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Research Article

Fundamental Results about the Fractional Integro-Differential Equation Described with Caputo Derivative

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In this paper, we study the existence and uniqueness of the mild solution of the fractional integro-differential with the nonlocal initial condition described by the Caputo fractional operator. Note that here the order of the Caputo derivative satisfies the condition that $\alpha \in (1, 2)$. The existence of α-resolvent operator in Banach space and fixed point theorem has been utilized in the proof of the existence of the mild solution. We have established in this paper the Hyers-Ulam stability of the mild solution of the considered fractional integro-differential equation. An illustrative example has been provided to support the main findings of the paper.

1. Introduction

This paper focuses on the application of fractional operators in modeling and studying fundamental mathematics. Note that there exist many fractional operators in the field of fractional calculus: Caputo derivative [1, 2], Caputo-Fabrio derivative [3, 4], see also in [4], where we can find the operator with exponential kernel. We have the Atangana-Baleanu derivative [5], the Riemann-Liouville fractional operator [1, 2], and many other fractional derivative operators similar to the previously cited operators [6, 7]. For paper supporting the applications of fractional calculus in the mathematical physics field see in [8, 9], in mathematical physics field see in [10–12], in biology [13–16], in statistical data [14], on vibration equation [10], on fluid modeling for getting analytical solutions via the Laplace transform method [9], in same direction see [17] and Khalid et al. in [18], the application of the Laplace transform is also applied by Sene in the following investigation [19] and in [20] where the Fourier transform is applied to the fractional operator to get the analytical solution of the fluid model. In the present paper, we will focus on the fractional-integro differential equation focussed in the literature [21], also studied in the literature with the integer-order derivative in the following works [22, 23]. In the present paper, we consider the fractional version defined by the equation

$$D^{\alpha}y(t) = Ay(t) + \int_0^t B(t-s)y(s) + h(t, y), \text{for } t \in [0, b], \quad (1)$$

$$y(0) - g(y) = y_0 \in X. \quad (2)$$

Here, X is a Banach space, where our investigation will be focussed. In this paper $A : D(A) \longrightarrow X$ is a sectorial operator of type $\Omega(\gamma, \theta)$; in other words, for $\lambda \in \rho(A)$ such that $|\lambda| > \gamma$, we have that $|\arg(\lambda)| < \theta$ defined in $D(A) \subset X$ into X, $B(t)$ is a closed linear operator with its domain satisfying the condition $D(B) \supset D(A)$. The function $h : [0, b] \times C \longrightarrow X$ is a continuous function, and the function $g : \mathscr{C}([0, b], X) \longrightarrow X$ is a given continuous function. For more piece of information on the definition of the same problem (1) in the integer-order version, see in [22]. The problem reported in Equation (1) is new in the context of fractional calculus because, at first, the Caputo fractional operator has been considered; second, some new resolvents will be introduced, and at last, the nonlocal condition gives the system interest audience. The studies of the Hyers Ulam stability will be a novel notion in the present investigation and can be considered a novel aspect of the present paper. In our problem, the order is considered into the interval $\alpha \in (1, 2)$. As it will be noticed later in the illustration section the present problem has many domains of applications notably in fractional heat processes. The same direction of investigations has been focussed in the literature, see the following paper [21]. In our information, there are not many investigations of the stability analysis of the solutions of the fractional integro-differential equations. We notice that the fractional integro-differential equation has begun to attract some researchers, and some results are well established in the literature like the definition and the existence of the α-resolvant operator; see more pieces of information in [24]. For short, the problem under investigation can be considered a new open problem in fractional calculus and merits more investigations in the future. The main importance of this investigation is to see the influence of the fractional operator in the investigation. The existence and the uniqueness of the mild solution of the model (1) will be focused on. For the findings related to the existence and the uniqueness of the mild solution, various fixed points theorems have been used, namely, the Banach fixed point theorem.

In fractional calculus, there exist works related to the integro-differential equations but many of them focus on the problems of the solutions, we mean the determination of the analytical solutions and the numerical solutions. In other words, in the applied mathematics view points of integro-differential equations, here, fundamental mathematics aspect is considered notably; we prove the existence of at least one mild solution for the problem considered with a novel initial condition, we mean nonlocal initial condition. For the literature review, we can cite the following papers. In fractional calculus, the existence of the analytic α-resolvant operator in the fractional integro-differential equation with Caputo derivative has been proprosed [24]. They consider the following the problem:

$$\begin{aligned} D^{\alpha} y(t) &= Ay(t) + \int_0^t B(t-s) y(s), \text{for } t \in [0, b], \\ y(0) &= y_0, y'(0) = 0. \end{aligned} \tag{3}$$

In [21], Agarwal et al. have proposed the existence of the mild solution of the fractional integro-differential equation described by Caputo derivative and with delay. In the same direction of the investigations, general results on the existence of a mild solution with fractional integro-differential equations with delay have been proposed by Santos et al. in [25]. For investigation related to the study of the upper bound of the α-resolvant operators, see Shu and Wang in [26]. In [27], Wang and Shu have investigated the existence of a positive mild solution for fractional differential evolution equations with nonlocal conditions, where the order describes the condition $1 < \alpha < 2$. The fractional differential equations used in their investigations were described by the Caputo derivative. For problems related to the existence of mild solutions using fractional resolvents, we have the following investigations [28, 29]. There also exist many investigations related to the integro-differential equation described by integer-order derivative; we cite the most relevant of them for the present investigations, refer to the papers by Ezzinbi et al. [22, 23] and Diop et al. [30, 31] and for the problem concerning the existence of mild solutions for a semilinear integro-differential equation with nonlocal initial conditions by Lizama and Pozo in [32].

In this paper, we consider fractional integro-differential equation with the nonlocal initial condition initiated studied in the literature by Lizama and Pozo in [32]. The difference between the problem (1) and Lizama and Pozo work is that in our present investigations, here, we consider the Caputo derivative with the order into $(1, 2)$ which is also a novel aspect in the fractional calculus. Many real problems in fractional calculus are considered in the interval $(0, 1)$. The second impact of the paper is the stability analysis provided in this paper. The existence of the mild solution for our considered problem will generate the definition of a new family of operators called the α-resolvant operators. The examples provided in this paper will provide the applicability of our findings in fractional diffusion processes.

The present paper is described in the following form: In Section 2, we define the fractional operators used in the present investigations. In Section 3, we recall the preliminary results established in the literature. In Section 4, we provide the existence and the uniqueness of the mild solution using the fixed point theorem. We also focus on the Hyers-Ulam stability of the solution of the considered model in this paper. In Section 5, we describe the example which illustrates our main findings. We finish with Section 6 with the concluding remarks and open a new door of investigations.

2. Fractional Operators

This section will be the part where we recall the fractional operator used in this section. For generalization, we consider the fractional-order derivative in their general forms. The attraction of this part will be the fractional integral, the Caputo fractional operator, and the Riemann-Liouville integral. There exist also recent fractional operators such as the Caputo-Fabrizio and Atangana-Baleanu fractional operators well established in the literature. At first, the following definition is the definition of the Rieman-Liouville integral which plays much interest in the determination of the form of the analytical solutions using the Volterra integral.

Definition 1 (see [1, 2]). We utilize a function represented by $f : [0, +\infty[\longrightarrow \mathbb{R}$, then we define the called in the literature the Riemann-Liouville integral of the considered function as the form that

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{4}$$

where $\Gamma(\cdots)$ symbolizes the Gamma Euler function and the order α verifies the condition as $\alpha > 0$.

The next definition is the definition of the Riemann-Liouville derivative associated with the previous integral.

The information on this derivative can be found in the paper in the literature in the following reference paper [1, 2].

Definition 2 (see [1, 2]). We utilize the Riemann-Liouville derivative of the considered function $f : [0,+\infty[\longrightarrow \mathbb{R}$, of order α as the form represented by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} f(s)(t-s)^{n-\alpha-1} ds, \quad (5)$$

for the time $t > 0$, the order of the derivative satisfies the condition that $\alpha \in (n-1, n)$, and where $\Gamma(\cdots)$ symbolizes the Gamma Euler function.

We continue with the Caputo derivative utilized in our investigation. As previously mentioned, we use the order of the derivative satisfying the condition that $\alpha \in (n-1, n)$. We have the following definition.

Definition 3 (see [1, 2]). We utilize the Caputo fractional derivative of the considered function $f : [0,+\infty[\longrightarrow \mathbb{R}$, of order α as the form represented by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} f^{(n)}(s)(t-s)^{n-\alpha-1} ds, \quad (6)$$

for the time $t > 0$, the order of the derivative satisfies the condition that $\alpha \in (n-1, n)$, and where $\Gamma(\cdots)$ symbolizes the Gamma Euler function.

Note that the previous definition is utilized in our investigation with the condition that $\alpha \in (1, 2)$ which means that $n = 2$. The previous definitions of the fractional operators are important because first, they permit the prediction of the future of the dynamic system using the information obtained from the past of the system. The memory property of the fractional operators is the main objective of their attraction to model real-world problems. In the literature, there also exist many controversies on the validity of the fractional operators, and there exist nowadays many questions without answers concerning the utility of the fractional operator. Apart from the classical derivative, there exist in the literature the so-called derivative with exponential kernel, namely, the Caputo-Fabrizio derivative and the derivative with Mittag-Leffler kernel, namely, the Atangana-Baleanu derivative, the definitions of these derivatives can be found in the literature [3–5].

Before closing this part, we recall the Laplace transform of the Caputo derivative in our context, which will be provided to get the form of the solution of our present model. We have the following representation for the Laplace transform

$$\mathscr{L}\{(D_c^{\alpha} f)(t)\} = s^{\alpha} \mathscr{L}\{f(t)\} - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0). \quad (7)$$

In the previous Equation (7), we considered $n = 2$ and applied the formulation of the Laplace transform reported in the literature on fractional calculus [1, 2].

3. Preliminaries Results

This section will be part where we describe we recall the existing results in the literature necessary for our extension. Let the fractional integro-differential be defined by the following equation, that is

$$D^{\alpha} y(t) = A y(t) + \int_{0}^{t} B(t-s) y(s), \text{for } t \in [0, a], \quad (8)$$

$$y(0) = y_0 \in X. \quad (9)$$

where A is a closed densely defined linear operator on a Banach space described by $(X, \|\cdots\|_X)$. Note that for the manupulation of $D(A)$, we will use the norm formulated as the form $\|x\|_Y = \|Ax\|_X + \|x\|_X$ for $x \in X$, and furthermore, it is considered as Banach space represented by $(Y, \|\cdots\|_Y)$. $(B(t))_{t\geq 0}$ is a family of a linear operator on X such that $B(t)$ is continuous from the set Y to the set X for almost all $t \geq 0$. The function $B(t)y$ is considered to be measurable and satisfying to the following inequality:

$$\|B(t)y\|_X \leq b(t) \|y\|_Y, \quad (10)$$

where $y \in Y$ and the function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is locally integrable function. Equation (8) can be found in [22, 23] in context of integer-order derivative. We introduce the following definition concerning an important operator known as the α-resolvent operator, which has been already defined in the literature.

Definition 4 (see [25]). The α-resolvent operator of the fractional integro-differential equation represented in Equation (5) is a bounded linear operator $(\mathscr{R}_{\alpha})_{t\geq 0}$ satisfying the following properties:

(a) The function represented by $\mathscr{R}_{\alpha} : \mathbb{R}^+ \longrightarrow \mathscr{L}(X)$ is strongly continuous and verifies the property that $\mathscr{R}_{\alpha}(0)x = x$ for all $x \in X$ and $\alpha \in (1, 2)$, and furthermore, we have $\|\mathscr{R}_{\alpha}(t)\|_{\mathscr{L}(X)} \leq M e^{\beta t}$, where M and β are constant

(b) for all $x \in D(A)$, $\mathscr{R}_{\alpha}(.)x \in \mathscr{C}([0,+\infty, [D(A)]) \cap \mathscr{C}^1((0,+\infty[, X)$ and the following equations held for every $t \geq 0$

$$\begin{aligned} D^{\alpha} \mathscr{R}_{\alpha}(t)x &= A \mathscr{R}_{\alpha}(t)x + \int_{0}^{t} B(t-s) \mathscr{R}_{\alpha}(s) x(s) ds, \\ D^{\alpha} \mathscr{R}_{\alpha}(t)x &= \mathscr{R}_{\alpha}(t) A x + \int_{0}^{t} B(t-s) \mathscr{R}_{\alpha}(s) x(s) ds. \end{aligned} \quad (11)$$

Definition 5 (see [25]). The existence of the unique α-resolvent operator for our considered fractional problem is possible under the following assumptions

(a) The operator $A : D(A) \subseteq X \longrightarrow X$ is a closed densely defined linear operator on a Banach space $(X, \|.\|_X)$

(b) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constant C_1, such that $A(D) \subseteq D(A)$, $\tilde{B}(\lambda)(D) \subseteq D(A)$, $\|A\tilde{B}(\lambda)y\|_X \le C_1\|y\|_X$ for every $y \in D$ and all $\lambda \in \Omega$, where

$$\Omega = \{\lambda \in \mathbb{C} : \arg(\lambda) \le \theta\}. \tag{12}$$

The detail of the present definition with the function used in its form can be found in the originated paper [25]. There exists a third condition for the existence of the α-resolvent operator but we advise the readers to refer to the paper [25]. The form of the alpha-resolvent operator in our paper will depend on the application of the Laplace transform as described later in the main results of this paper. It is now ready for our problem to be definite and to give the form of the potential mild solution. We now consider the fractional integro-differential equation with a nonlocal initial condition defined by the following equation

$$D^\alpha y(t) = Ay(t) + \int_0^t B(t-s)y(s) + h(t,y), \text{ for } t \in [0,a], \tag{13}$$

$$y(0) - g(y) = y_0 \in X. \tag{14}$$

The first objective will be to determine the form of its analytical solution via the α-resolvent operator. Before, we introduce a new family of fractional operators defined in the following definition.

Definition 6. We consider that $\alpha \in (1,2)$, and then, we represent the family $(\mathcal{S}_\alpha)_{t\ge0}$ for the fractional integro-differential Equation (13) under the condition (14) by the following form:

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\left(\lambda^{\alpha-1}I - A - \widehat{B}(\lambda)\right)^{-1} d\lambda, \tag{15}$$

for each $t \ge 0$ and i is an imaginary number in complex space and Ω being a suitable path.

It is straightforward to see that the new operator $\mathcal{S}_\alpha$ is admitting the upper bound which can be expressed via norm in $\mathcal{L}^p$ space. The upper bound of this operator will be determined and used later in our investigations. Note that the operator defined in Equation (15) will be obtained by the application of the inverse of the Laplace transform.

This section summarized the fundamental results already established in the literature. The main objective of our paper is to prove the existence and uniqueness of at least one mild solution for our considered problem in the introduction section. The second objective will be to study the stability analysis of the solution is Hyers-Ulam stability. Here, note that the Caputo derivative has been used in all investigations.

4. Main Findings

This section will be the part where the findings of the paper have been assigned. The first objective or finding of our paper is to determine the form of solution for fractional integro-differential (1) with a nonlocal initial condition considered in Equation (2). We describe our first finding in the following theorem.

Theorem 7. *We consider that the function $h : [0,b] \times C \longrightarrow X$, then the mild solution of the fractional integro-differential Equation (1) has the following form:*

$$y(t) = R_\alpha(t)(y_0 + g(y)) + \int_0^t S_\alpha(t-s)h(s,y(s))ds, \tag{16}$$

where the α-resolvent operator R_α, and the fractional operator S_α decribed by the following form:

$$\begin{aligned} R_\alpha(t) &= \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha I - A - \widehat{B}(\lambda)\right)^{-1} d\lambda, \\ S_\alpha(t) &= \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\left(\lambda^\alpha I - A - \widehat{B}(\lambda)\right)^{-1} d\lambda. \end{aligned} \tag{17}$$

Proof. The technique of the proof exists in the literature; here, we adapt it for our context. We just replace the initial condition with a nonlocal initial condition. Applying the Laplace transform to both sides of Equation (1), we get the following form

$$\begin{aligned} \lambda^\alpha\bar{y} - \lambda^{\alpha-1}y(0) - \lambda^{\alpha-2}y'(0) &= A\bar{y} + \bar{B}\bar{y} + \bar{h}, \\ \lambda^\alpha\bar{y} - \lambda^{\alpha-1}y(0) &= A\bar{y} + \bar{B}\bar{w} + \bar{h}, \\ \bar{y}\left(\lambda^\alpha I - A - \bar{B}\right) &= \lambda^{\alpha-1}w(0) + \bar{h}, \\ \lambda^{\alpha-1}\left(\lambda^\alpha I - A - \bar{B}\right)^{-1}w(0) + \left(\lambda^\alpha I - A - \bar{B}\right)^{-1}\bar{h} &= \bar{y}. \end{aligned} \tag{18}$$

From the inverse of the Laplace transform, it follows the solution takes the form described in Equation (16), and the α-resolvent operator and the second operator S_α are described by the forms

$$R_\alpha(t) = \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha I - A - \widehat{B}(\lambda)\right)^{-1} d\lambda, \tag{19}$$

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\left(\lambda^{\alpha-1}I - A - \widehat{B}(\lambda)\right)^{-1} d\lambda. \tag{20}$$

In Equations (19) and (20), the term i represents an imaginary variable. This first result is very important because all the findings in this paper come from the solution established in Equation (16). And we can note that the α-resolvent operator $R_\alpha(\cdots)$ and the second operator $S_\alpha(\cdots)$ is obtained via the applications of the inverse of the Laplace transform. □

4.1. Existence Results of the Mild Solution. This section is devoted to providing the existence of at least one mild solution for our fractional problem (1) with a nonlocal initial condition. Let the function $x : [0, b] \longrightarrow X$ according to the previous section, the mild solution of Equation (1) signified here can be expressed like the following form

$$y(t) = R_\alpha(t)(y_0 + g(y)) + \int_0^t S_\alpha(t-s)h(s, y(s))ds, \quad (21)$$

for all $t \in [0, b]$. For the existence and the uniqueness, we make the following assumptions which will be used to prove our finding.

(**H1**) We suppose that the α-resolvent operator R_α, and the fractional operator S_α are both bounded as the following relationships $\|\mathscr{R}_\alpha(t)\|_X \leq M_b$, and $\|\mathscr{S}_\alpha(t)\|_X \leq M_w$ where M_b and M_w are constants.

(**H2**) The function $h(.,y)$ is measurable for all $y \in X$ and satisfies the Caratheodory conditions. The function $h(t,.)$ is continuous with respect to almost all of the first argument with $t \in [0, b]$.

(**H3**) Let the existence of a function $\rho \in L^1([0, b], \mathbb{R}^+)$ and a continuous and nondecreasing function $\varpi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ verifying the relationship that $\|h(t, y)\|_X \leq \rho(t)\varpi(\|y\|)$ for all $t \in [0, b]$ and $y \in X$.

(**H4**) Let the existence of a nondecreasing continuous function defined by $\pi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ verifying for all $y \in \mathscr{C}([0, b], X)$ the condition that $\|g(y)\|_X \leq \pi(\|y\|_\infty)$. And for the rest in our present context, we define the ball as the form $B_r = \{y \in \mathscr{C}([0, b], X): \|y\|_{\mathscr{C}([0,b],X)} \leq r\}$.

(**H5**) Note that as previously mentioned, the function g is a continuous function and furthermore, we add the assumption that $g : \mathscr{C}([0, b], X) \longrightarrow X$ is lipschitz continuous. Let a constant k such that

$$\|g(y_1) - g(y_2)\|_X \leq k\|y_1 - y_2\|_X. \quad (22)$$

We now start proving the existence of at least one mild solution for our problem described by Caputo derivative with the novel initial condition which is a nonlocal condition. We describe one of the findings of the paper in the following theorem.

Theorem 8. *We assume that the assumptions (**H1**), (**H2**), (**H3**), (**H4**), and (**H5**) are hold. Thus, our model described by Equation (1) with nonlocal initial condition Equation (2) has at least one mild solution for all $t \in [0, b]$.*

Proof. The proof of the previous theorem combines the 3 steps described in the present proof. The first step will consist to prove our following operator is well defined.

We use Equation (21) and (**H4**) to define the following operator as the form described as $P : B_r \longrightarrow \mathscr{C}([0, b], X)$ such that

$$(Py)(t) = R_\alpha(t)(y_0 + g(y)) + \int_0^t S_\alpha(t-s)h(s, y(s))ds. \quad (23)$$

□

Procedure 1. The first step consists to prove that $P : B_r \longrightarrow B_r$. The proof is inspired from [22]. Before the proof, we assume the following condition that is hold

$$\max\{M_b, M_w\} \lim_{r\longrightarrow\infty} \left(\frac{\pi(r)}{r} + \frac{\varpi(r)}{r}\int_0^t \rho(s)ds\right) < 1. \quad (24)$$

To arrive at our end, we apply the norm considered in our Banach space, (**H1**), (**H3**), and (**H4**) and we use, we get that the following relationship

$$\begin{aligned}\|(Py)(t)\|_X &\leq \|R_\alpha(t)(y_0 + g(y))\|_X + \left\|\int_0^t S_\alpha(t-s)h(s, y(s))ds\right\|_X \\ &\leq M_b(\|y_0\|_X + \pi(r)) + M_w\varpi(r)\int_0^t \rho(s)ds.\end{aligned} \quad (25)$$

We proceed by contradiction; we want to prove that we can find $r > 0$ such that the following condition holds that is $Py \in B_r$. Now, by contradiction procedure, we suppose that the assumption is not held, and then, for each $r > 0$, we can find $y \in B_r$ satisfying the condition that $Py \notin B_r$. In other procedures, that means

$$r < \|(Py)(t)\|_X \leq M_b(\|y_0\|_X + \pi(r)) + M_w\varpi(r)\int_0^t \rho(s)ds, \quad (26)$$

and then we get the following relationship

$$1 < \max\{M_b, M_w\}\left(\frac{\|y_0\|_X}{r} + \frac{\pi(r)}{r}\right) + \frac{\max\{M_b, M_w\}}{r}\varpi(r)\int_0^t \rho(s)ds. \quad (27)$$

Applying the Infinium by taking $r \longrightarrow \infty$, we arrive to the following relationship

$$1 < \max\{M_b, M_w\} \liminf_{r\longrightarrow\infty} \left(\frac{\pi(r)}{r} + \varpi(r)\int_0^t \rho(s)ds\right). \quad (28)$$

We clearly observe that this Equation (28) contradicts the relation established in Equation (24) and then we conclude that $\|(Py)(t)\|_X \leq r$; thus, that means $Py \in B_r$ for all $y \in B_r$.

Procedure 2. The first step consists to prove that $P : B_r \longrightarrow B_r$ is an continuous function. We use the classical procedure to prove the continuity of the previous function, that is we take numerical suite family $\{(y_n)_{n\geq 0}\}$ included in B_r satisfying the property that $y_n \longrightarrow \bar{y}$ as n converges to ∞. For simplification, we decompose operator (23) in two sub-operator, we denote that

$$P_1 y = R_\alpha(t)(y_0 + g(y)). \quad (29)$$

We apply the considered norm in our Banach space, and we utilize the assumptions for our model in Equation (34), the reader can also refer to (**H1**) and (**H5**), that is, g is a continuous function, then we get the following relationship

$$\|(P_1 y_n - P_1\bar{y})(t)\|_X \le M_b \|(g(y_n) - g(\bar{y}))\|_X. \tag{30}$$

We continue our started decomposition in previous lines by letting that the function P_2 represented as the form

$$P_2 y = \int_0^t S_\alpha(t-s) h(s, y(s)) ds. \tag{31}$$

We apply the considered norm in our Banach space, and we use the assumptions (**H1**) and (**H2**) where it is stipulated that the h is a continuous function, then we get the following relationships:

$$\begin{aligned}\|(P_2 y_n - P_2\bar{y})(t)\|_X &\le \left\| \int_0^t S_\alpha(t-s)(h(s, y_n(s)) - h(s, \bar{y}(s))) ds \right\|_X \\ &\le \int_0^t \|S_\alpha(t-s)\|_X \|(h(s, y_n(s)) - h(s, \bar{y}(s)))\|_X ds \\ &\le M_w \int_0^t \|(h(s, y_n(s)) - h(s, \bar{y}(s)))\|_X ds.\end{aligned} \tag{32}$$

Combining Equation (30) and Equation (32) and using the theorem of convergence known as the Lebesgue-dominated convergence theorem and the continuity of the functions g and h described in the assumptions (**H2**) and (**H5**), we get the following relation

$$\begin{aligned}\|(P y_n - P\bar{y})(t)\|_X &\le M_b \|(g(y_n) - g(\bar{y}))\|_X \\ &\quad + M_w \int_0^t \|(h(s, y_n(s)) - h(s, \bar{y}(s)))\|_X ds \longrightarrow 0 \text{ as } n \longrightarrow \infty,\end{aligned} \tag{33}$$

where $Py = P_1 y + P_2 y$; thus, $\|(Py_n - P\bar{y})(t)\|_X \longrightarrow 0$. That means using the classical procedure that the function $P : B_r \longrightarrow B_r$ is a continuous function.

Procedure 3. This step consists to prove that $\{Py : y \in B_r\}$ is relatively compact. We conserve the decomposition considered in the previous procedures. We begin to prove that the function Py is an equicontinuous function and bounded in the set B_r. We have the following relationships that

$$\begin{aligned}\|(P_1 y(t_2) - P_1 y(t_1))(t)\|_X &= \|R_\alpha(t_2) g(y(t_2)) - R_\alpha(t_1) g(y(t_2))\|_X \\ &\le \|R_\alpha(t_2) - R_\alpha(t_1)\|_X \|g(y(t_2))\|_X \\ &\quad + \|R_\alpha(t_1)\|_X \|g(y(t_2)) - g(y(t_1))\|_X.\end{aligned} \tag{34}$$

Equation (1) needs the uniform continuity of the α-resolvent operator family in the set B_r, which is satisfied because the operator $R_\alpha(.)$ is continuous in a compact set. The same for the function g which is continuous in a compact set. Then, we can observe that when $t_2 \longrightarrow t_1$, then we have that $\|(P_1 y(t_2) - P_1 y(t_1))(t)\|_X \longrightarrow 0$. That is equivalent to the equicontinuity of the function Py_1. We now prove the boundedness of this operator using the assumptions (**H1**) and (**H4**), we have

$$\|P_1 y(t)\|_X = \|R_\alpha(t)(y_0 + g(y(t)))\|_X \le M_b \|y_0\|_X + \pi\left(\|r\|_\infty\right). \tag{35}$$

Finally, we confirm the relative compactness of the first operator Py_1. We continue with the operator Py_2. We have the following relationships

$$\begin{aligned}&\|(P_2 y(t_2) - P_2 y(t_1))(t)\|_X \\ &\le \left\| \int_0^{t_1} S_\alpha(t_2-s) h(s, y(s)) ds - \int_0^{t_1} S_\alpha(t_1-s) h(s, y(s)) ds \right\|_X \\ &\quad + \left\| \int_{t_1}^{t_2} S_\alpha(t_2-s) h(s, y(s)) ds \right\|_X.\end{aligned} \tag{36}$$

Let us consider the following transformation for simplification in the calculations, the first transformation is defined by

$$I_1 = \int_0^{t_1} (S_\alpha(t_2-s) - S_\alpha(t_1-s)) h(s, y(s)) ds. \tag{37}$$

Applying the considered norm in our Banach space, we find the following relationship is satisfied as well, thus

$$\|I_1\|_X \le \sup_{s \in [0, t_1]} \|S_\alpha(t_2-s) - S_\alpha(t_1-s)\| \left\| \int_0^{t_1} h(s, y(s)) ds \right\|. \tag{38}$$

By Equation (38), since the operator $S_\alpha(\cdots)$ is a continuous function, we can observe that when $t_2 \longrightarrow t_1$, then $I_1 \longrightarrow 0$. We repeat the same procedure with a second function that I_2 represented by the form that

$$I_2 = \left\| \int_{t_1}^{t_2} S_\alpha(t_2-s) h(s, y(s)) ds \right\|_X. \tag{39}$$

We adopt the same procedure and using the assumption (**H3**). Applying the classical norm adopted in this work, we get that the following relationship

$$\|I_2\|_X \le M_w \varpi(r) \|t_2 - t_1\|_X. \tag{40}$$

Utilizing Equation (40), we get that $\|(P_2 y(t_2) - P_2 y(t_1))(t)\|_X \longrightarrow 0$ as $t_2 \longrightarrow t_1$, and it forward to see the operator is bounded, and then, we conclude that $P y$ is an equicontinuous and bounded function in the set B_r. It follows from the Ascoli-Arzela theorem that $\{Py : y \in B_r\}$ is relatively compact. And then, we get from the three procedures that using Schauder fixed point theorem the operator P has at least one fixed point in the set B_r, that is, the system defined by Equation (1) has at least one mild solution.

4.2. Hyers-Ulam Stability Analysis. We continue this section with Ulam-Hyers stability analysis of the fractional integro-differential Equation (1) considered in this paper. We give the following definition related to the stability analysis. Note that in the context of fractional calculus, the stability in Hyers and Ulam's sense of the integro-differential equation in the form of Equation (1) under initial condition Equation (2) is novel in the literature.

Definition 9. The integro-fractional differential equation considered in this paper represented in Equation (1) with nonlocal initial condition (2) is Hyers-Ulam stable if we can find positive constant D satisfying the condition represented in the following form: for every λ, if

$$\left\| y(t) - R_\alpha(t)(y_0 + g(y)) + \int_0^t S_\alpha(t-s)h(s, y(s))ds \right\|_X \leq \lambda, \tag{41}$$

there exists y^* verifying the condition that

$$y^*(t) = R_\alpha(t)(y_0 + g(y^*)) + \int_0^t S_\alpha(t-s)h(s, y^*(s))ds, \tag{42}$$

such that

$$\|y(t) - y^*(t)\|_X \leq D\lambda. \tag{43}$$

Theorem 10. *We consider that the assumptions (**H1**), (**H2**), (**H3**), (**H4**), and (**H5**) are hold. Then, the solution of the fractional integro-differential Equation (1) under nonlocal initial condition (2) is Hyers-Ulam stable.*

Proof. In this part, we will be helped by the results established in the previous theorem. We first consider that y be the solution of the fractional integro-differential Equation (1) under nonlocal initial condition (2) and y^* be any other approximate solution of our considered fractional differential equation. Applying the considered norm in our present paper and with the aid of assumptions (**H1**), (**H2**), (**H3**), (**H4**), and (**H5**), from which we get in particular the Lipschitz continuous condition for the function g, we get the following relationships:

$$\begin{aligned} \|y_1 - y_1^*\|_X &= \|R_\alpha(t)g(y_1) - R_\alpha(t)g(y_1^*)\|_X \\ &\leq \|R_\alpha(t)\|_X \|g(y_1) - g(y_1^*)\|_X \\ &\leq M_b k \sup_{0\leq s\leq b} \|y_1(s) - y_1^*(s)\|_X. \end{aligned} \tag{44}$$

Note that we use decomposition of the solution y in two subsolution. We repeat the previous procedure for the second part

$$\begin{aligned} \|y_2 - y_2^*\|_X &= \left\| \int_0^t S_\alpha(t-s)h(s, y(s))ds - \int_0^t S_\alpha(t-s)h(s, y^*(s))ds \right\|_X \\ &\leq \|S_\alpha(t)\|_X \int_0^t \|h(s, y(s)) - h(s, y^*(s))\|_X ds. \end{aligned} \tag{45}$$

Under the second assumption $(H2)$ that we have the following relationship using the previous equation

$$\|y_2 - y_2^*\|_X \leq M_w L_{\rho(b)} \sup_{0\leq s\leq b} \|y_1(s) - y_1^*(s)\|_X. \tag{46}$$

Combining Equation (44) and Equation (46) and removing the indices, we get the following relationship for the stability analysis condition that is

$$\|y - y^*\|_X \leq \left[M_b k + M_w L_{\rho(b)}\right] \sup_{0\leq s\leq b} \|y(s) - y^*(s)\|_X. \tag{47}$$

The stability in sense of Hyers-Ulam for the solution of the fractional integro-differential Equation (1) with the nonlocal initial condition (2) follows by supposing that $D = M_b k + M_w L_{\rho(b)}$. □

5. Illustrative Example

In the present part, we give an illustrative example to illustrate the findings in this paper. The integer-order version of the example addressed in this can be found in the literature. Let the fractional integro-differential equation described by the Caputo derivative and represented as the following form

$$D_t^\alpha w(y, t) = Aw(y, t) + \int_0^t \beta e^{-\kappa(t-s)} Aw(y, s) + m_1(t) m_2(w(y, t)), \tag{48}$$

$$w(0, t) = w(2\pi, t), \tag{49}$$

$$w_0(y) = w(y, 0) + \int_0^{2\pi} k(y, s) w(0, s) ds, \tag{50}$$

where $t \in I$, $y \in [0, 2\pi]$, the function $k : [0, 2\pi] \times I \longrightarrow \mathbb{R}^+$ is a continuous function satisfying the condition that $k(2\pi, t) = 0$, c is positive constant and, furthermore, κ and β verify the condition that $-\kappa \leq \beta \leq 0 \leq \kappa$. Furthermore, in our modeling, the operator A is represented as the following form that

$$(Aw)(y, t) = a_1(y) \frac{\partial^2}{\partial y^2} w(y, t) + b_1(y) \frac{\partial}{\partial y} w(y, t) + c_1(y) w(y, t), \tag{51}$$

with the functions a_1, b_1, and c_1 verifying the usual uniform ellipticity conditions. In this present section, the set $D(A)$ can be represented as the form $D(A) = \{v \in X : v', v'' \in X, v(0) = v(2\pi)\}$. The objective is to rewrite the present Equation (49) in the form described by our fractional integro-differential Equation (1). Let the function $h : I \times [0, b] \longrightarrow I$ represented by $h(w, t) = m_1(t) m_2(w(y, t))$, where the function $m_1(.)$ is integrable function and in particular uniformly continuous and the function $m_2(\cdots)$ is supposed in our model to be Lipschitz continuous with constant L. Under these properties, we will prove our function h is also Lipschitz continuous and therefore satisfies the assumption (**H2**).

$$\|h(w_1,t)-h(w_1,t)\|_X \le \|m_1(t)\|\|m_2(w_1,t)-m_2(w_2,t)\|_X \le \|m_1(t)\|L\|w_1-w_2\|_X. \quad (52)$$

Note that Equation (52) is obtained by using the Lipschitz continuous of the function $m_2(\cdots)$. Note that from the condition of the Lipschitz continuous the following relationship is hold

$$\|h(w,t)\|_X \le \|m_1(t)\|[m_2(0)+L\|w\|_X], \quad (53)$$

by supposing that $\varpi(\|w\|)=m_2(0)+L\|w\|_X$, then the assumption (**H1**)-(**H2**) is held. We consider that the function $g(w)=\int_0^{2\pi}k(y,s)w(0,s)ds$. We try to prove the asumptions (**H3**)-(**H5**) are held. Applying the norm and using the Poincare inequality, we get the following relationships

$$\|g(w)\|_X \le \left[\int_0^{2\pi}k(y,s)ds\right]^{1/2}\left[\int_0^{2\pi}w(0,s)ds\right]^{1/2} \le \left[\int_0^{2\pi}k(y,s)ds\right]^{1/2}\|w\|^{1/2}, \quad (54)$$

and then the conditions established in (**H3**)-(**H5**) are held. Finally, we can affirm that Equation (49) can be rewritten as the form

$$D^\alpha w = Aw + \int_0^t B(t-s)w(s) + h(t,w), \text{ for } t \in I, \quad (55)$$

$$w(0)-g(w)=w_0 \in X = L^2([0,2\pi],\mathbb{R}), \quad (56)$$

where $B(t)=\beta e^{-\kappa t}A$. Then, as all assumptions are satisfied, thus, we can conclude about the existence of at least one mild solution for our fractional integro-differential Equation (1).

It is hard in the present form to see the impact of the fractional-order derivative. To solve this problem, we have just determine and prove the resolvents operators are bounded. Let the present context form of the solution of fractional differential equation described in Equation (49) be written as the following form:

$$w(y,t)=R_\alpha(t)(w_0+g(w))+\int_0^t S_\alpha(t-s)h(s,w(s))ds. \quad (57)$$

where the α-resolvent operator R_α, and S_α are the resolvents operators. In the present example, the α-resolvant operator R_α can be expressed as the following form:

$$R_\alpha(t)=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^{\alpha-1}I-A-\widehat{B}(\lambda)\right)^{-1}d\lambda. \quad (58)$$

It is obtained via the application of the inverse of the Laplace transform. Note that applying the Laplace transform to both sides of Equation (55), we get the following form:

$$\begin{aligned}
\lambda^\alpha\bar{w}-\lambda^{\alpha-1}w(0)-\lambda^{\alpha-2}w'(0)&=A\bar{w}+\bar{B}\bar{w}+\bar{h},\\
\lambda^\alpha\bar{w}-\lambda^{\alpha-1}w(0)&=A\bar{w}+\bar{B}\bar{w}+\bar{h},\\
\bar{w}\left(\lambda^\alpha I-A-\bar{B}\right)&=\lambda^{\alpha-1}w(0)+\bar{h},\\
\lambda^{\alpha-1}\left(\lambda^\alpha I-A-\bar{B}\right)^{-1}w(0)+\left(\lambda^\alpha I-A-\bar{B}\right)^{-1}\bar{h}&=\bar{w}.
\end{aligned} \quad (59)$$

From the inverse of the Laplace transform, it follows that the solution of Equation (55) under Equation (56) takes the form described in Equation (57) and the α-resolvant operator the form described in Equation (58); furthermore, the second operator S_α should be described by the form

$$S_\alpha(t)=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\left(\lambda^{\alpha-1}I-A-\widehat{B}(\lambda)\right)^{-1}d\lambda, \quad (60)$$

where in our context we have that $\widehat{B}(\lambda)=\beta A/\lambda+\kappa$. The objective to prove the impact of the fractional-order is to prove the previous operators R_α and S_α are bounded and then all assumptions will be held. Let A be a sectiorial operator in $\Omega(\gamma,\theta)$. The first remark is that the function $(\lambda^{\alpha-1}I-A-\widehat{B}(\lambda))^{-1}$ is an inverse continuous function and then there is positive constant M_b such that $\|\lambda^{\alpha-1}(\lambda^\alpha I-A-\widehat{B}(\lambda))^{-1}\|\le M_b/\lambda$, seen in [24]. Following the same investigations, we have the following relationships:

$$\|R_\alpha(t)\|=\left\|\frac{1}{2\pi i}\int_\Omega e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^{\alpha-1}I-A-\widehat{B}(\lambda)\right)^{-1}d\lambda\right\| \le M_b\left\|\frac{1}{2\pi i}\int_\Omega\lambda^{-1}e^{\lambda t}d\lambda\right\|=M_b\times 1,\le M_b. \quad (61)$$

We continue by finding the upper bound for the operator S_α; we have the following relationships:

$$\|S_\alpha(t)\|=\left\|\frac{1}{2\pi i}\int_\Omega e^{\lambda t}\left(\lambda^{\alpha-1}I-A-\widehat{B}(\lambda)\right)^{-1}d\lambda\right\| \le M_b\left\|\frac{1}{2\pi i}\int_\Omega\lambda^{-\alpha}e^{\lambda t}d\lambda\right\|=M_w. \quad (62)$$

The impact of the order of the fractional derivative can be noticed in the upper bound M_w which depends on the order of the Caputo fractional derivative. Then, we conclude that the present problem satisfies the assumptions (**H1**), (**H2**), (**H3**), (**H4**), and (**H5**), and then, it has at least one mild solution.

Except for the applications of the fractional integro-differential to heat equation as previously provided, the fractional integro-differential equations can also be applied in modeling fractional epidemic models, modeling physics phenomena, stochastic modeling, and others see the following investigations [30, 32].

6. Conclusion

In this paper, we have studied new results related to the fractional integro-differential equation described by the Caputo derivative. We have proved the existence of at least one mild solution for the considered fractional integro-differential equation. We have introduced the notion of Hyers-Ulam stability not previously mentioned in the literature with the fractional integro-differential equation under consideration in this paper. The main results of this paper have been illustrated with fractional heat equations which have many applications in real-world problems. This new paper contributes to the applications of fractional calculus in real-world problems. The problem of the existence and uniqueness of fractional integro-differential equations with local and nonlocal initial conditions constitutes a good challenge for future investigations; there exist many types of differential equations in this direction where the existence of a mild solution has not been proved. The present investigations will open new doors for future works. The stability analysis adopted in this paper will open news directions of investigations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest.

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