1. Introduction

Let $H(D)$ denote the space of all analytic functions in the open unit disk $D$. For $1 \leq p < \infty$, the Dirichlet type space $D^p$ is the set of all $f \in H(D)$ such that

$$\|f\|_{D^p} = \|f(0)\|^p + \int_D |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) < \infty, \quad (1)$$

where $dA(z) = (1/\pi)\,dx\,dy$ is the normalized Lebesgue area measure. $D^p$ is a Banach space under the norm $\|\cdot\|_{D^p}$. If $f' \in D^p$, we say that $f$ belongs to the Dirichlet-Zygmund space, denoted by $D^p$. To the best of our knowledge, this is the first work to study the Dirichlet-Zygmund space.

Recall that the space $B_1$, called the minimal Möbius invariant space, is the space of all $f \in H(D)$ that admit the representation $f(z) = \sum_{j=1}^{\infty} b_j \sigma_t(z)$ for some sequence $\{b_j\}$ in $l^1$ and $t_j \in D$. The norm on $f \in B_1$ is defined by

$$\|f\|_{B_1} = \inf \left\{ \sum_{j=1}^{\infty} |b_j| : f(z) = \sum_{j=1}^{\infty} b_j \sigma_t(z) \right\}. \quad (2)$$

Here, $\sigma_t(z) = (a - z)/(1 - \bar{a}z)$. For any $f \in B_1$, the authors in [1] showed that there exists a constant $C > 0$ such that

$$C^{-1} \int_D |f''(z)| dA(z) \leq \|f - f(0) - f'(0)z\|_{B_1} \leq C \int_D |f''(z)| dA(z). \quad (3)$$

Therefore, $Z_1$ is in fact the space $B_1$.

We call $v : D \rightarrow \mathbb{R}_+$ a weight, if $v$ is a continuous, strictly positive and bounded function. $v$ is called radial, if $v(z) = v(|z|)$ for all $z \in D$. Let $v$ be a radial weight. Recall that the Zygmund-type space $Z_v$ is the space that consists of all $f \in H(D)$ such that

$$\|f\|_{Z_v} = |f(0)| + |f'(0)| + \sup_{z \in D} v(z)|f''(z)| < \infty. \quad (4)$$

$Z_v$ is a Banach space under the norm $\|\cdot\|_{Z_v}$. We say that $f$ belongs to the Bloch-type space $R_v$, if

$$\|f\|_{R_v} = |f(0)| + \sup_{z \in D} v(z)|f'(z)| < \infty. \quad (5)$$

When $v(z) = 1 - |z|^2$, $Z_v = Z$ is called the Zygmund space, and $R_v = R$ is called the Bloch space, respectively. In particular, $Z_v$ is just the Bloch space when $v(z) = (1 - |z|^2)^2$. 

Recall that the space $B$ is defined by

$$B = \{ f \in H(D) : \|f\|_{B_1} < \infty \}.$$
When $\psi(z) = \psi_\alpha(z) = (1 - |z|^2)^\alpha (0 \leq \alpha < \infty)$, we denote $H^\infty_\alpha$ by $H^\infty_{\psi_\alpha}$. In particular, when $\alpha = 0$, $H^\infty_0 = H^\infty$ is just the bounded analytic function space.

We denote by $S(D)$ the set of all analytic self-maps of $D$ for simplicity. Let $\varphi \in S(D)$ and $\psi \in H(D)$. The weighted composition operator $\psi C_\varphi$ is defined as follows.

\[(\psi C_\varphi)(z) = \psi(z)f(\varphi(z)), f \in H(D), z \in D.\tag{7}\]

When $\psi = 1$, $\psi C_\varphi$ is called the composition operator, denoted by $C_\varphi$. See [2, 3] for more results about the theory of composition operators and weighted composition operators.

For any $\varphi \in S(D)$, by the Schwarz-Pick lemma, we see that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is bounded. It was shown in [4] that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\lim_{j \rightarrow \infty} \|\varphi_j\|_{\mathcal{B}} = 0$. Motivated by [4], Colonna and Li in [5, 6, 8–17] have studied the boundedness and compactness of weighted composition operators from the minimal Möbius invariant space $B_1(\mathcal{L}_1^2)$ to the Bloch space $\mathcal{B}$. In [8], Li studied the boundedness and compactness of the weighted composition operator $\psi C_\varphi : B_1(\mathcal{L}_1^2) \rightarrow \mathcal{L}$. See [5, 6, 8–17] for more results for composition operators, weighted composition operators, and related operators on the Zygmund space and Zygmund-type spaces.

In this paper, we follow the methods of [17] and give some characterizations for the boundedness, compactness, and essential norm of the operator $\psi C_\varphi : \mathcal{L}_{p-1} \rightarrow \mathcal{L}_{\mu}$ and $\psi C_\varphi : \mathcal{L}_{p-1} \rightarrow \mathcal{B}_{\alpha}$.

We denote by $C$ a positive constant which may differ from one occurrence to the next. In addition, we will use the following notations throughout this paper: $A \approx B$ means that there exists a constant $C$ such that $A \leq CB$, while $A \asymp B$ means that $A \leq B \leq A$.

2. Main Results and Proofs

In this section, we formulate and prove our main results in this paper.

**Lemma 1.** Suppose $1 < p < \infty$. Then, there exists a positive constant $C$ such that

\[|f' (z)| \leq \frac{C \|f\|_{\mathcal{L}_{p-1}^\mu}}{1 - |z|^2} \frac{1}{1/p}, \quad |f'' (z)| \leq \frac{C \|f\|_{\mathcal{L}_{p-1}^\mu}}{(1 - |z|^2)^{1/p}}, \tag{8}\]

and $\|f\|_{\infty} \leq C \|f\|_{\mathcal{L}_{p-1}^\mu}$, for every $f \in \mathcal{L}_{p-1}^\mu$.

**Proof.** Suppose $r > 0$ and $g \in H(D)$. Then, there exists a constant $C > 0$ such that

\[|g(z)|^p \leq \frac{C}{(1 - |z|^2)^{\alpha r}} \int_{D(z,r)} |g(w)|^p (1 - |w|^2)^\alpha dA(w), \tag{9}\]

which implies that

\[|g(z)| \leq \frac{C \|g\|_{\mathcal{L}_{p-1}^\mu}}{(1 - |z|^2)^{1/p}}, \quad |g'(z)| \leq \frac{C \|g\|_{\mathcal{L}_{p-1}^\mu}}{(1 - |z|^2)^{1/(1/p)}}, \tag{10}\]

The inequalities in (8) hold. Here, $D(z, r)$ is the hyperbolic disk (see [3]). From (8), we see that $\mathcal{L}_{p-1}^\mu$ are contained in the disk algebra for $p > 1$. Hence, we get that $\|f\|_{\infty} \leq C \|f\|_{\mathcal{L}_{p-1}^\mu}$.

**Lemma 2.** Let $1 < p < \infty$. If $f \in \mathcal{L}_{p-1}^\mu$, then for all $t \in (0, 1)$ and $z \in D \setminus \{0\}$, there exists a positive constant $C$ such that

\[|f(z) - f \left( \frac{t}{|z|} \right) | \leq C \|f\|_{\mathcal{L}_{p-1}^\mu} (1 - |z|)^{1-1/p}. \tag{11}\]

**Proof.** Fix $f \in \mathcal{L}_{p-1}^\mu$. Let $t \in (0, 1)$ and $z \in D \setminus \{0\}$. By Lemma 1,

\[|f(z) - f \left( \frac{t}{|z|} \right) | \leq \int_{|s| \leq \frac{1}{|z|}} |f'(sz)| \, ds \leq \int_{|s| \leq \frac{1}{|z|}} |z| |f'(sz)| \, ds \leq C \|f\|_{\mathcal{L}_{p-1}^\mu} (1 - |z|)^{1-1/p}, \tag{12}\]

as desired.

Using Lemma 2 and similarly to the proof of Lemma 7 in [18], we get the following lemma.

**Lemma 3.** Let $1 < p < \infty$. Every sequence in $\mathcal{L}_{p-1}^\mu$ bounded in norm has a subsequence which converges uniformly in $D$ to a function in $\mathcal{L}_{p-1}^\mu$.

**Lemma 4** (see [5]). Let $X$ be a Banach space that is continuously contained in the disk algebra, and let $Y$ be any Banach space of analytic functions on $D$. Suppose that

(i) The point evaluation functionals on $Y$ are continuous

(ii) For every sequence $\{f_n\}$ in the unit ball of $X$ that exists on $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \rightarrow f$ uniformly on $D$
(iii) The operator \( T : X \to Y \) is continuous if \( X \) has the\nuniform topology and \( Y \) is given by the topology of uniform\nconvergence on compact sets

Then, \( T \) is a compact operator if and only if, given a\nbounded sequence \( \{ f_n \} \) in \( X \) such that \( f_n \to 0 \)\nuniformly on \( \overline{D} \), then the sequence \( \| Tf_n \|_Y \to 0 \) as \( n \to \infty \).

The following result is a direct consequence of Lemmas 3\nand 4.

**Lemma 5.** Let \( 1 < p < \infty \) and \( \mu \) be a weight. If \( T : \mathcal{L}^p_{\mu \to \mathcal{L}_v} \)\nis bounded, then \( T \) is compact if and only if \( \| Tf_k \|_v \to 0 \)\nas \( k \to \infty \) for any sequence \( \{ f_k \} \) in \( \mathcal{L}^p_{\mu \to \mathcal{L}_v} \)\nbounded in norm which converge to \( \overline{D} \).

**Theorem 6.** Let \( \nu \) be a radial, nonincreasing weight tending\nto zero at the boundary of \( \overline{D} \). Let \( 1 < p < \infty \), \( \psi \in H(\overline{D}) \), and \( \varphi \in \mathcal{S}(\overline{D}) \). Then, the following statements are equivalent.

(i) The operator \( \psi C_{\psi} : \mathcal{L}^p_{\mu \to \mathcal{L}_v} \)\nis bounded

(ii) \( \psi \in \mathcal{L}_v \),

\[
P := \sup_{z \in \overline{D}} \left[ \frac{\nu(z) |\varphi(z)|}{\nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p}} \right] < \infty,
\]

and

\[
Q := \sup_{z \in \overline{D}} \left[ \frac{\nu(z) |\varphi(z)|^2}{\nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p+1}} \right] < \infty.
\]

(iii) \( \psi \in \mathcal{L}_v \),

\[
\sup_{j \geq 1} \left[ \frac{\nu(z) 2^{j+1} |\varphi_j(z)|}{\nu(z) \left( 1 - |\varphi_j(z)|^2 \right)^{1/p+1}} \right] < \infty \quad \text{and} \quad \sup_{j \geq 1} \left[ \frac{\nu(z) |\varphi_j(z)|^2}{\nu(z) \left( 1 - |\varphi_j(z)|^2 \right)^{1/p+1}} \right] < \infty.
\]

**Proof.** (ii) \(\Rightarrow\) (i). For any \( z \in \overline{D} \) and \( f \in \mathcal{L}^p_{\mu \to \mathcal{L}_v} \), by Lemma 1, we have

\[
| (\psi C_{\psi} f)(0) | \leq |\psi(0)| \| f \|_{\mathcal{L}^p_{\mu \to \mathcal{L}_v}},
\]

\[
| (\psi C_{\psi} f)'(0) | \leq \left( |\psi(0)| + \frac{1}{(1 - |\varphi(0)|^2)^{1/p}} \right) \| f \|_{\mathcal{L}^p_{\mu \to \mathcal{L}_v}},
\]

\[
| (\psi C_{\psi} f)'(0) | \leq \frac{\nu(z) |\varphi'(z)| + \nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p}}{\nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p}} \| f \|_{\mathcal{L}^p_{\mu \to \mathcal{L}_v}}.
\]

Then, by the boundedness of \( \psi C_{\psi} : \mathcal{L}^p_{\mu \to \mathcal{L}_v} \)\nand arbitrary of \( a \in \overline{D} \), we get

\[
\sup_{a \in \overline{D}} \left[ \frac{\nu(z) |\varphi'(z)| + \nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p}}{\nu(z) \left( 1 - |\varphi(z)|^2 \right)^{1/p}} \right] < \infty.
\]

For any \( a \in \overline{D} \), set

\[
f_a(z) = \frac{1 - |a|^2}{(1 - az)^{1/p}}, \quad g_a(z) = \frac{(1 - |a|^2)^2}{(1 - az)^{(1+p)/p}}, \quad z \in \overline{D}.
\]

It is easy to check that

\[
\sup_{a \in \overline{D}} \| f_a \|_{\mathcal{L}^p_{\mu \to \mathcal{L}_v}} < \infty \quad \text{and} \quad \sup_{a \in \overline{D}} \| g_a \|_{\mathcal{L}^p_{\mu \to \mathcal{L}_v}} < \infty.
\]

Therefore, by the boundedness of \( \psi C_{\psi} : \mathcal{L}^p_{\mu \to \mathcal{L}_v} \)\nand arbitrary of \( a \in \overline{D} \), we get

\[
\sup_{a \in \overline{D}} \| \psi C_{\psi} f_a \|_{\mathcal{L}_v} < \infty \quad \text{and} \quad \sup_{a \in \overline{D}} \| \psi C_{\psi} g_a \|_{\mathcal{L}_v} < \infty.
\]
For \( w \in D \), we get
\[
\left( \psi C_{\phi, \psi(w)} \right)''(w) = \psi''(w) (1 - |\phi(w)|^2)^{1-(1/p)} + \frac{2 \psi''(w) \phi''(w) + \psi(w) \phi'(w)}{p (1 - |\phi(w)|^2)^{1/p}} \]
\[
+ \frac{(1 + p) \psi(w) \phi'(w)}{p^2 (1 - |\phi(w)|^2)^{(1/p)+1}},
\]
valid for all \( w \in D \).

From (21) and (22), we obtain
\[
-\left( 1 + p \right) \left( \psi C_{\phi, \psi(w)} \right)''(w) + \left( \psi C_{\phi, \psi(w)} \right)''(w) \]
\[
+ \frac{2 \psi''(w) \phi''(w) + \psi(w) \phi'(w)}{p (1 - |\phi(w)|^2)^{1/(1/p)+1}} = \frac{1 + p) \psi(w) \phi'(w)}{p (1 - |\phi(w)|^2)^{1/(1/p)+1}},
\]
\[
\left( \psi C_{\phi, \psi(w)} \right)''(w) = \frac{2 \psi''(w) \phi''(w) + \psi(w) \phi'(w)}{p (1 - |\phi(w)|^2)^{1/(1/p)+1}}.
\]

On the other hand, from the fact that \( \psi, \psi' \in L_p \), we get
\[
\sup_{|\phi(w)| \geq 1/2} \frac{\psi(w)^2 |\phi'(w)|^2 + \phi(w) \phi''(w)}{1 - |\phi(w)|^2} < \infty.
\]

From (26) and (27), we see that \( P \) is finite. Using similar arguments, we see that \( Q \) is also finite.

Next, we consider the essential norm of \( \psi C_{\phi, \psi(w)} \) is equivalent to the operator \( (2 \psi' \phi' + \psi \phi''')C_{\psi} : H^0_{\psi} \rightarrow H^0_{\psi} \) is bounded. By [20], the boundedness of \( (2 \psi' \phi' + \psi \phi''')C_{\psi} \) is equivalent to
\[
\sup_{j \leq 1} \| (2 \psi' \phi' + \psi \phi''')^{j-1} \|_{H^0} < \infty.
\]

Similarly, the inequality in is equivalent to
\[
\sup_{j \leq 1} \| \psi \phi'' \phi'^{j-1} \|_{H^0} \leq \sup_{j \leq 1} \left( 1 + 2p \right) \| \psi C_{\phi, \psi(w)} \|_{L_p} + \frac{2 \psi''(w) \phi''(w) + \psi(w) \phi'(w)}{p (1 - |\phi(w)|^2)^{1/(1/p)+1}}.
\]

The proof is complete.

Next, we consider the essential norm of \( \psi C_{\phi} : X \rightarrow Y \) is its distance to the set of compact operators \( K : X \rightarrow Y \), that is,
\[
\| T \|_{L_p} = \inf \{ \| T - K \|_{X \rightarrow Y} : K \text{ is a compact operator} \}.
\]

Here, \( X, Y \) are Banach spaces, and \( T \) is a bounded linear operator.
Theorem 7. Let $\nu$ be a radial, nonincreasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $\psi \in H(\mathbb{D})$, and $\psi \in S(\mathbb{D})$. Suppose that $\Psi C_\nu : \mathcal{B}_{p-1}^p \to \mathcal{B}_p^p$ is bounded. Then,

$$\|\Psi C_\nu\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} = \max \{E, G\} \approx \max \{M, T\}. \quad (32)$$

Here,

$$E := \limsup_{|\psi(z)| \to 1} \frac{v(z)2|\psi'(z)\psi(z) + \psi(z)\psi''(z)|}{(1 - |\psi(z)|^2)^{1/p}},$$

$$G := \limsup_{|\psi(z)| \to 1} \frac{v(z)\|\psi'(z)\|}{(1 - |\psi(z)|^2)^{1/p+1}},$$

$$M := \limsup_{j \to \infty} j^{1/p} \left\| 2\psi' - \psi' \right\|_{H_q^p},$$

$$T := \limsup_{j \to \infty} j^{1/p+1} \left\| \psi' \psi^{-1} \right\|_{H_q^p}. \quad (33)$$

Proof. First we show that $\|\Psi C_\nu\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \approx \max \{E, G\}$. Let $\{z_j\}_{j \in \mathbb{N}}$ be a sequence in the unit disk such that $|\psi(z_j)| \to 1$ as $j \to \infty$. Define

$$k_j(z) = \frac{1 - |\psi(z_j)|^2}{(1 - |\psi(z_j)|^2)^{1/p}} - \frac{2(1 + 2p)}{(1 + 2p)^2} \left(1 - |\psi(z_j)|^2\right)^{1/(p+1)},$$

$$m_j(z) = \frac{1 - |\psi(z_j)|^2}{(1 - |\psi(z_j)|^2)^{1/p}} - \frac{2(1 + 2p)}{(1 + 2p)^2} \left(1 - |\psi(z_j)|^2\right)^{1/(p+1)},$$

After a calculation, we get all $k_j$ and $m_j$ belong to $\mathcal{B}_{p-1}^p$ and

$$k_j(\psi(z_j)) = 0, k''(\psi(z_j)) = 0, |k_j'(\psi(z_j))| = \frac{\rho(\psi(z_j))}{1 + 2p(1 - |\psi(z_j)|^2)^{1/p}},$$

$$m_j(\psi(z_j)) = 0, m''(\psi(z_j)) = 0, |m_j'(\psi(z_j))| = \frac{2(1 - |\psi(z_j)|^2)^{1/(p+1)}}{(1 - |\psi(z_j)|^2)^{1/(p+1)}}. \quad (35)$$

Moreover, $k_j$ and $m_j$ converge to 0 uniformly on $\mathbb{D}$ as $j \to \infty$. Hence, for any compact operator $K : \mathcal{B}_p^p \to \mathcal{B}_p^p$, by Lemma 5, we get

$$\|\Psi C_\nu - K\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \geq \limsup_{j \to \infty} \|\Psi C_\nu(k_j)\|_{\mathcal{B}_p^p} - \limsup_{j \to \infty} \|K(k_j)\|_{\mathcal{B}_p^p},$$

$$\approx \limsup_{j \to \infty} \frac{v(z_j)2|\psi'(z_j)\psi(z_j) + \psi(z_j)\psi''(z_j)|}{\left(1 - |\psi(z_j)|^2\right)^{1/p}}, \quad (36)$$

Hence,

$$\|\Psi C_\nu\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} = \inf_K \|\Psi C_\nu - K\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p},$$

$$\geq \limsup_{j \to \infty} \frac{v(z_j)2|\psi'(z_j)\psi(z_j) + \psi(z_j)\psi''(z_j)|}{\left(1 - |\psi(z_j)|^2\right)^{1/p}} = \frac{1 - |\psi(z_j)|^2}{(1 - |\psi(z_j)|^2)^{1/(p+1)}},$$

as desired.

Next, we show that

$$\|\Psi C_\nu\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \leq \max \{E, G\}. \quad (38)$$

Let $r \in [0, 1]$. Define $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$(K_r f)(z) = f(z), z \in \mathbb{D}. \quad (39)$$

It is clear that $K_r$ is compact on $\mathcal{B}_p^p$ and $\|K_r\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \leq 1$. Moreover, $f - f \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0, 1)$ such that $r_j \to 1$ as $j \to \infty$. Then, $\Psi C_\nu K_{r_j} : \mathcal{B}_{p-1}^p \to \mathcal{B}_p^p$ is compact for each $j \in \mathbb{N}$. Hence,

$$\|\Psi C_\nu\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \leq \limsup_{j \to \infty} \|\Psi C_\nu - \Psi C_\nu K_{r_j}\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p}. \quad (40)$$

Thus, we only need to prove that

$$\limsup_{j \to \infty} \|\Psi C_\nu - \Psi C_\nu K_{r_j}\|_{\mathcal{B}_{p-1}^p \to \mathcal{B}_p^p} \leq \max \{E, G\}. \quad (41)$$
For any $f \in \mathcal{Z}_p^{\mathbb{R}}$ with $\|f\|_{\mathcal{Z}_p^{\mathbb{R}}} \leq 1$, by the facts that

$$\lim_{j \to \infty} \|\psi(0)f(\psi(0)) - \psi(0)f(r, \psi(0))\| = 0,$$

$$\lim_{j \to \infty} \|\psi'(0)(f - f_r') + \psi(0)(f - f_r')\| \psi(0)\|f'\|_1 = 0,$$

we have

$$\limsup_{j \to \infty} \|\psi_{C_0} - \psi_{C_pK_r}\|_{\mathcal{Z}_p} = \limsup_{j \to \infty} v(z)\left(\psi \cdot (f - f_r')\right)(z) \leq \limsup_{j \to \infty} v(z)\left(\psi \cdot (f - f_r')\right)(\psi(z)) \|2\psi(z)f(r, \psi(z)) + \psi(0)f'\|_1$$

Similarly,

$$\|\psi_{C_p} - \psi_{C_pK_r}\|_{\mathcal{Z}_p} \leq \max\{E, G\},$$

where $t \in \mathbb{N}$ is large enough such that $r_j \geq 1/2$ for all $j \geq t$. Since $f_r' - f \to 0$, $r_j f_j' - f' \to 0$, and $r_j^2 f_j'' - f'' \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, by Lemma 3, we obtain

$$P_1 = \limsup_{j \to \infty} v(z)\left(\psi \cdot (f - f_r')\right)(\psi(z)) \|\psi'(z)\|_1 = 0,$$

Using Lemma 1 and $\|f\|_{\mathcal{Z}_p^{\mathbb{R}}} \leq 1$, we obtain

$$\|\psi_{C_p}\|_{\mathcal{Z}_p^{\mathbb{R}}} \approx \max\{M, T\}.$$
\[ \| \psi C_{\varphi} \|_{\mathcal{L}_{p}^{p} \rightarrow \mathcal{B}} \geq G = \| \psi \varphi^{2} C_{\varphi} \|_{\mathcal{L}_{1} \rightarrow \mathcal{B}}. \]

\[ = \limsup_{j \to \infty} \| \varphi^{2} \varphi^{j} \|_{\mathcal{B}} = T. \] (55)

Therefore,

\[ \| \psi C_{\varphi} \|_{\mathcal{L}_{p}^{p} \rightarrow \mathcal{B}} \geq \max \{ M, T \}. \] (56)

The proof is complete.

From Theorem 7 and the well-known result that \( \| T \|_{cX \rightarrow Y} = 0 \) if and only if \( T : X \rightarrow Y \) is compact, we get the following corollary.

**Corollary 8.** Let \( \varphi \) be a radial, nonincreasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( \psi \in H(\mathbb{D}) \), and \( \varphi \in \mathbb{S}(\mathbb{D}) \). Suppose that \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) is bounded. Then, the following statements are equivalent:

(i) The operator \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) is compact

(ii) \( \limsup_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\varphi(z)||\varphi'(z)|^{2}/(1 - |\varphi(z)|^{2})^{1/(p+1)}}{1 - |\varphi(z)|} = 0 \)

and \( \limsup_{|\varphi(z)| \rightarrow 1} \frac{(v(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|/(1 - |\varphi(z)|^{2})^{1/(p)}}{1 - |\varphi(z)|} = 0 \)

(iii) \( \liminf_{j \to \infty} j^{1/p} \| \psi \varphi^{j} \varphi^{j-1} \|_{\mathcal{B}_{\varphi}} = \limsup_{j \to \infty} j^{1/p} \| \varphi \varphi' \varphi^{j-1} \|_{\mathcal{B}_{\varphi}} = 0 \)

Similarly to the above proof, we can get the characterizations of the boundedness, compactness, and essential norm of the weighted composition operator \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) as follows. The details are left to the interested readers.

**Theorem 9.** Let \( \varphi \) be a radial, nonincreasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( \psi \in H(\mathbb{D}) \), and \( \varphi \in \mathbb{S}(\mathbb{D}) \). Then, the following statements are equivalent.

(i) \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) is bounded

(ii) \( \psi \in \mathcal{B}_{\varphi} \), and

\[ \sup_{z \in \mathbb{D}} \frac{v(z)|\varphi(z)||\varphi'(z)|}{(1 - |\varphi(z)|^{2})^{1/p}} < \infty. \] (57)

(iii) \( \psi \in \mathcal{B}_{\varphi} \) and \( \sup_{j \geq 1} j^{1/p} \| \psi \varphi^{j} \varphi^{j-1} \|_{\mathcal{B}_{\varphi}} < \infty \)

**Theorem 10.** Let \( \varphi \) be a radial, nonincreasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( \psi \in H(\mathbb{D}) \), and \( \varphi \in \mathbb{S}(\mathbb{D}) \). Suppose that \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) is bounded. Then,

\[ \| \psi C_{\varphi} \|_{\mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi}} = \limsup_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\varphi(z)|^{1/p}}{(1 - |\varphi(z)|^{-1})^{1/p}} \]

\[ = \limsup_{j \to \infty} j^{1/p} \| \psi \varphi^{j} \varphi^{j-1} \|_{\mathcal{B}_{\varphi}}. \] (58)

**Corollary 11.** Let \( \varphi \) be a radial, nonincreasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( \psi \in H(\mathbb{D}) \), and \( \varphi \in \mathbb{S}(\mathbb{D}) \). Suppose that \( \psi C_{\varphi} : \mathcal{L}_{p}^{p} \rightarrow \mathcal{B}_{\varphi} \) is bounded. Then, the following statements are equivalent:

(i) \( \psi \in \mathcal{B}_{\varphi} \), and

(ii) \( \limsup_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\varphi(z)||\varphi'(z)|^{2}/(1 - |\varphi(z)|^{2})^{1/(p+1)}}{1 - |\varphi(z)|} = 0 \)

(iii) \( \limsup_{j \to \infty} j^{1/p} \| \psi \varphi^{j} \varphi^{j-1} \|_{\mathcal{B}_{\varphi}} = 0 \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that she has no conflicts of interest.

**References**


