Research Article

Fekete–Szegő Inequality for Bi-Univalent Functions Subordinate to Horadam Polynomials

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Making use of Horadam polynomials, we propose a special family of regular functions of the type $g(z) = z + \sum_{j=2}^{\infty} d_j z^j$ which are bi-univalent (or bi-schlicht) in the disc $\{z \in \mathbb{C}: |z| < 1\}$. We find estimates on the coefficients $|d_2|$ and $|d_3|$ and the functional of Fekete–Szegő for functions in this subfamily. Relevant connections to existing results and new observations of the main result are also presented.

1. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\}$ be the sets of real numbers and positive integers, respectively. Let $\mathbb{C}$ be the set of all complex numbers, and let $\mathfrak{D}$ denote the disc $\{z \in \mathbb{C}: |z| < 1\}$. We denote by $\mathcal{A}$, the set of all regular functions in $\mathfrak{D}$ that has the series of the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \quad (1)$$

and $\mathcal{S}$ be the set of all members of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. According to the well-known Koebe theorem (see [1]), every univalent function g has an inverse defined by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2 \omega^2 + (2d_2^2 - d_3) \omega^3 - (5d_2^3 - 5d_2 d_3 + d_4) \omega^4 + \dots, \quad (2)$$

satisfying $z = g^{-1}(g(z))$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$.

A function g of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if both g and g^{-1} are univalent in $\mathfrak{D}$. Let Σ stand for the set of bi-univalent functions having form (1).

Lewin [2] investigated the family Σ and proved that $|d_2| < 1.51$. Brannan and Clunie [3] claimed that $|d_2| < \sqrt{2}$. Later, Tan [4] obtained initial coefficient estimates for bi-univalent functions. Subsequently, Brannan and Taha [5] examined certain well-known subfamilies of Σ in $\mathfrak{D}$. The momentum on the study of bi-univalent function family was gained recently, which is due to the work of Srivastava et al. [6]. This article has revived the topic apparently, and many researchers have investigated several interesting special families of Σ (see [7–10]).

Recently, Hörçum and Koçer [11] (see also Horadam and Mahon [12]) examined the Horadam polynomials $\mathcal{H}_j(x)$ (or $\mathcal{H}_j(x, a, b; p, q)$), which is defined by the recurrence relation

$$\begin{aligned} \mathcal{H}_j(x) &= px\mathcal{H}_{j-1}(x) + q\mathcal{H}_{j-2}(x), \\ \mathcal{H}_1(x) &= a, \\ \mathcal{H}_2(x) &= bx, \end{aligned} \quad (3)$$

where $j \in \mathbb{N} \setminus \{1, 2\}$, $x \in \mathbb{R}$, p, q, a , and b are real constants. It is seen from (3) that $\mathcal{H}_3(x) = pbx^2 + qa$. The generating function of the sequence $\mathcal{H}_j(x)$, $j \in \mathbb{N}$, is as follows (see [11]):

$$\mathcal{G}(x, z) := \sum_{j=1}^{\infty} \mathcal{H}_j(x) z^{j-1} = \frac{(b - ap)xz + a}{1 - pxz - qz^2}, \quad (4)$$

where $z \in \mathbb{C}$ is such that $\Re(z) \neq x$, $x \in \mathbb{R}$.

Few particular cases of $\mathcal{H}_j(x, a, b; p, q)$ are

- (1) The Fibonacci polynomials, $F_j(x) = \mathcal{H}_j(x, 1, 1; 1, 1)$
- (2) The second type Chebyshev polynomials, $U_j(x) = \mathcal{H}_j(x, 1, 2; 2, -1)$
- (3) The first type Chebyshev polynomials, $T_j(x) = \mathcal{H}_j(x, 1, 1; 2, -1)$
- (4) The Lucas polynomials, $L_j(x) = \mathcal{H}_j(x, 2, 1; 1, 1)$
- (5) The Pell – Lucas polynomials, $Q_j(x) = \mathcal{H}_j(x, 2, 2; 2, 1)$,
- (6) The Pell polynomials, $P_j(x) = \mathcal{H}_j(x, 1, 2; 2, 1)$

The estimates on $|d_2|$ and $|d_3|$ and the very popular Fekete–Szegő functional were determined for bi-univalent functions linked with certain polynomials like Lucas polynomials, Fibonacci polynomials, Chebyshev polynomials, Horadam polynomials, and Gegenbauer polynomials. It is well-known that these polynomials play a potentially important role in architecture, approximation theory, physics, statistics, mathematical, and engineering sciences.

The recent research trend is the study of functions in Σ linked with any of the abovementioned polynomials. Generally, interest was shown to obtain the initial coefficient bounds and the celebrated inequality of Fekete–Szegő for the special subfamilies of Σ . Recently, the Horadam polynomial was used by Abirami et al. [13] to find coefficient estimates for the families of bi-Bazilevic and λ -bi-starlike function, Frasin et al. [14] obtained coefficient estimates and Fekete–Szegő inequalities for certain subfamilies of Al-Oboudi-type bi-univalent functions related to k -Fibonacci numbers involving modified activation function, initial coefficient bounds for certain subsets of bi-univalent functions family subordinate to Horadam polynomials were obtained in [15, 16], Shaba and Wanas [17] obtained coefficient bounds which are sharp, for a family of bi-univalent functions using (U, V) -Lucas polynomials, Srivastava et al. [18] have proposed a methodology to estimate coefficient bounds and Fekete–Szegő problem for certain subsets of bi-univalent function family linked with Horadam polynomials, and Swamy [19] and Swamy et al. [20, 21] have initiated the study of some subfamilies of bi-univalent function family subordinate to Horadam polynomials involving modified activation function. Swamy and Sailaja [22] have used Horadam polynomials to investigate coefficient estimates for two families of bi-univalent functions, Swamy et al. [23] have introduced some subfamilies of Sălăgean type bi-univalent functions subordinate to (m, n) -Lucas polynomials and found initial coefficients, and Wanas and Alina [24] have fixed the Fekete–Szegő problem for Bazilevic bi-univalent function class linked with Horadam polynomials.

For functions g and f holomorphic in $\mathfrak{D}$, g is said to subordinate f , if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$, and $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. This

subordination is indicated as $g \prec f$. In particular, if $f \in \mathcal{S}$, then $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the article [25] and the recent trends on functions in Σ , we present a comprehensive family of Σ associated with Horadam polynomials $\mathcal{H}_j(x)$ as in (3) having the generating function (4).

Throughout this paper, the inverse function $g^{-1}(w) = f(w)$ is as in (2) and $\mathcal{G}(x, z)$ is as in (4).

Definition 1. A function g in Σ having the power series (1) is said to be in the set $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$, $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\tau \geq 1$ and $x \in \mathbb{R}$, if

$$\frac{z(g'(z))^{\tau} + \mu z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} < 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D}, \quad (5)$$

$$\frac{\omega(f'(\omega))^{\tau} + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} < 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D}.$$

The family $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$ is of interest as it contains many existing as well as new subfamilies of Σ for particular choices of γ , τ , and μ , as illustrated as follows:

- (1) $\mathfrak{F}_{\Sigma}^{\tau}(x, \mu) \equiv S\mathfrak{E}_{\Sigma}^{\tau}(x, 0, \mu)$, $\mu \geq 0$ and $x \in \mathbb{R}$, is the collection of functions $g \in \Sigma$ satisfying

$$\begin{aligned} (g'(z))^{\tau} + \mu z g''(z) &< 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D}, \\ (f'(\omega))^{\tau} + \mu \omega f''(\omega) &< 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D}. \end{aligned} \quad (6)$$

- (2) $\mathfrak{R}_{\Sigma}^{\tau}(x, \mu) \equiv S\mathfrak{E}_{\Sigma}^{\tau}(x, 1, \mu)$, $\mu \geq 0$ and $x \in \mathbb{R}$, is the collection of functions $g \in \Sigma$ satisfying

$$\begin{aligned} \frac{z(g'(z))^{\tau}}{g(z)} + \mu \left(\frac{z^2 g''(z)}{g(z)} \right) &< 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D}, \\ \frac{\omega(f'(\omega))^{\tau}}{f(\omega)} + \mu \left(\frac{\omega^2 f''(\omega)}{f(\omega)} \right) &< 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D}. \end{aligned} \quad (7)$$

- (3) $\mathfrak{L}_{\Sigma}^{\tau}(x, \gamma) \equiv S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, 1)$, $0 \leq \gamma \leq 1$ and $x \in \mathbb{R}$, is the collection of functions $g \in \Sigma$ satisfying

$$\begin{aligned} \frac{z(g'(z))^{\tau} + z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} &< 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D}, \\ \frac{\omega(f'(\omega))^{\tau} + \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} &< 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D}. \end{aligned} \quad (8)$$

- (4) The function classes $S\mathfrak{E}_{\Sigma}^1(x, \gamma, \mu)$ ($0 \leq \gamma \leq 1$, $\mu \geq 0$ and $x \in \mathbb{R}$) and $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, 0)$ ($0 \leq \gamma \leq 1$ and $x \in \mathbb{R}$) were investigated by the author in [19].

Remark 1. We note that

- (i) $\mathfrak{F}_{\Sigma}^{\tau}(x, 1) \equiv \mathfrak{L}_{\Sigma}^{\tau}(x, 0)$, $\tau \geq 1$ and $x \in \mathbb{R}$
- (ii) $\mathfrak{R}_{\Sigma}^{\tau}(x, 1) \equiv \mathfrak{L}_{\Sigma}^{\tau}(x, 1)$, $\tau \geq 1$ and $x \in \mathbb{R}$

Remark 2.

- (i) For $\tau = 1$, the family $\mathfrak{F}_{\Sigma}^1(x, \mu)$ was investigated by Swamy and Sailaja [22]
- (ii) $S\mathfrak{E}_{\Sigma}^{\tau}(x, 1, 0) \equiv \mathfrak{R}_{\Sigma}^{\tau}(x, 0)$ was due to Abirami et al. [13]
- (iii) $S\mathfrak{E}_{\Sigma}^1(x, 1, \mu) \equiv \mathfrak{R}_{\Sigma}(x, \mu)$ was introduced by Magesh et al. [16]

Remark 3.

- (i) For $\mu = 0$ and $\tau = 1$, the class $\mathfrak{F}_{\Sigma}^1(x, 0) \equiv \mathcal{H}_{\Sigma}(x)$ was studied by Alamoush [15]
- (ii) For $\mathfrak{F}_{\Sigma}^1(x, 0) \equiv \mathcal{H}_{\Sigma}(x)$ and $\tau = 1$, the family $\mathfrak{R}_{\Sigma}^1(x, 0) \equiv S_{\Sigma}^*(x)$ was introduced by Srivastava et al. [6]

In Section 2, we derive the estimates for $|d_2|$ and $|d_3|$ and the inequality of Fekete and Szegő [26] for functions in the class $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$. In Section 3, relevant connections to the existing results and few interesting consequences of the main result are presented.

2. Bi-Univalent Function Class $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$

We determine the initial coefficients bounds and the inequality of Fekete–Szegő for functions in $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$, in the following theorem.

Theorem 1. *Let the function $g(z)$ defined by (1) be in the family $S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$ and let $0 \leq \gamma \leq 1, \mu \geq 0, \tau \geq 1$ and $x \in \mathbb{R}$. Then,*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma)\right)(bx)^2 - (2(\mu + \tau) - \gamma)^2(pbx^2 + qa)}}, \quad (9)$$

$$|d_3| \leq \frac{(bx)^2}{(2(\mu + \tau) - \gamma)^2} + \frac{|bx|}{(2\mu + \tau) - \gamma}, \quad (10)$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau) - \gamma}, & |1 - \delta| \leq J, \\ \frac{|bx|^3|1 - \delta|}{\left(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma)\right)(bx)^2 - (2(\mu + \tau) - \gamma)^2(pbx^2 + qa)}, & |1 - \delta| \geq J, \end{cases} \quad (11)$$

where

$$J = \frac{1}{(3(2\mu + \tau) - \gamma)} \left| \gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma) \right| (bx)^2 - (2(\mu + \tau) - \gamma)^2 \left(\frac{pbx^2 + qa}{b^2x^2} \right). \quad (12)$$

Proof. Let $g \in S\mathfrak{E}_{\Sigma}^{\tau}(x, \gamma, \mu)$. Then, on account of Definition 1, we get

$$\frac{z(g'(z))^{\tau} + \mu z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} = 1 - a + \mathcal{G}(x, \mathfrak{M}(z)), \quad (13)$$

$$\frac{\omega(f'(\omega))^{\tau} + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} = 1 - a + \mathcal{G}(x, \mathfrak{N}(\omega)), \quad (14)$$

where

$$\mathfrak{M}(z) = m_1 z + m_2 z^2 + m_3 z^3 + \dots, \quad (15)$$

$$\mathfrak{N}(\omega) = n_1 \omega + n_2 \omega^2 + n_3 \omega^3 + \dots, \quad (16)$$

are some regular functions in $\mathfrak{D}$ with $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| < 1, \mathfrak{N}(0) = 0$, and $|\mathfrak{N}(\omega)| < 1, z, \omega \in \mathfrak{D}$. It follows from (13)–(16) with (4) that

$$\frac{z(g'(z))^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} = 1 - a + \mathcal{H}_1(x) + \mathcal{H}_2(x)\mathbf{m}(z) \quad |\mathbf{n}_i| \leq 1, \quad (i \in \mathbb{N}). \quad (22)$$

$$+ \mathcal{H}_3(x)\mathbf{m}^2(z) + \dots, \quad (17)$$

$$\frac{\omega(f'(\omega))^\tau + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} = 1 - a + \mathcal{H}_1(x) + \mathcal{H}_2(x)\mathbf{n}(\omega) + \mathcal{H}_3(x)\mathbf{n}^2(\omega) + \dots. \quad (18)$$

From (17) and (18), in view of (3), we find

$$\frac{z(g'(z))^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} = 1 + \mathcal{H}_2(x)\mathbf{m}_1 z + [\mathcal{H}_2(x)\mathbf{m}_2 + \mathcal{H}_3(x)\mathbf{m}_1^2]z^2 + \dots, \quad (19)$$

$$\frac{\omega(f'(\omega))^\tau + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} = 1 + \mathcal{H}_2(x)\mathbf{n}_1 \omega + [\mathcal{H}_2(x)\mathbf{n}_2 + \mathcal{H}_3(x)\mathbf{n}_1^2]\omega^2 + \dots. \quad (20)$$

It is known that if $|\mathfrak{M}(z)| = |\mathbf{m}_1 z + \mathbf{m}_2 z^2 + \mathbf{m}_3 z^3 + \dots| < 1, z \in \mathfrak{D}$, then

$$|\mathbf{m}_i| \leq 1, \quad (i \in \mathbb{N}). \quad (21)$$

Similarly, if $|\mathfrak{N}(\omega)| = |\mathbf{n}_1 \omega + \mathbf{n}_2 \omega^2 + \mathbf{n}_3 \omega^3 + \dots| < 1, \omega \in \mathfrak{D}$, then

Comparing (19) and (20), we have

$$(2(\mu + \tau) - \gamma)d_2 = \mathcal{H}_2(x)\mathbf{m}_1, \quad (23)$$

$$(3(2\mu + \tau) - \gamma)d_3 + (\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1))d_2^2 = \mathcal{H}_2(x)\mathbf{m}_2 + \mathcal{H}_3(x)\mathbf{m}_1^2, \quad (24)$$

$$-(2(\mu + \tau) - \gamma)d_2 = \mathcal{H}_2(x)\mathbf{n}_1, \quad (25)$$

$$(3(2\mu + \tau) - \gamma)(2d_2^2 - d_3) + (\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1))d_2^2 = \mathcal{H}_2(x)\mathbf{n}_2 + \mathcal{H}_3(x)\mathbf{n}_1^2. \quad (26)$$

From (23) and (25), we easily obtain

$$\mathbf{m}_1 = -\mathbf{n}_1, \quad (27)$$

and also

$$2(2(\mu + \tau) - \gamma)^2 d_2^2 = (\mathbf{m}_1^2 + \mathbf{n}_1^2)(\mathcal{H}_2(x))^2. \quad (28)$$

If we add (24) and (26), then we obtain

$$2(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))d_2^2 = \mathcal{H}_2(x)(\mathbf{m}_2 + \mathbf{n}_2) + \mathcal{H}_3(x)(\mathbf{m}_1^2 + \mathbf{n}_1^2). \quad (29)$$

Substituting the value of $\mathbf{m}_1^2 + \mathbf{n}_1^2$ from (28) in (29), we get

$$d_2^2 = \frac{\mathcal{H}_2^3(x)(\mathbf{m}_2 + \mathbf{n}_2)}{2[(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))\mathcal{H}_2^2(x) - (2(\mu + \tau) - \gamma)^2 \mathcal{H}_3(x)]}, \quad (30)$$

which yields (10) on using (21) and (22).

After subtracting (26) from (24) and then using (27), we obtain

$$d_3 = d_2^2 + \frac{\mathcal{H}_2(x)(\mathbf{m}_2 - \mathbf{n}_2)}{2(3(2\mu + \tau) - \gamma)}. \quad (31)$$

Then, in view of (28), (31) becomes

$$d_3 = \frac{\mathcal{H}_2^2(x)(\mathbf{m}_1^2 + \mathbf{n}_1^2)}{2(2(\mu + \tau) - \gamma)^2} + \frac{\mathcal{H}_2(x)(\mathbf{m}_2 - \mathbf{n}_2)}{2(3(2\mu + \tau) - \gamma)}, \quad (32)$$

which yields (10) using (21) and (22).

From (30) and (31), for $\delta \in \mathbb{R}$, we get in view of (3) that

$$|d_3 - \delta d_2^2| = |\mathcal{H}_2(x)| \left| \left(\mathfrak{B}(\delta, x) + \frac{1}{2(3(2\mu + \tau) - \gamma)} \right) \mathbf{m}_2 + \left(\mathfrak{B}(\delta, x) - \frac{1}{2(3(2\mu + \tau) - \gamma)} \right) \mathbf{n}_2 \right|, \quad (33)$$

where

$$\mathfrak{B}(\delta, x) = \frac{(1 - \delta)\mathcal{H}_2^2(x)}{2[(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))\mathcal{H}_2^2(x) - (2(\mu + \tau) - \gamma)^2 \mathcal{H}_3(x)]}. \quad (34)$$

Clearly,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\mathcal{H}_2(x)|}{(3(2\mu + \tau) - \gamma)}, & 0 \leq |\mathfrak{B}(\delta, x)| \leq \frac{1}{2(3(2\mu + \tau) - \gamma)}, \\ 2|\mathcal{H}_2(x)||\mathfrak{B}(\delta, x)|, & |\mathfrak{B}(\delta, x)| \geq \frac{1}{2(3(2\mu + \tau) - \gamma)}, \end{cases} \quad (35)$$

from which we conclude (9) with J as in (12). Thus, Theorem 1 is proved. $\square$

Remark 4. By taking $\tau = 1$ in Theorem 1, we get a result of the author (Corollary 2.4 [19]), and by letting $\mu = 0$ in Theorem 1, we obtain another result of the author (Corollary 3.3 in [19]).

3. Outcome of the Main Result

Theorem 1 would yield the following outcome when $\gamma = 0$.

Corollary 1. Let the function $g(z)$ defined by (1) be in the family $\mathfrak{F}_\Sigma^\tau(x, \mu) \equiv S\mathfrak{E}_\Sigma^\tau(x, 0, \mu)$, and let $\mu \geq 0, \tau \geq 1$ and $x \in \mathbb{R}$. Then,

$$|d_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{[(\tau(2\tau + 1) + 6\mu)(bx)^2 - 4(\mu + \tau)^2(pbx^2 + qa)]}},$$

$$|d_3| \leq \frac{b^2 x^2}{4(\mu + \tau)^2} + \frac{|bx|}{3(2\mu + \tau)}, \quad (36)$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau)}, & |1 - \delta| \leq J_1, \\ \frac{|bx|^3 |1 - \delta|}{[(\tau(2\tau + 1) + 6\mu)(bx)^2 - 4(\mu + \tau)^2(pbx^2 + qa)]}, & |1 - \delta| \geq J_1, \end{cases} \quad (37)$$

where

$$J_1 = \frac{1}{3(2\mu + \tau)} \left| \tau(2\tau + 1) + 6\mu - 4(\mu + \tau)^2 \left(\frac{pbx^2 + qa}{b^2 x^2} \right) \right|. \quad (38)$$

Remark 5. For $\tau = 1$, Corollary 1 reduces to Corollary 2.1 of Swamy and Sailaja [22]. Also, we obtain Theorem 2.2 of Alamoush [15] from Corollary 1 when $\mu = 0$ and $\tau = 1$.

Allowing $\gamma = 1$ in Theorem 1, we obtain the following.

Corollary 2. Let the function $g(z)$ defined by (1) be in the set $\mathfrak{R}_\Sigma^\tau(x, \mu) \equiv S\mathfrak{E}_\Sigma^\tau(x, 1, \mu)$, and let $\mu \geq 0, \tau \geq 1$ and $x \in \mathbb{R}$. Then,

$$|d_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{[(\tau(2\tau - 1) + 4\mu)(bx)^2 - (2(\mu + \tau) - 1)^2(pbx^2 + qa)]}},$$

$$|d_3| \leq \frac{(bx)^2}{(2(\mu + \tau) - 1)^2} + \frac{|bx|}{3(2\mu + \tau) - 1}, \quad (39)$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau) - 1}, & |1 - \delta| \leq J_2, \\ \frac{|bx|^3 |1 - \delta|}{[(\tau(2\tau + 1) + 4\mu)(bx)^2 - (2(\mu + \tau) - 1)^2(pbx^2 + qa)]}, & |1 - \delta| \geq J_2, \end{cases} \quad (40)$$

where

$$J_2 = \frac{1}{(3(2\mu + \tau) - 1)} \left| \tau(2\tau - 1) + 4\mu - (2(\mu + \tau) - 1)^2 \left(\frac{pbx^2 + qa}{b^2x^2} \right) \right|. \quad (41)$$

Remark 6.

- (i) Corollary 2 coincides with Theorem 2.1 of Abirami et al. [13] when $\mu = 0$
- (ii) Corollary 2 further coincides with Theorem 2.1 of Magesh et al. [16] when $\tau = 1$
- (iii) We obtain Corollary 1 and Corollary 3 of Srivastava et al. [18] from Corollary 1 when $\mu = 0$ and $\tau = 1$

Setting $\mu = 1$ in Theorem 1, we have the following.

Corollary 3. Let the function $g(z)$ defined by (1) be in the class $\mathfrak{L}_{\Sigma}^{\tau}(x, \gamma) \equiv S\mathfrak{G}_{\Sigma}^{\tau}(x, \gamma, 1)$ and let $0 \leq \gamma \leq 1$, $\tau \geq 1$, and $x \in \mathbb{R}$. Then,

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left| (1-\gamma)^2 + (\tau-\gamma)(2\tau+1) + 5-\gamma \right| (bx)^2 - (2(1+\tau)-\gamma)^2 (pbx^2 + qa)}}, \quad (42)$$

$$|d_3| \leq \frac{(bx)^2}{(2(1+\tau)-\gamma)^2} + \frac{|bx|}{3(2+\tau)-\gamma},$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau) - \gamma}, & |1 - \delta| \leq J_3, \\ \frac{|bx|^3 |1 - \delta|}{\left| (1-\gamma)^2 + (\tau-\gamma)(2\tau+1) + 5-\gamma \right| (bx)^2 - (2(1+\tau)-\gamma)^2 (pbx^2 + qa)}}, & |1 - \delta| \geq J_3, \end{cases} \quad (43)$$

where

$$J_3 = \frac{1}{(3(2+\tau) - \gamma)} \left| (1-\gamma)^2 + (\tau-\gamma)(2\tau+1) + 5-\gamma - (2(1+\tau)-\gamma)^2 \left(\frac{pbx^2 + qa}{b^2x^2} \right) \right|. \quad (44)$$

By choosing appropriate values for parameters γ , τ , and μ in Theorem 1 and above corollaries, we have effectively shown the similarity between many subfamilies of bi-univalent functions.

4. Conclusion

A subfamily of bi-univalent (or bi-schlicht) functions is examined by using Horadam polynomial. Bounds of the first two coefficients $|d_2|$ and $|d_3|$ and the Fekete-Szegő

functional have been fixed for this subfamily. We have presented relevant connections to the existing results and few interesting consequences of the main result.

A subfamily examined in this article could inspire researchers to focus on other aspects such as a family bi-univalent functions using q -derivative operator [27, 28], a family using q -integral operator [29], meromorphic bi-univalent function family based on Al-Oboudi differential operator [30], regular bi-univalent function family based on Frasin operator [31], and a family using integro-differential operator [32].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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