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Research Article

Generalized Hermite–Hadamard-Type Integral Inequalities for *h*-Godunova–Levin Functions

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The main objective of this article is to establish generalized fractional Hermite–Hadamard and related type integral inequalities for *h*-Godunova–Levin convexity and *h*-Godunova–Levin preinvexity with extended Wright generalized Bessel function acting as kernel. Moreover, Hermite–Hadamard-type and trapezoid-type inequalities for several known convexities including Godunova–Levin function, classical convex, *s*-Godunova–Levin function, *P*-function, and *s*-convex function are deduced as corollaries. These obtained results are analyzed in the form of generalization of fractional inequalities.

1. Introduction

The convexity, preinvexity, and their generalizations have been widely discussed by researchers due to its immense uses in different fields [1–11].

Many inequalities have been extensively analyzed and reported in research fields as a result of convexity and its generalizations in engineering and sciences [12–25]. Among them, a highly worked inequality is Hermite–Hadamard inequality is defined as

$$\Theta\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y \Theta(z)\mathrm{d}z \leq \frac{\Theta(x)+\Theta(y)}{2}, \tag{1}$$

for convex function [26–29]; $\Theta: J \longrightarrow R$, $x, y \in J \subseteq R$, $x<y$, $J \subseteq R$ which is playing a significant role in immense applications of inequalities and is widely used by researchers [30,31].

In recent years, the concept of convexity has been extended to *s*-Godunova–Levin type of convexity by Dragomir [32]. Moreover, *s*-Godunova–Levin-type convexity has been studied in [33]. The *h*-convexity was introduced by Varošanec in [34]. Ohud Almutari introduced *h*-Godunova–Levin convexity and *h*-Godunova–Levin preinvexity [35] by combining the concepts of Dragomir and Varošanec. In this study, we have considered *h*-Godunova–Levin convex and *h*-Godunova–Levin preinvex function to obtain generalized fractional version of Hermite–Hadamard-type inequality and trapezoid-type inequalities related to Hermite–Hadamard inequality.

Definition 1 (see [27, 36]). A function $\Theta: J \longrightarrow \mathbb{R}$ is called convex if the following inequality holds:

$$\Theta[\delta u + (1-\delta)v] \leq \delta\Theta(u) + (1-\delta)\Theta(v), \tag{2}$$

for $\delta \in [0,1]$, $\forall u, v \in J$

Definition 2 (see [4]). An invex set $J \subseteq \mathbb{R}$, with respect to a real bifunction $\Theta: J \times J \longrightarrow \mathbb{R}$, is defined for $u, v \in J$, $\lambda \in [0,1]$ as follows:

$$v + \lambda\Theta(u, v) \in J. \quad (3)$$

Definition 3 (see [4]). The preinvex function $\Theta: J \longrightarrow \mathbb{R}$ is defined for $x, y \in J$ and $\lambda \in [0,1]$ as follows:

$$\Theta(y + \lambda\zeta(x, y)) \le \lambda\Theta(x) + (1-\lambda)\Theta(y), \quad (4)$$

where J is an invex set with respect to ζ.

Definition 4 (see [37]). A positive valued function $\Theta: J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Godunova–Levin if

$$\Theta(\delta u + (1-\delta)v) \le \frac{\Theta(u)}{\delta} + \frac{\Theta(v)}{1-\delta}, \quad (5)$$

for all $u, v \in J$, $\delta \in (0,1)$

Definition 5 (see [35]). Suppose $h: (0,1) \longrightarrow \mathbb{R}$. A non-negative function $\Theta: J \longrightarrow \mathbb{R}$ is said to be h-Godunova–Levin, for all $u, v \in J$ and $\delta \in (0,1)$, if

$$\Theta(\delta u + (1-\delta)v) \le \frac{\Theta(u)}{h(\delta)} + \frac{\Theta(v)}{h(1-\delta)}. \quad (6)$$

Definition 6 (see [35]). A function $\Theta: J \longrightarrow \mathbb{R}$ is said to be h-Godunova–Levin preinvex with respect to ζ if, for all $u, v \in J$, $\phi \in (0,1)$,

$$\Theta(u + \phi\zeta(v, u)) \le \frac{\Theta(u)}{h(1-\phi)} + \frac{\Theta(v)}{h(\phi)}, \quad (7)$$

holds.

Definition 7 (see [38]). Pochammer's symbol is defined for $\delta \in \mathbb{N}$ as

$$\begin{aligned} (\gamma)_\delta &= \left\{ \begin{array}{cc} 1, & \text{for } \delta = 0, \gamma \ne 0, \\ \gamma(\gamma+1)\cdots(\gamma+\delta-1), & \text{for } \delta \ge 1, \end{array} \right\}, \\ (\gamma)_\delta &= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)}, \\ (\gamma)_{m\delta} &= \frac{\Gamma(\gamma+m\delta)}{\Gamma(\gamma)}, \end{aligned} \quad (8)$$

for $\gamma \in \mathbb{C}$ and $m \ge 0$, where Γ being the gamma function.

Definition 8 (see [38]). The integral representation of the gamma function is defined as

$$\Gamma(\delta) = \int_0^\infty z^{\delta-1} e^{-z} \mathrm{d}z, \quad (9)$$

for $\mathfrak{R}(t) > 0$.

Definition 9 (see [39–41]). The classical beta function is defined for $\mathfrak{R}(m) > 0$ and $\mathfrak{R}(n) > 0$ as

$$\begin{aligned} B(m, n) &= \int_0^1 \delta^{m-1}(1-\delta)^{n-1} \mathrm{d}\delta \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \end{aligned} \quad (10)$$

Definition 10 (see [42–44]). Extended beta functions are defined for $\mathfrak{R}(m) > 0$, $\mathfrak{R}(n) > 0$, and $\mathfrak{R}(p) > 0$ as follows:

$$B_p(m, n) = \int_0^1 z^{m-1}(1-z)^{n-1} \exp\left(\frac{-p}{z(1-z)}\right) \mathrm{d}z. \quad (11)$$

Definition 11 (see [45]). Ali et al. defined and investigated the generalized Bessel–Maitland function (eight parameters) with a new fractional integral operator and discussed its properties and relations with Mittag–Leffler functions. The function of generalized Bessel–Maitland is as follows:

$$J_{\phi,\xi,m,\sigma}^{\psi,\theta,\delta,\vartheta}(y) = \sum_{p=0}^{\infty} \frac{(\theta)_{\xi p} (\vartheta)_{\sigma p} (-y)^p}{\Gamma(\phi p + \psi + 1)(\delta)_{mp}}, \quad (12)$$

where $\phi, \psi, \theta, \delta, \vartheta \in \mathbb{C}$, $\mathfrak{R}(\phi) > 0$, $\mathfrak{R}(\nu) \ge -1$, $\mathfrak{R}(\theta) > 0$, $\mathfrak{R}(\delta) > 0$, $\mathfrak{R}(\vartheta) > 0$, $\xi, m, \sigma \ge 0$, and $m, \xi > \mathfrak{R}(\phi) + \sigma$.

Definition 12 (see [46]). The extended generalized Bessel–Maitland function is defined for $\mu, \nu, \eta, \rho, \gamma, c \in \mathbb{C}$, $\mathfrak{R}(\mu) > 0$, $\mathfrak{R}(\nu) \ge -1$, $\mathfrak{R}(\eta) > 0$, $\mathfrak{R}(\rho) > 0$, $\mathfrak{R}(\gamma) > 0$, $\xi, m, \sigma \ge 0$, and $m, \xi > \mathfrak{R}(\mu) + \sigma$ as follows:

$$J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + \nu + 1)(\rho)_{mn}} (-\omega)^n. \quad (13)$$

Generalized fractional integral operators are widely discussed, and many researchers have contributed to the field [47, 48]. Ali et al. defined a new generalized fractional operator as follows.

Definition 13 (see [46]). The generalized fractional integral operators, with extended generalized Bessel–Maitland function as kernel, are defined, for $\mu, \nu, \eta, \rho, \gamma, c \in \mathbb{C}$, $\mathfrak{R}(\mu) > 0$, $\mathfrak{R}(\nu) \ge -1$, $\mathfrak{R}(\eta) > 0$, $\mathfrak{R}(\rho) > 0$, $\mathfrak{R}(\gamma) > 0$, $\xi, m, \sigma \ge 0$, and $m, \xi > \mathfrak{R}(\mu) + \sigma$, as follows:

$$\left(\mathfrak{T}_{\nu,\eta,\rho,\gamma;p^+}^{\mu,\xi,m,\sigma,c} f\right)(x, r) = \int_p^x (x-t)^\nu J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(x-t)^\mu; r) f(t) \mathrm{d}t, \quad (14)$$

$$\left(\mathfrak{T}_{\nu,\eta,\rho,\gamma;q^-}^{\mu,\xi,m,\sigma,c} f\right)(x, r) = \int_x^q (t-x)^\nu J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t-x)^\mu; r) f(t) \mathrm{d}t. \quad (15)$$

In the paper, we obtain Hermite–Hadamard- and trapezoid-type inequalities using the generalized fractional integral operator with extended generalized Bessel–Maitland function as its nonsingular kernel.

The structure of the paper is as follows.

In Section 2, we present Hermite–Hadamard inequalities for h-Godunova–Levin convex function using the generalized fractional operator. Section 3 is devoted to trapezoid-type inequalities related to Hermite–Hadamard inequality for h-Godunova–Levin preinvex function using the generalized fractional operator.

In our work, we have frequently used the given notations:

$$\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega,\Theta)=\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;u^+}\Theta\right)(v,p), \tag{16}$$

$$\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega,\Theta)=\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;v^-}\Theta\right)(u,p). \tag{17}$$

2. Hermite–Hadamard Inequalities via h-Godunova–Levin Convex Function

In this section, we establish Hermite–Hadamard inequalities for h-Godunova–Levin convex function using the generalized fractional operator as follows.

Theorem 1. *Let Θ: $[u,v]\longrightarrow\mathbb{R}$ be a h-Godunova–Levin convex function, where $0<u<v$ and $\Theta\in L_1[u,v]$ with h: $(0,1)\longrightarrow\mathbb{R}$ is a positive function and $h(\delta)\neq0$; then, for the generalized fractional integral defined in (33), we have*

$$\begin{aligned}\frac{h(1/2)}{2}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)&\leq\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]\\&\leq\frac{\Theta(u)+\Theta(v)}{2}\int_0^1\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta,\end{aligned} \tag{18}$$

$$\omega'=\frac{\omega}{(v-u)^{\mu}}.$$

Proof. By the h-Godunova–Levin convexity of Θ on the interval $[u,v]$, let $x,y\in[u,v]$, and we have

$$\Theta(\xi x+(1-\xi)y)\leq\frac{\Theta(x)}{h(\xi)}+\frac{\Theta(y)}{h(1-\xi)}, \tag{19}$$

where if we take

$$x=\delta u+(1-\delta)v,\ y=(1-\delta)u+\delta v, \tag{20}$$

$$\xi=\frac{1}{2}, \tag{21}$$

leads to

$$\Theta\left(\frac{u+v}{2}\right)\leq\frac{1}{h(1/2)}[\Theta(\delta u+(1-\delta)v)+\Theta((1-\delta)u+\delta v)]. \tag{22}$$

Multiplying both sides by $\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)$ and integrating the resulting inequality on $[0,1]$ with respect to δ, we have

$$\begin{aligned}&\Theta\left(\frac{u+v}{2}\right)\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta\\&\leq\frac{1}{h(1/2)}\left[\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\Theta(\delta u+(1-\delta)v)\mathrm{d}\delta\right.\\&\left.\quad+\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\Theta((1-\delta)u+\delta v)\mathrm{d}\delta\right],\\&\Theta\left(\frac{u+v}{2}\right)\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\int_0^1\delta^{v'+\mu n}\mathrm{d}\delta\\&\leq\frac{1}{h(1/2)}\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\\&\quad\times\left[\int_0^1\delta^{v'+\mu n}\Theta(\delta u+(1-\delta)v)\mathrm{d}\delta+\int_0^1\delta^{v'+\mu n}\Theta((1-\delta)u+\delta v)\mathrm{d}\delta\right].\end{aligned} \tag{23}$$

Solving the integrals involved in inequality (23), we obtain

$$\frac{h(1/2)}{2}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)\leq\frac{1}{2}\left[\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega';\Theta)+\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega';\Theta)\right]. \tag{24}$$

For the second part of inequality, again using h-Godunova–Levin convexity of Θ, we have

$$\Theta(\delta u+(1-\delta)v)\leq\frac{\Theta(u)}{h(\delta)}+\frac{\Theta(v)}{h(1-\delta)}, \tag{25}$$

$$\Theta((1-\delta)u+\delta)\leq\frac{\Theta(u)}{h(1-\delta)}+\frac{\Theta(v)}{h(\delta)}. \tag{26}$$

Addition of these inequalities gives

$$\Theta(\delta u+(1-\delta)v)+\Theta((1-\delta)u+\delta v)\leq(\Theta(u)+\Theta(v))\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]. \tag{27}$$

Multiplying both sides by $\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)$ and integrating the resulting inequality on $[0,1]$ with respect to δ, we obtain

$$\begin{aligned}&\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\Theta(\delta u+(1-\delta)v)\mathrm{d}\delta+\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\Theta((1-\delta)u+\delta v)\mathrm{d}\delta\\&\leq(\Theta(u)+\Theta(v))\int_0^1\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta.\end{aligned} \tag{28}$$

Solving the integrals involved leads to

$$\frac{1}{2}\left[\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega';\Theta)+\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega';\Theta)\right]\leq\frac{\Theta(u)+\Theta(v)}{2}\int_0^1\left[\frac{1}{h(\delta)}+\frac{1}{h(1-\delta)}\right]\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta. \tag{29}$$

Combining (24) and (29), we reach to inequality. □

Corollary 1. *Choosing $h(\delta)=\delta^s$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for s-Godunova–Levin function:*

$$\begin{aligned}\frac{(1/2)^s}{2}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)&\leq\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]\\&\leq\frac{\Theta(u)+\Theta(v)}{2}\int_0^1\left[\frac{1}{\delta^s}+\frac{1}{(1-\delta)^s}\right]\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta.\end{aligned} \tag{30}$$

Corollary 2. *Choosing $h(\delta)=1$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for p function:*

$$\frac{1}{2}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)\le\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]$$
$$\le(\Theta(u)+\Theta(v))\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',1). \tag{31}$$

Corollary 3. *Choosing $h(\delta)=1/\delta$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for convex function:*

$$\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)\le\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]$$
$$\le\frac{\Theta(u)+\Theta(v)}{2}\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',1). \tag{32}$$

Corollary 4. *Choosing $h(\delta)=\delta$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for Godunova–Levin function:*

$$\frac{1}{4}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)\le\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]$$
$$\le\frac{\Theta(u)+\Theta(v)}{2}\int_0^1\left[\frac{\delta^{v'-1}}{1-\delta}\right]\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta. \tag{33}$$

Corollary 5. *Choosing $h(\delta)=1/\delta^s$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for s-convex function:*

$$2^{s-1}\Theta\left(\frac{u+v}{2}\right)\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',1)\le\frac{1}{2}\left[\left(\mathfrak{I}^{v^-}_{u,v'}\right)(\omega',\Theta)+\left(\mathfrak{I}^{u^+}_{v,v'}\right)(\omega',\Theta)\right]$$
$$\le\frac{\Theta(u)+\Theta(v)}{2}\int_0^1\left[\delta^s+(1-\delta)^s\right]\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)\mathrm{d}\delta. \tag{34}$$

3. Trapezoid-Type Inequalities Related to Hermite–Hadamard Inequalities for h-Godunova–Levin Preinvex Function

In this section, Wright generalized that the Bessel function is restricted to a real valued function. The trapezoid-type inequalities related to Hermite–Hadamard inequalities using fractional integral with Wright generalized Bessel function in its kernel can be obtained with the help of the following lemma.

Lemma 1. *Consider a function $\Theta: J=[u,u+\zeta(v,u)]\longrightarrow\mathbb{R}$ with $u,v\in\mathbb{R}$; let $\Theta\in L_1[u,u+\zeta(v,u)]$ be a differentiable function, where $J=[u,u+\zeta(v,u)]$ is taken to be an open invex set with respect to $\zeta: J\times J\longrightarrow\mathbb{R}$ with $\zeta(v,u)>0$, for $u,v\in J$; then, for the generalized fractional integral defined in (33), we have*

$$\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2\zeta(v,u)^{v'}}$$
$$\left[\left(\mathfrak{I}^{u^+}_{u+\zeta(v,u),v'-1}\right)(\omega';\Theta)+\left(\mathfrak{I}^{(u+\eta_1(v,u))^-}_{u,v'-1}\right)(\omega';\Theta)\right] \tag{35}$$
$$=\frac{\zeta(v,u)}{2}I,$$

where $I=\int_0^1\delta^{v'}\mathfrak{I}_{v',}\ \eta,\rho,\gamma^{\mu,\xi,m,\sigma,c}(\omega(\delta)^{\mu};p)\Theta'(u+\delta\zeta(v,u))\,d\delta+\int_0^1-(1-\delta)^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\Theta'(u+\delta\zeta(v,u))\,d\delta$, $\omega'=(\omega/\zeta(v,u)^{\mu})$.

Proof. Consider

$$I=\int_0^1\delta^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(\delta)^{\mu};p)\Theta'(u+\delta\zeta(v,u))\mathrm{d}\delta$$
$$+\int_0^1-(1-\delta)^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\Theta'(u+\delta\zeta(v,u))\mathrm{d}\delta. \tag{36}$$

Let

$$I=I_1+I_2. \tag{37}$$

First, we consider I_1:

$$I_1=\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n$$
$$\cdot\int_0^1\delta^{v'+\mu n}\Theta'(u+\delta\zeta(v,u))\mathrm{d}\delta. \tag{38}$$

Integrating by parts, we have

$$
\begin{aligned}
I_1 &= \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n \left[\delta^{v'+\mu n} \frac{\Theta(u+\delta\zeta(v,u))}{\zeta(v,u)}\Bigg|_0^1 \right. \\
&\quad \left. - \frac{v'+\mu n}{\zeta(v,u)} \int_0^1 \delta^{v'+\mu n-1} \Theta(u+\delta\zeta(v,u))\mathrm{d}\delta \right], \\
I_1 &= \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n \\
&\quad \times \left[\frac{\Theta(u+\zeta(v,u))}{\zeta(v,u)} - \frac{v'+\mu n}{\zeta(v,u)} \int_0^1 \delta^{v'+\mu n-1} \Theta(u+\delta\zeta(v,u))\mathrm{d}\delta \right], \\
I_1 &= \frac{\Theta(u+\zeta(v,u))}{\zeta(v,u)} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p) - \frac{1}{(\zeta(v,u))^{v'+1}} \left(\mathfrak{J}^{(u+\zeta(v,u))^-}_{u,v'-1} \right)(\omega';\Theta).
\end{aligned} \tag{39}
$$

Continuing in the same manner, we obtain

$$
\begin{aligned}
I_2 &= \frac{\Theta(u)}{\zeta(v,u)} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p) - \frac{1}{(\zeta(v,u))^{v'+1}} \left(\mathfrak{J}^{u^+}_{u+\zeta(v,u),v'-1} \right)(\omega',\Theta), \\
I &= \frac{\Theta(u)+\Theta(u+\zeta(v,u))}{\zeta(v,u)} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p) - \frac{1}{(\zeta(v,u))^{v'+1}} \\
&\quad \times \left[\left(\mathfrak{J}^{(u+\zeta(v,u))^-}_{u,v'-1} \right)(\omega',\Theta) + \left(\mathfrak{J}^{u^+}_{u+\zeta(v,u),v'-1} \right)(\omega',\Theta) \right].
\end{aligned} \tag{40}
$$

Multiplying by $\zeta(v,u)/2$, we get the required result. By Lemma 1, we present the following theorem.

Theorem 2. *Consider a function* $\Theta: J = [u, u+\zeta(v,u)] \longrightarrow (0,\infty)$ *with* $J \in \mathbb{R}$, *and let it be a differentiable function on* J. *Also, suppose that* $|\Theta'|$ *is a h-Godunova–Levin preinvex function on* J; *then, for the generalized fractional integral defined in (33) with the restricted Wright generalized Bessel function to a real valued function, we have*

$$
\begin{aligned}
&\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p) - \frac{1}{2\zeta(v,u)^{v'}} \\
&\quad \times \left[\left(\mathfrak{J}^{u^+}_{u+\zeta(v,u),v'-1} \right)(\omega',\Theta) + \left(\mathfrak{J}^{(u+\zeta(v,u))^-}_{u,v'-1} \right)(\omega',\Theta) \right] \\
&\leq \frac{\zeta(v,u)}{2} \left(|\Theta'(u)| + |\Theta'(v)| \right) \int_0^1 \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n \right| \\
&\quad \times \left| \frac{\delta^{v'+\mu n} - (1-\delta)^{v'+\mu n}}{h(\delta)} \right| \mathrm{d}\delta.
\end{aligned} \tag{41}
$$

Proof

$$
\begin{aligned}
&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2\zeta(v,u)^{v'}}\right.\\
&\quad\left.\times\left[\left(\mathfrak{I}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{I}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\
&=\left|\frac{\zeta(v,u)}{2}I\right|\\
&\leq\frac{\zeta(v,u)}{2}\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|\int_0^1\left|\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}\right|\left|\Theta'(u+\delta\zeta(v,u))\right|\mathrm{d}\delta\\
&\leq\frac{\zeta(v,u)}{2}\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|\\
&\quad\times\int_0^1\left|\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}\right|\left|\frac{\Theta'(u)}{h(\delta)}+\frac{\Theta'(v)}{h(1-\delta)}\right|\mathrm{d}\delta\\
&\leq\frac{\zeta(v,u)}{2}\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|\\
&\quad\times\left[\left|\Theta'(u)\right|\int_0^1\left|\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}\right|\left|\frac{1}{h(\delta)}\right|\mathrm{d}\delta+\left|\Theta'(v)\right|\int_0^1\left|\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}\right|\left|\frac{1}{h(1-\delta)}\right|\mathrm{d}\delta\right]\\
&=\frac{\zeta(v,u)}{2}\left(\left|\Theta'(u)\right|+\left|\Theta'(v)\right|\right)\int_0^1\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|\\
&\quad\times\left|\frac{\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}}{h(\delta)}\right|\mathrm{d}\delta.
\end{aligned}
\tag{42}
$$

□

Corollary 6. *Taking $\zeta(v,u)=v-u$ in Theorem 2, we obtain the following inequality:*

$$
\begin{aligned}
&\left|\frac{\Theta(u)+\Theta(v)}{2}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2(v-u)^{v'}}\right.\\
&\quad\left.\times\left[\left(\mathfrak{I}^{u^{+}}_{v,v'-1}\right)(\omega',\Theta)+\left(\mathfrak{I}^{v^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\
&\leq\frac{v-u}{2}\left(\left|\Theta'(u)\right|+\left|\Theta'(v)\right|\right)\int_0^1\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|\\
&\quad\times\left|\frac{\delta^{v'+\mu n}-(1-\delta)^{v'+\mu n}}{h(\delta)}\right|\mathrm{d}\delta.
\end{aligned}
\tag{43}
$$

Theorem 3. *Suppose that $\Theta: J=[u,u+\zeta(v,u)]\longrightarrow(0,\infty)$ with $J\in\mathbb{R}$, and let it be a differentiable function on J. Also, suppose that $|\Theta'|^q$ is a h-Godunova–Levin preinvex function on J with $p>1$ and $q=p/(p-1)$; then, for the generalized fractional integral defined in (33) with the restricted Wright generalized Bessel function to a real valued function, we have*

$$\begin{aligned}
&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2\zeta(v,u)^{v'}}\right.\\
&\quad\left.\times\left[\left(\mathfrak{J}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{J}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\
&\leq\frac{\zeta(v,u)}{2}\left(\left|\Theta'(u)\right|^{q}+\left|\Theta'(v)\right|^{q}\right)^{1/q}\\
&\left(\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|^{p}\mathrm{d}\delta\right)^{1/p}\left(\int_0^1\frac{1}{h(\delta)}\mathrm{d}\delta\right)^{1/q}.
\end{aligned}\tag{44}$$

Proof. Using Lemma 1, we have

$$\begin{aligned}
&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2\zeta(v,u)^{v'}}\right.\\
&\quad\left.\times\left[\left(\mathfrak{J}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{J}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\
&=\left|\frac{\zeta(v,u)}{2}I\right|\\
&\leq\frac{\zeta(v,u)}{2}\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|\left|\Theta'(u+\delta\zeta(v,u))\right|\mathrm{d}\delta.
\end{aligned}\tag{45}$$

Using Hölder's integral inequality, we have

$$\begin{aligned}
&\leq\frac{\zeta(v,u)}{2}\left(\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|^{p}\mathrm{d}\delta\right)^{1/p}\\
&\left(\int_0^1\left|\Theta'(u+\delta\zeta(v,u))\right|^{q}\mathrm{d}\delta\right)^{1/q},
\end{aligned}\tag{46}$$

where $(1/p)+(1/q)=1$.

Now, since $|\Theta'|^q$ is an $\underline{h}$-Godunova–Levin preinvex, we obtain

$$\begin{aligned}
\int_0^1\left|\Theta'(u+\delta\zeta(v,u))\right|^{q}\mathrm{d}\delta&\leq\int_0^1\left(\frac{\left|\Theta'(u)\right|^{q}}{h(\delta)}+\frac{\left|\Theta'(v)\right|^{q}}{h(1-\delta)}\right)\mathrm{d}\delta\\
&\leq\left(\left|\Theta'(u)\right|^{q}+\left|\Theta'(v)\right|^{q}\right)\int_0^1\frac{1}{h(\delta)}\mathrm{d}\delta.
\end{aligned}\tag{47}$$

Using (47) in (46) leads to the result. □

Theorem 4. *With the assumptions of Theorem 3, we get the following inequality related to Hermite–Hadamard inequality:*

$$\begin{aligned}&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)-\frac{1}{2\zeta(v,u)^{v'}}\left[\left(\mathfrak{J}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{J}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\&\leq\frac{\zeta(v,u)}{2^{1/q}}\left(\left|\Theta'(u)\right|^{q}+\left|\Theta'(v)\right|^{q}\right)^{1/q}\left[\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v'+1,\eta,\rho,\gamma}(\omega;p)-\left(\frac{1}{2}\right)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v'+1,\eta,\rho,\gamma}\left(\omega\left(\frac{1}{2}\right)^{\mu};p\right)\right]^{1-(1/q)}\\&\left[\int_0^1\frac{\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|}{h(\delta)}\mathrm{d}\delta\right]^{1/q},\end{aligned}\tag{48}$$

where $v',\mu\in\mathbb{R}^{+}$.

Proof. From Lemma 1, we have

$$\begin{aligned}&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)\right.\\&\left.-\frac{1}{2\zeta(v,u)^{v'}}\left[\left(\mathfrak{J}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{J}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|=\left|\frac{\zeta(v,u)}{2}I\right|\\&\leq\frac{\zeta(v,u)}{2}\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|\\&\left|\Theta'(u+\delta\zeta(v,u))\right|\mathrm{d}\delta.\end{aligned}\tag{49}$$

Applying power-mean inequality, we obtain

$$\begin{aligned}&\left|\frac{\Theta(u)+\Theta(u+\zeta(v,u))}{2}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;p)\right.\\&\left.-\frac{1}{2\zeta(v,u)^{v'}}\left[\left(\mathfrak{J}^{u^{+}}_{u+\zeta(v,u),v'-1}\right)(\omega',\Theta)+\left(\mathfrak{J}^{(u+\zeta(v,u))^{-}}_{u,v'-1}\right)(\omega',\Theta)\right]\right|\\&\leq\frac{\zeta(v,u)}{2}\left(\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|\mathrm{d}\delta\right)^{1-(1/q)}\\&\left(\int_0^1\left|\delta^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\delta^{\mu};p)-(1-\delta)^{v'}\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p)\right|\right.\\&\left.\times\left|\Theta'(u+\delta\zeta(v,u))\right|^{q}\mathrm{d}\delta\right)^{1/q}.\end{aligned}\tag{50}$$

Since $|\Theta'|^{q}$ is an h-Godunova–Levin preinvex, we obtain

$$\int_0^1 \left| \delta^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega\delta^{\mu};p) - (1-\delta)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p) \right| \left|\Theta'(u+\delta\zeta(\nu,u))\right|^q d\delta$$
$$\leq \int_0^1 \frac{\left| \delta^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega\delta^{\mu};p) - (1-\delta)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p) \right|}{h(\delta)} \left|\Theta'(u)\right|^q$$
$$\left. + \frac{\left| \delta^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega\delta^{\mu};p) - (1-\delta)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p) \right|}{h(1-\delta)} \left|\Theta'(\nu)\right|^q \right) d\delta \tag{51}$$
$$= \left(\left|\Theta'(u)\right|^q + \left|\Theta'(\nu)\right|^q \right) \int_0^1 \frac{\left| \delta^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega\delta^{\mu};p) - (1-\delta)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p) \right|}{h(\delta)}.$$

Now, consider

$$\int_0^1 \left| \delta^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega\delta^{\mu};p) - (1-\delta)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu',\eta,\rho,\gamma}(\omega(1-\delta)^{\mu};p) \right|$$
$$= \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+\nu'+1)(\rho)_{mn}} (-\omega)^n \right| \int_0^1 \left| \delta^{\nu'+\mu n} - (1-\delta)^{\nu'+\mu n} \right| d\delta$$
$$= \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+\nu'+1)(\rho)_{mn}} (-\omega)^n \right| \tag{52}$$
$$\times \left[\int_0^{1/2} \left((1-\delta)^{\nu'+\mu n} - \delta^{\nu'+\mu n} \right) d\delta + \int_{1/2}^1 \left(\delta^{\nu'+\mu n} - (1-\delta)^{\nu'+\mu n} \right) \right]$$
$$= 2\left[\mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu'+1,\eta,\rho,\gamma}(\omega;p) - \left(\frac{1}{2}\right)^{\nu'} \mathfrak{J}^{\mu,\xi,m,\sigma,c}_{\nu'+1,\eta,\rho,\gamma}\left(\omega\left(\frac{1}{2}\right)^{\mu};p\right) \right].$$

□

4. Conclusion

In the present paper, the advanced approach of the generalized fractional version of Hermite–Hadamard-type and trapezoid-type integral inequalities for a recently introduced function, h-Godunova–Levin convex, and h-Godunova–Levin preinvex have been established by using fractional integral operator with Wright generalized Bessel function as its kernel. Convexities and its different forms have remarkable uses in many fields and is extensively worked by researchers. Since h-Godunova–Levin convex function is generalization of several known convexities, so the results have been also deduced for them in the form of corollaries.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally, and they have read and approved the final manuscript for publication.

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