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Research Article

A Self-Adaptive Extragradient Algorithm for Solving Quasimonotone Variational Inequalities

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This article aims to research iterative schemes for searching a solution of a quasimonotone variational inequality in a Hilbert space. For solving this quasimonotone variational inequality, we propose an iterative procedure which combines a self-adaptive rule and the extragradient algorithm. We demonstrate that the procedure weakly converges to the solution of the investigated quasimonotone variational inequality provided the considered operator satisfies several additional conditions.

1. Introduction

Variational inequality emerged in 1964 arising from the study of mechanics has many applications in engineering, economics, operations research, etc. ([1–4]). Variational inequality theory acts as a tool for solving many problems, such as equilibrium problems ([5, 6]), optimization problems ([7–9]), fixed point problems ([10–12]), and split problems ([13–17]). There are numerous iterative schemes for solving variational inequalities in the existing results; see [18–25]. Next, we briefly review several valuable iterative methods.

Throughout, suppose that $\mathscr{H}$ is a Hilbert space. The symbols $\langle \cdot , \cdot \rangle$ and $\|\cdot\|$ are the inner and norm of $\mathscr{H}$, respectively. Let $\emptyset \neq \mathscr{D} \subset \mathscr{H}$ be a convex and closed set. For an operator $\phi : \mathscr{D} \longrightarrow \mathscr{H}$, the variational inequality aims to seek a point $\nu \in \mathscr{D}$ satisfying

$$\langle \phi(\nu), \nu - \hat{\nu} \rangle \leq 0, \forall \hat{\nu} \in \mathscr{D}. \tag{1}$$

We use $S(\phi, \mathscr{D})$ to indicate the set of solutions of (1).

A valuable algorithm for solving (1) is the projection algorithm ([26, 27]) which generates a procedure as follows:

$$s_0 \in \mathscr{D}, s_{n+1} = proj_{\mathscr{D}}(s_n - \rho \phi(s_n)), \text{for all } n \geq 0, \tag{2}$$

where $proj_{\mathscr{D}} : \mathscr{H} \longrightarrow \mathscr{D}$ stands for the orthogonal projection and $\rho > 0$ means the step-size.

When ϕ is strongly monotone, strongly pseudomonotone, or inverse strongly monotone, iterative scheme (2) is convergent ([28, 29]).

Another powerful method is extragradient method studied by Korpelevich [30] which generates a procedure starting from an initial point $s_0 \in \mathscr{D}$:

$$\begin{cases} t_n = proj_{\mathscr{D}}(s_n - \rho_n \phi(s_n)), \\ s_{n+1} = proj_{\mathscr{D}}(s_n - \rho_n \phi(t_n)), \text{for all } n \geq 0. \end{cases} \tag{3}$$

Thereafter, (3) has been discussed extensively for solving (1); see, e.g., [30–34]. The main reason why the extragradient method attracts so much attention is that extragradient method can be used to find a solution of plain monotone operators. In fact, extragradient algorithm can be used to solve (1) if ϕ is pseudomonotone and sequentially weakly continuous ([35–37]).

Very recently, iterative methods for solving quasimonotone variational inequality have been investigated in the literature [24, 38, 39]. Especially, Salahuddin [40] utilized (3) for solving a Lipschitz quasimonotone variational inequality and achieve the following result.

Conclusion 1 ([40]). Assume that the operator ϕ satisfies (i) quasimonotone on $\mathscr{H}$; (ii) sequentially weakly continuous on $\mathscr{D}$; and (iii) Lipschitz continuous on $\mathscr{D}$. Suppose that $S(\phi, \mathscr{D}) \neq \varnothing$ and $\rho_n \in [a, b] \subset (0, (1/L)), \forall n \geq 0$. Then, $\{s_n\}$ obtained from (3) weakly converges to $\hat{z} \in S(\phi, \mathscr{D})$.

In this article, we further utilize extragradient method (3) for solving quasimonotone variational inequality (1). For this task, we will make use of an auxiliary tool regarding the following dual variational inequality which is to find $\nu \in \mathscr{D}$ satisfying:

$$\langle \phi(\hat{\nu}), \nu - \hat{\nu} \rangle \leq 0, \forall \hat{\nu} \in \mathscr{D}. \tag{4}$$

We use $S_d(\phi, \mathscr{D})$ to indicate the set of solutions of (4).

Notice that $S_d(\phi, \mathscr{D})$ is closed convex. At the same time, we have $S_d(\phi, \mathscr{D}) \subset S(\phi, \mathscr{D})$ when ϕ is continuous and $\mathscr{D}$ is convex. However, to acquire the convergence of the constructed sequence, one has to add the following extra condition

$$S(\phi, \mathscr{D}) \subset S_d(\phi, \mathscr{D}), \tag{5}$$

which implies that

$$\left\langle \phi(z), z - z^{\dagger} \right\rangle \geq 0, \forall z^{\dagger} \in S(\phi, \mathscr{D}) \text{ and } z \in \mathscr{D}. \tag{6}$$

Note that the above condition (5) holds if ϕ is pseudomonotone. However, this condition (5) is not satisfied when ϕ is quasimonotone. Further, self-adaptive rule was applied for solving variational inequality problems, see [41–45]. In this paper, for solving quasimonotone variational inequality (1), we propose an iterative procedure which combines a self-adaptive method and extragradient method (3) without using condition (5). We show that the suggested iterative procedure is weakly convergent. Our result extends the above theorem (1) at two aspects: on the one hand "sequential weak continuity" imposed on ϕ can be replaced by a more general restriction and on the other hand a self-adaptive technique is used to relax Lipschitz condition of ϕ.

2. Notions and Lemmas

Throughout, suppose that $\mathscr{H}$ is a Hilbert space and $\varnothing \neq \mathscr{D} \subset \mathscr{H}$ is convex and closed. A map $\phi : \mathscr{D} \longrightarrow \mathscr{H}$ is called

(1) Monotone if

$$\langle \phi(q) - \phi(\hat{q}), q - \hat{q} \rangle \geq 0, \ \forall q, \hat{q} \in \mathscr{D}. \tag{7}$$

(2) Pseudomonotone if

$$\langle \phi(\hat{q}), q - \hat{q} \rangle \geq 0 \Longrightarrow \langle \phi(q), q - \hat{q} \rangle \geq 0, \ \forall q, \hat{q} \in \mathscr{D}. \tag{8}$$

(3) Quasimonotone if

$$\langle \phi(\hat{q}), q - \hat{\nu} \rangle > 0 \Longrightarrow \langle \phi(q), q - \hat{q} \rangle \geq 0, \ \forall q, \hat{q} \in \mathscr{D}. \tag{9}$$

By the above definition, we can deduce that if ϕ is pseudomonotone, then ϕ must be quasimonotone. However, the reverse conclusion may fail.

A map $\phi : \mathscr{D} \longrightarrow \mathscr{H}$ is called Lipschitz continuous if

$$\|\phi(q) - \phi(\hat{q})\| \leq \tau \|q - \hat{q}\|, \forall q, \hat{q} \in \mathscr{D}, \tag{10}$$

where τ is some positive constant. In this case, we call $\phi \tau$-Lipschitz. ϕ is called nonexpansive provided $\tau = 1$.

An orthogonal projection from $\mathscr{H}$ onto $\mathscr{D}$, denoted by $proj_{\mathscr{D}}$ fulfills

$$\nu \in \mathscr{H}, \|\nu - proj_{\mathscr{D}}(\nu)\| \leq \|\hat{\nu} - \nu\|, \forall \hat{\nu} \in \mathscr{D}. \tag{11}$$

$proj_{\mathscr{D}}$ possesses the following characteristic inequality:

$$\nu \in \mathscr{H}, \langle \nu - proj_{\mathscr{D}}(\nu), \hat{\nu} - proj_{\mathscr{D}}(\nu) \rangle \leq 0, \forall \hat{\nu} \in \mathscr{D}. \tag{12}$$

3. Algorithms and Convergence Results

First, we declare several related conditions. Suppose that $\mathscr{H}$ is a Hilbert space and $\varnothing \neq \mathscr{D} \subset \mathscr{H}$ is convex and closed. Suppose that the involved operator ϕ satisfies three restrictions:

(t1) $\phi : \mathscr{H} \longrightarrow \mathscr{H}$ is a quasimonotone operator

(t2) ϕ is μ-Lipschitz on $\mathscr{D}$

(t3) If $\underline{\lim}_{n \longrightarrow +\infty} \|\phi(s_n)\| = 0$ with $\{s_n\}$ being a sequence in $\mathscr{H}$ and $s_n \rightharpoonup s^{\ddagger}$, then $\phi(s^{\ddagger}) = 0$

In the sequel, assume that $S_d(\phi, \mathscr{D}) \neq \varnothing$ and the set $\{t^{\dagger} \in \mathscr{D} : \phi(t^{\dagger}) = 0\} \setminus S_d(\phi, \mathscr{D})$ is finite.

Suppose that ϖ, ζ and $\hat{\varsigma}$ are three constants in the open interval $(0, 1)$. Suppose that $\{\tau_n\}$ is a sequence in $(0, 2)$ satisfying $0 < \underline{\lim}_{n \longrightarrow \infty} \tau_n \leq \overline{\lim}_{n \longrightarrow \infty} \tau_n < 2$.

Next, we state our scheme for solving (1).

Algorithm 2. Select a fixed point u_0 in $\mathscr{D}$. Set $n = 0$.

Step 1. Assume u_n is presented. Compute

$$w_n = proj_{\mathscr{D}}[u_n - \varpi \zeta_n \phi(u_n)], \tag{13}$$

where $\zeta_n = \max \{1, \zeta, \zeta^2, \cdots\}$ fulfills

$$\zeta_n \|\phi(w_n) - \phi(u_n)\| \leq \frac{1 - \hat{\varsigma}}{\varpi} \|w_n - u_n\|. \tag{14}$$

Step 2. Compute $\widehat{w}_n = proj_{\mathscr{D}}[u_n - \phi(u_n)]$. (2a) If $\widehat{w}_n = u_n$, then stop. (2b) If $\widehat{w}_n \neq u_n$, then calculate

$$v_n = \frac{\widehat{v}_n}{\|\widehat{v}_n\|^2} \|w_n - u_n\|^2, \tag{15}$$

where $\widehat{v}_n = u_n - w_n + \varpi\zeta_n\phi(w_n)$ and compute

$$u_{n+1} = proj_{\mathscr{D}}(u_n - \widehat{\varsigma}\tau_n v_n). \tag{16}$$

Let $n := n + 1$ and return to Step 1.

Conclusion 3. Inequality (14) is well-defined. Moreover, $0 < (1 - \widehat{\varsigma})\zeta/\varpi\mu < \zeta_n \leq 1 - \widehat{\varsigma}/\varpi\mu$ or $\zeta_n = 1$.

Proof. Since ϕ is μ-Lipschitz, we obtain

$$\|\phi(proj_{\mathscr{D}}[u_n - \varpi\zeta_n\phi(u_n)]) - \phi(u_n)\| \leq \mu\|proj_{\mathscr{D}}[u_n - \varpi\zeta_n\phi(u_n)] - u_n\|, \tag{17}$$

which equals to

$$\frac{1 - \widehat{\varsigma}}{\mu\varpi}\|\phi(proj_{\mathscr{D}}[u_n - \varpi\zeta_n\phi(u_n)]) - \phi(u_n)\| \leq \frac{1 - \widehat{\varsigma}}{\varpi}\|proj_{\mathscr{D}}[u_n - \varpi\zeta_n\phi(u_n)] - u_n\|. \tag{18}$$

This implies that (14) holds for all $\zeta_n \leq 1 - \widehat{\varsigma}/\mu\varpi$.

It is obviously that

$$\varpi\frac{\zeta_n}{\zeta}\mu\|w_n - u_n\| \geq \varpi\frac{\zeta_n}{\zeta}\|\phi(w_n) - \phi(u_n)\| > (1 - \widehat{\varsigma})\|w_n - u_n\|, \tag{19}$$

which implies that $\zeta_n > (1 - \widehat{\varsigma})\zeta/\varpi\mu > 0$. □

Conclusion 4. (*i*) If $\widehat{w}_n = u_n$, then $u_n \in S(\phi, \mathscr{D})$. (ii) If $\widehat{w}_n \neq u_n$, then $\|\widehat{v}_n\| > 0$ and (15) is well-defined.

Proof.

(i) If $\widehat{w}_n = u_n$, that is $u_n = proj_{\mathscr{D}}[u_n - \phi(u_n)]$, by virtue of (12), we acquire

$$\langle u_n - [u_n - \phi(u_n)], v - u_n\rangle \geq 0, \forall v \in \mathscr{D}, \tag{20}$$

which results in that $u_n \in S(\phi, \mathscr{D})$.

(ii) Take $x^* \in S_d(\phi, \mathscr{D})$. Notice that

$$\begin{aligned}\langle\widehat{v}_n, u_n - x^*\rangle &= \langle u_n - w_n + \varpi\zeta_n\phi(w_n), u_n - x^*\rangle \\ &= \langle u_n - w_n - \varpi\zeta_n\phi(u_n), u_n - x^*\rangle \\ &\quad + \varpi\zeta_n\langle\phi(u_n), u_n - x^*\rangle \\ &\quad + \varpi\zeta_n\langle\phi(w_n), u_n - w_n\rangle \\ &\quad + \varpi\zeta_n\langle\phi(w_n), w_n - x^*\rangle.\end{aligned} \tag{21}$$

As a result of $x^* \in S_d(\phi, \mathscr{D})$ and $u_n \in \mathscr{D}$, we have

$$\langle\phi(u_n), u_n - x^*\rangle \geq 0. \tag{22}$$

As the same as (22), we obtain

$$\langle\phi(w_n), w_n - x^*\rangle \geq 0, \tag{23}$$

due to $w_n \in \mathscr{D}$.

In view of (21)-(23), we receive

$$\begin{aligned}\langle\widehat{v}_n, u_n - x^*\rangle &\geq \langle u_n - w_n - \varpi\zeta_n\phi(u_n), u_n - x^*\rangle \\ &\quad - \varpi\zeta_n\langle\phi(w_n), w_n - u_n\rangle \\ &= \langle w_n - u_n + \varpi\zeta_n(\phi(u_n) - \phi(w_n)), w_n - u_n\rangle \\ &\quad + \langle w_n - u_n + \varpi\zeta_n\phi(u_n), x^* - w_n\rangle \\ &= \|w_n - u_n\|^2 - \varpi\zeta_n\langle\phi(w_n) - \phi(u_n), w_n - u_n\rangle \\ &\quad + \langle w_n - u_n + \varpi\zeta_n\phi(u_n), x^* - w_n\rangle.\end{aligned} \tag{24}$$

Owing to $\zeta_n > (1 - \widehat{\varsigma})\zeta/\varpi\mu > 0$, from (14), we get

$$\begin{aligned}\langle\phi(w_n) - \phi(u_n), w_n - u_n\rangle &\leq \|\phi(w_n) - \phi(u_n)\|\|w_n - u_n\| \\ &\leq \frac{1 - \widehat{\varsigma}}{\varpi\zeta_n}\|u_n - w_n\|^2.\end{aligned} \tag{25}$$

Using (12) of $proj_{\mathscr{D}}$ and (14), we acquire

$$\langle u_n - w_n - \varpi\zeta_n\phi(u_n), w_n - x^*\rangle \geq 0. \tag{26}$$

In the light of (24), (25) and (26), we achieve

$$\langle\widehat{v}_n, u_n - x^*\rangle \geq \widehat{\varsigma}\|w_n - u_n\|^2. \tag{27}$$

If $\widehat{w}_n \neq u_n$, then $w_n \neq u_n$. Otherwise, by virtue of (12), $\langle u_n - [u_n - \varpi\zeta_n\phi(u_n)], v - u_n\rangle \geq 0, \forall v \in \mathscr{D}$ which results in that $u_n \in S(\phi, \mathscr{D})$ and hence $\widehat{w}_n = u_n$. This leads to a contradiction. So, $\|w_n - u_n\| > 0$. It follows from (27) that $\langle\widehat{v}_n, u_n - x^*\rangle > 0$ which yields that $\|\widehat{v}_n\| > 0$. Therefore, (15) is well-defined.

In this position, we prove a main theorem. □

Theorem 5. *$\{u_n\}$ defined by Algorithm 2 weakly converges to an element in $S(\phi, \mathscr{D})$.*

Proof. Let $x^* \in S_d(\phi, \mathscr{D})$. Since $proj_{\mathscr{D}}$ is nonexpansive and $x^* \in \mathscr{D}$, from (16) and (21), we have

$$\begin{aligned} \|u_{n+1} - x^*\|^2 &= \|proj_{\mathscr{D}}(u_n - \widehat{\varsigma}\tau_n v_n) - proj_{\mathscr{D}}(x^*)\|^{2\leq \left\|u_n - x^* - \widehat{\varsigma}\tau_n v_n\right\|^2} \\ &= (\widehat{\varsigma}\tau_n)^2\|v_n\|^2 - 2\widehat{\varsigma}\tau_n\langle v_n, u_n - x^*\rangle + \|u_n - x^*\|^2 \\ &= \|u_n - x^*\|^2 - 2\widehat{\varsigma}\tau_n \frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|^2}\langle \widehat{v}_n, u_n - x^*\rangle \\ &\quad + (\widehat{\varsigma}\tau_n)^2 \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2}. \end{aligned} \tag{28}$$

Substituting (27) into (28) to derive

$$\|u_{n+1} - x^*\|^2 \leq \|u_n - x^*\|^2 - \widehat{\varsigma}^2(2 - \tau_n)\tau_n \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2}. \tag{29}$$

Noting that $0 < \underline{\lim}_{n\longrightarrow\infty}\tau_n \leq \overline{\lim}_{n\longrightarrow\infty}\tau_n < 2$, by (29), we acquire

$$\|u_{n+1} - x^*\| \leq \|u_n - x^*\|, \tag{30}$$

which leads to that $\lim_{n\longrightarrow\infty}\|u_n - x^*\|$ exists. Then, $\{u_n\}$ is bounded and so is $\{\phi(u_n)\}$. From (13), we have

$$\|w_n - u_n\| = \|proj_{\mathscr{D}}[u_n - \varpi\zeta_n\phi(u_n)] - proj_{\mathscr{D}}[u_n]\| \leq \varpi\zeta_n\|\phi(u_n)\|. \tag{31}$$

Hence, $\{w_n\}$ and $\{\phi(w_n)\}$ are bounded. □

Taking into account (29), we gain

$$\widehat{\varsigma}^2(2 - \tau_n)\tau_n \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2} \leq -\|u_{n+1} - x^*\|^2 + \|u_n - x^*\|^2. \tag{32}$$

This leads to

$$\frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|} \longrightarrow 0. \tag{33}$$

Thanks to the boundedness of $\widehat{v}_n = u_n - w_n + \varpi\zeta_n\phi(w_n)$, by (33), we have

$$\lim_{n\longrightarrow\infty}\|w_n - u_n\| = 0. \tag{34}$$

With the help of the Lipschitz continuity of ϕ, from (34), we deduce

$$\lim_{n\longrightarrow\infty}\|\phi(w_n) - \phi(u_n)\| = 0. \tag{35}$$

Based on (15) and (16), we derive

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|proj_{\mathscr{D}}(u_n - \widehat{\varsigma}\tau_n v_n) - proj_{\mathscr{D}}(u_n)\| \leq \widehat{\varsigma}\tau_n\|v_n\| \\ &= \widehat{\varsigma}\tau_n \frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|}. \end{aligned} \tag{36}$$

In the light of (33) and (36), we achieve

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = 0. \tag{37}$$

By (12) and (13), we deduce

$$\langle u_n - \varpi\zeta_n\phi(u_n) - w_n, z - w_n\rangle \leq 0, \forall z \in \mathscr{D}. \tag{38}$$

So,

$$\frac{1}{\varpi\zeta_n}\langle u_n - w_n, z - w_n\rangle + \langle\phi(u_n), w_n - u_n\rangle \leq \langle\phi(u_n), z - u_n\rangle, \forall z \in \mathscr{D}. \tag{39}$$

Observe that $\{u_n\}$, $\{w_n\}$, and $\{\phi(u_n)\}$ are bounded. According to (34) and (39), we have

$$\underline{\lim}_{n\longrightarrow+\infty}\langle\phi(u_n), z - u_n\rangle \geq 0, \forall z \in \mathscr{D}. \tag{40}$$

Owing to $\{u_n\}$ is bounded, there is $\{u_{n_i}\} \subset \{u_n\}$ fulfilling $u_{n_i} \rightharpoonup b^\dagger \in \mathscr{D}$ as $i \longrightarrow +\infty$. Taking into account (40), we attain

$$\underline{\lim}_{i\longrightarrow+\infty}\left\langle\phi\left(u_{n_i}\right), z - u_{n_i}\right\rangle \geq 0, \forall z \in \mathscr{D}. \tag{41}$$

If $\underline{\lim}_{i\longrightarrow+\infty}\|\phi(u_{n_i})\| = 0$, by $u_{n_i} \rightharpoonup b^\dagger$ and ϕ verifying (t1), we get that $\phi(b^\dagger) = 0$. Then, $b^\dagger \in S(\phi, \mathscr{D})$.

Now, we assume that $\underline{\lim}_{i\longrightarrow+\infty}\|\phi(u_{n_i})\| > 0$. Then, there is an integer $m > 0$ fulfilling $\|\phi(u_{n_i})\| > 0$ for all $i \geq m$. By virtue of (41), we attain

$$\underline{\lim}_{i\longrightarrow+\infty}\left\langle \frac{\phi(u_{n_i})}{\|\phi(u_{n_i})\|}, z - u_{n_i}\right\rangle \geq 0, \forall z \in \mathscr{D}. \tag{42}$$

Let $\{\varepsilon_k\}$ be a real number sequence fulfilling $\varepsilon_k > 0, \varepsilon_{k+1} < \varepsilon_k$ and $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow +\infty$. Based on (42), there is $\{n_{i_k}\}$ of $\{n_i\}$ fulfilling $n_{i_k} \geq m (k \geq 0)$ and

$$\left\langle \frac{\phi\left(u_{n_{i_k}}\right)}{\left\|\phi\left(u_{n_{i_k}}\right)\right\|}, z - u_{n_{i_k}}\right\rangle + \varepsilon_k > 0, \forall k \geq 0, \forall z \in \mathscr{D}, \tag{43}$$

which results in that

$$\left\langle\phi\left(u_{n_{i_k}}\right), z - u_{n_{i_k}}\right\rangle + \varepsilon_k\left\|\phi\left(u_{n_{i_k}}\right)\right\| > 0, \forall k \geq 0, \forall z \in \mathscr{D}. \tag{44}$$

Put $b_k = \phi(u_{n_{i_k}})/\|\phi(u_{n_{i_k}})\|^2$ for all $k \geq 0$. It is easily seen that $\langle \phi(u_{n_{i_k}}), b_k \rangle = 1$ for all $k \geq 0$. With the help of (44), we achieve

$$\left\langle \phi\left(u_{n_{i_k}}\right), \varepsilon_k b_k \left\| \phi\left(u_{n_{i_k}}\right) \right\| + z - u_{n_{i_k}} \right\rangle > 0, \forall k \geq 0, \forall z \in \mathscr{D}. \tag{45}$$

Owing to (45) and using the quasimonotonicity of ϕ, we acquire

$$\left\langle \phi\left(z + \varepsilon_k b_k \left\| \phi\left(u_{n_{i_k}}\right) \right\| \right), \varepsilon_k b_k \left\| \phi\left(u_{n_{i_k}}\right) \right\| + z - u_{n_{i_k}} \right\rangle \geq 0, \forall k \geq 0, \forall z \in \mathscr{D}. \tag{46}$$

As a result of Lipschitz continuity of ϕ and $\lim_{k \longrightarrow +\infty} \varepsilon_k \|b_k\| \|\phi(u_{n_{i_k}})\| = \lim_{k \longrightarrow +\infty} \varepsilon_k = 0$, we deduce $\phi(z + \varepsilon_k b_k \|\phi(u_{n_{i_k}})\|) \longrightarrow \phi(z)$ as $k \longrightarrow +\infty$. In (46), letting $k \longrightarrow +\infty$, we receive

$$\left\langle \phi(z), z - b^\dagger \right\rangle \geq 0, \ \forall z \in \mathscr{D}. \tag{47}$$

Thus, $b^\dagger \in S_d(\phi, \mathscr{D})$. Therefore, $\omega_w(u_n) \subset (\{t^\dagger \in \mathscr{D} : \phi(t^\dagger) = 0\} \cup S_d(\phi, \mathscr{D})) \subset S(\phi, \mathscr{D})$. Next, we prove $\{u_n\}$ has no more than one weak cluster point in $S_d(\phi, \mathscr{D})$. Suppose that $b^\dagger \in S_d(\phi, \mathscr{D})$ and $c^\ddagger \in S_d(\phi, \mathscr{D})$ are two weak cluster points of $\{u_n\}$. Then, there exist two subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $\{u_{n_j}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup b^\dagger$ and $u_{n_j} \rightharpoonup c^\ddagger$.

It is obviously that

$$2\left\langle u_n, b^\dagger - c^\ddagger \right\rangle = \|u_n - c^\ddagger\|^2 - \|u_n - b^\dagger\|^2 + \|b^\dagger\|^2 - \|c^\ddagger\|^2, \forall n \geq 0. \tag{48}$$

Letting $n \longrightarrow \infty$ on both sides of (48), we have that $\lim_{n \longrightarrow +\infty} \langle u_n, b^\dagger - c^\ddagger \rangle$ exists, denoted by $\widehat{a}$. Therefore,

$$\lim_{i \longrightarrow +\infty} \left\langle u_{n_i}, b^\dagger - c^\ddagger \right\rangle = \widehat{a} = \lim_{j \longrightarrow +\infty} \left\langle u_{n_j}, b^\dagger - c^\ddagger \right\rangle \tag{49}$$

Note that $u_{n_i} \rightharpoonup b^\dagger$ and $u_{n_j} \rightharpoonup c^\ddagger$. By (45), we obtain

$$\left\langle b^\dagger, b^\dagger - c^\ddagger \right\rangle = \widehat{a} = \left\langle b^\dagger - c^\ddagger, c^\ddagger \right\rangle, \tag{50}$$

which yields that $c^\ddagger = b^\dagger$. So, $\{u_n\}$ has no more than one weak cluster point in $S_d(\phi, \mathscr{D})$. Since the set $\{t^\dagger \in \mathscr{D} : \phi(t^\dagger) = 0\} \setminus S_d(\phi, \mathscr{D})$ is finite, we deduce that $\{u_n\}$ has only finite weak cluster points in $S(\phi, \mathscr{D})$. Let $w_1, w_2, \cdots, w_p$ be p unequal weak cluster points of $\{u_n\}$ in $S(\phi, \mathscr{D})$. Let $\Gamma = \{1, 2, \cdots, p\}$ and

$$\alpha = \min \{\|w_r - w_s\|/4, r, s \in \Gamma, r \neq s\}. \tag{51}$$

For $w_r, r \in \Gamma$, there is $\{u^r_{n_i}\}$ of $\{u_n\}$ fulfilling $u^r_{n_i} \rightharpoonup w_r$ when $i \longrightarrow +\infty$. Hence,

$$\lim_{i \longrightarrow +\infty} \left\langle u^r_{n_i}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle = \left\langle w_r, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle, \forall s \in \Gamma. \tag{52}$$

For $\forall s \neq r$, we have

$$\begin{aligned} \langle w_r, (w_r - w_s)/\|w_r - w_s\| \rangle &= \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|) \\ &\quad + \|w_r - w_s\|/2 > \alpha + \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|). \end{aligned} \tag{53}$$

Thanks to (52) and (53), there exists an integer $N^r_i > 0$ satisfying for all $i \geq N^r_i$,

$$u^r_{n_i} \in \left\{ \widehat{u} : \left\langle \widehat{u}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > \alpha + \frac{\|w_r\|^2 - \|w_s\|^2}{2\|w_r - w_s\|} \right\}, s \in \Gamma, s \neq r. \tag{54}$$

Let

$$\Omega_r = \bigcap_{s=1, s \neq r}^{p} \left\{ \widehat{u} : \left\langle \widehat{u}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > \alpha + \frac{\|w_r\|^2 - \|w_s\|^2}{2\|w_r - w_s\|} \right\}. \tag{55}$$

Combining (54) with (55), we obtain $u^r_{n_i} \in \Omega_r, \forall i \geq \max \{N^r_i, r \in \Gamma\}$.

Next we demonstrate that if n is large enough, $u_n \in \cup_{r=1}^p \Omega_r$. Suppose that there is $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \notin \cup_{r=1}^p \Omega_r$. According to the boundedness of $\{u_{n_j}\}$, there exists a subsequence of $\{u_{n_j}\}$, without loss of generality, still denoted by $\{u_{n_j}\}$, which converges weakly to $\widehat{v}$. Hence $u_{n_j} \notin \Omega_r$ for any $r \in \Gamma$. Then, there is a subsequence $\{u_{n_{j_s}}\}$ of $\{u_{n_j}\}$ such that $\forall s \geq 0$:

$$\begin{aligned} u_{n_{j_s}} \notin \{\widehat{u} : &\langle \widehat{u}, (w_r - w_s)/\|w_r - w_s\| \rangle \\ &> \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|) + \alpha\}, s \in \Gamma, s \neq r. \end{aligned} \tag{56}$$

So,

$$\begin{aligned} \widehat{v} \notin \{\widehat{u} : &\langle \widehat{u}, (w_r - w_s)/\|w_r - w_s\| \rangle \\ &> \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|) + \alpha\}, s \in \Gamma, s \neq r, \end{aligned} \tag{57}$$

which results in that $\widehat{v} \neq w_r (\forall r \in \Gamma)$. It leads to a contradiction. Thus, there is a large enough integer $\tilde{N}$ such that $u_n \in \cup_{r=1}^p \Omega_r$ for all $n \geq \tilde{N}$.

Finally, we show that $\omega_w(u_n)$ is singleton in $S(\phi, \mathscr{D})$. Suppose that $p \geq 2$. Taking into account (37), there is $\widehat{N} \geq \tilde{N}$ fulfilling $\|u_{n+1} - u_n\| < \alpha$ when $n \geq \widehat{N}$. Hence, there is $m \geq \widehat{N}$ fulfilling $u_m \in \Omega_r$ and $u_{m+1} \in \Omega_s$, where $r, s \in \Gamma$ and $p \geq 2$, that is

$$u_m \in \Omega_r = \bigcap_{s=1,s\neq r}^{p} \{\hat{u} : \langle \hat{u}, (w_r - w_s)/\|w_r - w_s\|\rangle > \alpha + \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|)\}$$

$$u_{m+1} \in \Omega_s = \bigcap_{r=1,r\neq s}^{p} \{\hat{u} : \langle \hat{u}, (w_s - w_r)/\|w_s - w_r\|\rangle > \alpha + \left(\|w_s\|^2 - \|w_r\|^2\right)/(2\|w_s - w_r\|)\}. \quad (58)$$

Thus, we have

$$\langle u_m, (w_r - w_s)/\|w_r - w_s\|\rangle > \alpha + \left(\|w_r\|^2 - \|w_s\|^2\right)/(2\|w_r - w_s\|) \quad (59)$$

$$\langle u_{m+1}, (w_s - w_r)/\|w_s - w_r\|\rangle > \alpha + \left(\|w_s\|^2 - \|w_r\|^2\right)/(2\|w_s - w_r\|). \quad (60)$$

Thanks to (59) and (60), we acquire

$$\left\langle u_m - u_{m+1}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > 2\alpha. \quad (61)$$

Note that

$$\|u_{m+1} - u_n\| < \alpha. \quad (62)$$

By virtue of (61) and (62), we receive

$$2\alpha < \left\langle u_m - u_{m+1}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle \leq \|u_n - u_{m+1}\| < \alpha, \quad (63)$$

which is impossible. Then, $\omega_w(u_n)$ is singleton in $S(\phi, \mathscr{D})$. So, $\{u_n\}$ weakly converges to an element in $S(\phi, \mathscr{D})$.

Remark 6. A map $\phi : \mathscr{H} \longrightarrow \mathscr{H}$ is called weakly sequentially continuous, if $z_n \rightharpoonup \tilde{z} \Rightarrow \phi(z_n) \rightharpoonup \phi(\tilde{z})$, where $\{z_n\}$ is any sequence in $\mathscr{H}$.

To solve (1), many existing results have imposed the above "sequential weak-to-weak continuity" condition on ϕ; see, [35, 36, 40]. We can check if ϕ satisfies sequential weak continuity and then ϕ satisfies condition (t3).

4. Conclusions

The main purpose of this paper is to investigate iterative algorithms for solving variational inequality (1). A powerful method to solve (1) is extragradient method (3) introduced by Korpelevich [30] where the involved operator ϕ is pseudomonotone monotone. Based on the corresponding result of Salahuddin [40], we further apply extragradient method (3) to solve quasimonotone variational inequality (1).

We propose an iterative algorithm (Algorithm 2) which combines a self-adaptive rule and the extragradient algorithm. In general, in order to show $\omega(u_n)$ belongs to the solution set, ϕ should be sequentially weakly continuous. In this paper, we replace these conditions by a weaker condition (t3). We demonstrate that the procedure weakly converges to the solution of the investigated quasimonotone variational inequality under several additional conditions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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