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Research Article

A Novel Implementation of Krasnoselskii's Fixed-Point Theorem to a Class of Nonlinear Neutral Differential Equations

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In this work, we examine a class of nonlinear neutral differential equations. Krasnoselskii's fixed-point theorem is used to provide sufficient conditions for the existence of positive periodic solutions to this type of problem.

1. Introduction

In recent years, differential equations have garnered considerable interest (cf. [1, 2] and references therein). Important types of these problems include differential equations with delay. For instance, in [1, 3–10], the authors employed a variety of techniques to determine the existence of positive periodic solutions. The uniqueness and positivity of a first-order nonlinear periodic differential equation are investigated in [11]. The authors of [12] discussed nearly periodic solutions to nonlinear Duffing equations. Among them, the fixed-point principle has established itself as a critical tool for studying the existence and periodicity of positive solutions. Numerous studies, including [4, 6, 11], examined this method.

In this work, we investigate the following fourth-order nonlinear neutral differential equation:

$$\frac{d^4}{dt^4}(x(t)-g(t,x(t-\tau(t))))=-a(t)x(t)+f(t,x(t-\tau(t))). \tag{1}$$

Under the assumptions:

(i) $a,\tau\in C(\mathbb{R},(0,\infty))$

(ii) $g\in C(\mathbb{R}\times[0,\infty),\mathbb{R})$ and $f\in C(\mathbb{R}\times[0,\infty),[0,\infty))$

(iii) $a,\tau,g(t,x),f(t,x)$ are ω-periodic in t, ω is a positive constant

Krasnoselskii's fixed-point theorem offers sufficient conditions for the existence of positive periodic solutions to the aforesaid problem.

Neutral differential equations are employed in various technological and natural science applications. For example, they are widely employed to investigate distributed networks with lossless transmission lines (see [7]). Therefore, their qualitative qualities are significant.

It is worth noting that Krasnoselskii's fixed-point theorem was proposed in 2012 in [4] to show the existence of positive periodic solutions to the nonlinear neutral differential equation with variable delay of the form

$$\frac{d}{dt}(x(t)-g(t,x(t-\tau(t))))=r(t)x(t)-f(t,x(t-\tau(t))). \tag{2}$$

The same researchers evaluated the existence of positive periodic solutions for two types of second-order nonlinear neutral differential equations with variable delay the following year in [5].

$$\frac{d^2}{dt^2}(x(t) - g(t, x(t-\tau(t)))) = \pm a(t)x(t) \pm f(t, x(t-\tau(t))), \tag{3}$$

where Krasnoselskii's fixed-point theorem is also used as a tool. The authors of [10] investigated the following third-order nonlinear neutral differential equations with variable delay.

$$\frac{d^3}{dt^3}(x(t) - cx(t-\tau(t))) = -a(t)x(t) + f(t, x(t-\tau(t))). \tag{4}$$

The existence of positive periodic solutions is demonstrated using Krasnoselskii's fixed-point theorem. In [3], the authors investigated the fourth-order nonlinear neutral differential equations with variable delay of the form

$$\frac{d^4}{dt^4}(x(t) - g(t, x(t-\tau(t)))) = a(t)x(t) - f(t, x(t-\tau(t))). \tag{5}$$

Krasnoselskii's fixed-point theorem is used to derive some sufficient conditions for the existence of positive periodic solutions to the aforementioned problem.

The remainder of this paper is organized as follows: in the next Section, we deliver the definitions and lemmas required to prove our main results. In particular, we state some Green's function properties related to the problem (1). Section 3 establishes some necessary conditions for the existence of positive solutions to our problem (1).

2. Preliminaries

For a fixed $\omega > 0$, we consider a set P_ω of continuous scalar functions x which are periodic in t, with period ω. We recall that $x(t-\tau)$ and $x(t)$ are in P_ω and $(P_\omega, \|.\|)$ is a Banach space with the supremum norm [13, 14].

$$\|x\| \coloneqq \sup_{t\in\mathbb{R}}|x(t)| = \sup_{t\in[0,\omega]}|x(t)|. \tag{6}$$

Define

$$P_\omega^+ \coloneqq \{x \in P_\omega, x > 0\}, m \coloneqq \inf_{t\in[0,\omega]} a(t),$$
$$M \coloneqq \sup_{t\in[0,\omega]} a(t), \beta \coloneqq \sqrt[4]{M}. \tag{7}$$

Lemma 1. *The equation*

$$\frac{d^4}{dt^4}x(t) + Mx(t) = h(t), h(t) \in P_\omega^+ \tag{8}$$

has a unique ω-periodic solution

$$x(t) = \int_t^{t+\omega} G(t,s)h(s)ds, \tag{9}$$

where

$$G(t,s) = \frac{1}{4\gamma^3}\left(A\left(\sinh\gamma\left(s-t-\frac{\omega}{2}\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right) s \in [t, t+\omega], \right.$$
$$\left. +B\left(\cosh\gamma\left(s-t-\frac{\omega}{2}\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right)\right), \tag{10}$$

$$A \coloneqq \frac{\sin(\gamma\omega/2)\cosh(\gamma\omega/2) - \cos(\gamma\omega/2)\sinh(\gamma\omega/2)}{\cosh\gamma\omega - \cos\gamma\omega}, \tag{11}$$

$$B \coloneqq \frac{\cos(\gamma\omega/2)\sinh(\gamma\omega/2) + \sin(\gamma\omega/2)\cosh(\gamma\omega/2)}{\cosh\gamma\omega - \cos\gamma\omega}. \tag{12}$$

Proof. First, it is evident that the homogeneous equation associated with (8) has a solution

$$x(t) = c_1 e^{\gamma(1+i)t} + c_2 e^{\gamma(-1+i)t} + c_3 e^{\gamma(-1-i)t} + c_4 e^{\gamma(1-i)t}, \quad \text{where } \gamma = \frac{\sqrt{2}}{2}\beta. \tag{13}$$

Using the parameter variation method, we obtain

$$c_1'(t) = h(t)\frac{e^{-t\gamma(1+i)}}{-8(1-i)\gamma^3},$$
$$c_2'(t) = h(t)\frac{e^{t\gamma(1-i)}}{8(1+i)\gamma^3},$$
$$c_3'(t) = h(t)\frac{e^{t\gamma(1+i)}}{8(1-i)\gamma^3},$$
$$c_4'(t) = h(t)\frac{e^{-t\gamma(1-i)}}{-8(1+i)\gamma^3}. \tag{14}$$

Keeping in mind that $x(t)$, $x'(t)$, $x''(t)$, and $x'''(t)$ are periodic functions, we obtain

$$c_1(t) = t^{t+\omega} - \frac{e^{\gamma(1+i)(\omega-s)}}{8(1-i)\gamma^3\left[1-e^{\gamma(1+i)\omega}\right]}h(s)ds,$$
$$c_2(t) = t^{t+\omega} - \frac{e^{\gamma(1-i)s}}{8(1+i)\gamma^3\left[1-e^{\gamma(1-i)\omega}\right]}h(s)ds,$$
$$c_3(t) = t^{t+\omega} - \frac{e^{\gamma(1+i)s}}{8(1-i)\gamma^3\left[1-e^{\gamma(1+i)\omega}\right]}h(s)ds,$$
$$c_4(t) = t^{t+\omega} - \frac{e^{\gamma(1-i)(\omega-s)}}{8(1+i)\gamma^3\left[1-e^{\gamma(1-i)\omega}\right]}h(s)ds. \tag{15}$$

Hence,

$$x(t) = c_1(t)e^{\gamma(1+i)t} + c_2(t)e^{\gamma(-1+i)t} + c_3(t)e^{\gamma(-1-i)t} + c_4(t)e^{\gamma(1-i)t} = \int_t^{t+\omega} G(t,s)h(s)ds, \tag{16}$$

where $G(t,s)$ is identified by (10). □

Lemma 2. *Assume that*

$$\int_t^{t+\omega} G(t,s)h(s)ds = \frac{1}{M}, \quad \max\{a(t)\colon t\in[0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4. \tag{17}$$

Then,

$$0 < \alpha_1 < G(t,s) < \alpha_2, \quad \text{for all } t\in[0,\omega], s\in[t,t+\omega]. \tag{18}$$

Proof. The definition of $G(t,s)$ gives

$$\begin{aligned}
\int_t^{t+\omega} G(t,s)ds &= \frac{1}{4\gamma^3}\left[\int_t^{t+\omega} A\left(\sinh\gamma\left(s-t-\frac{\omega}{2}\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right)ds\right.\\
&\quad\left.+\int_t^{t+\omega} B\left(\cosh\gamma\left(s-t-\frac{\omega}{2}\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right)ds\right]\\
&= \frac{A}{16\gamma^4}e^{\gamma(s-t-\omega/2)}\left[\left(e^{\gamma(2s-2t+\omega)}-1\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right.\\
&\quad\left.++\left(e^{\gamma(2s-2t+\omega)}+1\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right]\Big|_t^{t+\omega}\\
&\quad++\frac{B}{16\gamma^4}e^{\gamma(s-t-\omega/2)}\left[\left(e^{\gamma(2s-2t+\omega)}+1\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right.\\
&\quad\left.--\left(e^{\gamma(2s-2t+\omega)}-1\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right]\Big|_t^{t+\omega}\\
&= \frac{1}{4\gamma^4}A\left[\cosh\frac{\gamma\omega}{2}\sin\frac{\gamma\omega}{2}-\sinh\frac{\gamma\omega}{2}\cos\frac{\gamma\omega}{2}\right]\\
&\quad+\frac{1}{4\gamma^4}B\left[\cosh\frac{\gamma\omega}{2}\sin\frac{\gamma\omega}{2}+\sinh\frac{\gamma\omega}{2}\cos\frac{\gamma\omega}{2}\right]\\
&= \frac{1}{4\gamma^4}\left[2\frac{(\cosh(\gamma\omega/2)\sin(\gamma\omega/2))^2+(\sinh(\gamma\omega/2)\cos(\gamma\omega/2))^2}{\cosh\gamma\omega-\cos\gamma\omega}\right]\\
&= \frac{1}{4\gamma^4}\left[2\frac{(1/2)\cosh\gamma\omega-(1/2)\cos\gamma\omega}{\cosh\gamma\omega-\cos\gamma\omega}\right] = \frac{1}{4\gamma^4} = \frac{1}{\beta^4} = \frac{1}{M}.
\end{aligned} \tag{19}$$

On the other hand, it is simple to demonstrate that $(d/ds)G(t,s)=0$ only if $s=t+\omega/2$.
Hence,

$$\begin{aligned}
G(t,t) = G(t,t+\omega) &= \frac{1}{4\gamma^3}\frac{\sinh(\gamma\omega/2)\cosh(\gamma\omega/2)+\sin(\gamma\omega/2)\cos(\gamma\omega/2)}{\cosh(\gamma\omega/2)-\cos(\gamma\omega/2)} = \alpha_2,\\
G\left(t,t+\frac{\omega}{2}\right) &= \frac{1}{4\gamma^3}\frac{\cos(\gamma\omega/2)\sinh(\gamma\omega/2)+\sin(\gamma\omega/2)\cosh(\gamma\omega/2)}{\cosh\gamma\omega-\cos\gamma\omega} = \alpha_1.
\end{aligned} \tag{20}$$

Since

$$\{a(t)\colon t\in[0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4, \tag{21}$$

we get

$$0 < \frac{\gamma\omega}{2} < \frac{\pi}{2}. \tag{22}$$

So,

$$\sin\frac{\gamma\omega}{2} > 0, 1 > \cos\frac{\gamma\omega}{2} > 0, \sinh\frac{\gamma\omega}{2} > 0. \tag{23}$$

Consequently,

$$0 < \alpha_1 < G(t,s) < \alpha_2, \quad \text{for all } t\in[0,\omega], s\in[t,t+\omega]. \tag{24}$$

□

Lemma 3. *If*

$$\max\{a(t)\colon t\in[0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4, F(t,x) > 0. \tag{25}$$

Then, $x\in P_\omega$ solves equation (1) if and only if

$$x(t) = g(t,x(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds. \tag{26}$$

Proof. Let $x\in P_\omega$ be a solution of (1). Equation (1) reads as

$$\begin{aligned}
&\frac{d^4}{dt^4}(x(t)-g(t,x(t-\tau(t)))) + M(x(t)-g(t,x(t-\tau(t))))\\
&\quad= -a(t)x(t) + f(t,x(t-\tau(t))) + M(x(t)-g(t,x(t-\tau(t))))\\
&\quad= (M-a(t))x(t) + f(t,x(t-\tau(t))) - Mg(t,x(t-\tau(t))).
\end{aligned} \tag{27}$$

According to Lemma 1, we obtain

$$\begin{aligned}
x(t) - g(t,x(t-\tau(t))) &= \int_t^{t+\omega} G(t,s)(M-a(t)x(t) + f(t,x(t-\tau(t))))ds\\
&\quad- \int_t^{t+\omega} G(t,s)(Mg(t,x(t-\tau(t))))ds,
\end{aligned} \tag{28}$$

which implies that

$$x(t) = g(t,x(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds. \tag{29}$$

This completes the proof. Let us define the two operators $\mathcal{T}_1, \mathcal{T}_2 : P_\omega \longrightarrow P_\omega$ as follows:

$$\mathcal{T}_1(\varphi)(t) := g(t, \varphi(t-\tau(t))),$$
$$\mathcal{T}_2(\varphi)(t) := \int_{t_i}^{t_i+\omega} G(t_i, s)((M-a(s))\varphi(s) + f(s, \varphi(s-\tau(s))) - Mg(t, \varphi(s-\tau(s))))ds. \quad (30)$$

We formulate equation (26) in Lemma 3 as follows:

$$\varphi(t) = (\mathcal{T}_1\varphi)(t) + (\mathcal{T}_2\varphi)(t). \quad (31)$$

□

Remark 4. Any solution to equation (31) is a solution to problem (1).

Let us introduce the following hypotheses, which are assumed hereafter:

The function $g(t, x)$ is Lipschitz continuous in x. That is to say, there exists a positive constant k such that

$$\|g(t,x) - g(t,y)\| \le k\|x-y\|, \quad \text{for all } t \in [0,\omega], x, y \in P_\omega. \quad (32)$$

Lemma 5. *Assume that (32) holds and*

$$k < 1. \quad (33)$$

Then, $\mathcal{T}_1$ is a contraction.

Proof. It is evident that $\mathcal{T}_1\varphi$ is continuous for all $\varphi \in \mathcal{D}$. Moreover,

$$(\mathcal{T}_1\varphi)(t+\omega) = (\mathcal{T}_1\varphi)(t). \quad (34)$$

So, for all $\varphi, \psi \in \mathcal{D}$, we have

$$|(\mathcal{T}_1\varphi)(t) - (\mathcal{T}_1\psi)(t)| = |g(t, \varphi(t-\tau(t))) - g(t, \psi(t-\tau(t)))| \le \sup_{t\in[0,\omega]} |g(t, \varphi(t-\tau(t))) - g(t, \psi(t-\tau(t)))| \le k\|\varphi - \psi\|. \quad (35)$$

Thus,

$$\|\mathcal{T}_1\varphi - \mathcal{T}_1\psi\| \le k\|\varphi-\psi\|. \quad (36)$$

Consequently, it follows from (33) that $\mathcal{T}_1 : P_\omega \longrightarrow P_\omega$ is a contraction. □

Lemma 6. *Assume that $M < 4(\pi/\omega)^4$ and $0 < F(t,x) \le C$. Then, $\mathcal{T}_2$ is completely continuous.*

Proof. Firstly, we show that $\mathcal{T}_2$ is continuous. To this end, let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in P_ω. We have

$$|\mathcal{T}_2(y_n)(t) - \mathcal{T}_2(y)(t)| \le \int_t^{t+\omega} G(t,s)((M-a(s))|y_n(s) - y(s)| + |f(s, y_n(s-\tau(s))) - f(s, y(s-\tau(s)))| + |Mg(t, y_n(s-\tau(s))) - Mg(t, y(s-\tau(s)))|)ds. \quad (37)$$

It follows from the continuity of f and g that

$$\|\mathcal{T}_2(y_n) - \mathcal{T}_2(y)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (38)$$

Thus, $\mathcal{T}_2$ is continuous.

Secondly, we prove that $\mathcal{T}_2$ maps bounded sets into bounded sets in $(P_\omega, \|.\|)$. To this end, let $B_r = \{(x \in P_\omega, \|x\| < r)\}$ be a bounded ball in $(P_\omega, \|.\|)$, we have

$$|\mathcal{T}_2(x)(t)| = \left|\int_t^{t+\omega} G(t,s)((M-a(s))x(s) + f(s, x(s-\tau(s))) - Mg(t, x(s-\tau(s))))ds\right| \le \int_t^{t+\omega} G(t,s)|(M-a(s))x(s) + f(s, x(s-\tau(s))) - Mg(t, x(s-\tau(s)))|ds. \quad (39)$$

From Lemma 2 and since $F(t,x) \le C$, we get

$$|\mathcal{T}_2(x)(t)| \le \alpha_2 \int_t^{t+\omega} (M-m)r + f(s, x(s-\tau(s))) - Mg(t, x(s-\tau(s)))ds \le \alpha_2\omega((M-m)r + C). \quad (40)$$

The estimation of $\|\mathcal{T}_2(x)\|$ implies

$$\|\mathcal{T}_2(x)\| \le \alpha_2\omega((M-m)r + C). \quad (41)$$

This shows that $\mathcal{T}_2$ is uniformly bounded. □

Finally, we prove that $\mathcal{T}_2$ sends bounded sets into equi-continuous sets. Let t_1, $t_2 \in [0,\omega]$, $t_1 < t_2$, and B_r be a bounded set of P_ω. For all $i \in \{1,2\}$, we have

$$|\mathcal{T}_2(x)(t_i)| = \left|\int_{t_i}^{t_i+\omega} G(t_i,s)((M-a(s))x(s) + f(s, x(s-\tau(s))) - Mg(t, x(s-\tau(s))))ds\right|, \quad i = 1, 2. \quad (42)$$

Denote $\mathcal{T}_3 = |\mathcal{T}_2(x)(t_2) - \mathcal{T}_2(x)(t_1)|$. So, we obtain

$$\begin{aligned}
\mathcal{T}_3 &= \Bigg|\int_{t_2}^{t_2+\omega} G(t_2,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) \\
&\quad - Mg(t,x(s-\tau(s))))ds - \int_{t_1}^{t_1+\omega} G(t_1,s)((M-a(s))x(s) \\
&\quad + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds\Bigg| \\
&= \Bigg|\int_{t_2}^{t_1+\omega} G(t_2,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) \\
&\quad - Mg(t,x(s-\tau(s))))ds + \int_{t_1+\omega}^{t_2+\omega} G(t_2,s)((M-a(s))x(s) \\
&\quad + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds, \\
&\quad - \int_{t_1}^{t_2} G(t_1,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) \\
&\quad - Mg(t,x(s-\tau(s))))s - \int_{t_2}^{t_1+\omega} G(t_1,s)((M-a(s))x(s) \\
&\quad + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds\Bigg| \\
&= \Bigg|\int_{t_2}^{t_1+\omega} (G(t_2,s) - G(t_1,s))((M-a(s))x(s) \\
&\quad + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds \\
&\quad + \int_{t_1+\omega}^{t_2+\omega} G(t_2,s)((M-a(s))x(s) + f(s,x(s-\tau(s))) \\
&\quad - Mg(t,x(s-\tau(s))))ds - \int_{t_1}^{t_2} G(t_1,s)((M-a(s))x(s) \\
&\quad + f(s,x(s-\tau(s))) - Mg(t,x(s-\tau(s))))ds\Bigg| \\
&\leq (M-m)r\left(\int_{t_2}^{t_1+\omega} |(G(t_2,s) - G(t_1,s))| ds \right. \\
&\quad \left. + \int_{t_1+\omega}^{t_2+\omega} G(t_2,s)ds + \int_{t_1}^{t_2} G(t_1,s)ds\right).
\end{aligned} \tag{43}$$

As $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero. By the Arzela-Ascoli theorem, we conclude that $\mathcal{T}_2$ is a completely continuous operator. This completes the proof. This section will be concluded by referring to Krasnoselskii's fixed-point theorem (see [9]).

Theorem 7. (Krasnoselskii). *Let $\mathcal{D}$ be a closed convex nonempty subset of a Banach space $(B, \|.\|)$. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ map $\mathcal{D}$ into B such that*

(i) $x, y \in \mathcal{D}$, implies $\mathcal{T}_1 x + \mathcal{T}_2 y \in \mathcal{D}$

(ii) $\mathcal{T}_1$ is a contraction mapping

(iii) $\mathcal{T}_2$ is completely continuous

Then, there exists $z \in \mathcal{D}$ with $z = \mathcal{T}_1 z + \mathcal{T}_2 z$.

3. Existence of Positive Periodic Solutions

We will examine the existence of positive periodic solutions to problem (1) using Krasnoselskii's fixed-point theorem. For this purpose, we consider $(\mathbf{B}, \|.\|) = (P_\omega, \|.\|)$ and for some positive constant K and L. Moreover, define the set $\mathcal{D} = \{\varphi \in P_\omega : K \leq \varphi \leq L\}$, which is a closed convex and bounded subset of the Banach space P_ω.

By looking at the three cases $g(t,x) < 0, g(t,x) = 0$, and $g(t,x) > 0$ for all $t \in \mathbb{R}, x \in \mathcal{D}$, we can prove the existence of a positive periodic solution of (1).

3.1. The Case $g(t,x) < 0$. We assume that there exist nonpositive constants k_1 and k_2 such that

$$k_1 \leq g(t,x) \leq k_2, t \text{ or all } t \in [0.\omega], x \in \mathcal{D}. \tag{44}$$

Theorem 8. *Assume that $M < 4(\pi/\omega)^4$ and the function f satisfies*

$$\frac{K-k_1}{\alpha_1\omega} \leq f(t, x(t-\tau(t))) \leq \frac{L}{\alpha_2\omega} - (M-m)L + Mk_1. \tag{45}$$

Then, problem (1) has a positive ω-periodic solution x in the subset $\mathcal{D}$.

Proof. Let us start by proving that

$$\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) \in \mathcal{D}, \quad \text{for all } \varphi, \phi \in \mathcal{D}. \tag{46}$$

In fact,

$$\begin{aligned}
\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) &= g(t,\varphi(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))\varphi(s) \\
&\quad + f(s,\varphi(s-\tau(s))) - Mg(t,\varphi(s-\tau(s))))ds \\
&\leq \alpha_2\omega((M-m)L - Mk_1) + \alpha_2\int_t^{t+\omega} f(s,\varphi(s-\tau(s)))ds \\
&\leq \alpha_2\omega((M-m)L - Mk_1) + \alpha_2\omega\left(\frac{L}{\alpha_2\omega} - (M-m)L + Mk_1\right) = L.
\end{aligned} \tag{47}$$

On the other hand,

$$\begin{aligned}
\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) &= g(t,\varphi(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))\varphi(s) \\
&\quad + f(s,\varphi(s-\tau(s))) - Mg(t,\varphi(s-\tau(s))))ds, \\
&\geq k_1 + \alpha_1\omega\int_t^{t+\omega} f(s,\varphi(s-\tau(s)))ds, \geq k_1 + \alpha_1\omega\left(\frac{K-k_1}{\alpha_1\omega}\right) = K,
\end{aligned} \tag{48}$$

which leads to

$$\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) \in \mathcal{D}, \quad \text{for all } \varphi, \phi \in \mathcal{D}. \tag{49}$$

We conclude from Lemma 5 that $\mathcal{T}_1$ is a contraction. Also, Lemma 6 implies that the operator $\mathcal{T}_2$ is completely continuous.

We deduce from Krasnoselskii's fixed-point theorem (see [15], p.~31) that $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ has a fixed point $\varphi \in \mathcal{D}$ which is a solution to (31). As a result of Remark 4, φ is a solution to problem (1). This completes the proof. □

3.2. The Case $g(t, x) = 0$

Theorem 9. *Assume that $M < 4(\pi/\omega)^4$, and*

$$\frac{K}{\alpha_1\omega} \le f(t, x(t-\tau(t))) \le \frac{L}{\alpha_2\omega} - (M-m)L, \quad \text{for all } t \in [0.\omega], x \in \mathcal{D}. \tag{50}$$

Then, equation (1) has a positive ω-periodic solution x in the subset $\mathcal{D}$.

Proof. According to [16], we have $\mathcal{T}_1 = 0$. Similarly to the proof of Theorem 8, we show that (1) has a nonnegative ω -periodic solution $x \in \mathcal{D}$. Since $F(t, x) > 0$, it is easy to see that $x(t) > 0$; i.e., (1) has a positive ω-periodic solution $x \in \mathcal{D}$. □

3.3. The Case $g(t, x) > 0$. We assume that there exist nonnegative constants k_3 and k_4 such that

$$k_3 \le g(t, x) \le k_4, t \quad \text{or all t} \in [0.\omega], x \in \mathcal{D}. \tag{51}$$

Theorem 10. *Assume that $M < 4(\pi/\omega)^4$ and the function f satisfies*

$$\frac{K-k_1}{\alpha_1\omega} \le f(t, x(t-\tau(t))) \le \frac{L-k_4}{\alpha_2\omega} - (M-m)L + Mk_3. \tag{52}$$

Then, problem (1) has a positive ω-periodic solution x in the subset $\mathcal{D}$.

Proof. According to Lemma 5, it follows that the operator $\mathcal{T}_1$ is a contraction, and from Lemma 6, the operator $\mathcal{T}_2$ is completely continuous.

Now, we prove that

$$\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) \in \mathcal{D}, \quad \text{for all } \varphi, \phi \in \mathcal{D}. \tag{53}$$

We have

$$\begin{aligned} \mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) &= g(t, \varphi(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))\varphi(s) \\ &\quad + f(s, \varphi(s-\tau(s))) - Mg(t, \varphi(s-\tau(s))))ds \\ &\le k_4 + \alpha_2\omega((M-m)L - Mk_3) + \alpha_2 \int_t^{t+\omega} f(s, \varphi(s-\tau(s)))ds \\ &\le k_4 + \alpha_2\omega((M-m)L - Mk_3) \\ &\quad + \alpha_2\omega\left(\frac{L-k_4}{\alpha_2\omega} - (M-m)L + Mk_3\right) = L. \end{aligned} \tag{54}$$

Also,

$$\begin{aligned} \mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) &= g(t, \varphi(t-\tau(t))) + \int_t^{t+\omega} G(t,s)((M-a(s))\varphi(s) \\ &\quad + f(s, \varphi(s-\tau(s))) - Mg(t, \varphi(s-\tau(s))))ds \\ &\ge k_3 + \alpha_1\omega \int_t^{t+\omega} f(s, \varphi(s-\tau(s)))ds \ge k_3 + \alpha_1\omega\left(\frac{K-k_3}{\alpha_1\omega}\right) = K. \end{aligned} \tag{55}$$

Thus,

$$\mathcal{T}_1(\varphi) + \mathcal{T}_2(\phi) \in \mathcal{D}, \quad \text{for all } \varphi, \phi \in \mathcal{D}. \tag{56}$$

By Krasnoselskii's theorem (see [15], p. 31), we deduce that $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ has a fixed point which is a solution to (31), so problem (1) has a positive ω-periodic solution x in the subset $\mathcal{D}$. □

4. Conclusion

In this work, we established sufficient conditions for the existence of positive periodic solutions to the fourth-order nonlinear neutral differential equations with variable delay. Our proof relies on Krasnoselskii's fixed-point theorem, which is an excellent tool when the conditions of the Banach or Schauder fixed-point theorems are not fulfilled.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed to the design and implementation of the research, to the analysis of the results, and to the writing of the manuscript.

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