Research Article

A Comparative Study of Fractional-Order Diffusion Model within Atangana-Baleanu-Caputo Operator

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We provided two different approaches for solving fractional-order diffusion equations in this article. The fractional Atangana-Baleanu derivative operator in addition to the Laplace transform is used to generate several new approximate-analytical solutions to the time-fractional diffusion equations. The implementation of a sophisticated and straightforward approach to solving diffusion equations having a fractional-order derivative is the motivation and uniqueness behind the current work. The solutions to some illustrative problems are calculated to ensure that the actual and approximate solutions to the targeted problems are in close contact. The results we obtained have a higher rate of convergence and provide a closed-form solution, according to analysis. The proposed method's key advantage is the small amount of calculations required. The suggested techniques can be applied to nonlinear fractional-order problems in a variety of applied science areas due to their simple and straightforward implementation. It can be used to overcome specific fractional-order physical problems in a variety of fields of applied sciences.

1. Introduction

Fractional calculus (FC) is a branch of calculus that is an extension of ordinary calculus. In fact, the origins of FC may be traced back over 300 years. The idea of fractional calculus comes from Leibniz's usage of the notation $d^n y/dx^n$ in his publication for the nth derivative. L'Hopital asked Leibniz about the derivative of order 1/2 in 1695 [1]. In nature, integer-order derivatives are local, whereas fractional-order derivatives are nonlocal. Several mathematicians, including Iqbal et al. [2], Alesemi et al. [3], Riemann [4], Alesemi et al. [5], and Podlubny [6], collaborated to establish the mathematical framework for fractional-order derivatives. Later on, a number of mathematicians focused their efforts on this topic. Because of its potential applications in several scientific fields such as biology, viscoelasticity, engineering, fluid mechanics, and other branches of science, fractional calculus has attracted a lot of attention [7–10].

Mathematicians used the method proposed above to solve various fractional differential equations (FDEs) in general, as well as fractional partial differential equations (FPDEs). FPDEs are regarded to be one of the most basic mathematical tools for modelling many physical phenomena with greater accuracy than integer-order. Different phenomena in engineering and applied sciences are determined by nonlinear FPDEs. Nonlinear FPDEs are useful tools in a variety of fields, including material sciences, physics, fluid dynamics, chemistry, chemical kinetics, and a variety of other physical processes [11–15]. In literature, different methods are found for solving differential equations having fractional-order such as the modified Adomian decomposition method (MADM) [16], variational iteration method [17], optimal homotopy asymptotic method (OHAM) [18], differential transform method (DTM) [19], and homotopy perturbation method (HPM) [20–22].

In this paper, we used the Atangana-Baleanu fractional derivative operator [23–26] and the Laplace transform to

solve fractional-order diffusion equations using two analytical methodologies. The first methodology shows how the Laplace transform may be used to approximate the solution of a nonlinear differential using Adomian's decomposition method. The Laplace decomposition method has been demonstrated to work with a wide range of physical problems and has been used for a large number of functional equations [27, 28]. The Laplace Variational Iteration Approach (LVIM), which combines the Laplace transform (LT) and the variational iteration method, is the second approach. Lagrange multiplies are chosen to solve nonlinear equations in this method based on the nature of the problem [29, 30]. He was the first to introduce VIM [31], a powerful method for solving a wide range of issues in applied sciences [32–34]. This method of solving linear and nonlinear equations has lately gained popularity among scholars. Unlike other methodologies, VIM does not require discretization, linearization, or perturbation.

We use LTDM and VITM to solve diffusion equations of the form in this article.

(1) Fractional diffusion equation in one dimension as

$$\frac{\partial^\rho \mu}{\partial \kappa^\rho} = \frac{\partial^2 \mu}{\partial \kappa^2} - \frac{\partial \mu}{\partial \kappa} + \mu \frac{\partial^2 \mu}{\partial \kappa^2} - \mu^2 + \mu, 0 < \rho \leq 1, \vartheta > 0, \quad (1)$$

with initial source

$$\mu(\kappa, 0) = e^\kappa \quad (2)$$

(2) Fractional diffusion equation in one dimension as

$$\frac{\partial^\rho \mu}{\partial \kappa^\rho} = \frac{\partial^2 \mu}{\partial \kappa^2} + \mu \frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu, 0 < \rho \leq 1, \vartheta > 0, \quad (3)$$

with initial source

$$\mu(\kappa, 0) = 1 + e^\kappa \quad (4)$$

(3) Fractional diffusion equation in three dimensions as

$$\frac{\partial^\rho \mu}{\partial \kappa^\rho} = \frac{\partial^2 \mu}{\partial \kappa^2} + \frac{\partial^2 \mu}{\partial \ell^2} + \frac{\partial^2 \mu}{\partial \psi^2}, \quad 0 < \rho \leq 1, \vartheta \geq 0, \quad (5)$$

with initial source

$$\mu(\kappa, \ell, \psi, 0) = \sin \kappa \sin \ell \sin \psi. \quad (6)$$

The diffusion equation is the partial differential equation. The movement of atoms or molecules from a location of higher concentration to a region of lower concentration is referred to as diffusion. Adolf Fick, a physiologist, pro-posed Fick's law of diffusion in 1885. Fick's second law was later renamed the diffusion equation. Slow diffusion, diffusion-wave hybrid, classical diffusion, and the classical wave equation [35] were developed as generalisations of classical diffusion and wave equations. Diffusion equation applications include filtration, electromagnetism, electrochemistry, phase transitions, cosmology, biochemistry, acoustics, and biological group dynamics [36–38].

2. Preliminaries

In this part, we discussed some basic definitions of fractional calculus.

Definition 1. The fractional derivative in Caputo manner as where

$$^{LC}D_\vartheta^\rho\{g(\vartheta)\} = \frac{1}{(n-\rho)} \int_0^\vartheta (\vartheta - k)^{n-\rho-1} g^n(k) dk, \quad (7)$$
$$n < \rho \leq n + 1.$$

Definition 2. The Laplace transform in connection with Caputo derivative $^{LC}D_\vartheta^\rho\{g(\vartheta)\}$ is given as

$$\mathbb{L}\{^{LC}D_\vartheta^\rho\{g(\vartheta)\}\}(\nu)$$
$$= \frac{1}{\nu^{n-\rho}} \left[\nu^n \mathbb{L}\{g(\kappa, \vartheta)\}(\nu) - \nu^{n-1} g(\kappa, 0) - \cdots - g^{n-1}(\kappa, 0)\right]. \quad (8)$$

Definition 3. The Atangana-Baleanu derivative in Caputo manner is as

$$^{ABC}D_\vartheta^\rho\{g(\vartheta)\} = \frac{A(\rho)}{1-\rho} \int_a^\vartheta g'(k) E_\rho\left[-\frac{\rho}{1-\rho}(1-k)^\rho\right] dk, \quad (9)$$

where $A(\gamma)$ represent normalization function with $A(0) = A(1) = 1$, $g \in H^1(a, b)$, $b > a$, $\rho \in [0, 1]$ and E_γ illustrates the Mittag-Leffler function.

Definition 4. The Atangana-Baleanu derivative in Riemann-Liouville manner is as

$$^{ABC}D_\vartheta^\rho\{g(\vartheta)\} = \frac{A(\rho)}{1-\rho} \frac{d}{d\vartheta} \int_a^\vartheta g(k) E_\rho\left[-\frac{\gamma}{1-\rho}(1-k)^\rho\right] dk. \quad (10)$$

Definition 5. The Laplace transform connected with the Atangana-Baleanu operator is given as

$$^{AB}D_\vartheta^\rho\{g(\vartheta)\}(\nu) = \frac{A(\gamma)\nu^\rho \mathbb{L}\{g(\vartheta)\}(\nu) - \nu^{\rho-1} g(0)}{(1-\rho)(\nu^\rho + \rho/1 - \gamma)}. \quad (11)$$

Definition 6. Consider $0 < \rho < 1$, and g is a function of ρ, then the fractional integral operator of ρ is as

$$^{ABC}I_\vartheta^\rho\{g(\vartheta)\} = \frac{1-\rho}{A(\rho)}g(\vartheta) + \frac{\rho}{A(\rho)\Gamma(\rho)}\int_a^\vartheta g(k)(\vartheta - k)^{\rho-1}dk.$$

$$(12)$$

3. Idea of LTDM

Here, we discuss the general methodology of LTDM for solving the fractional partial differential equation:

$$D_\vartheta^\rho\mu(\kappa, \vartheta) + \bar{\mathscr{G}}_1(\kappa, \vartheta) + \mathscr{N}_1(\kappa, \vartheta) = \mathscr{F}(\kappa, \vartheta), \quad 0 < \rho \le 1,$$

$$(13)$$

with initial conditions

$$\mu(\kappa, 0) = \phi(\kappa), \quad \frac{\partial}{\partial\vartheta}\mu(\kappa, 0) = \zeta(\kappa),$$

$$(14)$$

where $D_\vartheta^\rho = \partial^\rho/\partial\vartheta^\rho$ represents fractional AB operator of order ρ. $\bar{\mathscr{G}}_1, \mathscr{N}_1$ are linear and nonlinear terms, and $\mathscr{F}(\kappa, \vartheta)$ demonstrates the source term.

On applying Laplace transform to (13),

$$L\left[D_\vartheta^\rho\mu(\kappa, \vartheta) + \bar{\mathscr{G}}_1(\kappa, \vartheta) + \mathscr{N}_1(\kappa, \vartheta)\right] = L[\mathscr{F}(\kappa, \vartheta)]. \quad (15)$$

By the Laplace differentiation property, we get

$$L[\mu(\kappa, \vartheta)] = \Theta(\kappa, \nu) - \frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}L\left[\bar{\mathscr{G}}_1(\kappa, \vartheta) + \mathscr{N}_1(\kappa, \vartheta)\right],$$

$$(16)$$

where $\Theta(\kappa, \nu) = 1/\nu^{\rho+1}[\nu^\rho g_0(\kappa) + \nu^{\rho-1}g_1(\kappa) + \cdots + g_1(\kappa)] + \nu^\rho + \rho(1-\rho)/\nu^\rho\mathscr{F}(\kappa, \vartheta)$.

On taking Laplace inverse transform,

$$\mu(\kappa, \vartheta) = \Theta(\kappa, \nu) - L^{-1}\left\{\frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}L\left[\bar{\mathscr{G}}_1(\kappa, \vartheta) + \mathscr{N}_1(\kappa, \vartheta)\right]\right\},$$

$$(17)$$

where $\Theta(\kappa, \nu)$ represent the term coming from the source term. Let $\mu(\kappa, \vartheta)$ have series form solution as

$$\mu(\kappa, \vartheta) = \sum_{m=0}^\infty \mu_m(\kappa, \vartheta), \quad (18)$$

and the decomposition of nonlinear term $\mathscr{N}_1$ is as

$$\mathscr{N}_1(\kappa, \vartheta) = \sum_{m=0}^\infty \mathscr{A}_m, \quad (19)$$

where $\mathscr{A}_m$ represents Adomian polynomials as

$$\mathscr{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathscr{N}_1\left(\sum_{k=0}^\infty \ell^k\kappa_k, \sum_{k=0}^\infty \ell^k\vartheta_k\right)\right\}\right]_{\ell=0}. \quad (20)$$

By substituting Equations (18) and (20) into (17), we get

$$\sum_{m=0}^\infty \mu_m(\kappa, \vartheta) = \Theta(\kappa, \nu) - L^{-1}\left\{\frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}L \right.$$
$$\left. \cdot \left[\bar{\mathscr{G}}_1\left(\sum_{m=0}^\infty \kappa_m, \sum_{m=0}^\infty \vartheta_m\right) + \sum_{m=0}^\infty \mathscr{A}_m\right]\right\}.$$

$$(21)$$

The following terms are described:

$$\mu_0(\kappa, \vartheta) = \Theta(\kappa, \nu),$$

$$\mu_1(\kappa, \vartheta) = L^{-1}\left\{\frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}L\left[\bar{\mathscr{G}}_1(\kappa_0, \vartheta_0) + \mathscr{A}_0\right]\right\}.$$

$$(22)$$

Thus, all the terms for $m \ge 1$ are determined as

$$\mu_{m+1}(\kappa, \vartheta) = L^{-1}\left\{\frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}L\left[\bar{\mathscr{G}}_1(\kappa_m, \vartheta_m) + \mathscr{A}_m\right]\right\}.$$

$$(23)$$

4. VITM Formulation

Here, we discuss the general methodology of VITM for solving a fractional partial differential equation:

$$D_\vartheta^\rho\mu(\kappa, \vartheta) + \mathscr{M}\mu(\kappa, \vartheta) + \mathscr{N}\mu(\kappa, \vartheta) - \mathscr{P}(\kappa, \vartheta) = 0, \quad m-1 < \rho \le m,$$

$$(24)$$

with initial source

$$\mu(\kappa, 0) = g_1(\kappa), \quad (25)$$

where $D_\vartheta^\rho = \partial^\rho/\partial\vartheta^\rho$ represents fractional AB operator. $\mathscr{M}, \mathscr{N}$ are linear and nonlinear terms, and $\mathscr{P}$ demonstrates the source term.

On applying the Laplace transform to (24),

$$L\left[D_\vartheta^\rho\mu(\kappa, \vartheta)\right] + L[\mathscr{M}\mu(\kappa, \vartheta) + \mathscr{N}\mu(\kappa, \vartheta) - \mathscr{P}(\kappa, \vartheta)] = 0.$$

$$(26)$$

By the Laplace differentiation property, we get

$$L[\mu(\kappa, \vartheta)] = \frac{\nu^\rho}{\nu^\rho + \rho(1-\rho)}L[\mathscr{M}\mu(\kappa, \vartheta) + \mathscr{N}\mu(\kappa, \vartheta) - \mathscr{P}(\kappa, \vartheta)].$$

$$(27)$$

Thus, for Equation (27), the iteration technique is

$$\mu_{m+1}(\kappa, \vartheta) = \mu_m(\kappa, \vartheta) + \rho(\nu)\left[\frac{\nu^\rho}{\nu^\rho + \rho(1-\rho)}L[\mathscr{M}\mu(\kappa, \vartheta) \right.$$
$$\left. + \mathscr{N}\mu(\kappa, \vartheta) - \mathscr{P}(\kappa, \vartheta)]\right].$$

$$(28)$$

$\rho(\nu)$ is Lagrange multiplier and

$$\rho(\nu) = -\frac{\nu^\rho + \rho(1-\rho)}{\nu^\rho}. \tag{29}$$

On taking Laplace inverse transform, the series form solution for Equation (28) is

$$\mu_0(\kappa, \vartheta) = \mu(0) + L^{-1}[\rho(s)L[-\mathscr{P}(\kappa, \vartheta)]],$$

$$\mu_1(\kappa, \vartheta) = L^{-1}[\rho(s)L[\mathscr{M}\mu(\kappa, \vartheta) + \mathscr{N}\mu(\kappa, \vartheta)]],$$

$$\vdots$$

$$\mu_{n+1}(\kappa, \vartheta) = L^{-1}[\rho(s)L[\mathscr{M}[\mu_0(\kappa, \vartheta) + \mu_1(\kappa, \vartheta) + \cdots, \mu_n(\kappa, \vartheta)]] + \mathscr{N}[\mu_0(\kappa, \vartheta) + \mu_1(\kappa, \vartheta), \cdots, \mu_n(\kappa, \vartheta)]]. \tag{30}$$

5. Applications

The solutions to different time-fractional diffusion equations are derived using LTDM and VITM in this section.

5.1. Example. Consider fractional diffusion equation in one dimension as

$$\frac{\partial^\rho \mu}{\partial \kappa^\rho} = \frac{\partial^2 \mu}{\partial \kappa^2} - \frac{\partial \mu}{\partial \kappa} + \mu\frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu \quad 0 < \rho \le 1, \quad \vartheta > 0, \tag{31}$$

with initial source

$$\mu(\kappa, 0) = e^\kappa. \tag{32}$$

On applying the Laplace transform to (31),

$$\frac{\nu^\rho L[\mu(\kappa, \vartheta)] - \nu^{-1}\mu(\kappa, 0)}{\nu^\rho + \rho(1-\nu^\rho)} = L\left[\frac{\partial^2 \mu}{\partial \kappa^2} - \frac{\partial \mu}{\partial \kappa} + \mu\frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu\right]. \tag{33}$$

On taking Laplace inverse transform,

$$\mu(\kappa, \vartheta) = \exp(\kappa) + L^{-1}\left[\frac{\nu^\rho + \rho(1-\nu^\rho)}{\nu^\rho}L\left[\frac{\partial^2 \mu}{\partial \kappa^2} - \frac{\partial \mu}{\partial \kappa} + \mu\frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu\right]\right]. \tag{34}$$

Let $\mu(\kappa, \vartheta)$ have series form solution as

$$\mu(\kappa, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \vartheta), \tag{35}$$

where $\mu(\mu)_\kappa = \sum_{k=0}^{\infty} \mathscr{A}_k$ and $\mu^2 = \sum_{k=0}^{\infty} \mathscr{B}_k$ illustrates Adomian polynomials that show nonlinear terms; thus, on putting certain terms in Equation (34), we get

$$\sum_{m=0}^{\infty} \mu_m(\kappa, \vartheta) = \exp(\kappa) + L^{-1}\left[\frac{\nu^\rho + \rho(1-\nu^\rho)}{\nu^\rho}\right.$$
$$\left. \cdot L\left[\frac{\partial^2 \mu}{\partial \kappa^2} - \frac{\partial \mu}{\partial \kappa} + \sum_{k=0}^{\infty} \mathscr{A}_k - \sum_{k=0}^{\infty} \mathscr{B}_k + \mu\right]\right]. \tag{36}$$

The nonlinear term decomposition by means of Adomian polynomials is given in Equation (20) as

$$\mathscr{A}_0 = \mu_0(\mu_0)_\kappa, \mathscr{A}_1 = \mu_1(\mu_0)_\kappa + \mu_0(\mu_1)_\kappa,$$
$$\mathscr{A}_2 = \mu_2(\mu_0)_\kappa + \mu_1(\mu_1)_\kappa + \mu_0(\mu_2)_\kappa,$$

$$\mathscr{B}_0 = \mu_0^2, \mathscr{B}_1 = 2\mu_0\mu_1, \mathscr{B}_2 = 2\mu_0\mu_2 + (\mu_2)^2. \tag{37}$$

By the comparison of Equation (36), both sides

$$\mu_0(\kappa, \vartheta) = e^\kappa. \tag{38}$$

For $m = 0$,

$$\mu_1(\kappa, \vartheta) = e^\kappa\left[\frac{\rho\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)\right]. \tag{39}$$

For $m = 1$,

$$\mu_2(\kappa, \vartheta) = e^\kappa\left[\frac{\rho^2\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 2\rho(1-\rho)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^2\right]. \tag{40}$$

For $m = 2$,

$$\mu_3(\kappa, \vartheta) = e^\kappa\left[\frac{\rho^3\vartheta^{3\rho}}{\Gamma(3\rho+1)} + 3\rho^2(1-\rho)\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}\right.$$
$$\left. + 3\rho(1-\rho)^2\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^3\right]. \tag{41}$$

The series form solution is

$$\mu(\kappa, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \vartheta)$$
$$= \mu_0(\kappa, \vartheta) + \mu_1(\kappa, \vartheta) + \mu_2(\kappa, \vartheta) + \mu_3(\kappa, \vartheta) + \cdots,$$

$$\mu(\kappa, \vartheta) = e^\kappa + e^\kappa\left[\frac{\rho\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)\right] + e^\kappa\left[\frac{\rho^2\vartheta^{2\rho}}{\Gamma(2\rho+1)}\right.$$
$$+ 2\rho(1-\rho)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^2\right] + e^\kappa\left[\frac{\rho^3\vartheta^{3\rho}}{\Gamma(3\rho+1)}\right.$$
$$+ 3\rho^2(1-\rho)\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 3\rho(1-\rho)^2\frac{\vartheta^\rho}{\Gamma(\rho+1)}$$
$$\left. + (1-\rho)^3\right] + \cdots. \tag{42}$$

Thus, we get the solution at $\rho = 1$ as $\mu(\kappa, \vartheta) = e^{(\kappa+\vartheta)}$.
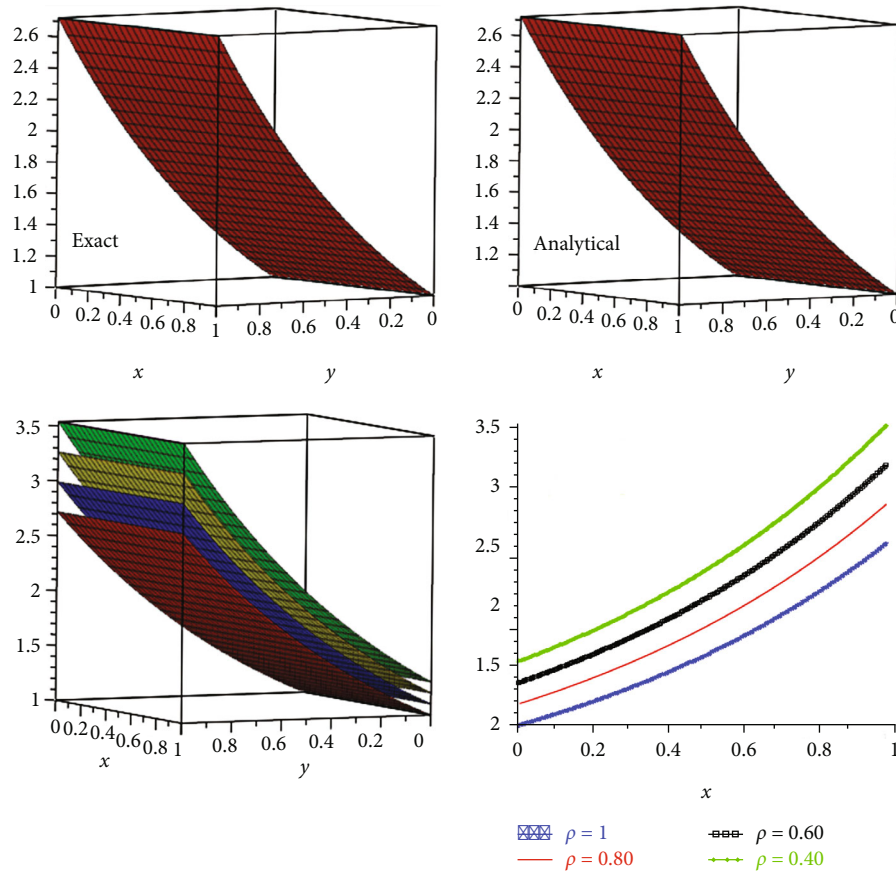
FIGURE 1: The graphical layout for problem 1.

The analytical solution is by VITM.

In the iterative formula for Equation (31), we get

$$\mu_{m+1}(\kappa, \vartheta) = \mu_m(\kappa, \vartheta) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L\left\{\frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)}\right.\right.$$
$$\left.\left. \cdot \frac{\partial^2 \mu_m}{\partial \kappa^2} - \frac{\partial \mu_m}{\partial \kappa} + \mu_m \frac{\partial \mu_m}{\partial \kappa} - \mu_m^2 + \mu_m\right\}\right],$$

$$(43)$$

where

$$\mu_0(\kappa, \vartheta) = e^\kappa. \qquad (44)$$

For $m = 0, 1, 2, \cdots$,

$$\mu_1(\kappa, \vartheta) = \mu_0(\kappa, \vartheta) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L\left\{\frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)}\frac{\partial^2 \mu_0}{\partial \kappa^2} - \frac{\partial \mu_0}{\partial \kappa} + \mu_0\frac{\partial \mu_0}{\partial \kappa} - \mu_0^2 + \mu_0\right\}\right],$$
$$\mu_1(\kappa, \vartheta) = e^\kappa + e^\kappa\left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)\right],$$

$$\mu_2(\kappa, \vartheta) = \mu_1(\kappa, \vartheta) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L\left\{\frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)}\frac{\partial^2 \mu_1}{\partial \kappa^2} - \frac{\partial \mu_1}{\partial \kappa} + \mu_1\frac{\partial \mu_1}{\partial \kappa} - \mu_1^2 + \mu_1\right\}\right],$$
$$\mu_2(\kappa, \vartheta) = e^\kappa + e^\kappa\left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)\right] + e^\kappa\left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho+1)} + 2\rho(1 - \rho)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)^2\right],$$

$$\mu_3(\kappa, \vartheta) = \mu_2(\kappa, \vartheta) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L\left\{\frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)}\frac{\partial^2 \mu_2}{\partial \kappa^2} - \frac{\partial \mu_2}{\partial \kappa} + \mu_2\frac{\partial \mu_2}{\partial \kappa} - \mu_2^2 + \mu_2\right\}\right],$$

$$\mu_3(\kappa, \vartheta) = e^\kappa + e^\kappa\left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)\right] + e^\kappa\left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho+1)} + 2\rho(1 - \rho)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)^2\right]$$
$$+ e^\kappa\left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho+1)} + 3\rho^2(1 - \rho)\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 3\rho(1 - \rho)^2\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)^3\right],$$

$$\mu(\kappa, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \vartheta) = e^\kappa + e^\kappa\left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)\right]$$
$$+ e^\kappa\left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho+1)} + 2\rho(1 - \rho)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)^2\right]$$
$$+ e^\kappa\left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho+1)} + 3\rho^2(1 - \rho)\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 3\rho(1 - \rho)^2\frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1 - \rho)^3\right] + \cdots.$$

$$(45)$$

Thus, we get the solution at $\rho = 1$ as $\mu(\kappa, \vartheta) = e^{(\kappa + \vartheta)}$.

At $\rho = 1$, the graphs in Figure 1 demonstrate the exact and approximate solutions in (AB fractional derivative) sense. For example 1, Figure 1 demonstrates our method's solution for various fractional orders of $\rho = 1, 0.8, 0.6, 0.4$ and $0 \leq \kappa \geq 1$. Our result is in good agreement with the actual solution, as shown by the graphical illustration. Table 1 also shows a comparison of the exact solution and our approaches solution at various fractional orders using absolute error.

TABLE 1: Comparison of the exact and suggested method solution with the aid of absolute errors for problem 1.

$\vartheta = 0.01$	Exact solution	Our method solution	AE of our methods	AE of our methods	AE of our methods
κ	$\rho = 1$	$\rho = 1$	$\rho = 1$	$\rho = 0.9$	$\rho = 0.8$
0	1.001000500000000	1.001000500000000	$0.0000000000E + 00$	$7.3650000000E - 06$	$7.6109000000E - 05$
0.1	1.106276642000000	1.106276642000000	$0.0000000000E + 00$	$8.1390000000E - 06$	$8.4113000000E - 05$
0.2	1.222624772000000	1.222624771000000	$1.0000000000E - 09$	$8.9950000000E - 06$	$9.2959000000E - 05$
0.3	1.351209342000000	1.351209342000000	$0.0000000000E + 00$	$9.9410000000E - 06$	$1.0273600000E - 04$
0.4	1.493317268000000	1.493317269000000	$1.0000000000E - 09$	$1.0988000000E - 05$	$1.1354200000E - 04$
0.5	1.650370817000000	1.650370817000000	$0.0000000000E + 00$	$1.2142000000E - 05$	$1.2548200000E - 04$
0.6	1.823941831000000	1.823941830000000	$1.0000000000E - 09$	$1.3419000000E - 05$	$1.3867800000E - 04$
0.7	2.015767467000000	2.015767467000000	$0.0000000000E + 00$	$1.4831000000E - 05$	$1.5326400000E - 04$
0.8	2.227767583000000	2.227767582000000	$1.0000000000E - 09$	$1.6390000000E - 05$	$1.6938200000E - 04$
0.9	2.462063944000000	2.462063944000000	$0.0000000000E + 00$	$1.8115000000E - 05$	$1.8719800000E - 04$
1.0	2.721001470000000	2.721001469000000	$1.0000000000E - 09$	$2.0019000000E - 05$	$2.0688500000E - 04$

5.2. *Example.* Consider fractional diffusion equation in one dimension as

$$\frac{\partial^\rho \mu}{\partial \kappa^\rho} = \frac{\partial^2 \mu}{\partial \kappa^2} + \mu \frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu \quad 0 < \rho \le 1, \quad \vartheta > 0, \qquad (46)$$

with initial source

$$\mu(\kappa, 0) = 1 + e^\kappa. \qquad (47)$$

On applying Laplace transform to (46),

$$\frac{\nu^\rho L[\mu(\kappa, \vartheta)] - \nu^{-1}\mu(\kappa, 0)}{\nu^\rho + \rho(1 - \nu^\rho)} = L\left[\frac{\partial^2 \mu}{\partial \kappa^2} + \mu \frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu\right]. \quad (48)$$

Applying Laplace inverse transformation,

$$\mu(\kappa, \vartheta) = (1 + \exp(\kappa)) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L\left[\frac{\partial^2 \mu}{\partial \kappa^2} + \mu \frac{\partial \mu}{\partial \kappa} - \mu^2 + \mu\right]\right]. \qquad (49)$$

Let $\mu(\kappa, \vartheta)$ have a series form solution as

$$\mu(\kappa, \vartheta) = \sum_{m=0}^\infty \mu_m(\kappa, \vartheta), \qquad (50)$$

where $\mu(\mu)_\kappa = \sum_{k=0}^\infty \mathscr{A}_k$ and $\mu^2 = \sum_{k=0}^\infty \mathscr{B}_k$ illustrates Adomian polynomials that show nonlinear terms; thus, on putting certain terms in Equation (49), we get

$$\sum_{m=0}^\infty \mu_m(\kappa, \vartheta) = (1 + \exp(\kappa)) + L^{-1}\left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho}\right.$$
$$\left. \cdot L\left[\frac{\partial^2 \mu}{\partial \kappa^2} + \sum_{k=0}^\infty \mathscr{A}_k - \sum_{k=0}^\infty \mathscr{B}_k + \mu\right]\right]. \qquad (51)$$

The nonlinear term decomposition by means of Ado-mian polynomials is given in Equation (20) as

$$\mathscr{A}_0 = \mu_0(\mu_0)_\kappa, \mathscr{A}_1 = \mu_1(\mu_0)_\kappa + \mu_0(\mu_1)_\kappa,$$
$$\mathscr{A}_2 = \mu_2(\mu_0)_\kappa + \mu_1(\mu_1)_\kappa + \mu_0(\mu_2)_\kappa,$$

$$\mathscr{B}_0 = \mu_0^2, \mathscr{B}_1 = 2\mu_0\mu_1, \mathscr{B}_2 = 2\mu_0\mu_2 + (\mu_2)^2. \qquad (52)$$

By the comparison of Equation (51), both sides

$$\mu_0(\kappa, \vartheta) = (1 + e^\kappa). \qquad (53)$$

For $m = 0$,

$$\mu_1(\kappa, \vartheta) = (1 + e^\kappa)\left[\frac{\rho\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)\right]. \qquad (54)$$

For $m = 1$,

$$\mu_2(\kappa, \vartheta) = (1 + e^\kappa)\left[\frac{\rho^2\vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho)\frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^2\right]. \qquad (55)$$

For $m = 2$,

$$\mu_3(\kappa, \vartheta) = (1 + e^\kappa)\left[\frac{\rho^3\vartheta^{3\rho}}{\Gamma(3\rho + 1)} + 3\rho^2(1 - \rho)\frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)}\right.$$
$$\left. + 3\rho(1 - \rho)^2\frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^3\right]. \qquad (56)$$

The series form solution is as

$$\mu(\kappa, \vartheta) = \sum_{m=0}^\infty \mu_m(\kappa, \vartheta)$$
$$= \mu_0(\kappa, \vartheta) + \mu_1(\kappa, \vartheta) + \mu_2(\kappa, \vartheta) + \mu_3(\kappa, \vartheta) + \cdots,$$

$$\mu(\kappa, \vartheta) = (1 + e^{\kappa}) + (1 + e^{\kappa}) \left[\frac{\rho \vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho) \right]$$
$$+ (1 + e^{\kappa}) \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right]$$
$$+ (1 + e^{\kappa}) \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho + 1)} + 3\rho^2(1 - \rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)} \right.$$
$$\left. + 3\rho(1 - \rho)^2 \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^3 \right] + \cdots.$$

$$(57)$$

Thus, we get the solution at $\rho = 1$ as $\mu(\kappa, \vartheta) = 1 + e^{(\kappa + \vartheta)}$.
The analytical solution is by VITM.
In the iterative formula for Equation (46), we get

$$\mu_{m+1}(\kappa, \vartheta) = \mu_m(\kappa, \vartheta) + L^{-1} \left[\frac{\nu^{\rho} + \rho(1 - \nu^{\rho})}{\nu^{\rho}} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^{\rho}}{\nu^{\rho} + \rho(1 - \nu^{\rho})} \frac{\partial^2 \mu_m}{\partial \kappa^2} + \mu_m \frac{\partial \mu_m}{\partial \kappa} - \mu_m^2 + \mu_m \right\} \right],$$

$$(58)$$

where

$$\mu_0(\kappa, \vartheta) = (1 + e^{\kappa}).$$

$$(59)$$

For $m = 0, 1, 2, \cdots,$

$$\mu_1(\kappa, \vartheta) = \mu_0(\kappa, \vartheta) + L^{-1} \left[\frac{\nu^{\rho} + \rho(1 - \nu^{\rho})}{\nu^{\rho}} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^{\rho}}{\nu^{\rho} + \rho(1 - \nu^{\rho})} \frac{\partial^2 \mu_0}{\partial \kappa^2} + \mu_0 \frac{\partial \mu_0}{\partial \kappa} - \mu_0^2 + \mu_0 \right\} \right],$$

$$\mu_1(\kappa, \vartheta) = (1 + e^{\kappa}) + (1 + e^{\kappa}) \left[\frac{\rho \vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho) \right],$$

$$\mu_2(\kappa, \vartheta) = \mu_1(\kappa, \vartheta) + L^{-1} \left[\frac{\nu^{\rho} + \rho(1 - \nu^{\rho})}{\nu^{\rho}} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^{\rho}}{\nu^{\rho} + \rho(1 - \nu^{\rho})} \frac{\partial^2 \mu_1}{\partial \kappa^2} + \mu_1 \frac{\partial \mu_1}{\partial \kappa} - \mu_1^2 + \mu_1 \right\} \right],$$

$$\mu_2(\kappa, \vartheta) = (1 + e^{\kappa}) + (1 + e^{\kappa}) \left[\frac{\rho \vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho) \right]$$
$$+ (1 + e^{\kappa}) \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \right.$$
$$\left. \cdot \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right],$$

$$\mu_3(\kappa, \vartheta) = \mu_2(\kappa, \vartheta) + L^{-1} \left[\frac{\nu^{\rho} + \rho(1 - \nu^{\rho})}{\nu^{\rho}} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^{\rho}}{\nu^{\rho} + \rho(1 - \nu^{\rho})} \frac{\partial^2 \mu_2}{\partial \kappa^2} + \mu_2 \frac{\partial \mu_2}{\partial \kappa} - \mu_2^2 + \mu_2 \right\} \right],$$

$$\mu_3(\kappa, \vartheta) = (1 + e^{\kappa}) + (1 + e^{\kappa}) \left[\frac{\rho \vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho) \right]$$
$$+ (1 + e^{\kappa}) \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \right.$$
$$\left. \cdot \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right] + (1 + e^{\kappa}) \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho + 1)} \right.$$
$$+ 3\rho^2(1 - \rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 3\rho(1 - \rho)^2$$
$$\left. \cdot \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^3 \right],$$

$$\mu(\kappa, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \vartheta) = (1 + e^{\kappa}) + (1 + e^{\kappa})$$
$$\cdot \left[\frac{\rho \vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho) \right] + (1 + e^{\kappa}) \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} \right.$$
$$\left. + 2\rho(1 - \rho) \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right]$$
$$+ (1 + e^{\kappa}) \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho + 1)} + 3\rho^2(1 - \rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)} \right.$$
$$\left. + 3\rho(1 - \rho)^2 \frac{\vartheta^{\rho}}{\Gamma(\rho + 1)} + (1 - \rho)^3 \right] + \cdots.$$

$$(60)$$

Thus, we get the solution at $\rho = 1$ as $\mu(\kappa, \vartheta) = 1 + e^{(\kappa + \vartheta)}$.

Figure 2 depicts the accurate and approximate solutions in the (AB fractional derivative) sense for $\rho = 1$ and at various fractional-orders inside the domain $0 \leq \kappa \geq 1$. Our result is in good agreement with the actual solution, as shown by the graphical depiction. Table 2 shows a comparison of the exact solution and the solution obtained using our approaches, demonstrating the correctness of the presented methods. Table 2 also includes a comparison of absolute errors at various fractional orders, indicating that results improve as fractional order moves to integer order.

5.3. *Example.* Consider the fractional diffusion equation in three dimensions as

$$\frac{\partial^{\rho} \mu}{\partial \kappa^{\rho}} = \frac{\partial^2 \mu}{\partial \kappa^2} + \frac{\partial^2 \mu}{\partial \ell^2} + \frac{\partial^2 \mu}{\partial \psi^2}, \quad 0 < \rho \leq 1, \vartheta \geq 0, \quad (61)$$

with initial source

$$\mu(\kappa, \ell, \psi, 0) = \sin \kappa \sin \ell \sin \psi. \quad (62)$$

On applying Laplace transform to (61),

$$\frac{\nu^{\rho} L[\mu(\kappa, \ell, \psi, \vartheta)] - \nu^{-1} \mu(\kappa, 0)}{\nu^{\rho} + \rho(1 - \nu^{\rho})} = L \left[\frac{\partial^2 \mu}{\partial \kappa^2} + \frac{\partial^2 \mu}{\partial \ell^2} + \frac{\partial^2 \mu}{\partial \psi^2} \right].$$
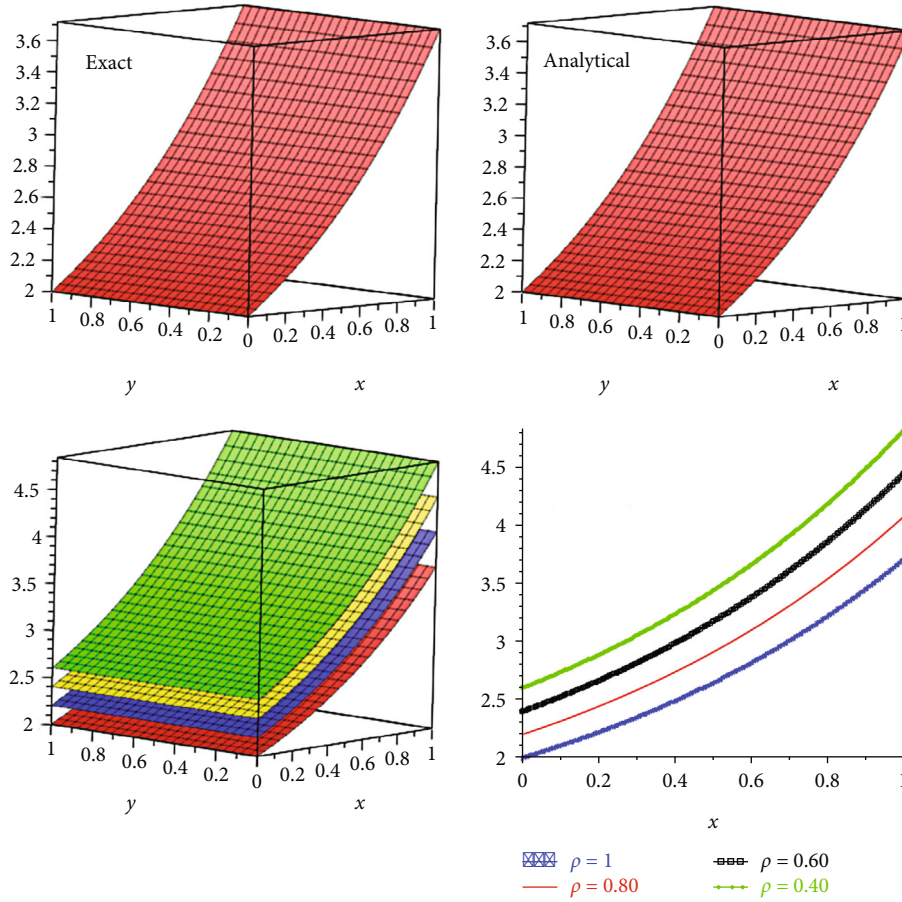
$$(63)$$

FIGURE 2: The graphical layout for problem 2.

TABLE 2: Comparison of the exact and suggested method solution with the aid of absolute errors for problem 2.

$\vartheta = 0.01$	Exact solution	Our methods solution	AE of our methods	AE of our methods	AE of our methods
κ	$\rho = 1$	$\rho = 1$	$\rho = 1$	$\rho = 0.9$	$\rho = 0.8$
0	2.000000100000000	2.000000200000000	$1.0000000000E - 07$	$1.0280000000E - 06$	$1.2534000000E - 05$
0.1	2.105171029000000	2.105171129000000	$1.0000000000E - 07$	$1.0300000000E - 06$	$1.2667000000E - 05$
0.2	2.221402880000000	2.221402980000000	$1.0000000000E - 07$	$1.0310000000E - 06$	$1.2815000000E - 05$
0.3	2.349858943000000	2.349859043000000	$1.0000000000E - 07$	$1.0340000000E - 06$	$1.2978000000E - 05$
0.4	2.491824847000000	2.491824947000000	$1.0000000000E - 07$	$1.0360000000E - 06$	$1.3157000000E - 05$
0.5	2.648721436000000	2.648721536000000	$1.0000000000E - 07$	$1.0380000000E - 06$	$1.3356000000E - 05$
0.6	2.822118983000000	2.822119082000000	$9.9000000000E - 08$	$1.0390000000E - 06$	$1.3575000000E - 05$
0.7	3.013752909000000	3.013753008000000	$9.9000000000E - 08$	$1.0420000000E - 06$	$1.3818000000E - 05$
0.8	3.225541151000000	3.225541251000000	$1.0000000000E - 07$	$1.0450000000E - 06$	$1.4086000000E - 05$
0.9	3.459603357000000	3.459603457000000	$1.0000000000E - 07$	$1.0480000000E - 06$	$1.4383000000E - 05$
1.0	3.718282100000000	3.718282200000000	$1.0000000000E - 07$	$1.0510000000E - 06$	$1.4711000000E - 05$

On taking Laplace inverse transform,

$$\mu(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi + L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L \left[\frac{\partial^2 \mu}{\partial \kappa^2} + \frac{\partial^2 \mu}{\partial \ell^2} + \frac{\partial^2 \mu}{\partial \psi^2} \right] \right].$$

(64)

Let $\mu(\kappa, \ell, \psi, \vartheta)$ have series form solution as

$$\mu(\kappa, \ell, \psi, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \ell, \psi, \vartheta),$$

(65)

$$\sum_{m=0}^{\infty} \mu_m(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi$$
$$+ L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} L \left[\frac{\partial^2 \mu}{\partial \kappa^2} + \frac{\partial^2 \mu}{\partial \ell^2} + \frac{\partial^2 \mu}{\partial \psi^2} \right] \right].$$

(66)

By the comparison of Equation (66), both sides

$$\mu_0(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi.$$

(67)

For $m = 0$,

$$\mu_1(\kappa, \ell, \psi, \vartheta) = -3 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho) \right].$$

(68)

For $m = 1$,

$$\mu_2(\kappa, \ell, \psi, \vartheta) = (-3)^2 \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right].$$

(69)

For $m = 2$,

$$\mu_3(\kappa, \ell, \psi, \vartheta) = (-3)^3 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho + 1)} \right.$$
$$+ 3\rho^2(1 - \rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 3\rho(1 - \rho)^2$$
$$\left. \cdot \frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^3 \right].$$

(70)

The series form solution is as

$$\mu(\kappa, \ell, \psi, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \ell, \psi, \vartheta) = \mu_0(\kappa, \ell, \psi, \vartheta)$$
$$+ \mu_1(\kappa, \ell, \psi, \vartheta) + \mu_2(\kappa, \ell, \psi, \vartheta)$$
$$+ \mu_3(\kappa, \ell, \psi, \vartheta) + \cdots,$$

$$\mu(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi + (-3) \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho) \right] + (-3)^2 \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right]$$
$$+ (-3)^3 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho + 1)} \right.$$
$$+ 3\rho^2(1 - \rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho + 1)}$$
$$\left. + 3\rho(1 - \rho)^2 \frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^3 \right] + \cdots.$$

(71)

Thus, we get the solution at $\rho = 1$ as $\mu(\kappa, \ell, \psi, \vartheta) = \exp^{-3\vartheta} \sin \kappa \sin \ell \sin \psi$.

The analytical solution is by VITM.

The iterative formula for Equation (61), we get

$$\mu_{m+1}(\kappa, \ell, \psi, \vartheta) = \mu_m(\kappa, \ell, \psi, \vartheta) + L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)} \frac{\partial^2 \mu_m}{\partial \kappa^2} + \frac{\partial^2 \mu_m}{\partial \ell^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} \right\} \right],$$

(72)

where

$$\mu_0(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi.$$

(73)

For $m = 0, 1, 2, \cdots$,

$$\mu_1(\kappa, \ell, \psi, \vartheta) = \mu_0(\kappa, \ell, \psi, \vartheta) + L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)} \frac{\partial^2 \mu_0}{\partial \kappa^2} + \frac{\partial^2 \mu_0}{\partial \ell^2} + \frac{\partial^2 \mu_0}{\partial \psi^2} \right\} \right],$$

$$\mu_1(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi + (-3) \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho) \right],$$

$$\mu_2(\kappa, \ell, \psi, \vartheta) = \mu_1(\kappa, \ell, \psi, \vartheta) + L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)} \frac{\partial^2 \mu_1}{\partial \kappa^2} + \frac{\partial^2 \mu_1}{\partial \ell^2} + \frac{\partial^2 \mu_1}{\partial \psi^2} \right\} \right],$$

$$\mu_2(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi + (-3) \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho) \right] + (-3)^2 \sin \kappa \sin \ell \sin \psi$$
$$\cdot \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho + 1)} + 2\rho(1 - \rho) \frac{\vartheta^\rho}{\Gamma(\rho + 1)} + (1 - \rho)^2 \right],$$

$$\mu_3(\kappa, \ell, \psi, \vartheta) = \mu_2(\kappa, \ell, \psi, \vartheta) + L^{-1} \left[\frac{\nu^\rho + \rho(1 - \nu^\rho)}{\nu^\rho} \right.$$
$$\left. \cdot L \left\{ \frac{\nu^\rho}{\nu^\rho + \rho(1 - \nu^\rho)} \frac{\partial^2 \mu_2}{\partial \kappa^2} + \frac{\partial^2 \mu_2}{\partial \ell^2} + \frac{\partial^2 \mu_2}{\partial \psi^2} \right\} \right],$$
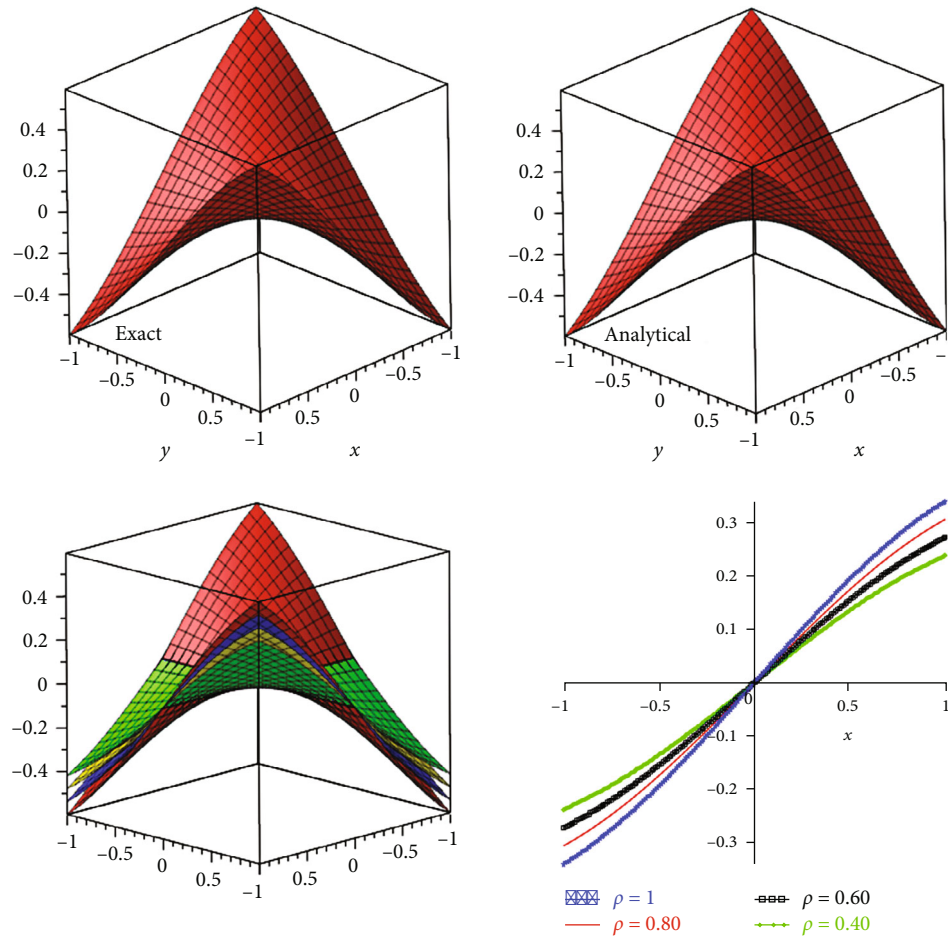
Figure 3: The graphical layout for problem 3.

Table 3: Comparison of the exact and suggested method solution with the aid of absolute errors for problem 3.

$\vartheta = 0.01$	Exact solution	Our method solution	AE of our methods	AE of our methods	AE of our methods
κ	$\rho = 1$	$\rho = 1$	$\rho = 1$	$\rho = 0.9$	$\rho = 0.8$
0	0.000000000000000	0.000000000000000	$0.0000000000E + 00$	$1.1470000000E - 04$	$2.3021230000E - 03$
0.1	0.040275052430000	0.040275060490000	$8.0612920340E - 09$	$4.0267087740E - 05$	$4.0274327760E - 04$
0.2	0.080147689840000	0.080147705880000	$1.6038436970E - 08$	$8.0131842970E - 05$	$8.0146248110E - 04$
0.3	0.119219518100000	0.119219541900000	$2.3813628870E - 08$	$1.1919596380E - 04$	$1.1921737920E - 03$
0.4	0.157100144200000	0.157100175600000	$3.1386867730E - 08$	$1.5706905820E - 04$	$1.5709731480E - 03$
0.5	0.193411077700000	0.193411116400000	$3.8707665300E - 08$	$1.9337280920E - 04$	$1.9340759560E - 03$
0.6	0.227789511600000	0.227789557200000	$4.5607727380E - 08$	$2.2774445860E - 04$	$2.2778541300E - 03$
0.7	0.259891948000000	0.259892000000000	$5.2002906860E - 08$	$2.5984051810E - 04$	$2.5988727020E - 03$
0.8	0.289397630000000	0.289397687900000	$5.7893203750E - 08$	$2.8934038720E - 04$	$2.8939242370E - 03$
0.9	0.316011746500000	0.316011809700000	$6.3194470960E - 08$	$3.1594929850E - 04$	$3.1600606410E - 03$
1.0	0.339468378100000	0.339468446000000	$6.7906708470E - 08$	$3.3940126310E - 04$	$3.3946226980E - 03$

$$\mu_3(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi + (-3) \sin \kappa \sin \ell \sin \psi$$

$$\cdot \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho) \right] + (-3)^2 \sin \kappa \sin \ell \sin \psi$$

$$\cdot \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho+1)} + 2\rho(1-\rho) \frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^2 \right]$$

$$+ (-3)^3 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho+1)} + 3\rho^2(1-\rho) \right.$$

$$\left. \cdot \frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 3\rho(1-\rho)^2 \frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^3 \right],$$

$$\mu(\kappa, \ell, \psi, \vartheta) = \sum_{m=0}^{\infty} \mu_m(\kappa, \ell, \psi, \vartheta) = \sin \kappa \sin \ell \sin \psi$$

$$+ (-3) \sin \kappa \sin \ell \sin \psi \left[\frac{\rho \vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho) \right]$$

$$+ (-3)^2 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho^2 \vartheta^{2\rho}}{\Gamma(2\rho+1)} \right.$$

$$\left. + 2\rho(1-\rho) \frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^2 \right]$$

$$+ (-3)^3 \sin \kappa \sin \ell \sin \psi \left[\frac{\rho^3 \vartheta^{3\rho}}{\Gamma(3\rho+1)} \right.$$

$$+ 3\rho^2(1-\rho) \frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + 3\rho(1-\rho)^2$$

$$\left. \cdot \frac{\vartheta^\rho}{\Gamma(\rho+1)} + (1-\rho)^3 \right] + \cdots.$$

$$(74)$$

Thus, we get solution at $\rho = 1$ as $\mu(\kappa, \ell, \psi, \vartheta) = \exp^{-3\vartheta}$ $\sin \kappa \sin \ell \sin \psi$.

Figure 3 depicts the accurate and approximate solutions in the (AB fractional derivative) sense for $\rho = 1$ and at various fractional orders inside the domain $-1 \leq \kappa \geq 1$. Our result is in good agreement with the actual solution, as shown by the graphical depiction. The LTDM and VITM solutions are closely connected to the exact solution, as seen in Figure 3 and Table 3.

6. Conclusion

In this article, we employed the LTDM and VITM to investigate the importance of the well-studied AB derivative of fractional order. The proposed methods were implemented to solve fractional-order diffusion equations. Following our analysis, we discovered that the current strategies are the most effective tool for solving nonlinear fractional-order problems. The accurate and achieved solutions in closed contact have confirmed the technique's reliability. Also, the graphical representation has confirmed the convergence of approximation results to exact solutions, as well as providing closed-form solutions to the targeted problems. The reliability of the proposed method has been confirmed by the con-vergence phenomenon. Furthermore, LTDM and VITM are often regarded as highly effective and efficient tools for handling a variety of real-world problems.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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