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Research Article

On the 3D Incompressible Boussinesq Equations in a Class of Variant Spherical Coordinates

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This paper investigates the global stabilizing effects of the geometry of the domain at which the flow locates and of the geometric structure of the solution to the incompressible flows by studying the three-dimensional (3D) incompressible, viscosity, and diffusivity Boussinesq system in spherical coordinates. We establish the global existence and uniqueness of the smooth solution to the Cauchy problem for a full 3D incompressible Boussinesq system in a class of variant spherical coordinates for a class of smooth large initial data. We also construct one class of nonempty bounded domains in the three-dimensional space $\mathbb{R}^3$, in which the initial boundary value problem for the full 3D Boussinesq system in a class of variant spherical coordinates with a class of large smooth initial data with swirl has a unique global strong or smooth solution with exponential decay rate in time.

1. Introduction and Main Results

In this paper, we consider the Cauchy problem for the three-dimensional(3D) incompressible Boussinesq $(\nu, \mu > 0)$ equations

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \rho\mathbf{e}_3, \mathbf{x}\in\mathbb{R}^3, & t>0,\\ \rho_t + \mathbf{u}\cdot\nabla\rho = \mu\Delta\rho, \mathbf{x}\in\mathbb{R}^3, & t>0,\\ \operatorname{div}\ \mathbf{u} = 0, \mathbf{x}\in\mathbb{R}^3, & t>0,\\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}), \mathbf{x}\in\mathbb{R}^3, \end{cases} \tag{1}$$

and the initial boundary value problem for the 3D incompressible Boussinesq $(\nu, \mu > 0)$ equations in the bounded domain

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \rho\mathbf{e}_3, \mathbf{x}\in\Omega, & t>0,\\ \rho_t + \mathbf{u}\cdot\nabla\rho = \mu\Delta\rho, \mathbf{x}\in\Omega, & t>0,\\ \operatorname{div}\ \mathbf{u} = 0, \mathbf{x}\in\Omega, & t>0,\\ \mathbf{u} = 0, \rho = 0, \mathbf{x}\in\partial\Omega, & t>0,\\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}), \mathbf{x}\in\Omega, \end{cases} \tag{2}$$

respectively. Here, $\mathbf{x} = (x_1, x_2, x_3)$; the unknowns $\mathbf{u} = (u^1(t,\mathrm{x}), u^2(t,\mathrm{x}), u^3(t,\mathrm{x}))^T$ denote the fluid velocity vector field; $P = P(t, \mathbf{x})$ is the scalar pressure and $\rho = \rho(t, x)$ is the scalar density; ν, μ are viscosity and thermal diffusivity, respectively; $\mathbf{n}$ is the unit outer normal vector of bounded domain Ω; $\mathbf{e}_3 = (0,0,1)^T$ is the unit vector in the vertical direction; and $\mathbf{u}_0$ and ρ_0 are the given initial velocity and initial density, respectively, with div $\mathbf{u}_0 = 0$. It should be noted that, if $\rho \equiv 0$, (1) comes back to the classical 3D incompressible Navier-Stokes equations.

It is well known that the 3D incompressible Navier-Stokes equations have at least one global weak solution with the finite energy [1, 2]. However, the issue of the regularity and uniqueness for the global weak solution is still a challenging open problem in the field of mathematical fluid dynamics [3–8].

Recently, motivated by the studies on the axisymmetric flow (see [6–11] and the references therein), the helical flow (see [12] and the references therein), and the 3D incompressible Euler and the SQG (surface quasigeostrophic) equations [13–15], we investigate further the global dynamical stabilizing effects of the geometry of the domain at which the flow locates and of the geometry structure of the solution to the 3D incompressible Navier-Stokes equations. As an example, we study the 3D incompressible Navier-Stokes and Euler equations in the spherical coordinate system, see S. Wang and Y.X. Wang [16], where the existence and uniqueness of the global strong solution of the 3D incompressible Navier-Stokes and Euler equations in the spherical coordinates are obtained for a class of large smooth initial data with swirl or without swirl.

As stated in the beginning, the present paper is focused on the Boussinesq system, which plays an important role in the atmospheric and oceanographic sciences [11, 17–20]. Considering the 2D standard Boussinesq equations with the viscosity or diffusive coefficient, Hou and Li [21] and Chae [22] obtain the global well-posedness results similar to the 2D incompressible Navier-Stokes equations [6]. On global regularity on the smooth solution for the 2D Boussinesq system, see also, e.g., [23–25] and the references therein. On the other hand, comparing with the magnitude of research conducted on the Boussinesq equations on Euclidean domains, the qualitative behaviour of the model on Riemannian manifolds has been investigated relatively little, see [26], in which the convergence of the average of weak solutions of the 3D equations to a 2D problem is proved by Saito, and see [27], in which the nondegenerate and partially degenerate Boussinesq equations on a closed surface are studied by Li et al..

The global well-posedness for a 3D axisymmetric Boussinesq system without swirl and with partial viscosity or thermal diffusivity in the system of cylindrical coordinates is obtained by Abidi et al. in [28], Hmidi and Keraani in [29], and Hmidi et al. [30, 31], respectively. For the general 3D Boussinesq system, there exist some results on the local well-posedness problem, partial regularity, or the global regularity with respect to small initial data; see [32–37], etc.

In this paper, we further investigate the global stabilizing effects of the geometry of the domain and the solution to the three-dimensional incompressible flows by studying the 3D incompressible axisymmetric Boussinesq system in the system of a class of variant spherical coordinates.

Let the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{3}$$

be a real orthogonal matrix, i.e., $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix and $\mathbf{A}^T$ is a transpose of the matrix $\mathbf{A}$. For the given

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in \mathbb{R}^3, \tag{4}$$

and the constant $a > 0$, introduce *a class of variant spherical coordinates* (r, θ, φ) defined as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + a\mathbf{A} \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{5}$$

Because the matrix $\mathbf{A}$ is an orthogonal one, we have

$$\begin{aligned} r &= \frac{1}{a}\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})} \\ &= \frac{1}{a}\sqrt{(x_1-\alpha_1)^2 + (x_2-\alpha_2)^2 + (x_3-\alpha_3)^2} \ge 0, \\ 0 \le \theta &= \arctan\frac{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})}}{\zeta(\mathbf{x})} \le \pi, \quad 0 \le \varphi = \arctan\frac{\eta(\mathbf{x})}{\xi(\mathbf{x})} < 2\pi, \end{aligned} \tag{6}$$

where

$$\begin{cases} \xi(\mathbf{x}) = a_{11}(x_1-\alpha_1) + a_{21}(x_2-\alpha_2) + a_{31}(x_3-\alpha_3), \\ \eta(\mathbf{x}) = a_{12}(x_1-\alpha_1) + a_{22}(x_2-\alpha_2) + a_{32}(x_3-\alpha_3), \\ \zeta(\mathbf{x}) = a_{13}(x_1-\alpha_1) + a_{23}(x_2-\alpha_2) + a_{33}(x_3-\alpha_3). \end{cases} \tag{7}$$

Note that, for variant spherical coordinates (r, θ, φ), the r coordinate is spherical symmetric in $\mathbb{R}^3$, but the θ coordinate and φ coordinate are not axisymmetric with respect to the Cartesian coordinates $\mathbf{x} \in \mathbb{R}^3$ except that $\mathbf{A} = \mathbf{I}$. Denote

$$\begin{aligned} \mathrm{e}_r &= \mathbf{A}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \frac{1}{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})}}\begin{pmatrix} x_1-\alpha_1 \\ x_2-\alpha_2 \\ x_3-\alpha_3 \end{pmatrix}, \\ \mathrm{e}_\theta &= \mathbf{A}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \\ &= \frac{\begin{pmatrix} a_{11}\xi^2(\mathbf{x}) + a_{12}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{13}\left(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})\right) \\ a_{21}\xi^2(\mathbf{x}) + a_{22}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{23}\left(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})\right) \\ a_{31}\xi^2(\mathbf{x}) + a_{32}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{33}\left(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})\right) \end{pmatrix}}{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})}\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})}}, \end{aligned}$$

$$e_\varphi = \mathbf{A}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\xi^2(\mathbf{x})+\eta^2(\mathbf{x})}}\begin{pmatrix} -a_{11}\xi(\mathbf{x})+a_{12}\eta(\mathbf{x}) \\ -a_{21}\xi(\mathbf{x})+a_{22}\eta(\mathbf{x}) \\ -a_{31}\xi(\mathbf{x})+a_{32}\eta(\mathbf{x}) \end{pmatrix}. \quad (8)$$

Also, denote the special bounded domain $\tilde{\Omega}$ described by variant spherical coordinates by

$$\begin{aligned}\tilde{\Omega} = \{&(x_1, x_2, x_3) \\ &= (\alpha_1 + aa_{11} r\sin\theta\cos\varphi + aa_{12} r\sin\theta\sin\varphi \\ &\quad + aa_{13} r\cos\theta, \alpha_2 + aa_{21} r\sin\theta\cos\varphi + aa_{22} r\sin\theta\sin\varphi \\ &\quad + aa_{23} r\cos\theta, \alpha_3 + aa_{31} r\sin\theta\cos\varphi + aa_{32} r\sin\theta\sin\varphi \\ &\quad + aa_{33} r\cos\theta) \\ &\in \mathbb{R}^3 : 0 < r_0 \le r \le R_0 < \infty, 0 < \theta_0 \le \theta \le \theta_1 < \pi, 0 \le \varphi < 2\pi\}, \end{aligned} \quad (9)$$

where $r_0, R_0, \theta_0, \theta_1$ are given fixed positive constants. Here, we give an explicit example for the domain

$$\begin{aligned}\tilde{\Omega} = \Big\{&(x_1, x_2, x_3) \\ &= \left(\sqrt{2} r\sin\theta\sin\varphi, -r\sin\theta\cos\varphi + r\cos\theta, r\sin\theta\cos\varphi \right. \\ &\quad \left. + r\cos\theta\right) \\ &\in \mathbb{R}^3 : 1 \le r \le 10, \frac{\pi}{8} \le \theta \le \frac{3\pi}{4}, 0 \le \varphi < 2\pi\Big\}, \end{aligned} \quad (10)$$

by taking

$$a = \sqrt{2}, \alpha = 0, \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad r_0 = 1, R_0 = 10, \theta_0 = \frac{\pi}{8}, \theta_1 = \frac{3\pi}{4}. \quad (11)$$

Now, we consider the 3D incompressible Boussinesq equations (1) and (2) with the form

$$\begin{aligned} &\mathbf{u}(t,\mathbf{x}) = u^r(t,r,\theta)\mathbf{e}_r + u^\theta(t,r,\theta)\mathbf{e}_\theta + u^\varphi(t,r,\theta)\mathbf{e}_\varphi, \\ &P(t,\mathbf{x}) = P(t,r,\theta), \rho(t,\mathbf{x}) = \rho(t,r,\theta), \end{aligned} \quad (12)$$

with

$$\mathbf{u}_0(\mathbf{x}) = u_0^r(t,r,\theta)\mathbf{e}_r + u_0^\theta(t,r,\theta)\mathbf{e}_\theta + u_0^\varphi(t,r,\theta)\mathbf{e}_\varphi, \rho_0(\mathbf{x}) = \rho_0(r,\theta). \quad (13)$$

When the matrix $\mathbf{A}$ is an orthogonal matrix, the gradient operator ∇ and Laplacian Δ have the expression

$$\begin{aligned} &\nabla = e_r \frac{1}{a}\partial_r + \frac{1}{ar} e_\theta \partial_\theta + \frac{1}{ar\sin\theta} e_\varphi \partial_\varphi, \\ &\Delta = \frac{1}{a^2}\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\right), \end{aligned} \quad (14)$$

respectively.

Then, one can derive the evolution equations for $(u^r, u^\theta, u^\varphi \rho)(t,r,\theta)$ for 3D incompressible Boussinesq equations as follows:

$$\begin{cases} \partial_t u^r + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})u^r + \frac{1}{a}\partial_r P = \nu\left[\left(\tilde{\Delta} - \frac{2}{a^2r^2}\right)u^r - \frac{2\cos\theta}{a^2r^2\sin\theta}u^\theta - \frac{2}{a^2r^2}\partial_\theta u^\theta\right] + \frac{(u^\theta)^2 + (u^\varphi)^2}{ar} + \rho\cos\theta, \\ \partial_t u^\theta + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})u^\theta + \frac{1}{ar}\partial_\theta P = \nu\left[\left(\tilde{\Delta} - \frac{1}{a^2r^2\sin^2\theta}\right)u^\theta + \frac{2}{a^2r^2}\partial_\theta u^r\right] - \frac{u^r u^\theta}{ar} + \frac{\cos\theta}{ar\sin\theta}(u^\varphi)^2 - \rho\sin\theta, \\ \partial_t u^\varphi + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})u^\varphi = \nu\left(\tilde{\Delta} - \frac{1}{a^2r^2\sin^2\theta}\right)u^\varphi - \frac{u^r u^\varphi}{ar} - \frac{\cos\theta}{ar\sin\theta}u^\theta u^\varphi, \\ \partial_t \rho + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\rho = \nu\tilde{\Delta}\rho, \\ \partial_r u^r + \frac{2}{r}u^r + \frac{1}{r}\partial_\theta u^\theta + \frac{\cos\theta}{r\sin\theta}u^\theta = 0, \end{cases} \quad (15)$$

where $\tilde{\mathbf{u}} = u^r \mathrm{e}_r + u^\theta \mathrm{e}_\theta$, and

$$\tilde{\nabla} = \mathbf{e}_r \partial_r + \mathbf{e}_\theta \frac{1}{r} \partial_\theta, \tilde{\Delta} = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos\theta}{r^2 \sin\theta} \partial_\theta. \quad (16)$$

Note that equations (15) completely determine the evolution of the 3D Boussinesq equations in a class of variant spherical coordinates once the initial conditions and/or the boundary value conditions are given. Also, the 3D incompressible Boussinesq system in a class of variant spherical coordinates is completely different from the one in cylindrical coordinates because of the complexity of the last equation in system (15) and of Laplace operator $\tilde{\Delta}$ given by (16).

We take the initial condition for system (15) as follows:

$$\left(u^r, u^\theta, u^\varphi, \rho\right)(t = 0, r, \theta) = \left(u_0^r, u_0^\theta, u_0^\varphi\right)(r, \theta). \quad (17)$$

Moreover, the boundary condition $\mathbf{u}|_{\partial\Omega} = 0, t \geq 0$ is equivalent to the following condition:

$$\left(u^r, u^\theta, u^\varphi, \rho\right)\Big|_{\partial\Omega} = 0, t \geq 0. \quad (18)$$

It is easy to know, by direct computation, that the vorticity $\omega = \nabla\times \mathbf{u}$ can be expressed as

$$\omega(t, \mathbf{x}) = \omega^r(t, r, \theta)\mathbf{e}_r + \omega^\theta(t, r, \theta)\mathbf{e}_\theta + \omega^\varphi(t, r, \theta)\mathbf{e}_\varphi, \quad (19)$$

with the initial vorticity

$$\omega_0 = \omega(0, \mathbf{x}) = \omega_0^r(r, \theta)\mathbf{e}_r + \omega_0^\theta(r, \theta)\mathbf{e}_\theta + \omega_0^\varphi(r, \theta)\mathbf{e}_\varphi, \quad (20)$$

where

$$\begin{aligned} \omega^r &= \frac{1}{ar\sin\theta} \partial_\theta(\sin\theta u^\varphi), \omega^\theta = -\frac{1}{ar} \partial_r(r u^\varphi), \\ \omega^\varphi &= \frac{1}{a}\left(\partial_r u^\theta + \frac{u^\theta}{r} - \frac{\partial_\theta u^r}{r}\right). \end{aligned} \quad (21)$$

It is clear that

$$\operatorname{div} \omega = \frac{1}{a}\left(\partial_r \omega^r + \frac{2}{r}\omega^r + \frac{1}{r}\partial_\theta \omega^\theta + \frac{\cos\theta}{r\sin\theta}\omega^\theta\right) \equiv 0. \quad (22)$$

In addition, we can obtain the equation of ω^φ from (15) as

$$\begin{aligned} &\partial_t \omega^\varphi + \left(\tilde{\mathbf{u}} \cdot \tilde{\nabla}\right)\omega^\varphi \\ &\quad = \nu\left(\tilde{\Delta} - \frac{1}{a^2 r^2 \sin^2\theta}\right)\omega^\varphi + \frac{u^r \omega^\varphi}{ar} + \frac{\cos\theta}{ar\sin\theta} u^\theta \omega^\varphi \\ &\quad + \left(\frac{\cos\theta}{ar\sin\theta}\partial_r - \frac{1}{ar^2}\partial_\theta\right)|u^\varphi|^2 \\ &\quad - \left(\sin\theta \partial_r \rho + \cos\theta \frac{1}{r}\partial_\theta \rho\right). \end{aligned} \quad (23)$$

We now state our main results as follows:

Theorem 1 (the case of 3D incompressible Boussinesq equations in $\mathbb{R}^3$ without swirl in the sense of spherical coordinates). *Assume that $\nu > 0$ and $\mu > 0$. Let $(\mathbf{u}_0, \rho_0)(t, \mathbf{x})$ be given by (13) with $u_0^\varphi = 0$. Let $\omega_0^\varphi = \partial_r u_0^\theta + u_0^\theta/r - \partial_\theta u_0^r/r$. If $(\mathbf{u}_0, \rho_0) \in H^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$ and $\omega_0^\varphi/r\sin\theta \in L^2(\mathbb{R}^3)$, then the Cauchy problems (15) and (17) have a unique global strong solution $(u^r, u^\theta, u^\varphi, P, \rho)(t, r, \theta, \varphi)$ with $u^\varphi \equiv 0$ satisfying $\mathbf{u} \in L^\infty(0,+\infty;H^1(\mathbb{R}^3))$, given by (12). Moreover, assume that $\mathbf{u}_0(\mathbf{x}) = u_0^r(r, \theta)\mathbf{e}_r + u_0^\theta(r, \theta)\mathbf{e}_\theta$ is smooth with $u_0^r(0, \theta) = u_0^\theta(0, \theta) = \rho_0(0, \theta)|_{\theta=0,\pi} = 0$, and furthermore, with some compatibility conditions for the initial data with respect to $\theta = 0, \pi$ and $r = 0$, then the Cauchy problem (1) to the 3D incompressible Boussinesq equations has a unique global smooth solution in time.*

Theorem 2 (the exponential decay rate in time and the global strong solution of 3D incompressible Boussinesq equations in the special bounded domain of $\mathbb{R}^3$ with swirl in the sense of spherical coordinates). *Assume that $\nu > 0$ and $\mu > 0$. Let $\Omega = \tilde{\Omega} \subset \mathbb{R}^3$ in (2), given by (9). Let $(\mathbf{u}_0, \rho_0)$ $(t, \mathbf{x})$ be given by (13) with $u_0^\varphi \not\equiv 0$. If $(\mathbf{u}_0, \rho_0) \in H^2(\Omega)$ with $\operatorname{div} \mathbf{u}_0 = 0$ and $(\mathbf{u}_0, \rho_0)|_{\partial\Omega} = 0$, then the initial-boundary value problems (15), (17), and (18) to the incompressible Boussinesq equation (2) have a unique global strong solution $(u^r, u^\theta, u^\varphi, P, \rho)(t, r, \theta, \varphi)$ satisfying $\partial_t^i(\mathbf{u}, \rho) \in L^\infty(0,+\infty;$ $H^{1-i}(\Omega))$, $i = 0, 1$, given by (12), and the exponential decay rate in time*

$$\|(\mathbf{u}, \rho)(t, \cdot)\|_{H^1(\Omega)}^2 + \|(\mathbf{u}_t, \rho_t)(t, \cdot)\|_{L^2(\Omega)}^2 \leq Ce^{-\alpha t}, \quad 0 \leq t \leq +\infty, \quad (24)$$

for some constants $C = C(\Omega, \nu, \mu, \|(\mathbf{u}_0, \rho_0)\|_{H^2(\Omega)}) > 0$ and $\alpha = \alpha(\Omega, \nu, \mu) > 0$, independent of $t : 0 \leq t \leq \infty$. Moreover, any Leray-Hopf-type global weak solution $(\mathbf{u}, \rho, P)$, given by (12), to the initial-boundary value problem (2) is globally smooth in $(0, T] \times \Omega_1$ for any $0 \leq T \leq \infty$ and any smooth domain $\Omega_1 \subset\subset \Omega \subset \mathbb{R}^3$.

Remark 3. The assumptions $\nu > 0$ and $\mu > 0$ are key in the proofs of Theorems 1 and 2. The key point of the proof of Theorem 1 is to establish the a priori estimate on the quality $\omega^\varphi/r\sin\theta - (1/2\nu)\rho$ and then to use the special geometry structure (12) of the solutions $(\mathbf{u}, \rho)(t, \mathbf{x})$, which guarantees that there exist some kinds of cancelation regimes so that we can deal with the vortex stretching term $\omega \cdot \nabla \mathbf{u}$ in the vorticity equation for ω. The present method used in this paper cannot be extended to the case of $\nu = 0$ or $\mu = 0$. The global well-posedness problem on the 3D incompressible Boussinesq system with partial viscosity or diffusivity and without swirl in spherical coordinates is complex because each component of the velocity field in spherical coordinates in the Boussinesq system given by the classical Biot-Savart law is very complex, which will be discussed in the future. The classical Biot-Savart law expresses the velocity field that transports the vorticity in terms of the vorticity itself; see [38] and the references therein. The assumption in Theorem 2 on the domain $\Omega = \tilde{\Omega}$ with the special geometry structure given by (9) is key for one to prove our global regularity for the strong solution and global interior regularity for the smooth solution in time for 3D Boussinesq equations with large smooth initial data, which yields to one inequality of Ladyzhenskaya's type (see [3] and Lemma 6 for details), close to a two-dimensional case, for the function $(\mathbf{u}, \rho)(t, \mathbf{x})$ having the special geometry structure (12) for $\mathbf{x} \in \tilde{\Omega} \subset \mathbb{R}^3$. Also, if we replace the domain $\tilde{\Omega}$ in Theorem 2 by one smooth domain $\Omega_2 \subset \mathbb{R}^3$ satisfying that there exists one positive constant ϵ_1, ϵ_2, ϵ_3, and ϵ_4 such that $\Omega_2 \subset \Omega_\epsilon \subset \mathbb{R}^3$ with

$$\begin{aligned}\Omega_\epsilon = \Big\{ &(x_1, x_2, x_3) \\ &= (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta) \\ &\in \mathbb{R}^3 : 0 < \epsilon_1 \le r \le \epsilon_2 < \infty, 0 < \epsilon_3 \le \theta \le \epsilon_4 < \pi, 0 \le \varphi < 2\pi \Big\},\end{aligned} \tag{25}$$

then the global strong solution obtained in Theorem 2 is also smooth in $(0,\infty) \times \Omega_2$.

Remark 4. The axisymmetric flow makes the 3D flow close to the 2D flow; that is, all velocity components (radial, angular (or swirl) and x_3 component) as well as the pressure are independent of the angular variable in the cylindrical coordinates. As a kind of fluid with special geometry structure, we know that the 1D parabolic Hausdorff measure of the set of possible singular points to the suitable weak solutions of the incompressible Navier-Stokes or Boussinesq system is zero; see [7, 8, 39] for details. This implies that the incompressible axisymmetric Navier-Stokes or Boussinesq equations cannot develop finite time singularities away from the symmetry axis. Based on this fact, it is not clear whether the potential finite-time-blow-up set for 3D incompressible Boussinesq equations in spherical coordinates is only one point set, where the flow is a special variant of axisymmetric, i.e., spherically symmetric, in $\mathbb{R}^3$. This is the main motivation of the current paper.

The rest of this paper is organized as follows. In Section 2, we introduce some technical lemmas used for the proof of the main theorems. In Section 3, we prove Theorems 1 and 2.

2. Preliminaries

In this section, we provide some lemmas used for the proof of the main theorems.

Lemma 5. (see [40]). *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ be a velocity field with its divergence free and vorticity ω; then, the inequality*

$$\|\nabla u\|_{L^p} \le C(p) \|\omega\|_{L^p}, \tag{26}$$

holds for any $p \in (1,\infty)$, where the constant $C(p)$ depends only on p.

Lemma 6 (see [3]). *Let $D \subseteq \mathbb{R}^2$; then, there exists a constant $C(D)$ such that, for any $f \in H_0^1(D)$,*

$$\|f\|_{L^4(D)} \le C(D) \|f\|_{L^2(D)}^{1/2} \|\nabla f\|_{L^2(D)}^{1/2}. \tag{27}$$

Lemma 7 (see [41]). *Suppose that the initial data $(\mathbf{u}_0, \rho_0) \in H^2(\mathbb{R}^3)$ with div $\mathbf{u}_0 = 0$ in (1); then, any Leray-Hopf weak solution $\mathbf{u}$ of 3D incompressible Boussinesq equation (1) is also a smooth solution in $(0, T] \times \mathbb{R}^3$ if there holds that*

$$u \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right), \tag{28}$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \le 1 \quad \text{with } 3 < q < \infty, 2 < p \le \infty. \tag{29}$$

Lemma 8 (see [42]). *Suppose that Ω is smooth and the initial data $(\mathbf{u}_0, \rho_0)$ in (2) satisfies $(\mathbf{u}_0, \rho_0) \in H^2(\Omega)$ with div $\mathbf{u}_0 = 0$ and $(\mathbf{u}_0, \rho_0)|_{\partial\Omega} = 0$; then, any Leray-Hopf weak solution $\mathbf{u}$ of 3D incompressible Boussinesq equation (2) is also a smooth solution in $(0, T] \times \Omega$ if there holds that*

$$u \in L^p(0, T; L^q(\Omega)), \tag{30}$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \le 1 \quad \text{with } 3 < q \le \infty, 2 \le p \le \infty. \tag{31}$$

3. Proof of Main Results

In this section, we give the proofs of Theorems 1 and 2.

Proof of Theorem 1. From (1), for any $T > 0$, we have the energy inequality

$$\begin{aligned}&\sup_{0\le t\le T} \left(\|\mathbf{u}\|_{L^2(\mathbb{R}^3)}^2 + \|\rho\|_{L^2(\mathbb{R}^3)}^2 \right) \\ &\quad + 2\nu \int_0^T \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3)}^2 dt + 2\mu \int_0^T \|\nabla \rho\|_{L^2(\mathbb{R}^3)}^2 dt \\ &\le C\left(T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)}\right).\end{aligned} \tag{32}$$

By the existence and uniqueness of the local smooth solution to the Cauchy problem (1) for the 3D Boussinesq equations, it is easy to get that $u^\varphi \equiv 0$ for the case of no swirl initial data $u_0^\varphi \equiv 0$. In this kind of case of no swirl, the velocity and vorticity satisfy the following special form:

$$\mathbf{u}(t,\mathbf{x}) = u^r(t,r,\theta)\mathbf{e}_r + u^\theta(t,r,\theta)\mathbf{e}_\theta, \omega(t,\mathbf{x}) = \omega^\varphi(t,r,\theta)\mathbf{e}_\varphi, \tag{33}$$

and hence, equation (23) for ω^φ is simplified as

$$\begin{aligned}\partial_t\omega^\varphi + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\omega^\varphi &= \nu\left(\tilde{\Delta} - \frac{1}{r^2\sin^2\theta}\right)\omega^\varphi + \frac{u^r\omega^\varphi}{r} + \frac{\cos\theta}{r\sin\theta}u^\theta\omega^\varphi \\ &\quad - \left(\sin\theta\partial_r\rho + \cos\theta\frac{1}{r}\partial_\theta\rho\right).\end{aligned} \tag{34}$$

Multiplying (34) by $r^2\sin^2\theta$ and then letting $r = 0$, $\theta = 0$, or $\theta = \pi$ and by the existence and uniqueness of the local smooth solution to the Cauchy problem (1) or (15)–(17) for the 3D Boussinesq equations, it is easy to see that

$$\omega^\varphi(t,0,\theta) = \omega^\varphi(t,r,0) = \omega^\varphi(t,r,\pi) = 0. \tag{35}$$

Similarly, we have

$$\left.\frac{\omega^\varphi(t,r,\theta)}{r\sin\theta}\right|_{r=0} = \left.\frac{\omega^\varphi(t,r,\theta)}{r\sin\theta}\right|_{\theta=0,\pi} = 0. \tag{36}$$

Taking $\omega^\varphi(t,r,\theta) = g(t,r,\theta)r\sin\theta$, i.e., $g(t,r,\theta) = \omega^\varphi(t,r,\theta)/r\sin\theta$, satisfying $g(t,0,\theta) = g(t,r,0) = g(t,r,\pi) = 0$, then we have

$$\begin{aligned}(\tilde{\mathbf{u}}\cdot\tilde{\nabla})(gr\sin\theta) &= \left(u^r\partial_r + \frac{u^\theta}{r}\partial_\theta\right)(gr\sin\theta) \\ &= u^r\sin\theta(g + r\partial_r g) + u^\theta(g\cos\theta + \sin\theta\partial_\theta g) \\ &= r\sin\theta(\tilde{\mathbf{u}}\cdot\tilde{\nabla})g + u^r g\sin\theta + u^\theta g\cos\theta,\end{aligned} \tag{37}$$

$$\begin{aligned}&\left(\tilde{\Delta} - \frac{1}{r^2\sin^2\theta}\right)(gr\sin\theta) \\ &= \left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta - \frac{1}{r^2\sin^2\theta}\right)(gr\sin\theta) \\ &= \sin\theta\left[\partial_r^2(rg) + \frac{2}{r}\partial_r(rg)\right] + \frac{1}{r}\partial_\theta^2(g\sin\theta) \\ &\quad + \frac{\cos\theta}{r\sin\theta}\partial_\theta(g\sin\theta) - \frac{g}{r\sin\theta} \\ &= \sin\theta\left(r\partial_r^2 g + 4\partial_r g + \frac{2}{r}g\right) \\ &\quad + \frac{1}{r}\left(3\cos\theta\partial_\theta g - g\sin\theta + \sin\theta\partial_\theta^2 g + \frac{\cos^2\theta}{\sin\theta}g\right) \\ &\quad - \frac{g}{r\sin\theta} \\ &= r\sin\theta\left(\partial_r^2 + \frac{4}{r}\partial_r\right)g + \frac{2\sin\theta}{r}g \\ &\quad + r\sin\theta\left(\frac{1}{r^2}\partial_\theta^2 g + \frac{3\cos\theta}{r^2\sin\theta}\partial_\theta g\right) \\ &\quad + \left(\frac{\cos^2\theta}{r\sin\theta} - \frac{\sin\theta}{r}\right)g - \frac{g}{r\sin\theta} \\ &= r\sin\theta\left(\partial_r^2 + \frac{4}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{3\cos\theta}{r^2\sin\theta}\partial_\theta\right)g \\ &\quad + \left(\frac{2\sin\theta}{r} + \frac{1-2\sin^2\theta}{r\sin\theta} - \frac{1}{r\sin\theta}\right)g \\ &= r\sin\theta\left(\tilde{\Delta} + \frac{2}{r}\partial_r + \frac{2\cos\theta}{r^2\sin\theta}\partial_\theta\right)g.\end{aligned} \tag{38}$$

Now putting $\omega^\varphi(t,r,\theta) = g(t,r,\theta)r\sin\theta$ into (34) and using (37)–(38), we obtain the following equation for $g(t,r,\theta)$:

$$\begin{aligned}\partial_t g + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})g - \nu\tilde{\Delta}g &= 2\nu\left(\frac{1}{r}\partial_r + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\right)g \\ &\quad - \frac{1}{r\sin\theta}\left(\sin\theta\partial_r\rho + \cos\theta\frac{1}{r}\partial_\theta\rho\right).\end{aligned} \tag{39}$$

To deal with the more singular second term in the right-hand side of (39), we decompose g into $g = G + (1/2\nu)\rho$; then, $G(t,r,\theta) = g(t,r,\theta) - (1/2\nu)\rho(t,r,\theta)$ satisfies

$$G(t,0,\theta) = G(t,r,0) = G(t,r,\pi) = 0, \tag{40}$$

and the following equation

$$\begin{aligned}&\partial_t\left(G + \frac{1}{2\nu}\rho\right) + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\left(G + \frac{1}{2\nu}\rho\right) - \nu\tilde{\Delta}\left(G + \frac{1}{2\nu}\rho\right) \\ &= 2\nu\left(\frac{1}{r}\partial_r + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\right)\left(G + \frac{1}{2\nu}\rho\right) \\ &\quad - \frac{1}{r\sin\theta}\left(\sin\theta\partial_r\rho + \cos\frac{1}{r}\partial_\theta\rho\right),\end{aligned} \tag{41}$$

which implies that

$$\partial_t G + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})G - \nu\tilde{\Delta}G = 2\nu\left(\frac{1}{r}\partial_r + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\right)G - \frac{\mu-\nu}{2\nu}\tilde{\Delta}\rho. \tag{42}$$

Multiplying equation (42) by G and integrating the resulting equation on $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|G\|_{L^2(\mathbb{R}^3)}^2 + \nu\|\tilde{\nabla}G\|_{L^2(\mathbb{R}^3)}^2 \\ &= \nu\left(\int_{\mathbb{R}^3}\frac{1}{r}\partial_r|G|^2 d\mathbf{x} + \int_{\mathbb{R}^3}\frac{\cos\theta}{r^2\sin\theta}\partial_\theta|G|^2 d\mathbf{x}\right) \\ &\quad - \frac{\mu-\nu}{2\nu}\int_{\mathbb{R}^3}G\tilde{\Delta}\rho d\mathbf{x} = I_1 + I_2,\end{aligned} \tag{43}$$

where I_1 and I_2 are defined by and can be estimated as follows:

$$\begin{aligned} I_1 &= \nu\left(\int_{\mathbb{R}^3} \frac{1}{r}\partial_r |G|^2 d\boldsymbol{x} + \int_{\mathbb{R}^3} \frac{\cos\theta}{r^2 \sin\theta}\partial_\theta |G|^2 d\mathbf{x}\right) \\ &= \nu\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{1}{r}\partial_r |G|^2\right) r^2 \sin\theta dr d\theta d\varphi \right. \\ &\quad \left. + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{\cos\theta}{r^2\sin\theta}\partial_\theta |G|^2\right) r^2 \sin\theta dr d\theta d\varphi\right) \\ &= \nu\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r\sin\theta \partial_r |G|^2 dr d\theta d\varphi \right. \\ &\quad \left. + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \cos\theta \partial_\theta |G|^2 dr d\theta d\varphi\right) \\ &= -\nu\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |G|^2 \sin\theta (\partial_r r) dr d\theta d\varphi \right. \\ &\quad \left. + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |G|^2 \partial_\theta(\cos\theta) dr d\theta d\varphi\right) \equiv 0, \end{aligned} \tag{44}$$

$$\begin{aligned} I_2 &= -\frac{\mu-\nu}{2\nu}\int_{\mathbb{R}^3} G\tilde{\Delta}\rho d\mathbf{x} = \frac{\mu-\nu}{2\nu}\int_{\mathbb{R}^3} \tilde{\nabla} G \cdot \tilde{\nabla}\rho d\mathbf{x} \\ &\le \frac{\nu}{2}\left\|\tilde{\nabla} G\right\|^2_{L^2(\mathbb{R}^3)} + C\left\|\tilde{\nabla}\rho\right\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{45}$$

Putting (44) and (45) into (43), we get

$$\frac{1}{2}\frac{d}{dt}\|G\|^2_{L^2(\mathbb{R}^3)} + \nu\|\tilde{\nabla} G\|^2_{L^2(\mathbb{R}^3)} \le \frac{\nu}{2}\|\tilde{\nabla} G\|^2_{L^2(\mathbb{R}^3)} + C\|\tilde{\nabla}\rho\|^2_{L^2(\mathbb{R}^3)}, \tag{46}$$

which, together with (32), yields to the following estimate for $G(t, r, \theta)$:

$$\begin{aligned} &\|G(t, r, \theta)\|_{L^2(\mathbb{R}^3)} \\ &\quad \le C\left(T, \|G(0, r, \theta)\|_{L^2(\mathbb{R}^3)}, \|\mathrm{u}_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)}\right). \end{aligned} \tag{47}$$

Thus, we have

$$\begin{aligned} &\|g(t, r, \theta)\|_{L^2(\mathbb{R}^3)} \\ &\quad = \left\|G + \frac{1}{2\nu}\rho\right\|_{L^2(\mathbb{R}^3)} \\ &\quad \le \|G\|_{L^2(\mathbb{R}^3)} + C\|\rho\|_{L^2(\mathbb{R}^3)} \\ &\quad \le C\left(T, \left\|\frac{\omega_0^\varphi(r,\theta)}{r\sin\theta}\right\|_{L^2(\mathbb{R}^3)}, \|\mathrm{u}_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)}\right). \end{aligned} \tag{48}$$

Next, we obtain the estimate for the vorticity $\omega = \omega^\varphi (t, r, \theta)\mathbf{e}_\varphi$, given by (33) in the case of no swirl for the 3D incompressible Boussinesq equation in the spherical coordinate system.

It is known that the vorticity equation for the vorticity $\omega = \nabla\times\mathbf{u}$ for the 3D incompressible Boussinesq equation is the following:

$$\partial_t\omega + (\mathbf{u}\cdot\nabla)\omega - \nu\Delta\omega = \omega\cdot\nabla\mathbf{u} - \left(\sin\theta\partial_r\rho + \cos\theta\frac{1}{r}\partial_\theta\rho\right)\mathbf{e}_\varphi. \tag{49}$$

Multiplying equation (49) by ω and integrating the resulting equation on $\mathbb{R}^3$, we have, for any $T > 0$, $0 \le t \le T$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2(\mathbb{R}^3)} + \nu\|\nabla\omega\|^2_{L^2(\mathbb{R}^3)} \\ &\quad = \int_{\mathbb{R}^3} (\omega\cdot\nabla)\mathbf{u}\cdot\omega d\mathbf{x} - \int_{\mathbb{R}^3} \omega^\varphi\left(\sin\theta\partial_r\rho + \cos\theta\frac{1}{r}\partial_\theta\rho\right) d\mathbf{x} \\ &\quad = J_1 + J_2, \end{aligned} \tag{50}$$

where $J_1 = \int_{\mathbb{R}^3}(\omega\cdot\nabla)\mathbf{u}\cdot\omega d\mathbf{x}$ and $J_2 = -\int_{\mathbb{R}^3}\omega^\varphi(\sin\theta\partial_r\rho + \cos\theta(1/r)\partial_\theta\rho)d\mathbf{x}$ can be estimated as follows by using the special structure (33) of the velocity $\mathbf{u}$ and the vorticity ω. Using (33), with the help of the Hölder inequality, Gagliardo-Nirenberg inequality, and Young inequality, we have

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3}\left[\omega^\varphi\mathbf{e}_\varphi\cdot\left(\mathbf{e}_r\partial_r + \mathbf{e}_\theta\frac{1}{r}\partial_\theta + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi\right)\right] \\ &\quad \cdot\left(u^r\mathbf{e}_r + u^\theta\mathbf{e}_\theta\right)\cdot\left(\omega^\varphi\mathbf{e}_\varphi\right)d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\left[\frac{\omega^\varphi}{r\sin\theta}\partial_\varphi\left(u^r\mathbf{e}_r + u^\theta\mathbf{e}_\theta\right)\right]\cdot\left(\omega^\varphi\mathbf{e}_\varphi\right)d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\left[\frac{\omega^\varphi}{r\sin\theta}\left(u^r\sin\theta + u^\theta\cos\theta\right)\mathbf{e}_\varphi\right]\cdot\left(\omega^\varphi\mathbf{e}_\varphi\right)d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\frac{1}{r}u^r\omega^\varphi\omega^\varphi d\mathbf{x} + \int_{\mathbb{R}^3}\frac{\cos\theta}{r\sin\theta}u^\theta\omega^\varphi\omega^\varphi d\mathbf{x} \\ &= \int_{\mathbb{R}^3} u^r g\omega^\varphi\sin\theta d\mathbf{x} + \int_{\mathbb{R}^3} u^\theta g\omega^\varphi\cos\theta d\mathbf{x} \\ &\le \int_{\mathbb{R}^3}|u^r g\omega^\varphi|d\mathbf{x} + \int_{\mathbb{R}^3}\left|u^\theta g\omega^\varphi\right|d\mathbf{x} \\ &\le \left(\|u^r\|_{L^3(\mathbb{R}^3)} + \left\|u^\theta\right\|_{L^3(\mathbb{R}^3)}\right)\|g\|_{L^2(\mathbb{R}^3)}\|\omega\|_{L^6(\mathbb{R}^3)} \\ &\le C\|\mathrm{u}\|^{1/2}_{L^2(\mathbb{R}^3)}\|\nabla\mathrm{u}\|^{1/2}_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}\|\nabla\omega\|_{L^2(\mathbb{R}^3)} \\ &\le C\|\mathrm{u}\|^2_{H^1(\mathbb{R}^3)} + \frac{\nu}{2}\|\nabla\omega\|^2_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{51}$$

$$J_2 \le \int_{\mathbb{R}^3}|\omega^\varphi|\left(|\partial_r\rho| + \left|\frac{1}{r}\partial_\theta\rho\right|\right)d\mathbf{x} \le \|\omega\|^2_{L^2(\mathbb{R}^3)} + \left\|\tilde{\nabla}\rho\right\|^2_{L^2(\mathbb{R}^3)}. \tag{52}$$

Putting (51) and (52) into (50), we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2(\mathbb{R}^3)}+\frac{\nu}{2}\|\nabla\omega\|^2_{L^2(\mathbb{R}^3)}\le C\|\mathbf{u}\|^2_{H^1(\mathbb{R}^3)}+\|\omega\|^2_{L^2(\mathbb{R}^3)}+\left\|\tilde{\nabla}\rho\right\|^2_{L^2(\mathbb{R}^3)}, \tag{53}$$

which, by applying Gronwall's inequality and by using (32), yields to, for any $T>0$,

$$\|\omega(t,\cdot)\|_{L^2(\mathbb{R}^3)}\le C\left(T,\|\mathbf{u}_0\|_{H^1(\mathbb{R}^3)},\left\|\frac{\omega_0^{\varphi}}{r\sin\theta}\right\|_{L^2(\mathbb{R}^3)},\|\rho_0\|_{L^2(\mathbb{R}^3)}\right),\quad 0\le t\le T, \tag{54}$$

$$\int_0^t\|\nabla\omega(s,\cdot)\|^2_{L^2(\mathbb{R}^3)}ds\le C\left(T,\|\mathbf{u}_0\|_{H^1(\mathbb{R}^3)},\left\|\frac{\omega_0^{\varphi}}{r\sin\theta}\right\|_{L^2(\mathbb{R}^3)},\|\rho_0\|_{L^2(\mathbb{R}^3)}\right),\quad 0\le t\le T. \tag{55}$$

Using Lemma 5, we get from (54) that, for any $0\le T\le\infty$,

$$\nabla\mathbf{u}\in L^{\infty}\left(0,T;L^2\left(\mathbb{R}^3\right)\right), \tag{56}$$

and hence, by Sobolev's imbedding theorem, we have, for any $0\le T\le\infty$,

$$\mathbf{u}\in L^{\infty}\left(0,T;L^6\left(\mathbb{R}^3\right)\right). \tag{57}$$

Now, the desired regularity estimate for the 3D incompressible Boussinesq equation (1) is obtained; hence, by applying Lemma 7, we obtain the results stated in Theorem 1.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. We take $\Omega=\tilde{\Omega}$ in Theorem 2, where $\tilde{\Omega}$ is given by (9) having one special geometry structure. Also, $\mathbf{u}$ in Theorem 2 is given by (12), which satisfies that $|\mathbf{u}|^2=u_1^2+u_2^2+u_3^2=(u^r)^2+(u^\theta)^2+(u^\varphi)^2$ by using the orthogonality of three spherical coordinate unit vectors. Firstly, for the system (2), we have the following basic energy estimates, for $0\le t\le+\infty$,

$$\frac{d}{dt}\int_\Omega|\mathbf{u}(t,\cdot)|^2d\mathbf{x}+2\nu\int_\Omega|\nabla\mathbf{u}(t,\cdot)|^2d\mathbf{x}\le\delta\int_\Omega|\mathbf{u}(t,\cdot)|^2d\mathbf{x}+C(\delta)\int_\Omega|\rho(t,\cdot)|^2d\mathbf{x}, \tag{58}$$

$$\frac{d}{dt}\int_\Omega|\rho(t,\cdot)|^2d\mathbf{x}+2\mu\int_\Omega|\nabla\rho(t,\cdot)|^2d\mathbf{x}=0, \tag{59}$$

for some constant $C(\delta)>0$ and any $\delta>0$, which, together with Poincare's inequality for $\mathbf{u}$ and ρ, yield the energy estimate, for $0\le t\le+\infty$,

$$\left(\|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega)}+\|\rho(t,\cdot)\|^2_{L^2(\Omega)}\right)+2\nu\int_0^t\|\nabla\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega)}dt+2\mu\int_0^t\|\nabla\rho(t,\cdot)\|^2_{L^2(\Omega)}dt\le C\left(\Omega,\nu,\mu,\|\mathbf{u}_0\|_{L^2(\Omega)},\|\rho_0\|_{L^2(\Omega)}\right), \tag{60}$$

$$\|(\mathbf{u},\rho)(t,\cdot)\|^2_{L^2(\Omega)}\le C\left(\Omega,\nu,\mu,\|\mathbf{u}_0\|_{L^2(\Omega)},\|\rho_0\|_{L^2(\Omega)}\right)e^{-\alpha t}, \tag{61}$$

for some constants $C=C(\Omega,\nu,\mu,\|\mathbf{u}_0\|_{L^2(\Omega)},\|\rho_0\|_{L^2(\Omega)})>0$ and $\alpha=\alpha(\Omega,\nu,\mu)>0$.

Next, we give the estimates of $\|\partial_t(\mathbf{u},\rho)(t,\cdot)\|^2_{L^2(\Omega)}$.

Differentiating (2) with respect to t, one gets

$$\begin{cases}\mathbf{u}_{tt}+\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t+\nabla P_t=\nu\Delta\mathbf{u}_t+\rho_t\mathbf{e}_3,\\ \rho_{tt}+\mathbf{u}\cdot\nabla\rho_t+\mathbf{u}_t\cdot\nabla\rho=\mu\Delta\rho_t,\\ \operatorname{div}\mathbf{u}_t=0,\\ \mathbf{u}_t|_{\partial\Omega}=0,\ \rho_t|_{\partial\Omega}=0,\\ \mathbf{u}_t(0,\mathbf{x})=\mathbf{v}_0(\mathbf{x}),\ \rho_t(0,\mathbf{x})=\rho_0(\mathbf{x}),\end{cases} \tag{62}$$

where $\mathbf{v}_0$ and ρ_0 satisfy, by using (2), that

$$\mathbf{v}_0+(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0+\nabla P_0=\nu\Delta\mathbf{u}_0+\rho_0\mathbf{e}_3,\ \operatorname{div}\mathbf{v}_0=0, \tag{63}$$

$$\rho_0+(\mathbf{u}_0\cdot\nabla)\rho_0=\mu\Delta\rho_0. \tag{64}$$

It is easy to get that

$$\|\mathbf{v}_0\|_{L^2(\Omega)}\le C\left(\|\mathbf{u}_0\|_{H^2(\Omega)},\|\rho_0\|_{H^2(\Omega)}\right), \tag{65}$$

$$\|\rho_0\|_{L^2(\Omega)}\le C\left(\|\mathbf{u}_0\|_{H^2(\Omega)},\|\rho_0\|_{H^2(\Omega)}\right). \tag{66}$$

In fact, multiplying (64) by $\mathbf{v}_0$ and integrating the resulting equation on Ω, applying the Hölder inequality, Gagliardo-Nirenberg inequality, and Young inequality, we have

$$\begin{aligned}\|\mathbf{v}_0\|^2_{L^2(\Omega)}&=-\int_\Omega(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0\cdot\mathbf{v}_0d\mathbf{x}+\nu\int_\Omega\Delta\mathbf{u}_0\cdot\mathbf{v}_0d\mathbf{x}+\int_\Omega\rho_0\mathbf{e}_3\cdot\mathbf{v}_0d\mathbf{x}\\&\le C\|\mathbf{u}_0\|_{L^3(\Omega)}\|\nabla\mathbf{u}_0\|_{L^6(\Omega)}\|\mathbf{v}_0\|_{L^2(\Omega)}+C\|\Delta\mathbf{u}_0\|_{L^2(\Omega)}\|\mathbf{v}_0\|_{L^2(\Omega)}+\|\rho_0\|_{L^2(\Omega)}\|\mathbf{v}_0\|_{L^2(\Omega)}\\&\le C\left(\|\mathbf{u}_0\|_{H^2(\Omega)}+\|\rho_0\|_{L^2(\Omega)}\right)\|\mathbf{v}_0\|_{L^2(\Omega)},\end{aligned} \tag{67}$$

which implies (65). Similarly, we have (66).

Multiplying the first equation in (62) by $\mathbf{u}_t$ and integrating the resulting equation on Ω, with the help of the Hölder inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|_{L^2(\Omega)}^2+\nu\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}^2 \\ &\quad=\int_\Omega \rho_t\mathbf{e}_3\cdot\mathbf{u}_t d\mathbf{x}-\int_\Omega(\mathbf{u}_t\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t d\mathbf{x}\le C(\delta)\|\rho_t\|_{L^2(\Omega)}^2 \\ &\qquad+\delta\|\mathbf{u}_t\|_{L^2(\Omega)}^2+C\|\mathbf{u}_t\|_{L^4(\Omega)}^2\|\nabla\mathbf{u}\|_{L^2(\Omega)}. \end{aligned} \tag{68}$$

Multiplying the second equation in (62) by ρ_t and integrating the resulting equation on Ω, with the help of the Hölder inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\rho_t\|_{L^2(\Omega)}^2+\mu\|\nabla\rho_t\|_{L^2(\Omega)}^2 \\ &\quad=-\int_\Omega(\mathbf{u}_t\cdot\nabla)\rho\cdot\rho_t d\mathbf{x}\le C\|\mathbf{u}_t\|_{L^4(\Omega)}\|\nabla\rho\|_{L^2(\Omega)}\|\rho_t\|_{L^4(\Omega)}. \end{aligned} \tag{69}$$

In the following, we use the special geometry structure (9) of the domain $\Omega=\tilde{\Omega}\subset\mathbb{R}^3$ and the special geometry structure (12) of the functions $(\mathbf{u},\rho)(t,\mathbf{x})$ in spherical coordinates in $\mathbb{R}^3$ to obtain the following inequality for $(\mathbf{u}_t,\rho_t)(t,\mathbf{x})$ defined in $[0,t)\times\tilde{\Omega}\subset[0,\infty)\times\subset\mathbb{R}^3$ with $(\mathbf{u},\rho)|_{\partial\Omega}=0$: there exists a constant $C=C(\Omega)>0$ such that

$$\|\mathbf{u}_t(t,\cdot)\|_{L^4(\Omega)}\le C(\Omega)\|\mathbf{u}_t(t,\cdot)\|_{L^2(\Omega)}^{1/2}\|\nabla\mathbf{u}_t(t,\cdot)\|_{L^2(\Omega)}^{1/2}, \tag{70}$$

$$\|\rho_t(t,\cdot)\|_{L^4(\Omega)}\le C(\Omega)\|\rho_t(t,\cdot)\|_{L^2(\Omega)}^{1/2}\|\nabla\rho_t(t,\cdot)\|_{L^2(\Omega)}^{1/2}, \tag{71}$$

where

$$\nabla=\mathbf{e}_r\partial_r+\frac{1}{r}\mathbf{e}_\theta\partial_\theta+\frac{1}{r\sin\theta}\mathbf{e}_\varphi\partial_\varphi. \tag{72}$$

We note that the inequalities (70) are the same as in the two-dimensional case, which are, in general, not true for the general functions $\mathbf{u}(t,\mathbf{x})$ or $\rho(t,\mathbf{x})$ when $\mathbf{x}\in\Omega$ if Ω is the general bounded domain of $\mathbb{R}^3$. However, the equalities (70) are true under the assumption of Theorem 2 because of the special geometry structures (9) and (12) for the domain Ω and the functions $(\mathbf{u},\rho)(t,\mathbf{x})$, especially $\rho(t,\mathbf{x})=\rho(t,r,\theta)$ independent of φ and $\mathbf{u}(t,\mathbf{x})=\mathbf{u}(t,r,\theta,\varphi)$ depending upon φ only by the three orthogonal unit vectors in spherical coordinates (the combination of the functions $\cos\varphi$ and $\sin\varphi$). In fact, the inequalities (70) and (71) has been proven by S. Wang and Y.X. Wang in [16] for the domain $\Omega\in\mathbb{R}^3$ having the special geometry structure (9) and for the function $\mathbf{u}(t,\mathbf{x}),\mathbf{x}\in\Omega\subset\mathbb{R}^3$, with $\mathbf{u}(t,\mathbf{x})|_{\partial\Omega}=0$ having the special geometry structure (12).

For completeness, we give the proof of the inequalities (70) and (71). Taking $f(t,r,\theta)=\rho_t r^{1/2}\sin^{1/4}\theta$ in Lemma 6 with the domain $D=(r_0,r_1)\times(\theta_0,\theta_1)\subset\mathbb{R}^2$ and $\rho(t,r,\theta)|_{\partial D}=0$, we get

$$\begin{aligned} \|\rho_t\|_{L^4(\Omega)}^4&=\int_\Omega|\rho_t|^4(t,r,\theta)d\mathbf{x} \\ &=\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}(\rho_t)^4(t,r,\theta)r^2\sin\theta drd\theta d\varphi \\ &=2\pi\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\rho_t r^{1/2}\sin^{1/4}\theta\right)^4 drd\theta \\ &\le C\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(|\rho_t|r^{1/2}\sin^{1/4}\theta\right)^2 drd\theta \\ &\quad\cdot\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left|\nabla_{r,\theta}\left(\rho_t r^{1/2}sin^{1/4}\theta\right)\right|^2 drd\theta, \end{aligned} \tag{73}$$

where $C=C(\Omega)$ is a constant depending upon the domain $\Omega=\tilde{\Omega}$, and $\nabla_{r,\theta}=(\partial_r,\partial_\theta)$. It is clear that

$$\begin{aligned} &\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(|\rho_t|r^{1/2}\sin^{1/4}\theta\right)^2 drd\theta \\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\frac{1}{r\sqrt{\sin\theta}}|\rho_t|^2 r^2\sin\theta drd\theta d\varphi \\ &\quad\le\frac{1}{2\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\frac{1}{r_0\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}|\rho_t|^2 r^2\sin\theta drd\theta d\varphi \\ &\quad=\frac{1}{2\pi r_0\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\|\rho_t\|_{L^2(\Omega)}^2, \end{aligned} \tag{74}$$

$$\begin{aligned} &\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left|\nabla_{r,\theta}\left(\rho_t r^{1/2}\sin^{1/4}\theta\right)\right|^2 drd\theta \\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\left|\partial_r\left(\rho_t r^{1/2}\sin^{1/4}\theta\right)\right|^2\right. \\ &\qquad\left.+\left|\partial_\theta\left(\rho_t r^{1/2}\sin^{1/4}\theta\right)\right|^2\right)drd\theta d\varphi \\ &\quad\le\frac{1}{\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left[r\sin^{1/2}\theta\left(|\partial_r\rho_t|^2+|\partial_\theta\rho_t|^2\right)\right. \\ &\qquad\left.+\left(\frac{1}{4r}\sin^{1/2}\theta+\frac{\cos^2\theta}{16}r\sin^{-3/2}\theta\right)|\rho_t|^2\right]drd\theta d\varphi \\ &\quad=K_1+K_2, \end{aligned} \tag{75}$$

where

$$\begin{aligned} K_1&=\frac{1}{\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\frac{1}{r\sqrt{\sin\theta}}|\partial_r\rho_t|^2+\frac{r}{\sqrt{\sin\theta}}\left|\frac{1}{r}\partial_\theta\rho_t\right|^2\right) \\ &\quad\cdot r^2\sin\theta drd\theta d\varphi \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\frac{1}{r_0\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}|\partial_r\rho_t|^2 + \frac{R_0}{\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\left|\frac{1}{r}\partial_\theta\rho_t\right|^2\right) r^2\sin\theta drd\theta d\varphi \\
&\leq \frac{1}{\pi\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\max\left\{\frac{1}{r_0},R_0\right\} \cdot \int_\Omega\left(|\partial_r\rho_t|^2+\left|\frac{1}{r}\partial_\theta\rho_t\right|^2\right)d\mathbf{x} \\
&\leq \frac{1}{\pi\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\max\left\{\frac{1}{r_0},R_0\right\} \cdot \int_\Omega\left(|\partial_r\rho_t|^2+\left|\frac{1}{r}\partial_\theta\rho_t\right|^2+\left|\frac{1}{rsin\theta}\partial_\varphi\rho_t\right|^2\right)dx \\
&= \frac{1}{\pi\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\max\left\{\frac{1}{r_0},R_0\right\}\|\nabla\rho_t\|^2_{L^2(\Omega)},
\end{aligned} \tag{76}$$

$$\begin{aligned}
K_2 &= \frac{1}{\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\frac{1}{4r^3\sqrt{\sin\theta}}+\frac{1}{16r\sin^2\theta\sqrt{\sin\theta}}\right) \cdot |\rho_t|^2 r^2\sin\theta drd\theta d\varphi \\
&\leq \frac{1}{\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left(\frac{1}{4r_0^3\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}+\frac{1}{16r_0\left(\min_{\theta\in(\theta_0,\theta_1)}\sin\theta\right)^2\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\right) \cdot |\rho_t|^2 r^2\sin\theta drd\theta d\varphi \\
&\leq \frac{1}{\pi}\left(\frac{1}{4r_0^3\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}+\frac{1}{16r_0\left(\min_{\theta\in(\theta_0,\theta_1)}\sin\theta\right)^2\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\right)\int_\Omega|\rho_t|^2d\mathbf{x} \\
&\leq \frac{1}{\pi}\left(\frac{1}{4r_0^3\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}+\frac{1}{16r_0\left(\min_{\theta\in(\theta_0,\theta_1)}\sin\theta\right)^2\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\right)\|\rho_t\|^2_{L^2(\Omega)} \\
&\leq C(\Omega)\left(\frac{1}{4r_0^3\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}+\frac{1}{16r_0\left(\min_{\theta\in(\theta_0,\theta_1)}\sin\theta\right)^2\sqrt{\min_{\theta\in(\theta_0,\theta_1)}\sin\theta}}\right)\|\nabla\rho_t\|^2_{L^2(\Omega)},
\end{aligned} \tag{77}$$

with the help of Poincare's inequality and the fact that $|\nabla\rho(r,\theta,\varphi,t)|^2 = |\partial_r\rho|^2 + |\partial_\theta\rho/r|^2 + |\partial_\varphi\rho/r\sin\theta|^2$.

Combining (75) together with (76)–(77), we have

$$\int_{\theta_0}^{\theta_1}\int_{r_0}^{R_0}\left|\nabla_{r,\theta}\left(\rho_t r^{1/2}\sin^{1/4}\theta\right)\right|^2 drd\theta \leq C(\Omega)\|\nabla\rho_t\|^2_{L^2(\Omega)}. \tag{78}$$

Thus, putting (74) and (78) into (73), we get (70) and (71).

Combining (68), (69), and (70), with the help of the Young inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}_t\|^2_{L^2(\Omega)}+M_1\|\rho_t\|^2_{L^2(\Omega)}\right)+\nu\|\nabla\mathbf{u}_t\|^2_{L^2(\Omega)}+M_1\mu\|\nabla\rho_t\|^2_{L^2(\Omega)} \\
&\quad\leq C(\delta)\|\rho_t\|^2_{L^2(\Omega)}+\delta\|\mathbf{u}_t\|^2_{L^2(\Omega)}+C\|\mathbf{u}_t\|^2_{L^4(\Omega)}\|\nabla \mathrm{u}\|_{L^2(\Omega)} \\
&\qquad+C\|\mathbf{u}_t\|_{L^4(\Omega)}\|\nabla\rho\|_{L^2(\Omega)}\|\rho_t\|_{L^4(\Omega)} \\
&\quad\leq C(\delta)\|\rho_t\|^2_{L^2(\Omega)}+\delta\|\mathbf{u}_t\|^2_{L^2(\Omega)} \\
&\qquad+C\|\mathbf{u}_t\|_{L^2(\Omega)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)} \\
&\qquad+C\left(\|\mathbf{u}_t\|^{1/2}_{L^2(\Omega)}\|\nabla\mathbf{u}_t\|^{1/2}_{L^2(\Omega)}\|\nabla\rho\|^{1/2}_{L^2(\Omega)}\right) \\
&\qquad\cdot\left(\|\rho_t\|^{1/2}_{L^2(\Omega)}\|\nabla\rho_t\|^{1/2}_{L^2(\Omega)}\|\nabla\rho\|^{1/2}_{L^2(\Omega)}\right) \\
&\quad\leq C(\delta)\|\rho_t\|^2_{L^2(\Omega)}+\delta\|\mathbf{u}_t\|^2_{L^2(\Omega)}+C\|\nabla \mathrm{u}\|^2_{L^2(\Omega)}\|\mathbf{u}_t\|^2_{L^2(\Omega)} \\
&\qquad+C\|\nabla\rho\|^2_{L^2(\Omega)}\|\mathbf{u}_t\|^2_{L^2(\Omega)}+C\|\nabla\rho\|^2_{L^2(\Omega)}\|\rho_t\|^2_{L^2(\Omega)} \\
&\qquad+\frac{\nu}{2}\|\nabla\mathbf{u}_t\|^2_{L^2(\Omega)}+\frac{\mu}{2}\|\nabla\rho_t\|^2_{L^2(\Omega)}
\end{aligned} \tag{79}$$

for any $\delta>0$, for suitably large constant $M_1>0$ and for some constants $C(\delta)>0$ and $C=C(M_1,\Omega,\nu,\mu)>0$, which yields, by applying Gronwall's inequality and Poincare's inequality, using the estimate (60), to the decay exponentially in time

$$\begin{aligned}
&\|\mathbf{u}_t(t,\cdot)\|_{L^2(\Omega)}+\|\rho_t(t,\cdot)\|_{L^2(\Omega)} \\
&\quad\leq C\left(\|\mathbf{u}_0\|_{H^2(\Omega)},\|\rho_0\|_{H^2(\Omega)}\right)e^{-\alpha t}, \quad 0\leq t\leq+\infty.
\end{aligned} \tag{80}$$

Finally, we obtain the estimate $L^\infty([0,T];\|\nabla\mathbf{u}\|^2_{L^2(\Omega)})$ for $\mathbf{u}$ and for any $T>0$.

Multiplying (2) by $\mathbf{u}$ and integrating the resulting equation on Ω, and by using the estimates (61) and (80), we have, for $0 \le t \le +\infty$,

$$\begin{aligned}\|\nabla \mathbf{u}(t,\cdot)\|_{L^2(\Omega)}^2 &= \frac{1}{\nu}\int_\Omega \rho \mathbf{e}_3 \cdot \mathbf{u} d\mathbf{x} - \frac{1}{\nu}\int_\Omega \mathbf{u}\cdot \mathbf{u}_t d\mathbf{x} \\ &\le C(\nu)\left(\|\rho\|_{L^2(\Omega)}^2 + \|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\mathbf{u}_t\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}\right) \\ &\le C\left(\|\mathbf{u}_0\|_{H^2(\Omega)}, \|\rho_0\|_{L^2(\Omega)}\right)e^{-\alpha t}.\end{aligned} \tag{81}$$

Combining (60) and (81) together, we have

$$\mathbf{u} \in L^\infty\left(0, T; H_0^1(\Omega)\right), \tag{82}$$

which gives, by Sobolev's imbedding theorem, that

$$\mathbf{u} \in L^\infty\left(0, T; L^6(\Omega)\right). \tag{83}$$

Thus, we can obtain the desired global regularity estimate for strong solution $(\mathbf{u}, \rho)$ and the global smooth interior regularity for the solution $(\mathbf{u}, \rho)$ by choosing the suitable cutoff function with compact subset of the domain Ω, and we can conclude the regularity results on Theorem 2 by using Lemma 8.

Also, it is easy from (59) to get that

$$\begin{aligned}\|\nabla \rho(t,\cdot)\|_{L^2(\Omega)}^2 &\le C\|\rho(t,\cdot)\|_{L^2(\Omega)}\|\rho_t(t,\cdot)\|_{L^2(\Omega)} \\ &\le Ce^{-\alpha t}, \quad 0 \le t \le +\infty.\end{aligned} \tag{84}$$

Thus, the decay rate (24) in Theorem 2 can be obtained from the estimates (61) and (80)–(84).

The proof of Theorem 2 is complete. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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