Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9332406, 8 pages
https://doi.org/10.1155/2022/9332406

Research Article

Weighted Composition Operators from Dirichlet-Zygmund Spaces into Zygmund-Type Spaces and Bloch-Type Spaces

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Received 9 April 2022; Accepted 4 June 2022; Published 20 June 2022

Academic Editor: Andrea Scapellato

The boundedness, compactness, and essential norm of weighted composition operators from Dirichlet-Zygmund spaces into Zygmund-type spaces and Bloch-type spaces are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disk $\mathbb{D}$. For $1 \leq p < \infty$, the Dirichlet type space $\mathcal{D}^p_{p-1}$ is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{D}^p_{p-1}} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1} dA(z) < \infty, \quad (1)$$

where $dA(z) = (1/\pi)dxdy$ is the normalized Lebesgue area measure. $\mathcal{D}^p_{p-1}$ is a Banach space under the norm $\|\cdot\|_{\mathcal{D}^p_{p-1}}$. If $f' \in \mathcal{D}^p_{p-1}$, we say that f belongs to the Dirichlet-Zygmund space, denoted by $\mathcal{Z}^p_{p-1}$. To the best of our knowledge, this is the first work to study the Dirichlet-Zygmund space.

Recall that the space B_1, called the minimal Möbius invariant space, is the space of all $f \in H(\mathbb{D})$ that admit the representation $f(z) = \sum_{j=1}^{\infty} b_j \sigma_{t_j}(z)$ for some sequence $\{b_j\}$ in l^1 and $t_j \in \mathbb{D}$. The norm on $f \in B_1$ is defined by

$$\|f\|_{B_1} = \inf \left\{ \sum_{j=1}^{\infty} |b_j| : f(z) = \sum_{j=1}^{\infty} b_j \sigma_{t_j}(z) \right\}. \quad (2)$$

Here, $\sigma_a(z) = (a-z)/(1-\bar{a}z)$. For any $f \in B_1$, the authors in [1] showed that there exists a constant $C > 0$ such that

$$C^{-1} \int_{\mathbb{D}} |f''(z)| dA(z) \leq \|f - f(0) - f'(0)z\|_{B_1} \leq C \int_{\mathbb{D}} |f''(z)| dA(z). \quad (3)$$

Therefore, $\mathcal{Z}^1_0$ is in fact the space B_1.

We call $v : \mathbb{D} \longrightarrow \mathbb{R}_+$ a weight, if v is a continuous, strictly positive and bounded function. v is called radial, if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. Let v be a radial weight. Recall that the Zygmund-type space $\mathcal{Z}_v$ is the space that consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}_v} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} v(z)|f''(z)| < \infty. \quad (4)$$

$\mathcal{Z}_v$ is a Banach space under the norm $\|\cdot\|_{\mathcal{Z}_v}$. We say that f belongs to the Bloch-type space $\mathcal{B}_v$, if

$$\|f\|_{\mathcal{B}_v} = |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty. \quad (5)$$

When $v(z) = 1 - |z|^2$, $\mathcal{Z}_v = \mathcal{Z}$ is called the Zygmund space, and $\mathcal{B}_v = \mathcal{B}$ is called the Bloch space, respectively. In particular, $\mathcal{Z}_v$ is just the Bloch space when $v(z) = (1-|z|^2)^2$.

The weighted space, denoted by H_v^∞, is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty. \quad (6)$$

When $v(z) = v_\alpha(z) = (1-|z|^2)^\alpha (0 \le \alpha < \infty)$, we denote H_v^∞ by $H_{v_\alpha}^\infty$. In particular, when $\alpha = 0$, $H_{v_0}^\infty = H^\infty$ is just the bounded analytic function space.

We denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$ for simplicity. Let $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$. The weighted composition operator ψC_φ is defined as follows.

$$(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z)), f \in H(\mathbb{D}), z \in \mathbb{D}. \quad (7)$$

When $\psi = 1$, ψC_φ is called the composition operator, denoted by C_φ. See [2, 3] for more results about the theory of composition operators and weighted composition operators.

For any $\varphi \in S(\mathbb{D})$, by the Schwarz-Pick lemma, we see that $C_\varphi : \mathscr{B} \longrightarrow \mathscr{B}$ is bounded. It was shown in [4] that $C_\varphi : \mathscr{B} \longrightarrow \mathscr{B}$ is compact if and only if $\lim_{j\longrightarrow\infty} \|\varphi^j\|_{\mathscr{B}} = 0$. Motivated by [4], Colonna and Li in [5, 6] studied the operators $\psi C_\varphi : H^\infty \longrightarrow \mathscr{Z}$ and $\psi C_\varphi : Lip_\alpha \longrightarrow \mathscr{Z}$ by $\|\psi\varphi^j\|_{\mathscr{Z}}$ and $\|j^{-\alpha}\psi\varphi^j\|_{\mathscr{Z}}$, respectively. Here, Lip_α is the Lipschtiz space. The composition operator on the space B_1 was extensively studied in [1]. In [7], Colonna and Li studied the boundedness and compactness of weighted composition operators from the minimal Möbius invariant space $B_1(\mathscr{Z}_0^1)$ to the Bloch space $\mathscr{B}$. In [8], Li studied the boundedness and compactness of the weighted composition operator $\psi C_\varphi : B_1(\mathscr{Z}_0^1) \longrightarrow \mathscr{Z}$. See [5, 6, 8–17] for more results for composition operators, weighted composition operators, and related operators on the Zygmund space and Zygmund-type spaces.

In this paper, we follow the methods of [17] and give some characterizations for the boundedness, compactness, and essential norm of the operator $\psi C_\varphi : \mathscr{Z}_{p-1}^p \longrightarrow \mathscr{Z}_\mu$ and $\psi C_\varphi : \mathscr{Z}_{p-1}^p \longrightarrow \mathscr{B}_\mu$.

We denoted by C a positive constant which may differ from one occurrence to the next. In addition, we will use the following notations throughout this paper: $A \approx B$ means that there exists a constant C such that $A \le CB$, while $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main Results and Proofs

In this section, we formulate and prove our main results in this paper.

Lemma 1. *Suppose $1 < p < \infty$. Then, there exists a positive constant C such that*

$$|f'(z)| \le \frac{C\|f\|_{\mathscr{Z}_{p-1}^p}}{(1-|z|^2)^{1/p}}, |f''(z)| \le \frac{C\|f\|_{\mathscr{Z}_{p-1}^p}}{(1-|z|^2)^{1+(1/p)}}, \quad (8)$$

and $\|f\|_\infty \le C\|f\|_{\mathscr{Z}_{p-1}^p}$ for every $f \in \mathscr{Z}_{p-1}^p$.

Proof. Suppose $r > 0$ and $g \in H(\mathbb{D})$. Then, there exists a constant $C > 0$ such that

$$|g(z)|^p \le \frac{C}{(1-|z|^2)^{\alpha+2}} \int_{D(z,r)} |g(w)|^p (1-|w|^2)^\alpha dA(w), \quad (9)$$

which implies that

$$|g(z)| \le \frac{C\|g\|_{\mathscr{D}_{p-1}^p}}{(1-|z|^2)^{1/p}} \text{and} |g'(z)| \le \frac{C\|g\|_{\mathscr{D}_{p-1}^p}}{(1-|z|^2)^{1+(1/p)}}. \quad (10)$$

The inequalities in (8) hold. Here, $D(z, r)$ is the hyperbolic disk (see [3]). From (8), we see that $\mathscr{Z}_{p-1}^p$ are contained in the disk algebra for $p > 1$. Hence, we get that $\|f\|_\infty \le C \|f\|_{\mathscr{Z}_{p-1}^p}$. □

Lemma 2. *Let $1 < p < \infty$. If $f \in \mathscr{Z}_{p-1}^p$, then for all $t \in (0, 1)$ and $z \in \mathbb{D} \setminus \{0\}$, there exists a positive constant C such that*

$$\left| f(z) - f\left(\frac{t}{|z|} z\right) \right| \le C\|f\|_{\mathscr{Z}_{p-1}^p} (1-|z|)^{1-1/p}. \quad (11)$$

Proof. Fix $f \in \mathscr{Z}_{p-1}^p$. Let $t \in (0, 1)$ and $z \in \mathbb{D} \setminus \{0\}$. By Lemma 1,

$$\begin{aligned} \left| f(z) - f\left(\frac{t}{|z|} z\right) \right| &\le \left| \int_1^{t/|z|} z f'(sz) ds \right| \le \int_1^{1/|z|} |z||f'(sz)| ds \\ &\le C\|f\|_{\mathscr{Z}_{p-1}^p} \int_1^{1/|z|} \frac{|z|}{(1-s^2|z|^2)^{1/p}} ds \\ &\le C\|f\|_{\mathscr{Z}_{p-1}^p} (1-|z|)^{1-1/p}, \end{aligned} \quad (12)$$

as desired. □

Using Lemma 2 and similarly to the proof of Lemma 7 in [18], we get the following lemma.

Lemma 3. *Let $1 < p < \infty$. Every sequence in $\mathscr{Z}_{p-1}^p$ bounded in norm has a subsequence which converges uniformly in $\bar{\mathbb{D}}$ to a function in $\mathscr{Z}_{p-1}^p$.*

Lemma 4 (see [5]). *Let X be a Banach space that is continuously contained in the disk algebra, and let Y be any Banach space of analytic functions on $\mathbb{D}$. Suppose that*

(i) The point evaluation functionals on Y are continuous

(ii) For every sequence $\{f_n\}$ in the unit ball of X that exists an $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \longrightarrow f$ uniformly on $\bar{\mathbb{D}}$

(iii) *The operator $T : X \longrightarrow Y$ is continuous if X has the supremum norm and Y is given by the topology of uniform convergence on compact sets*

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \longrightarrow 0$ uniformly on $\bar{\mathbb{D}}$, then the sequence $\|Tf_n\|_Y \longrightarrow 0$ as $n \longrightarrow \infty$.

The following result is a direct consequence of Lemmas 3 and 4.

Lemma 5. *Let $1 < p < \infty$ and μ be a weight. If $T : \mathscr{Z}^p_{p-1} \longrightarrow \mathscr{Z}_v$ is bounded, then T is compact if and only if $\|Tf_k\|_{\mathscr{Z}_v} \longrightarrow 0$ as $k \longrightarrow \infty$ for any sequence $\{f_k\}$ in $\mathscr{Z}^p_{p-1}$ bounded in norm which converge to 0 uniformly in $\bar{\mathbb{D}}$.*

Theorem 6. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(i) *The operator $\psi C_\varphi : \mathscr{Z}^p_{p-1} \longrightarrow \mathscr{Z}_v$ is bounded*

(ii) $\psi \in \mathscr{Z}_v$,

$$P := \sup_{z\in\mathbb{D}} \frac{v(z)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|}{\left(1 - |\varphi(z)|^2\right)^{1/p}} < \infty, \tag{13}$$

and

$$Q := \sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)||\varphi'(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{(1/p)+1}} < \infty, \tag{14}$$

(iii) $\psi \in \mathscr{Z}_v$,

$\sup_{j\geq 1} j^{1/p}\|(2\psi'\varphi' + \psi\varphi'')\varphi^{j-1}\|_{H^\infty_v} < \infty$ *and* $\sup_{j\geq 1} j^{(1/p)+1}\|\psi\varphi'^2\varphi^{j-1}\|_{H^\infty_v} < \infty$.

Proof. $(ii) \Rightarrow (i)$. For any $z \in \mathbb{D}$ and $f \in \mathscr{Z}^p_{p-1}$, by Lemma 1, we have

$$\left|(\psi C_\varphi f)(0)\right| \lesssim |\psi(0)|\|f\|_{\mathscr{Z}^p_{p-1}},$$

$$\left|(\psi C_\varphi f)'(0)\right| \lesssim \left(\left|\psi'(0)\right| + \frac{\left|\psi(0)\varphi'(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{1/p}}\right)\|f\|_{\mathscr{Z}^p_{p-1}},$$

$$\begin{aligned} v(z)\left|(\psi C_\varphi f)''(z)\right| &\leq v(z)\left|\psi''(z)\|f(\varphi(z))\right| \\ &\quad + v(z)\left|f''(\varphi(z))\|\psi(z)\left(\varphi'(z)\right)^2\right| \\ &\quad + v(z)\left|f'(\varphi(z))\|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right| \\ &\lesssim v(z)\left|\psi''(z)\right|\|f\|_{\mathscr{Z}^p_{p-1}} \\ &\quad + \frac{v(z)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|}{\left(1 - |\varphi(z)|^2\right)^{1/p}}\|\varphi(z)\|_{\mathscr{Z}^p_{p-1}} \\ &\quad + \frac{v(z)|\psi(z)||\varphi'(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{(1/p)+1}}\|f\|_{\mathscr{Z}^p_{p-1}}. \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} \|\psi C_\varphi f\|_{\mathscr{Z}_v} &= \left|(\psi C_\varphi f)(0)\right| + \left|(\psi C_\varphi f)'(0)\right| + \sup_{z\in\mathbb{D}} v(z)\left|(\psi C_\varphi f)''(z)\right| \\ &\lesssim \left(|\psi(0)| + \left|\psi'(0)\right| + \frac{\left|\psi(0)\varphi'(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{1/p}} + P + Q\right)\|f\|_{\mathscr{Z}^p_{p-1}} \\ &< \infty. \end{aligned} \tag{16}$$

Therefore, $\psi C_\varphi : \mathscr{Z}^p_{p-1} \longrightarrow \mathscr{Z}_v$ is bounded.

$(i) \Rightarrow (ii)$. Applying the operator ψC_φ to z^j with $j = 0, 1, 2$ and using the boundedness of ψC_φ, we get that $\psi \in \mathscr{Z}_v$, $\psi\varphi \in \mathscr{Z}_v$, and $\psi\varphi^2 \in \mathscr{Z}_v$. Hence, we obtain

$$\begin{aligned} &\sup_{z\in\mathbb{D}} v(z)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right| < \infty, \\ &\sup_{z\in\mathbb{D}} v(z)\left|\psi(z)\left(\varphi'(z)\right)^2\right| < \infty. \end{aligned} \tag{17}$$

For any $a \in \mathbb{D}$, set

$$f_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^{1/p}},\ g_a(z) = \frac{\left(1 - |a|^2\right)^2}{(1 - \bar{a}z)^{(1+p)/p}},\ z \in \mathbb{D}. \tag{18}$$

It is easy to check that

$$\sup_{a\in\mathbb{D}}\|f_a\|_{\mathscr{Z}^p_{p-1}} < \infty \text{and } \sup_{a\in\mathbb{D}}\|g_a\|_{\mathscr{Z}^p_{p-1}} < \infty. \tag{19}$$

Therefore, by the boundedness of $\psi C_\varphi : \mathscr{Z}^p_{p-1} \longrightarrow \mathscr{Z}_v$ and arbitrary of $a \in \mathbb{D}$, we get

$$\sup_{a\in\mathbb{D}}\left\|\psi C_\varphi f_a\right\|_{\mathscr{Z}_v} < \infty \text{and } \sup_{a\in\mathbb{D}}\left\|\psi C_\varphi g_a\right\|_{\mathscr{Z}_v} < \infty. \tag{20}$$

For $w \in \mathbb{D}$, we get

$$\begin{aligned}\left(\psi C_\varphi f_{\varphi(w)}\right)''(w) = {} & \psi''(w)\left(1-|\varphi(w)|^2\right)^{1-(1/p)} \\ & + \frac{\left(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right)\bar{\varphi(w)}}{p\left(1-|\varphi(w)|^2\right)^{1/p}} \\ & + \frac{(1+p)\psi(w)\left(\varphi'(w)\right)^2\bar{\varphi(w)}^2}{p^2\left(1-|\varphi(w)|^2\right)^{(1/p)+1}},\end{aligned} \tag{21}$$

$$\begin{aligned}\left(\psi C_\varphi g_{\varphi(w)}\right)''(w) = {} & \psi''(w)\left(1-|\varphi(w)|^2\right)^{1-(1/p)} \\ & + \frac{(1+p)}{p}\frac{\left(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right)\bar{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{1/p}} \\ & + \frac{(1+p)(1+2p)}{p^2}\frac{\psi(w)\left(\varphi'(w)\right)^2\bar{\varphi(w)}^2}{\left(1-|\varphi(w)|^2\right)^{(1/p)+1}}.\end{aligned} \tag{22}$$

From (21) and (22), we obtain

$$\begin{aligned} & -(1+p)\left(\psi C_\varphi f_{\varphi(w)}\right)''(w) + \left(\psi C_\varphi g_{\varphi(w)}\right)''(w) \\ & \quad + p\psi''(w)\left(1-|\varphi(w)|^2\right)^{1-(1/p)} \\ & = \frac{(1+p)\psi(w)\left(\varphi'(w)\right)^2\bar{\varphi(w)}^2}{p\left(1-|\varphi(w)|^2\right)^{(1/p)+1}},\end{aligned} \tag{23}$$

$$\begin{aligned} & \frac{\left(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right)\bar{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{1/p}} \\ & = -\left(\psi C_\varphi f_{\varphi(w)}\right)''(w) + \left(\psi C_\varphi g_{\varphi(w)}\right)''(w) \\ & \quad - \frac{2(1+p)\psi(w)\left(\varphi'(w)\right)^2\bar{\varphi(w)}^2}{p\left(1-|\varphi(w)|^2\right)^{(1/p)+1}} \\ & = (1+2p)\left(\psi C_\varphi f_{\varphi(w)}\right)''(w) - \left(\psi C_\varphi g_{\varphi(w)}\right)''(w) \\ & \quad - 2p\psi''(w)\left(1-|\varphi(w)|^2\right)^{1-(1/p)}.\end{aligned} \tag{24}$$

From (24), we get

$$\begin{aligned} & \sup_{w\in\mathbb{D}} \frac{v(w)\left|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right|\left|\varphi(w)\right|}{\left(1-|\varphi(w)|^2\right)^{1/p}} \\ & \le (1+2p)\left\|\psi C_\varphi f_{\varphi(w)}\right\|_{\mathscr{Z}_v} + \left\|\psi C_\varphi g_{\varphi(w)}\right\|_{\mathscr{Z}_v} + 2p\|\psi\|_{\mathscr{Z}_v} < \infty.\end{aligned} \tag{25}$$

On one hand, from (25), we obtain

$$\sup_{|\varphi(w)|>1/2} \frac{v(w)\left|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right|}{\left(1-|\varphi(w)|^2\right)^{1/p}} < \infty. \tag{26}$$

On the other hand, from the fact that $\psi, \psi\varphi \in \mathscr{Z}_v$, we get

$$\begin{aligned} & \sup_{|\varphi(w)|\le 1/2} \frac{v(w)\left|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right|}{\left(1-|\varphi(w)|^2\right)^{1/p}} \\ & \le \left(\frac{4}{3}\right)^{1/p} \sup_{z\in\mathbb{D}} v(w)\left|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\right| \\ & \lesssim \sup_{z\in\mathbb{D}}\left(\|\psi\varphi\|_{\mathscr{Z}_v} + 2\|\psi\|_{\mathscr{Z}_v}\right) < \infty.\end{aligned} \tag{27}$$

From (26) and (27), we see that P is finite. Using similar arguments, we see that Q is also finite.

$(ii) \Leftrightarrow (iii)$. From [19], we see that the inequality in is equivalent to the operator $(2\psi'\varphi' + \psi\varphi'')C_\varphi : H^\infty_{v_{1/p}} \longrightarrow H^\infty_v$ is bounded. By [20], the boundedness of $(2\psi'\varphi' + \psi\varphi'')C_\varphi$ is equivalent to

$$\sup_{j\ge 1} \frac{\left\|\left(2\psi'\varphi' + \psi\varphi''\right)\varphi^{j-1}\right\|_{H^\infty_v}}{\|z^{j-1}\|_{H^\infty_{v_{1/p}}}} < \infty. \tag{28}$$

From [21], we get $\lim_{j\longrightarrow\infty} j^{1/p}\|z^{j-1}\|_{H^\infty_{v_{1/p}}} = \sqrt[p]{2/pe}$, which together with (28) imply that

$$\begin{aligned} & \sup_{j\ge 1} j^{1/p}\left\|\left(2\psi'\varphi' + \psi\varphi''\right)\varphi^{j-1}\right\|_{H^\infty_v} \\ & \approx \sup_{j\ge 1} \frac{j^{1/p}\left\|\left(2\psi'\varphi' + \psi\varphi''\right)\varphi^{j-1}\right\|_{H^\infty_v}}{j^{1/p}\|z^{j-1}\|_{H^\infty_{v_{1/p}}}} < \infty.\end{aligned} \tag{29}$$

Similarly, the inequality in is equivalent to

$$\begin{aligned} \sup_{j\ge 1} j^{(1/p)+1}\left\|\psi\varphi'^2\varphi^{j-1}\right\|_{H^\infty_v} & \approx \sup_{j\ge 1} \frac{j^{(1/p)+1}\left\|\psi\varphi'^2\varphi^{j-1}\right\|_{H^\infty_v}}{j^{(1/p)+1}\|z^{j-1}\|_{H^\infty_{v_{(1/p)+1}}}} \\ & = \sup_{j\ge 1} \frac{\left\|\psi\varphi'^2\varphi^{j-1}\right\|_{H^\infty_v}}{\|z^{j-1}\|_{H^\infty_{v_{(1/p)+1}}}} < \infty.\end{aligned} \tag{30}$$

The proof is complete. □

Next, we consider the essential norm of $\psi C_\varphi : \mathscr{Z}^p_{p-1} \longrightarrow \mathscr{Z}_v$. Recall that the essential norm of $T : X \longrightarrow Y$ is its distance to the set of compact operators $K : X \longrightarrow Y$, that is,

$$\|T\|_{e,X\longrightarrow Y} = \inf\left\{\|T-K\|_{X\longrightarrow Y} : K \text{ is a compact operator}\right\}. \tag{31}$$

Here, X, Y are Banach spaces, and T is a bounded linear operator.

Theorem 7. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1<p<\infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Suppose that $\psi C_\varphi : \mathcal{Z}_{p-1}^{p} \longrightarrow \mathcal{Z}_v$ is bounded. Then,*

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow \mathcal{Z}_v} \approx \max\{E,G\} \approx \max\{M,T\}. \tag{32}$$

Here,

$$E := \limsup_{|\varphi(z)|\longrightarrow 1} \frac{v(z)\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{1/p}},$$

$$G := \limsup_{|\varphi(z)|\longrightarrow 1} \frac{v(z)\left|\psi(z)\right|\left|\varphi'(z)\right|^2}{\left(1-|\varphi(z)|^2\right)^{(1/p)+1}},$$

$$\begin{aligned} M &:= \limsup_{j\longrightarrow\infty} j^{1/p}\left\|\left(2\psi'\varphi'+\psi\varphi''\right)\varphi^{j-1}\right\|_{H_v^\infty}, T \\ &:= \limsup_{j\longrightarrow\infty} j^{(1/p)+1}\left\|\psi\left(\varphi'\right)^2\varphi^{j-1}\right\|_{H_v^\infty}. \end{aligned} \tag{33}$$

Proof. First we show that $\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow \mathcal{Z}_v} \gtrsim \max\{E,G\}$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in the unit disk such that $|\varphi(z_j)| \longrightarrow 1$ as $j\longrightarrow\infty$. Define

$$\begin{aligned} k_j(z) &= \frac{1-|\varphi(z_j)|^2}{\left(1-\bar{\varphi}(z_j)z\right)^{1/p}} - \frac{(1+2p)(1+3p)-(1+p)}{2p(1+2p)}\frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\bar{\varphi}(z_j)z\right)^{(1/p)+1}} \\ &\quad + \frac{1+p}{1+2p}\frac{\left(1-|\varphi(z_j)|^2\right)^3}{\left(1-\bar{\varphi}(z_j)z\right)^{(1/p)+2}}, \end{aligned}$$

$$\begin{aligned} m_j(z) &= \frac{1-|\varphi(z_j)|^2}{\left(1-\bar{\varphi}(z_j)z\right)^{1/p}} - 2\frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\bar{\varphi}(z_j)z\right)^{(1/p)+1}} \\ &\quad + \frac{\left(1-|\varphi(z_j)|^2\right)^3}{\left(1-\bar{\varphi}(z_j)z\right)^{(1/p)+2}}. \end{aligned} \tag{34}$$

After a calculation, we get all k_j and m_j belong to $\mathcal{Z}_{p-1}^{p}$ and

$$\begin{aligned} k_j(\varphi(z_j)) = 0, k''_j(\varphi(z_j)) = 0, \left|k'_j(\varphi(z_j))\right| \\ = \frac{p}{1+2p}\frac{|\varphi(z_j)|}{\left(1-|\varphi(z_j)|^2\right)^{1/p}}, \end{aligned}$$

$$\begin{aligned} m_j(\varphi(z_j)) = 0, m'_j(\varphi(z_j)) = 0, \left|m''_j(\varphi(z_j))\right| \\ = \frac{2|\varphi(z_j)|^2}{\left(1-|\varphi(z_j)|^2\right)^{(1/p)+1}}. \end{aligned} \tag{35}$$

Moreover, k_j and m_j converge to 0 uniformly on $\bar{\mathbb{D}}$ as $j\longrightarrow\infty$. Hence, for any compact operator $K:\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v$, by Lemma 5, we get

$$\begin{aligned} \left\|\psi C_\varphi - K\right\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} &\gtrsim \limsup_{j\longrightarrow\infty}\left\|\psi C_\varphi(k_j)\right\|_{\mathcal{Z}_v} - \limsup_{j\longrightarrow\infty}\left\|K(k_j)\right\|_{\mathcal{Z}_v} \\ &\gtrsim \limsup_{j\longrightarrow\infty}\frac{v(z_j)\left|2\psi'(z_j)\varphi'(z_j)+\psi(z_j)\varphi''(z_j)\right|\left|\varphi(z_j)\right|}{\left(1-|\varphi(z_j)|^2\right)^{1/p}}, \end{aligned}$$

$$\begin{aligned} \left\|\psi C_\varphi - K\right\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} &\gtrsim \limsup_{j\longrightarrow\infty}\left\|\psi C_\varphi(m_j)\right\|_{\mathcal{Z}_v} - \limsup_{j\longrightarrow\infty}\left\|K(m_j)\right\|_{\mathcal{Z}_v} \\ &\gtrsim \limsup_{j\longrightarrow\infty}\frac{v(z_j)\left|\psi(z_j)\right|\left|\varphi'(z_j)\right|^2\left|\varphi(z_j)\right|^2}{\left(1-|\varphi(z_j)|^2\right)^{(1/p)+1}}. \end{aligned} \tag{36}$$

Hence,

$$\begin{aligned} \left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} &= \inf_K\left\|\psi C_\varphi - K\right\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} \\ &\gtrsim \limsup_{j\longrightarrow\infty}\frac{v(z_j)\left|2\psi'(z_j)\varphi'(z_j)+\psi(z_j)\varphi''(z_j)\right|\left|\varphi(z_j)\right|}{\left(1-|\varphi(z_j)|^2\right)^{1/p}} \\ &= \limsup_{|\varphi(z)|\longrightarrow 1}\frac{v(z)\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{1/p}} = E, \end{aligned}$$

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} \gtrsim \limsup_{|\varphi(z)|\longrightarrow 1}\frac{v(z)\left|\psi(z)\right|\left|\varphi'(z)\right|^2}{\left(1-|\varphi(z)|^2\right)^{(1/p)+1}} = G, \tag{37}$$

as desired.

Next, we show that

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} \lesssim \max\{E,G\}. \tag{38}$$

Let $r\in[0,1)$. Define $K_r : H(\mathbb{D})\longrightarrow H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), f\in H(\mathbb{D}). \tag{39}$$

It is clear that K_r is compact on $\mathcal{Z}_{p-1}^{p}$ and $\|K_r\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_{p-1}^{p}} \leq 1$. Moreover, $f_r - f\longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $r\longrightarrow 1$. Let $\{r_j\}\subset(0,1)$ such that $r_j\longrightarrow 1$ as $j\longrightarrow\infty$. Then, $\psi C_\varphi K_{r_j} : \mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v$ is compact for each $j\in\mathbb{N}$. Hence,

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} \leq \limsup_{j\longrightarrow\infty}\left\|\psi C_\varphi - \psi C_\varphi K_{r_j}\right\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v}. \tag{40}$$

Thus, we only need to prove that

$$\limsup_{j\longrightarrow\infty}\left\|\psi C_\varphi - \psi C_\varphi K_{r_j}\right\|_{\mathcal{Z}_{p-1}^{p}\longrightarrow\mathcal{Z}_v} \lesssim \max\{E,G\}. \tag{41}$$

For any $f \in \mathscr{Z}^p_{p-1}$ with $\|f\|_{\mathscr{Z}^p_{p-1}} \le 1$, by the facts that

$$\lim_{j\longrightarrow\infty}\left|\psi(0)f(\varphi(0)) - \psi(0)f\left(r_j\varphi(0)\right)\right| = 0,$$
$$\lim_{j\longrightarrow\infty}\left|\psi'(0)\left(f-f_{r_j}\right)'(\varphi(0)) + \psi(0)\left(f-f_{r_j}\right)'(\varphi(0))\varphi'(0)\right| = 0, \tag{42}$$

we have

$$\begin{aligned}\limsup_{j\longrightarrow\infty}\left\|\left(\psi C_\varphi - \psi C_\varphi K_{r_j}\right)f\right\|_{\mathscr{Z}_v} &= \limsup_{j\longrightarrow\infty} v(z)\left|\left(\psi\cdot\left(f-f_{r_j}\right)\circ\varphi\right)''(z)\right|\\ &\le \limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|\le r_t} v(z)\left|\left(f-f_{r_j}\right)'(\varphi(z))\right|\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|\\ &\quad+\limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|> r_t} v(z)\left|\left(f-f_{r_j}\right)'(\varphi(z))\right|\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|\\ &\quad+\limsup_{j\longrightarrow\infty}\sup_{z\in\mathbb{D}} v(z)\left|\left(f-f_{r_j}\right)(\varphi(z))\right|\left|\psi''(z)\right|\\ &\quad+\limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|\le r_t} v(z)\left|\left(f-f_{r_j}\right)''(\varphi(z))\right|\left|\varphi'(z)\right|^2|\psi(z)|\\ &\quad+\limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|> r_t} v(z)\left|\left(f-f_{r_j}\right)''(\varphi(z))\right|\left|\varphi'(z)\right|^2|\psi(z)|\\ &\coloneqq P_1+P_2+P_3+P_4+P_5,\end{aligned} \tag{43}$$

where $t\in\mathbb{N}$ is large enough such that $r_j \ge 1/2$ for all $j \ge t$. Since $f_{r_j} - f \longrightarrow 0$, $r_j f'_{r_j} - f' \longrightarrow 0$, and $r_j^2 f''_{r_j} - f'' \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\longrightarrow\infty$, by Lemma 3, we obtain

$$\begin{aligned}P_3 &= \limsup_{j\longrightarrow\infty}\sup_{z\in\mathbb{D}} v(z)\left|\left(f-f_{r_j}\right)(\varphi(z))\right|\left|\psi''(z)\right|\\ &\le \|\psi\|_{\mathscr{Z}_v}\limsup_{j\longrightarrow\infty}\sup_{w\in\mathbb{D}}\left|f(w)-f\left(r_j w\right)\right| = 0,\end{aligned} \tag{44}$$

$$\begin{aligned}P_1 &= \limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|\le r_t} v(z)\left|\left(f-f_{r_j}\right)'(\varphi(z))\right|\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|\\ &\lesssim \left(\|\psi\varphi\|_{\mathscr{Z}_v}+\|\psi\|_{\mathscr{Z}_v}\right)\limsup_{j\longrightarrow\infty}\sup_{|w|\le r_t}\left|f'(w)-r_j f'\left(r_j w\right)\right| = 0,\end{aligned} \tag{45}$$

$$\begin{aligned}P_4 &= \limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|\le r_t} v(z)\left|\left(f-f_{r_j}\right)''(\varphi(z))\right|\left|\varphi'(z)\right|^2|\psi(z)|\\ &\lesssim \left(\|\psi\varphi^2\|_{\mathscr{Z}_v}+\|\psi\varphi\|_{\mathscr{Z}_v}+\|\psi\|_{\mathscr{Z}_v}\right)\limsup_{j\longrightarrow\infty}\sup_{|w|\le r_t}\left|f''(w)-r_j^2 f''\left(r_j w\right)\right| = 0.\end{aligned} \tag{46}$$

Using Lemma 1 and $\|f\|_{\mathscr{Z}^p_{p-1}} \le 1$, we obtain

$$\begin{aligned}P_2 &= \limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|> r_t} v(z)\left|\left(f-f_{r_j}\right)'(\varphi(z))\right|\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|\\ &\lesssim \limsup_{j\longrightarrow\infty}\left\|f-f_{r_j}\right\|_{\mathscr{Z}^p_{p-1}}\sup_{|\varphi(z)|> r_t}\frac{v(z)\left|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\right|}{\left(1-|\varphi(z)|^2\right)^{1/p}}.\end{aligned} \tag{47}$$

Taking the limit as $t\longrightarrow\infty$, we get

$$P_2 \lesssim E. \tag{48}$$

Similarly,

$$\begin{aligned}P_5 &= \limsup_{j\longrightarrow\infty}\sup_{|\varphi(z)|> r_t} v(z)\left|\left(f-f_{r_j}\right)''(\varphi(z))\right|\left|\varphi'(z)\right|^2|\psi(z)|\\ &\lesssim \limsup_{j\longrightarrow\infty}\left\|f-f_{r_j}\right\|_{\mathscr{Z}^p_{p-1}}\sup_{|\varphi(z)|> r_t}\frac{v(z)\left|\varphi'(z)\right|^2|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{(1/p)+1}}.\end{aligned} \tag{49}$$

Taking the limit as $t\longrightarrow\infty$, we get

$$P_5 \lesssim G. \tag{50}$$

Hence, by (43), (44), (45), (46), (48), and (50), we get

$$\limsup_{j\longrightarrow\infty}\left\|\psi C_\varphi - \psi C_\varphi K_{r_j}\right\|_{\mathscr{Z}^p_{p-1}\longrightarrow\mathscr{Z}_v} \lesssim \max\{E, G\}, \tag{51}$$

which with (40) implies the desired result.

Finally, we prove that

$$\left\|\psi C_\varphi\right\|_{e,\mathscr{Z}^p_{p-1}\longrightarrow\mathscr{Z}_v} \approx \max\{M, T\}. \tag{52}$$

On one hand, by the proof of Theorem 6, we see that the boundedness of $\psi C_\varphi : \mathscr{Z}^p_{p-1}\longrightarrow\mathscr{Z}_v$ is equivalent to the boundedness of $(2\psi'\varphi'+\psi\varphi'')C_\varphi : H^\infty_{v_{1/p}}\longrightarrow H^\infty_v$ and $\psi\varphi'^2 C_\varphi : H^\infty_{v_{(1/p)+1}}\longrightarrow H^\infty_v$. From [19, 20], we have

$$\left\|\left(2\psi'\varphi'+\psi\varphi''\right)C_\varphi\right\|_{e,H^\infty_{v_{1/p}}\longrightarrow H^\infty_v} \approx M,\ \left\|\psi\varphi'^2 C_\varphi\right\|_{e,H^\infty_{v_{(1/p)+1}}\longrightarrow H^\infty_v} \approx T. \tag{53}$$

Hence,

$$\begin{aligned}\left\|\psi C_\varphi\right\|_{e,\mathscr{Z}^p_{p-1}\longrightarrow\mathscr{Z}_v} &\lesssim \left\|\left(2\psi'\varphi'+\psi\varphi''\right)C_\varphi\right\|_{e,H^\infty_{v_{1/p}}\longrightarrow H^\infty_v}\\ &\quad+\left\|\psi\varphi'^2 C_\varphi\right\|_{e,H^\infty_{v_{(1/p)+1}}\longrightarrow H^\infty_v} \lesssim M+T\\ &\lesssim \max\{M, T\}.\end{aligned} \tag{54}$$

On the other hand, from [19, 21], we have

$$\begin{aligned}\left\|\psi C_\varphi\right\|_{e,\mathscr{Z}^p_{p-1}\longrightarrow\mathscr{Z}_v} \gtrsim E &= \left\|\left(2\psi'\varphi'+\psi\varphi''\right)C_\varphi\right\|_{e,H^\infty_{v_{1/p}}\longrightarrow H^\infty_v}\\ &= \limsup_{j\longrightarrow\infty}\frac{\left\|\left(2\psi'\varphi'+\psi\varphi''\right)\varphi^{j-1}\right\|_{H^\infty_v}}{\|z^{j-1}\|_{H^\infty_{v_{1/p}}}} \approx M,\end{aligned}$$

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}^p_{p-1}\longrightarrow\mathcal{Z}_v} \gtrsim G = \left\|\psi\varphi'^2 C_\varphi\right\|_{e,H^\infty_{v_{(1/p)+1}}\longrightarrow H^\infty_v} = \limsup_{j\longrightarrow\infty}\frac{\left\|\psi\varphi'^2\varphi^{j-1}\right\|_{H^\infty_v}}{\|z^{j-1}\|_{H^\infty_{v_{(1/p)+1}}}} \approx T. \quad (55)$$

Therefore,

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}^p_{p-1}\longrightarrow\mathcal{Z}_v} \gtrsim \max\{M, T\}. \quad (56)$$

The proof is complete. □

From Theorem 7 and the well-known result that $\|T\|_{e,X\longrightarrow Y} = 0$ if and only if $T : X \longrightarrow Y$ is compact, we get the following corollary.

Corollary 8. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Suppose that $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{Z}_v$ is bounded. Then, the following statements are equivalent:*

(i) The operator $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{Z}_v$ is compact

(ii) $\limsup_{|\varphi(z)|\longrightarrow 1}\left(v(z)|\psi(z)||\varphi'(z)|^2/(1-|\varphi(z)|^2)^{(1/p)+1}\right) = 0$ and $\limsup_{|\varphi(z)|\longrightarrow 1}\left(v(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|/(1-|\varphi(z)|^2)^{1/p}\right) = 0$

(iii) $\limsup_{j\longrightarrow\infty} j^{(1/p)+1}\|\psi(\varphi')^2\varphi^{j-1}\|_{H^\infty_v} = \limsup_{j\longrightarrow\infty} j^{1/p}\|(2\psi'\varphi' + \psi\varphi'')\varphi^{j-1}\|_{H^\infty_v} = 0$

Similarly to the above proof, we can get the characterizations of the boundedness, compactness, and essential norm of the weighted composition operator $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{B}_v$ as follows. The details are left to the interested readers.

Theorem 9. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(i) $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{B}_v$ is bounded

(ii) $\psi \in \mathcal{B}_v$ and

$$\sup_{z\in\mathbb{D}}\frac{v(z)\left|\psi(z)\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{1/p}} < \infty. \quad (57)$$

(iii) $\psi \in \mathcal{B}_v$ and $\sup_{j\geq 1} j^{1/p}\|\psi\varphi'\varphi^{j-1}\|_{H^\infty_v} < \infty$

Theorem 10. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Suppose that $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{B}_v$ is bounded. Then,*

$$\left\|\psi C_\varphi\right\|_{e,\mathcal{Z}^p_{p-1}\longrightarrow\mathcal{B}_v} \approx \limsup_{|\varphi(z)|\longrightarrow 1}\frac{v(z)\left|\psi(z)\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{1/p}} \approx \limsup_{j\longrightarrow\infty} j^{1/p}\|\psi\varphi'\varphi^{j-1}\|_{H^\infty_v}. \quad (58)$$

Corollary 11. *Let v be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Suppose that $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{B}_v$ is bounded. Then, the following statements are equivalent:*

(i) The operator $\psi C_\varphi : \mathcal{Z}^p_{p-1} \longrightarrow \mathcal{B}_v$ is compact

(ii) $\limsup_{|\varphi(z)|\longrightarrow 1}\left(v(z)|\psi(z)\varphi'(z)|/(1-|\varphi(z)|^2)^{1/p}\right) = 0$

(iii) $\limsup_{j\longrightarrow\infty} j^{1/p}\|\psi\varphi'\varphi^{j-1}\|_{H^\infty_v} = 0$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that she has no conflicts of interest.

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