	$0.447436^{\{9\}}$
		$\hat{\theta}$	$0.427527^{\{2\}}$	$0.450167^{\{4\}}$	$0.523862^{\{9\}}$	$0.422169^{\{1\}}$	$0.504418^{\{8\}}$	$0.445308^{\{3\}}$	$0.481925^{\{6\}}$	$0.473978^{\{5\}}$	$0.495872^{\{7\}}$
	MSE	$\hat{\alpha}$	$0.202636^{\{3\}}$	$0.220043^{\{4\}}$	$0.367905^{\{9\}}$	$0.162656^{\{1\}}$	$0.282437^{\{7\}}$	$0.174738^{\{2\}}$	$0.235339^{\{5\}}$	$0.254841^{\{6\}}$	$0.367708^{\{8\}}$
		$\hat{\theta}$	$0.299391^{\{2\}}$	$0.316274^{\{4\}}$	$0.477481^{\{9\}}$	$0.271901^{\{1\}}$	$0.410023^{\{8\}}$	$0.30061^{\{3\}}$	$0.379802^{\{6\}}$	$0.375685^{\{5\}}$	$0.406432^{\{7\}}$
	MRE	$\hat{\alpha}$	$0.112265^{\{3\}}$	$0.117491^{\{4\}}$	$0.143143^{\{8\}}$	$0.10802^{\{1\}}$	$0.13369^{\{7\}}$	$0.109838^{\{2\}}$	$0.11925^{\{5\}}$	$0.124709^{\{6\}}$	$0.149145^{\{9\}}$
		$\hat{\theta}$	$0.085505^{\{2\}}$	$0.090033^{\{4\}}$	$0.104772^{\{9\}}$	$0.084434^{\{1\}}$	$0.100884^{\{8\}}$	$0.089062^{\{3\}}$	$0.096385^{\{6\}}$	$0.094796^{\{5\}}$	$0.099174^{\{7\}}$
	$\sum$ *Ranks*		$15^{\{2.5\}}$	$24^{\{4\}}$	$52^{\{9\}}$	$6^{\{1\}}$	$45^{\{7\}}$	$15^{\{2.5\}}$	$33^{\{5.5\}}$	$33^{\{5.5\}}$	$47^{\{8\}}$
100	BIAS	$\hat{\alpha}$	$0.241288^{\{1\}}$	$0.266587^{\{5\}}$	$0.328484^{\{8\}}$	$0.250573^{\{2\}}$	$0.29527^{\{7\}}$	$0.257789^{\{3\}}$	$0.260292^{\{4\}}$	$0.284411^{\{6\}}$	$0.357^{\{9\}}$
		$\hat{\theta}$	$0.316796^{\{1\}}$	$0.33826^{\{2\}}$	$0.409549^{\{9\}}$	$0.341089^{\{3\}}$	$0.374399^{\{7\}}$	$0.342775^{\{4\}}$	$0.357668^{\{5\}}$	$0.363199^{\{6\}}$	$0.396579^{\{8\}}$
	MSE	$\hat{\alpha}$	$0.105701^{\{3\}}$	$0.119815^{\{5\}}$	$0.186885^{\{8\}}$	$0.098055^{\{1\}}$	$0.142812^{\{7\}}$	$0.104898^{\{2\}}$	$0.110255^{\{4\}}$	$0.139001^{\{6\}}$	$0.223422^{\{9\}}$
		$\hat{\theta}$	$0.161836^{\{1\}}$	$0.181868^{\{3\}}$	$0.272851^{\{9\}}$	$0.172578^{\{2\}}$	$0.23128^{\{7\}}$	$0.18437^{\{4\}}$	$0.21176^{\{5\}}$	$0.212133^{\{6\}}$	$0.255802^{\{8\}}$
	MRE	$\hat{\alpha}$	$0.080429^{\{1\}}$	$0.088862^{\{5\}}$	$0.109495^{\{8\}}$	$0.083524^{\{2\}}$	$0.098423^{\{7\}}$	$0.08593^{\{3\}}$	$0.086764^{\{4\}}$	$0.094804^{\{6\}}$	$0.119^{\{9\}}$
		$\hat{\theta}$	$0.063359^{\{1\}}$	$0.067652^{\{2\}}$	$0.08191^{\{9\}}$	$0.068218^{\{3\}}$	$0.07488^{\{7\}}$	$0.068555^{\{4\}}$	$0.071534^{\{5\}}$	$0.07264^{\{6\}}$	$0.079316^{\{8\}}$
	$\sum$ *Ranks*		$8^{\{1\}}$	$22^{\{4\}}$	$51^{\{8.5\}}$	$13^{\{2\}}$	$42^{\{7\}}$	$20^{\{3\}}$	$27^{\{5\}}$	$36^{\{6\}}$	$51^{\{8.5\}}$
200	BIAS	$\hat{\alpha}$	$0.175226^{\{3\}}$	$0.199898^{\{6\}}$	$0.221033^{\{8\}}$	$0.17111^{\{1\}}$	$0.208835^{\{7\}}$	$0.173908^{\{2\}}$	$0.185609^{\{4\}}$	$0.193912^{\{5\}}$	$0.242327^{\{9\}}$
		$\hat{\theta}$	$0.224783^{\{2\}}$	$0.245806^{\{5\}}$	$0.27591^{\{9\}}$	$0.22348^{\{1\}}$	$0.273244^{\{8\}}$	$0.231358^{\{3\}}$	$0.258757^{\{6\}}$	$0.242492^{\{4\}}$	$0.271779^{\{7\}}$
	MSE	$\hat{\alpha}$	$0.048735^{\{3\}}$	$0.065673^{\{6\}}$	$0.078958^{\{8\}}$	$0.045382^{\{1\}}$	$0.068861^{\{7\}}$	$0.046069^{\{2\}}$	$0.056114^{\{4\}}$	$0.062625^{\{5\}}$	$0.096021^{\{9\}}$
		$\hat{\theta}$	$0.079223^{\{2\}}$	$0.097761^{\{5\}}$	$0.121375^{\{9\}}$	$0.078737^{\{1\}}$	$0.117729^{\{8\}}$	$0.082417^{\{3\}}$	$0.103162^{\{6\}}$	$0.092745^{\{4\}}$	$0.117672^{\{7\}}$
	MRE	$\hat{\alpha}$	$0.058409^{\{3\}}$	$0.066633^{\{6\}}$	$0.073678^{\{8\}}$	$0.057037^{\{1\}}$	$0.069612^{\{7\}}$	$0.057969^{\{2\}}$	$0.06187^{\{4\}}$	$0.064637^{\{5\}}$	$0.080776^{\{9\}}$
		$\hat{\theta}$	$0.044957^{\{2\}}$	$0.049161^{\{5\}}$	$0.055182^{\{9\}}$	$0.044696^{\{1\}}$	$0.054649^{\{8\}}$	$0.046272^{\{3\}}$	$0.051751^{\{6\}}$	$0.048498^{\{4\}}$	$0.054356^{\{7\}}$
	$\sum$ *Ranks*		$15^{\{2.5\}}$	$33^{\{6\}}$	$51^{\{9\}}$	$6^{\{1\}}$	$45^{\{7\}}$	$15^{\{2.5\}}$	$30^{\{5\}}$	$27^{\{4\}}$	$48^{\{8\}}$
350	BIAS	$\hat{\alpha}$	$0.132268^{\{2\}}$	$0.13862^{\{5\}}$	$0.16235^{\{8\}}$	$0.130753^{\{1\}}$	$0.161735^{\{7\}}$	$0.132516^{\{3\}}$	$0.135826^{\{4\}}$	$0.144216^{\{6\}}$	$0.179334^{\{9\}}$
		$\hat{\theta}$	$0.169065^{\{1\}}$	$0.17603^{\{4\}}$	$0.206035^{\{8\}}$	$0.176016^{\{3\}}$	$0.211213^{\{9\}}$	$0.174479^{\{2\}}$	$0.198401^{\{6\}}$	$0.184569^{\{5\}}$	$0.204219^{\{7\}}$
	MSE	$\hat{\alpha}$	$0.028086^{\{3\}}$	$0.030201^{\{5\}}$	$0.041446^{\{7\}}$	$0.026003^{\{1\}}$	$0.041732^{\{8\}}$	$0.027581^{\{2\}}$	$0.029765^{\{4\}}$	$0.032407^{\{6\}}$	$0.05196^{\{9\}}$
		$\hat{\theta}$	$0.04502^{\{1\}}$	$0.049537^{\{4\}}$	$0.067428^{\{8\}}$	$0.047436^{\{2\}}$	$0.069066^{\{9\}}$	$0.048152^{\{3\}}$	$0.060185^{\{6\}}$	$0.053221^{\{5\}}$	$0.066214^{\{7\}}$
	MRE	$\hat{\alpha}$	$0.044089^{\{2\}}$	$0.046207^{\{5\}}$	$0.054117^{\{8\}}$	$0.043584^{\{1\}}$	$0.053912^{\{7\}}$	$0.044172^{\{3\}}$	$0.045275^{\{4\}}$	$0.048072^{\{6\}}$	$0.059778^{\{9\}}$
		$\hat{\theta}$	$0.033813^{\{1\}}$	$0.035206^{\{4\}}$	$0.041207^{\{8\}}$	$0.035203^{\{3\}}$	$0.042243^{\{9\}}$	$0.034896^{\{2\}}$	$0.03968^{\{6\}}$	$0.036914^{\{5\}}$	$0.040844^{\{7\}}$
	$\sum$ *Ranks*		$10^{\{1\}}$	$27^{\{4\}}$	$47^{\{7\}}$	$11^{\{2\}}$	$49^{\{9\}}$	$15^{\{3\}}$	$30^{\{5\}}$	$33^{\{6\}}$	$48^{\{8\}}$

Table 6: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
500	BIAS	$\widehat{\alpha}$	0.10917$^{\{1\}}$	0.118367$^{\{5\}}$	0.135449$^{\{7\}}$	0.111372$^{\{2\}}$	0.136309$^{\{8\}}$	0.111958$^{\{3\}}$	0.117893$^{\{4\}}$	0.121807$^{\{6\}}$	0.155991$^{\{9\}}$
		$\widehat{\theta}$	0.136107$^{\{1\}}$	0.152479$^{\{5\}}$	0.161434$^{\{6\}}$	0.142899$^{\{2\}}$	0.172049$^{\{8\}}$	0.145944$^{\{3\}}$	0.16638$^{\{7\}}$	0.15149$^{\{4\}}$	0.182506$^{\{9\}}$
	MSE	$\widehat{\alpha}$	0.018978$^{\{1\}}$	0.022797$^{\{5\}}$	0.029026$^{\{7\}}$	0.019146$^{\{2\}}$	0.029311$^{\{8\}}$	0.019362$^{\{3\}}$	0.021941$^{\{4\}}$	0.023526$^{\{6\}}$	0.040674$^{\{9\}}$
		$\widehat{\theta}$	0.028912$^{\{1\}}$	0.036719$^{\{5\}}$	0.041851$^{\{6\}}$	0.033342$^{\{2\}}$	0.046981$^{\{8\}}$	0.03375$^{\{3\}}$	0.042394$^{\{7\}}$	0.035251$^{\{4\}}$	0.052224$^{\{9\}}$
	MRE	$\widehat{\alpha}$	0.03639$^{\{1\}}$	0.039456$^{\{5\}}$	0.04515$^{\{7\}}$	0.037124$^{\{2\}}$	0.045436$^{\{8\}}$	0.037319$^{\{3\}}$	0.039298$^{\{4\}}$	0.040602$^{\{6\}}$	0.051997$^{\{9\}}$
		$\widehat{\theta}$	0.027221$^{\{1\}}$	0.030496$^{\{5\}}$	0.032287$^{\{6\}}$	0.02858$^{\{2\}}$	0.03441$^{\{8\}}$	0.029189$^{\{3\}}$	0.033276$^{\{7\}}$	0.030298$^{\{4\}}$	0.036501$^{\{9\}}$
	$\sum$*Ranks*		6$^{\{1\}}$	30$^{\{4.5\}}$	39$^{\{7\}}$	12$^{\{2\}}$	48$^{\{8\}}$	18$^{\{3\}}$	33$^{\{6\}}$	30$^{\{4.5\}}$	54$^{\{9\}}$

Table 7: Partial and overall ranks of all estimation methods of PXL distribution.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE
$\alpha = 0.25$, $\theta = 0.75$	20	1.0	2.0	4.5	3.0	8.0	9.0	4.5	6.0	7.0
	60	1.0	4.5	8.0	3.0	7.0	9.0	4.5	2.0	6.0
	100	1.0	3.0	6.0	5.0	7.0	9.0	4.0	2.0	8.0
	200	1.0	4.0	7.0	3.0	7.0	9.0	7.0	2.0	5.0
	350	1.0	4.0	7.0	3.0	7.0	9.0	5.0	2.0	7.0
	500	1.0	4.0	7.5	2.0	5.0	9.0	6.0	3.0	7.5
$\alpha = 0.75$, $\theta = 1.5$	20	5.0	4.0	9.0	1.0	6.0	2.0	7.0	3.0	8.0
	60	2.0	4.0	5.0	1.0	9.0	6.5	6.5	3.0	8.0
	100	4.0	2.0	7.0	1.0	5.0	8.0	6.0	3.0	9.0
	200	4.0	2.0	8.0	1.0	9.0	6.0	5.0	3.0	7.0
	350	2.0	4.0	9.0	1.0	5.0	8.0	3.0	6.0	7.0
	500	5.0	2.5	8.0	1.0	6.0	4.0	9.0	2.5	7.0
$\alpha = 2.5$, $\theta = 0.5$	20	3.5	1.5	9.0	1.5	6.0	7.0	5.0	3.5	8.0
	60	2.0	4.0	8.5	1.0	7.0	6.0	5.0	3.0	8.5
	100	2.0	3.0	6.0	1.0	7.0	8.0	4.0	5.0	9.0
	200	2.0	4.5	8.5	1.0	6.0	7.0	3.0	4.5	8.5
	350	2.0	5.0	6.5	1.0	6.5	9.0	3.5	3.5	8.0
	500	1.0	5.0	6.5	2.0	6.5	9.0	4.0	3.0	8.0
$\alpha = 1.5$, $\theta = 2.5$	20	5.0	3.5	8.0	1.0	7.0	2.0	3.5	6.0	9.0
	60	1.0	4.0	8.0	3.0	6.0	2.0	7.0	5.0	9.0
	100	3.0	4.0	9.0	2.0	7.0	1.0	5.5	5.5	8.0
	200	3.0	4.0	7.5	1.5	7.5	1.5	6.0	5.0	9.0
	350	1.0	4.0	8.0	3.0	7.0	2.0	6.0	5.0	9.0
	500	2.0	3.0	8.0	4.0	7.0	1.0	5.0	6.0	9.0
$\alpha = 3$, $\theta = 5$	20	4.0	3.0	8.5	1.0	7.0	2.0	5.0	6.0	8.5
	60	2.5	4.0	9.0	1.0	7.0	2.5	5.5	5.5	8.0
	100	1.0	4.0	8.5	2.0	7.0	3.0	5.0	6.0	8.5
	200	2.5	6.0	9.0	1.0	7.0	2.5	5.0	4.0	8.0
	350	1.0	4.0	7.0	2.0	9.0	3.0	5.0	6.0	8.0
	500	1.0	4.5	7.0	2.0	8.0	3.0	6.0	4.5	9.0
$\sum$ ranks		67.5	111.0	228.5	56.0	206.5	160.0	156.5	124.5	239.5
Overall rank		2	3	8	1	7	6	5	4	9

Table 8: The analytical measures and MLEs.

Model	AIC	CAIC	BIC	HQIC	AD	CM	KS	KS p value	Est. parameters (SEs)
PXL	158.27	158.714	161.072	159.166	0.271483	0.0428458	0.0966882	0.941823	$\widehat{\alpha} = 1.44195(0.162533)$ $\widehat{\theta} = 0.119134(0.0399148)$
XL	164.523	164.665	165.924	164.971	1.45961	0.241631	0.201536	0.17473	$\widehat{\theta} = 0.263157(0.0347212)$
L	161.928	162.071	163.329	162.376	1.06265	0.170447	0.17938	0.28923	$\widehat{\alpha} = 0.288492(0.0377199)$
WL	160.068	160.992	164.272	161.413	0.280903	0.0461283	0.099987	0.925156	$\widehat{\alpha} = 1.87999(0.262921)$ $\widehat{\theta} = 1.0 \times 10^{-9}(0.1701)$ $\widehat{\beta} = 0.143602(0.0147442)$
TPL	160.097	160.541	162.899	160.993	0.5336	0.0843296	0.143206	0.569768	$\widehat{\alpha} = 1.07737 \times 10^{12}\left(4.8327 \times 10^{17}\right)$ $\widehat{\theta} = 0.32486(0.0419392)$
QL	160.097	160.541	162.899	160.993	0.5333	0.0842624	0.143158	0.570205	$\widehat{\alpha} = 3.04027(0.268359)$ $\widehat{\theta} = 0.324822(0.0275723)$
GL	188.517	188.961	191.319	189.413	3.9992	0.73173	0.301807	0.00846255	$\widehat{\alpha} = 0.169894(0.038391)$ $\widehat{a} = 0.658107(0.139477)$
EXL	191.089	191.232	192.49	191.537	15.9315	3.10258	0.514012	<0.000001	$\widehat{\theta} = 0.627259(0.0788995)$ $\widehat{\alpha} = 0.918778(0.638821)$ $\widehat{\beta} = 0.841764(0.076182)$

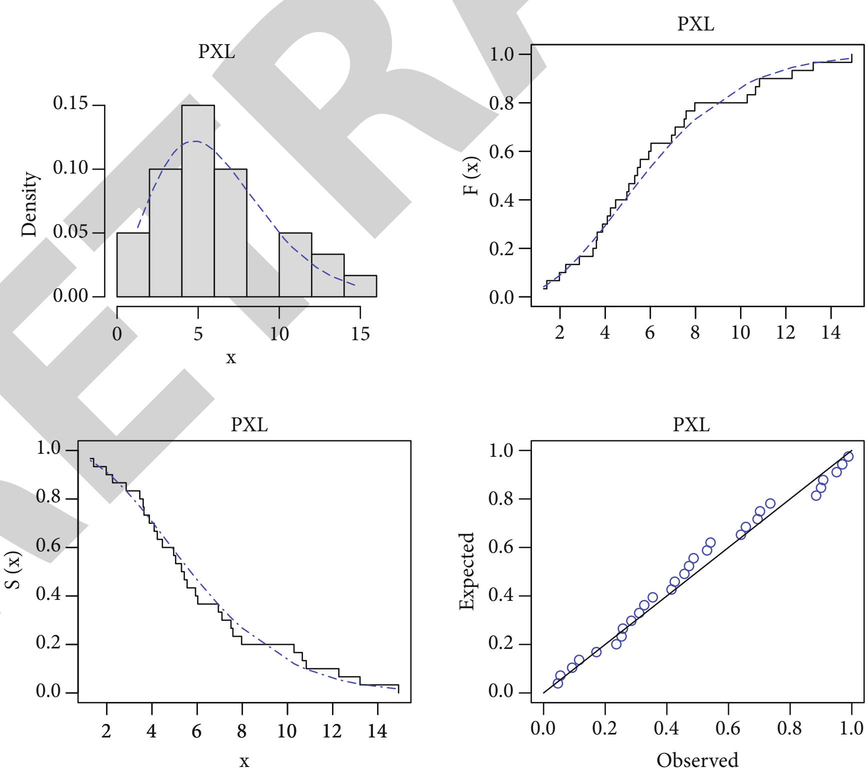

Figure 4: Histogram with the estimated PDF, CDF, SF, and P-P plots.

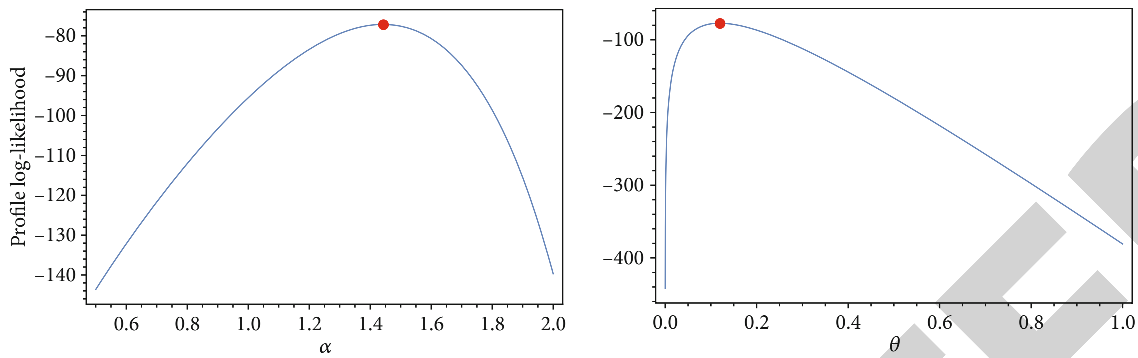

Figure 5: Plots of log-likelihood function for each estimated parameters.

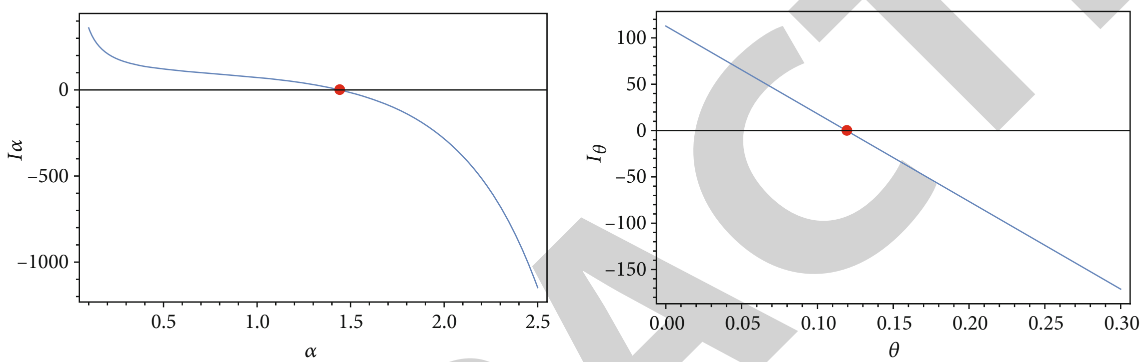

Figure 6: Existence and uniqueness plots of PXL model parameters.

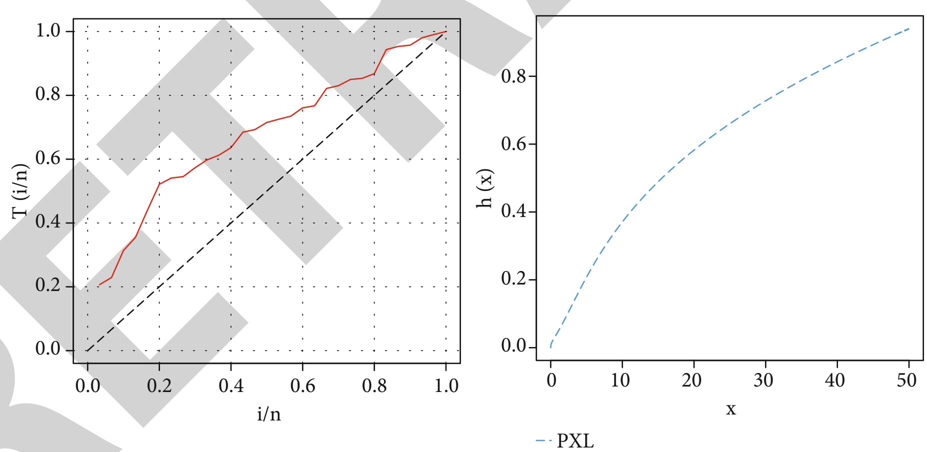

Figure 7: TTT plot and estimated HRF of PXL model.

the following equation:

$$G = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \left(D_i\right). \tag{28}$$

The suggested distribution's RTADE is calculated by minimizing the following mechanisms.

$$R = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1) \log S(x_{i:n}). \tag{29}$$

The LTADE of the proposed distribution is obtained by minimizing the following function:

$$L = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1) \log F(x_{i:n}), \tag{30}$$

Let $p_i = i/(n+1)$ be an estimate of $F(x_{i:n})$; then the PCE of the proposed model parameters α and θ is obtained by

Table 9: The estimates of PXL distribution parameters and goodness-of-fit measures by different estimation methods.

	$\widehat{\alpha}$	$\widehat{\theta}$	AD	CM	KS	KSP
MLE	1.44195	0.119134	0.271483	0.0428458	0.0966882	0.941823
ADE	1.4008	0.129651	0.253678	0.0370985	0.0868119	0.977502
CVME	1.44895	0.123329	0.289468	0.033287	0.0958577	0.945644
MPSE	1.44895	0.123329	0.289468	0.033287	0.0958577	0.945644
OLSE	1.38306	0.13604	0.265164	0.0357559	0.0893374	0.970367
PCE	1.48306	0.12604	0.592078	0.0586034	0.123824	0.747137
RTADE	1.32553	0.148498	0.293833	0.0423738	0.100008	0.925045
WLSE	1.36043	0.138312	0.263805	0.0396358	0.0871226	0.976695
LTADE	1.54176	0.107267	0.42138	0.0375144	0.115516	0.818245

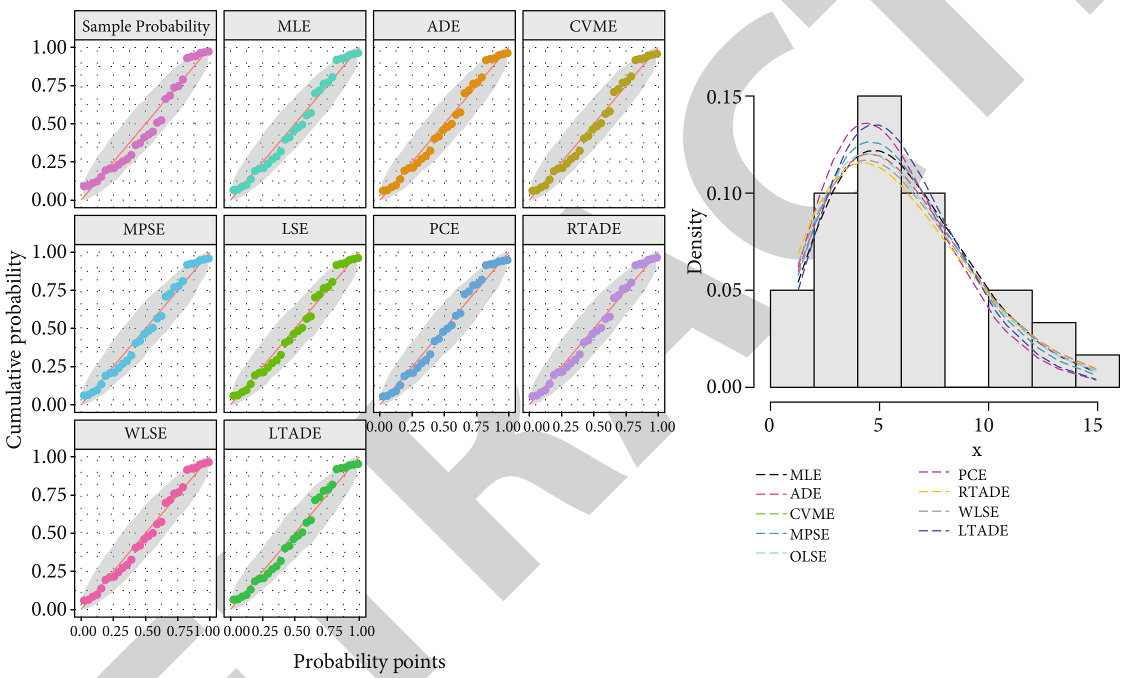

Figure 8: P-P plots and the estimated PDFs of the PXL model.

minimizing the following equation:

$$PCE = \sum_{i=1}^{n} \left[x_{i:n} - Q(p_i)\right]^2. \quad (31)$$

6. Numerical Simulation

This section investigates the performance of the offered estimation strategies for parameter estimation in the proposed model using comprehensive simulation data. Numerous sample sizes are available. $n = \{20,60,100,200,350,500\}$, and several values of the parameters, $\alpha = \{0.25,0.75,2.5,1.5,3\}$ and $\theta = \{0.5,1.5,0.5,2.5,5\}$, by using the quantile function of the suggested distribution; $N = 5000$ random samples are generated. The average absolute biases ($|BIAS|$), mean square errors (MSEs), and mean relative errors (MREs) are calculated, where $\vartheta = \alpha, \theta$.

Tables 2–6 show the simulated results for the proposed model parameters using the nine estimation techniques. It should be noted that, practically, all estimates of the proposed distribution's parameters are quite reliable and very close to their true values, with relatively small biases, MSEs, and MREs under all conditions. The estimating techniques used are precise. The BIAS, MSEs, and MREs decrease as n increases. Each of the several estimating techniques does an excellent job of identifying the proposed model parameters. According to the values of the factors assessed in our study, we discover that MPSE is the best (total score of 56.0), as seen in Table 7. Table 7 summarises the aggregate ranks of residual estimation approaches.

7. Application

We use a real data set in this section to demonstrate the versatility of the suggested distribution. The true data set shown is COVID-19 data from the Netherlands, covering 30 days from March 31 to April 30, 2020, and is available at url https://covid19.who.int/. This information is made up of an approximate mortality rate.

The proposed distribution is compablack with X-Lindley (XL) [19], Lindley (L) [8], Weibull–Lindley (WL) [23], two-parameter Lindley (TPL) [24], quasi-Lindley (QL) [25], gamma Lindely (GL), and extend Lindley (EXL) [26] distributions.

Version 12.0 of the Wolfram Mathematica software is used to calculate the analytical measures to determine the best fitting model, which depends on likelihood (ML) estimates. We used the MLE method in this section, which we studied in the estimation section. For the real data set, Table 8 lists the analytical measures, as well as ML estimates and their standard errors (SEs) in parenthesis. This table shows that the proposed distribution fits the considered real data set better than other competing models. We used the MLE technique in this section, which we learned about in the estimating section. For the real data set, Table 8 lists the analytical measurements, as well as ML estimates and associated standard errors (SEs) in parentheses. This table shows that the suggested distribution fits the data better than other competing models.

P-P plots with fitted PDF, CDF, and SF of PXL distribution are illustrated in Figure 4. From Table 8, we note that the PXL distribution is an excellent match for the real data set. For the real data set, Figure 5 shows the profile-likelihood graphs of the PXL model parameters. For all parameter estimates, these charts demonstrate the unimodality of profile-likelihood functions. Figure 6 illustrates the existence and uniqueness of estimated parameters of the PXL model. Figure 7 presented the TTT plot along with the estimated HRF of the PXL model.

Table 9 presents studies on used data by different estimation methods, and Figure 8 presents fitted PDFs with P-P plots based on the results of these methods.

8. Conclusion

We introduced the PXL distribution, a novel, enhanced, and highly adaptable continuous model. The PDF and HRF of the suggested model were presented, and their various forms were mathematically investigated. Numerous statistical aspects of the suggested model were also deduced analytically in an explicit manner, including the quantile function, moments with related measures, stochastic ordering, and research on the limits of maximum and minimum order statistics. The fuzzy reliability function for the proposed model was deduced theoretically, and numerical values were presented in this section. We developed nine distinct estimate strategies for estimating unknown parameters in the proposed model. Additionally, we conducted simulation research using random samples produced from a full sample and discovered that parameter values and sample size had an effect on the estimator's effectiveness. We find that the MPSE strategy outperforms all other strategies. Additionally, real-world data analysis was undertaken to demonstrate that the suggested model beats many of its competitors, meaning that the new distribution more closely fits the data than the prior competing distributions. Additionally, we found this by graphing the log-likelihood against its roots, indicating that the root systems are both global maximum and unique.

9. Future Work

We will perform an accelerated life test on the CT-Lindley in the next article, utilising a Type-II censored sample and a number of classical and Bayesian approach estimators. We will run accelerated life tests using data from Nelson's book; we will use a variety of acceleration models, including constant and partially accelerated trials; and we will build on our previous work to find the optimal censoring approach and sample size for the experiment. We may assess the proposed model's adaptability to the data by comparing it to competitor risk statistics.

Data Availability

All data is available in the paper.

Conflicts of Interest

The authors disclose that they have no conflicting interests.

Acknowledgments

This research has been funded by the Research Deanship in University of Ha'il, Saudi Arabia, through project number RG-21 012.

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