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Research Article

A Generalized Decision-Making Technique Based on Bipolar-Valued Multivague Soft Sets

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The decision-making technique, launched by Roy and Maji, is considered an effective method to overcome uncertainty and fuzziness in decision-making problems, though, adapting it to reflect the problem parameters' vagueness, as well as multibipolarity, is very difficult. So, in this article, the bipolarity is interpolated into the multivague soft set of order n. This gives a new more generalized, flexible, and applicable extension than the fuzzy soft model, or any previous hybrid model, which is the bipolar-valued multivague soft model of dimension n. Moreover, types of bipolar-valued multivague soft sets of dimension n, as well as some new associated concepts and operations, are investigated with examples. Furthermore, properties of bipolar-valued multivague soft sets of dimension n including absorption, commutative, associative, and distributive properties, as well as De Morgan's laws, are provided in detail. Finally, a bipolar-valued multivague soft set-designed decision-making algorithm, as well as a real-life example, are discussed generalizing the Roy and Maji method.

1. Introduction, Motivation, and Related Works

Nowadays, making decisions has risen to be one of the most significant areas of human endeavor, because of its strong connection to efficiency and sustainability. Making wise and effective judgments is how successful individuals accomplish their personal and professional aims. Individual perspectives, attitudes, and beliefs, along with ideas, are regularly utilized to guide decision-making. While a person can make judgments based on a variety of concepts, he should be very careful to choose one that is effective and adds to great achievement. The difficulty of making decisions in an uncertain environment has attracted attention recently. The most regular sources of complication while trying to make judgments in the actual world are uncertainty and confusion. Uncertain data is frequent and implicit in many basic applications, particularly where decision-making is needed. This ambiguous information is indeed a consequence of missing or delayed retrieval of information, information unpredictability, calculation tool restrictions, etc. One can find several kinds of research and uses of various specialized mathematical methods in the literature.

In 1937, Black [1] introduced the vagueness of a term as it is shown by producing "borderline cases" to model uncertain data in many important applications. In 1965, Zadeh [2] introduced an extension of ordinary (crisp) sets for describing uncertainty and dealing with it, namely the theory of the fuzzy sets. Just as an ordinary set on the initial universal set X is determined by its membership function from X to $\{0, 1\}$, in fuzzy set theory, an element's membership degree is determined by its membership percentage (the characteristic function from the domain X to the interval $[0, 1]$). The fuzzy set is considered a useful mathematical approach for working with ambiguity, but it has the following limitation. The particular number (membership percentage) includes advice for and against object belonging without indicating how much more of each there really is, i.e., this single number gives us very little information about its accuracy. In 1986, Atanassov [3] gave the notion "intuitionistic fuzzy set," which is a generalization of the word "fuzzy set," along with an example.

In 1993, Gau and Buehrer [4] developed a new extension of the set theory called vague set theory, based on Black's idea of vagueness in 1937 [1], to overcome the difficulties posed by fuzzy set theory. Instead of just a single value, they gave each object a membership degree, which is a subinterval of $[0, 1]$. The soft set idea was first suggested by Molodtsov [5], in 1999, when it was claimed that one of the causes of the aforementioned problems might be the inadequate parametrization tools of the aforementioned theories. The softness concept or the soft set concept is a mathematical tool, free from those above difficulties, for dealing easier with uncertainties. In 2002, Maji et al. ([6, 7]) looked into and explored the Molodtsov-initiated soft set theory. They examined a variety of concepts related to soft sets, created a comprehensive theoretical overview of soft set theory, and then used it to solve a problem involving decision-making.

Numerous attempts were made to create the soft version of all known crisp concepts and many more results in the soft set environment, as the following: Ali et al. [8] introduced many new definitions and concepts in the soft set theory. In addition, Sezgin and Atagün [9] established several novel theoretical operations in the soft set theory. In addition, Majumdar and Samanta [10] established the soft mappings concept, and also, Choudhure et al. worked on the soft relation concept and consequently used it for solving various decision-making issues. Moreover, Aktas and Cağman [11] extended the softness concept to group theory and defined the soft group concept. Furthermore, Feng et al. [12] applied and extended the soft set concept to semirings, Acar [13] initiated the soft rings, and Jun et al. extended the softness concept to BCK/BCI-algebras ([14–16]). Also, Sezgin and Atagü n [17] introduced the normalistic soft groups concept, Zhan et al. [18] defined the concept of soft ideal of BL-algebras and Kazanci et al. [19] applied the softness concept to BCH-algebras. In addition, Sezgin et al. [20] worked on soft near-rings and Cağman et al. [21] defined the group soft union and group soft intersection of a group (for more details, one can refer also to [22]). In addition, many other researchers introduced new extended concepts based on soft sets in recent years, providing examples and studying their properties, such as soft point [23], soft real numbers ([24, 25]), soft complex numbers [26], soft metric spaces [27], soft normed spaces [28], soft inner product spaces [29], and soft Hilbert spaces [30]. After that, Cağman et al. [31] established the soft matrix theory to facilitate dealing with soft sets and organized a model of soft decision-making.

However, it appeared that it is very important to consider the bipolarity of knowledge in decision-making problems because it is a very useful factor when developing a mathematical framework for most situations in decision-making problems. Within the bipolarity idea, it is believed that bipolar subjectivity beliefs include a variety of individual decision-making processes. Examples of several decision analysis directions involve happiness and sadness, positive effects and negative effects, sweet and salty, and hunger and satisfaction. For instance, an inefficient medication might not have any negative effects. So, in 2000, Lee [32] introduced the bipolar-valued fuzzy set concept as a novel extension or generalization of the fuzzy set concept. In this instance, the element's membership interval range value is extended from $[0, 1]$ to $[-1, 1]$. For more information about appearing bipolarity in fuzzy sets and soft sets, one can refer to [33–36].

After that, Maji et al. [37] defined the fuzzy soft set theory, in 2001, by entering the fuzzy sets ideas in the softness concept. In addition, Roy and Maji [38] proposed a decision-making technique to determine the optimal (best) choice of an object to buy among many objects based on fuzzy soft set. Based on fuzzy soft set concept, Yang et al. [39] introduced a fuzzy soft set matrix representation and Cağman et al. [40] studied the fuzzy soft matrices, several algebraic operations and made a theoretical study in fuzzy soft settings. Basu et al. [41] and Kumar and Kaur [42] studied fuzzy soft matrices and established some new notions and operations on them. Later on, Faried et al. developed a fuzzy soft version of functional analysis by introducing a series of results as follows: FS inner product space [43], FS Hilbert space [44], FS linear operators [45], and FS spectral theory ([46–50]).

Maji et al. [51] introduced intuitionistic fuzzy soft sets, in 2004, as a new extension of soft sets. On intuitionistic fuzzy soft sets, additional operations were also provided, and several of their characteristics were identified. In addition, a simple example was given to demonstrate how such a mathematical tool can be used. Then, Chetia et al. [52] initiated the intuitionistic fuzzy soft matrices concepts to represent the intuitionistic fuzzy soft sets easily. In order to perform theoretical study in the intuitionistic fuzzy soft set environment and generate some findings, they, additionally, specified their higher functional operations.

In 2010, Xu et al. [53] proposed the vague soft set concept and presented its general properties. In reality, the vague soft set theory makes element world characteristics more precise, useful, and genuine, trying to make it, in particular in some instances, a flexible tool. Recently, Wang [54] introduced many results on vague soft set theory and studied its associated properties and potential applications. Moreover, Alhazaymeh and Hassan [55] initiated the vague soft set relations and functions concepts. Also, Varol et al. [56] defined vague soft groups and Yin et al. [57] studied vague soft hemirings. Furthermore, Selvachandran and Salleh ([58, 59]) introduced rings and ideals in the vague soft sets settings and established some algebraic hyper structures of the vague soft set theory related to hyperrings and hyperideals. At present, Inthumathi and Pavithra [60], as well as Faried et al. [61], introduced the vague soft matrix concept, examined its general properties, and investigated its unique uses. Furthermore, Abdullah et al. [62] investigated the bipolar fuzzy soft set concept, in 2014, and introduced their basic characteristics. Basic operations of bipolar fuzzy soft sets were also established. Furthermore, they used the bipolar fuzzy soft set to overcome decision-making difficulties. Contrary to appearances, there is a distinction between the intuitionistic fuzzy soft sets and the bipolar fuzzy soft sets.

After that, Sebastian and Ramakrishnan [63] initiated the multifuzzy set concept with multimembership function, which is an ordered sequence of membership functions stated above. In addition, Zhou et al. [64] introduced the multivague set concept and the multivague soft set concept with their properties. In fact, multifuzzy set and multivague set can overcome some specific problems that are very difficult to be represented by other fuzzy set extensions or vague set extensions. For instance, the color of pixels in a two-dimensional image cannot be described by a characteristic function of a regular fuzzy set, but it can be described by a three-dimensional characteristic function, whose components are the characteristic functions representing the primary known colors: red, green, and blue. Consequently, one can roughly represent any image by a collection of ordered pixels with a multicharacteristic function.

Furthermore, Santhi and Shyamala [65] defined the bipolar-valued multifuzzy set and made some notes on bipolar-valued multifuzzy subgroups of a group. Moreover, Yang et al. [66] proposed the bipolar-valued multifuzzy soft set concept and introduced some decision-making applications using it. Finally, Sakr et al. [67] used the soft decision-making technique in bipolarity problems containing vague information by proposing bipolar vague soft sets. They introduced a real-life application to choose the optimal object based on bipolar vague soft information using decision-making techniques.

Despite this progress, there is still another limitation: how we can deal with a multibipolarity problem involving too vague information by applying soft decision-making techniques. From here, the need to introduce another novel extension arose. The chief motivation of this article is to overcome the above-stated limitations of the previously used tools in decision-making problems. So, in this paper, the bipolar-valued multivague soft set of dimension n, its types, operations, properties, and applications are introduced with an illustrative example of each. A bipolar-valued multivague soft set-based decision-making technique is designed, which extends Roy and Maji's strategy and helps us to make more successful conclusions when picking the correct choice.

The rest of this research is constructed as follows: Section 2 is set up to give the main preliminary needed definitions. After that, in Section 3, the bipolar-valued multivague soft set is defined with its types and some new related concepts and operations. In addition, Section 3 gives general properties, absorption properties, commutative properties, associative properties, distributive laws, and De Morgan's laws. Furthermore, the aim of Section 4 is to establish a generalized algorithm for Roy and Maji method based on the bipolar-valued multivague soft sets to determine the optimal alternative among others given in a decision-making problem. Finally, Section 5 gives open questions for further investigations and concluding remarks.

2. Preliminary Definitions

The purpose of this section is to introduce the basic preliminary definitions needed in the following investigations.

Definition 1. (see [2]). Let Π be a universal set (space of points or objects). A fuzzy set (class) F over Π is a set described by a function $\eta_F : \Pi \longrightarrow [0,1]$. η_F is said to be the characteristic, indicator, or membership function of the fuzzy set F, and $\eta_F(u)$ is said to be the membership grade value of $\pi \in \Pi$ in F. A fuzzy set F over a universal set Π can be written as $F = \{(\eta_F(\pi)/\pi): \pi \in \Pi, \eta_F(\pi) \in [0,1]\}$ or $F = \{(\pi, \eta_F(\pi)): \pi \in \Pi, \eta_F(\pi) \in [0,1]\}$.

Definition 2. (see [32]). A bipolar-valued fuzzy set $\mathscr{A}$ on Π can be represented as the following formula: $\mathscr{A} = \{(\pi, \eta^+_{\mathscr{A}}(\pi), \eta^-_{\mathscr{A}}(\pi)): \pi \in \Pi\}$, taking into account that $\eta^+_{\mathscr{A}} : \Pi \longrightarrow [0,1]$ expresses the positive characteristic function describing the satisfaction amount of π to the feature corresponding to $\mathscr{A}$, and $\eta^-_{\mathscr{A}} : \Pi \longrightarrow [-1,0]$ expresses the negative characteristic function describing the satisfaction amount of π to the counter-feature of $\mathscr{A}$.

Definition 3. (see [63]). A multifuzzy set $\mathscr{N}$ of dimension n over Π is represented by a set of ordered sequences as follows: $\mathscr{N} = \{(\pi, \eta_{1\mathscr{N}}(\pi), \eta_{2\mathscr{N}}(\pi), \cdots, \eta_{n\mathscr{N}}(\pi)): \pi \in \Pi\}$, where for $i = 1, 2, \cdots, n$, we have $\eta_{i\mathscr{N}} : \Pi \longrightarrow [0,1]$ representing the membership functions. The function $\eta_{\mathscr{N}} = (\eta_{1\mathscr{N}}, \eta_{2\mathscr{N}}, \cdots, \eta_{n\mathscr{N}})$ is said to be a fuzzy multimembership function of a multifuzzy set $\mathscr{N}$ of dimension n.

Definition 4. (see [65]). A bipolar-valued multifuzzy set $\mathscr{B}$ of dimension n over Π is given by the following structure: $\mathscr{B} = \{(\pi, \eta^+_{1\mathscr{B}}(\pi), \eta^+_{2\mathscr{B}}(\pi), \cdots, \eta^+_{n\mathscr{B}}(\pi), \eta^-_{1\mathscr{B}}(\pi), \eta^-_{2\mathscr{B}}(\pi), \cdots, \eta^-_{n\mathscr{B}}(\pi)): \pi \in \Pi\}$, where for $i = 1, 2, \cdots, n$, we have $\eta^+_{i\mathscr{B}} : \Pi \longrightarrow [0,1]$ representing the positive membership degrees denoting the satisfaction degrees of π to some properties corresponding to $\mathscr{B}$ and $\eta^-_{i\mathscr{B}} : \Pi \longrightarrow [-1,0]$ representing the negative membership degrees denoting the satisfaction degrees of π to some implicit counter-properties of $\mathscr{B}$.

Definition 5. (see [4]). Given the universal set $\Pi = \{\pi_1, \pi_2, \cdots, \pi_n\}$, a vague set V over Π is a set determined by a truth membership function τ_V and a false membership function η_V. The exact grade of membership of $\pi \in \Pi$ ($\mu_V(\pi)$) belongs to an interval $[\tau_V(\pi), 1 - \eta_V(\pi)] \subseteq [0,1]$, i.e., $\mu_V(\pi)$ may be unknown, but it is bounded by $\tau_V(\pi) \leq \mu_V(\pi) \leq 1 - \eta_V(\pi)$, where $\tau_V(\pi) + \eta_V(\pi) \leq 1$ and $\tau_V, \eta_V : \Pi \longrightarrow [0,1]$. V can be written as $V = \{([\tau_V(\pi), 1 - \eta_V(\pi)]/\pi): \pi \in \Pi, \tau_V(\pi), \eta_V(\pi) \in [0,1]\}$, or $V = \{(\pi, [\tau_V(\pi), 1 - \eta_V(\pi)]): \pi \in \Pi, \tau_V(\pi), \eta_V(\pi) \in [0,1]\}$.

Definition 6. (see [68]). A bipolar-valued vague set $\mathscr{M}$ over the universal set Π is defined by $\mathscr{M} = \{(\pi, [\tau^+_{\mathscr{M}}(\pi), 1 - \eta^+_{\mathscr{M}}(\pi)], [-1 - \eta^-_{\mathscr{M}}(\pi), \tau^-_{\mathscr{M}}(\pi)]): \pi \in \Pi\}$, where $\tau^+_{\mathscr{M}}, \eta^+_{\mathscr{M}} : \Pi \longrightarrow [0,1]$ are the positive truth and false membership functions denoting the satisfaction degree of an element π to the property corresponding to $\mathscr{M}$, such that $\tau^+_{\mathscr{M}} + \eta^+_{\mathscr{M}} \leq 1$ and $\tau^-_{\mathscr{M}}, \eta^-_{\mathscr{M}} : \Pi \longrightarrow [-1,0]$ are the negative truth and false membership functions denoting the satisfaction degree of π to some implicit counter-property of $\mathscr{M}$, such that $\tau^-_{\mathscr{M}} + \eta^-_{\mathscr{M}} \geq -1$, i.e., the intervals $[\tau^+_{\mathscr{M}}(\pi), 1 - \eta^+_{\mathscr{M}}(\pi)]$ and $[-1$

$-\eta^-_{\mathscr{M}}(\pi), \tau^-_{\mathscr{M}}(\pi)]$ denote the satisfaction region of π to the property corresponding to $\mathscr{M}$ and to some implicit counter-property of $\mathscr{M}$, respectively.

Definition 7. (see [64]). A multivague set $\mathscr{D}$ of dimension n over Π is represented by a set of ordered sequences as follows: $\mathscr{D} = \{(\pi, [\tau_{1\mathscr{D}}(\pi), 1-\eta_{1\mathscr{D}}(\pi)], [\tau_{2\mathscr{D}}(\pi), 1-\eta_{2\mathscr{D}}(\pi)], \cdots, [\tau_{n\mathscr{D}}(\pi), 1-\eta_{n\mathscr{D}}(\pi)]): \pi \in \Pi\}$, where for $i = 1, 2, \cdots, n$, we have $\tau_{i\mathscr{D}}$ and $\eta_{i\mathscr{D}} : \Pi \longrightarrow [0, 1]$, the truth and false membership functions, respectively, such that $\tau_{i\mathscr{D}} + \eta_{i\mathscr{D}} \leq 1$, i.e., the intervals $[\tau_{i\mathscr{D}}(\pi), 1-\eta_{i\mathscr{D}}(\pi)]$ denote the satisfaction intervals of π to some properties corresponding to $\mathscr{D}$. The function $\mu_{\mathscr{D}} = [\tau_{\mathscr{D}}, 1-\eta_{\mathscr{D}}] = ([\tau_{1\mathscr{D}}, 1-\eta_{1\mathscr{D}}], [\tau_{2\mathscr{D}}, 1-\eta_{2\mathscr{D}}], \cdots, [\tau_{n\mathscr{D}}, 1-\eta_{n\mathscr{D}}])$ is said to be a vague multimembership function of a multivague set $\mathscr{D}$ of dimension n.

Definition 8. (see [5]). Suppose that Π is a universal set, Y is a set of attributes (or parameters) and $\Lambda \subseteq Y$. The power set of Π is given by $P(\Pi) = 2^{\Pi}$. A pair (Γ, Λ) or Γ_Λ is said to be a soft set over Π, where Γ is a mapping characterized by $\Gamma : \Lambda \longrightarrow P(\Pi)$. Also, Γ_Λ may be written as a set of ordered pairs $\Gamma_\Lambda = \{(\lambda, \Gamma_\Lambda(\lambda)): \lambda \in \Lambda, \Gamma_\Lambda(\lambda) \in P(\Pi)\}$. Λ is called the support of Γ_Λ, and we have $\Gamma_\Lambda(\lambda) \neq \phi$ for all $\lambda \in \Lambda$ and $\Gamma_\Lambda(\lambda) = \phi$ for all $\lambda \notin \Lambda$. In other words, a soft set (Γ, Λ) over Π is a parameterized family of subsets of the set Π.

Definition 9. (see [64]). A multivague soft set of dimension n over the universal set Π is a parameterized family of multivague sets of the universal set Π, stated above in Definition 7.

3. Operations and Properties on Bipolar-Valued Multivague Soft Sets

The purpose of this section is to introduce the definition of the bipolar-valued multivague soft sets of dimension n and many related new concepts and operations on them with illustrative examples on each item. In addition, some important properties of bipolar-valued multivague soft sets of dimension n are introduced with proofs such as absorption properties, commutative properties, associative properties, distributive laws, and De Morgan's laws.

Definition 10. A bipolar-valued multivague set $\mathscr{V}$ of dimension n over the universal set Π is defined by $\mathscr{V} = \{(\pi, [\tau^+_{1\mathscr{V}}(\pi), 1-\eta^+_{1\mathscr{V}}(\pi)], [\tau^+_{2\mathscr{V}}(\pi), 1-\eta^+_{2\mathscr{V}}(\pi)], \cdots, [\tau^+_{n\mathscr{V}}(\pi), 1-\eta^+_{n\mathscr{V}}(\pi)], [-1-\eta^-_{1\mathscr{V}}(\pi), \tau^-_{1\mathscr{V}}(\pi)], [-1-\eta^-_{2\mathscr{V}}(\pi), \tau^-_{2\mathscr{V}}(\pi)], \cdots, [-1-\eta^-_{n\mathscr{V}}(\pi), \tau^-_{n\mathscr{V}}(\pi)]): \pi \in \Pi\}$, where for $\mathrm{i} = 1, 2, \cdots, \mathrm{n}$, we have $\tau^+_{i\mathscr{V}}, \eta^+_{i\mathscr{V}} : \Pi \longrightarrow [0, 1]$, the positive truth and false membership functions denoting the satisfaction degrees of an element π to some properties corresponding to $\mathscr{V}$, such that $\tau^+_{i\mathscr{V}} + \eta^+_{i\mathscr{V}} \leq 1$ and $\tau^-_{i\mathscr{V}}, \eta^-_{i\mathscr{V}} : \Pi \longrightarrow [-1, 0]$ are the negative truth and false membership functions denoting the satisfaction degrees of π to some implicit counter-properties of $\mathscr{V}$, such that $\tau^-_{i\mathscr{V}} + \eta^-_{i\mathscr{V}} \geq -1$, i.e., the intervals $[\tau^+_{i\mathscr{V}}(\pi), 1-\eta^+_{i\mathscr{V}}(\pi)]$ and $[-1-\eta^-_{i\mathscr{V}}(\pi), \tau^-_{i\mathscr{V}}(\pi)]$ denote the satisfaction regions of π to some properties corresponding to $\mathscr{V}$ and to some implicit counter-properties of $\mathscr{V}$, respectively.

Definition 11. Assume that Π is a universal set, Y is a parameter set and $\Lambda \subseteq Y$. Then, a pair $(\sqsupset, \Lambda)$ given by $\sqsupset_\Lambda = \{(\lambda, \sqsupset_\Lambda(\lambda)): \lambda \in \Lambda, \sqsupset_\Lambda(\lambda) \in \mathscr{BV}^n(\Pi)\}$ is said to be a bipolar-valued multivague soft set of dimension n over the universal set Π, taking into account that $\sqsupset$ is a mapping defined as $\sqsupset : \Lambda \longrightarrow \mathscr{BV}^n(\Pi)$, where $\mathscr{BV}^n(\Pi)$ is the power set of bipolar-valued multivague sets of dimension n on Π (i.e., the family of all bipolar-valued multivague subsets of dimension n of Π) and the bipolar-valued multivague subset of dimension n of Π is as stated above in Definition 10.

Definition 12. Any bipolar-valued multivague soft set $(\sqsupset, \Lambda)$ of dimension n on the universal set Π is said to be complete (or absolute), which stands for $\mathscr{C}^n_\Lambda$, if for all $\lambda \in \Lambda$, we have $\sqsupset_\Lambda(\lambda) = \mathscr{BV}^n(\Pi)$. That is to say that, for $i = 1, 2, \cdots, n$, we have $\tau^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 1$, $1-\eta^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 1$ (i.e., $\eta^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$), $-1-\eta^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = -1$ (i.e., $\eta^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$), and $\tau^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = -1$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. According to those assumptions, we have $\mathscr{C}^n_\Lambda = \{(\lambda, \{(\pi, [1, 1], \cdots^{n-times}, [1, 1], [-1,-1], \cdots^{n-times}, [-1,-1])\}): \lambda \in \Lambda, \pi \in \Pi\}$.

Definition 13. Any bipolar-valued multivague soft set $(\sqsupset, \Lambda)$ of dimension n on the universal set Π is said to be null (or empty), which stands for ϕ^n_Λ, if for all $\lambda \in \Lambda$, we have $\sqsupset_\Lambda(\lambda) = \phi$. That is to say that, for $\mathrm{i} = 1, 2, \cdots, \mathrm{n}$, we have $\tau^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$, $1-\eta^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$ (i.e., $\eta^+_{i\sqsupset_\Lambda(\lambda)}(\pi) = 1$), $-1-\eta^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$ (i.e., $\eta^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = -1$), and $\tau^-_{i\sqsupset_\Lambda(\lambda)}(\pi) = 0$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. According to those assumptions, we have $\phi^n_\Lambda = \{(\lambda, \{(\pi, [0, 0], \cdots^{2n-times}, [0, 0])\}): \lambda \in \Lambda, \pi \in \Pi\}$.

Definition 14. The complement of a bipolar-valued multivague soft set $(\sqsupset, \Lambda)$ of dimension n is defined by $(\sqsupset, \Lambda)^c = (\sqsupset^c, \Lambda)$, where $\sqsupset^c : \Lambda \longrightarrow \mathscr{BV}^n(\Pi)$ is a mapping given, for $i = 1, 2, \cdots, n$, by the following: $\tau^+_{i\sqsupset^c(\lambda)}(\pi) = \eta^+_{i\sqsupset(\lambda)}(\pi)$, $\eta^+_{i\sqsupset^c(\lambda)}(\pi) = \tau^+_{i\sqsupset(\lambda)}(\pi)$, $\tau^-_{i\sqsupset^c(\lambda)}(\pi) = \eta^-_{i\sqsupset(\lambda)}(\pi)$, and $\eta^-_{i\sqsupset^c(\lambda)}(\pi) = \tau^-_{i\sqsupset(\lambda)}(\pi)$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. Then, we have

$$\begin{aligned}\sqsupset^c_\Lambda = \Big\{\lambda, \Big\{\pi, &\left[\eta^+_{1\sqsupset_\Lambda(\lambda)}(\pi), 1-\tau^+_{1\sqsupset_\Lambda(\lambda)}(\pi)\right], \\ &\left[\eta^+_{2\sqsupset_\Lambda(\lambda)}(\pi), 1-\tau^+_{2\sqsupset_\Lambda(\lambda)}(\pi)\right], \cdots, \\ &\left[\eta^+_{n\sqsupset_\Lambda(\lambda)}(\pi), 1-\tau^+_{n\sqsupset_\Lambda(\lambda)}(\pi)\right],\end{aligned} \tag{1}$$

$$\begin{aligned}&\left[-1-\tau^-_{1\sqsupset_\Lambda(\lambda)}(\pi), \eta^-_{1\sqsupset_\Lambda(\lambda)}(\pi)\right], \left[-1-\tau^-_{2\sqsupset_\Lambda(\lambda)}(\pi), \eta^-_{2\sqsupset_\Lambda(\lambda)}(\pi)\right], \\ &\cdots, \left[-1-\tau^-_{n\sqsupset_\Lambda(\lambda)}(\pi), \eta^-_{n\sqsupset_\Lambda(\lambda)}(\pi)\right]\Big)\Big\}\Big): \lambda \in \Lambda, \pi \in \Pi\Big\}.\end{aligned} \tag{2}$$

Definition 15. Assume that (Γ, Λ) and (Ψ, Θ) are two bipolar-valued multivague soft sets of dimension n on a universal set Π. Then, (Γ, Λ) is called a bipolar-valued multivague soft subset of (Ψ, Θ) if $\Lambda \subseteq \Theta$, and $\Gamma(\lambda) \subseteq \Psi(\lambda)$ for all $\lambda \in \Lambda$, that is to say that, for $i = 1, 2, \cdots, n$, we have $\tau^+_{i\Gamma(\lambda)}(\pi) \leq \tau^+_{i\Psi(\lambda)}(\pi)$, $1-\eta^+_{i\Gamma(\lambda)}(\pi) \leq 1-\eta^+_{i\Psi(\lambda)}(\pi)$, $-1-\eta^-_{i\Gamma(\lambda)}(\pi)$

$\geq -1-\eta^{-}_{i\Psi(\lambda)}(\pi)$, and $\tau^{-}_{i\Gamma(\lambda)}(\pi)\geq \tau^{-}_{i\Psi(\lambda)}(\pi)$, i.e., for $i=1,2,\cdots,n$, we have $\tau^{+}_{i\Gamma(\lambda)}(\pi)\leq \tau^{+}_{i\Psi(\lambda)}(\pi)$, $\eta^{+}_{i\Gamma(\lambda)}(\pi)\geq \eta^{+}_{i\Psi(\lambda)}(\pi)$, $\eta^{-}_{i\Gamma(\lambda)}(\pi)\leq \eta^{-}_{i\Psi(\lambda)}(\pi)$, and $\tau^{-}_{i\Gamma(\lambda)}(\pi)\geq \tau^{-}_{i\Psi(\lambda)}(\pi)$, for all $\lambda\in\Lambda$ and for all $\pi\in\Pi$. One can write $(\Gamma,\Lambda)\tilde{\subseteq}(\Psi,\Theta)$. In this case, (Ψ,Θ) is called a bipolar-valued multivague soft superset of (Γ,Λ), denoted by $(\Psi,\Theta)\tilde{\supseteq}(\Gamma,\Lambda)$.

Definition 16. Two bipolar-valued multivague soft sets (Γ,Λ) and (Ψ,Θ) of dimension n on a common universal set Π are called bipolar-valued multivague soft equal if they are bipolar-valued multivague soft subsets of each other, i.e., $(\Gamma,\Lambda)\tilde{\subseteq}(\Psi,\Theta)$ and $(\Psi,\Theta)\tilde{\subseteq}(\Gamma,\Lambda)$.

Definition 17. The union of two bipolar-valued multivague soft sets (Γ,Λ) and (Ψ,Θ) of dimension n on a common universal set Π is a bipolar-valued multivague soft set $(\sqsupset,\Delta)$ of dimension n, written as $(\sqsupset,\Delta)=(\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta)$, where $\Delta=\Lambda\cup\Theta$ and for all $\delta\in\Delta$:

$$\begin{aligned}(\sqsupset,\Delta)=\Big\{\Big(\delta,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Gamma(\delta)}(\pi),1-\eta^{+}_{1\Gamma(\delta)}(\pi)\right],\\&\left[\tau^{+}_{2\Gamma(\delta)}(\pi),1-\eta^{+}_{2\Gamma(\delta)}(\pi)\right],\cdots,\\&\left[\tau^{+}_{n\Gamma(\delta)}(\pi),1-\eta^{+}_{n\Gamma(\delta)}(\pi)\right],\left[-1-\eta^{-}_{1\Gamma(\delta)}(\pi),\tau^{-}_{1\Gamma(\delta)}(\pi)\right],\\&\left[-1-\eta^{-}_{2\Gamma(\delta)}(\pi),\tau^{-}_{2\Gamma(\delta)}(\pi)\right],\cdots,\\&\left[-1-\eta^{-}_{n\Gamma(\delta)}(\pi),\tau^{-}_{n\Gamma(\delta)}(\pi)\right]\Big)\Big\}\Big),\pi\in\Pi\Big\},\text{ if }\delta\in\Lambda-\Theta,\end{aligned}\tag{3}$$

$$\begin{aligned}(\sqsupset,\Delta)=\Big\{\Big(\delta,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Psi(\delta)}(\pi),1-\eta^{+}_{1\Psi(\delta)}(\pi)\right],\\&\left[\tau^{+}_{2\Psi(\delta)}(\pi),1-\eta^{+}_{2\Psi(\delta)}(\pi)\right],\cdots,\\&\left[\tau^{+}_{n\Psi(\delta)}(\pi),1-\eta^{+}_{n\Psi(\delta)}(\pi)\right],\left[-1-\eta^{-}_{1\Psi(\delta)}(\pi),\tau^{-}_{1\Psi(\delta)}(\pi)\right],\\&\left[-1-\eta^{-}_{2\Psi(\delta)}(\pi),\tau^{-}_{2\Psi(\delta)}(\pi)\right],\cdots,\\&\left[-1-\eta^{-}_{n\Psi(\delta)}(\pi),\tau^{-}_{n\Psi(\delta)}(\pi)\right]\Big)\Big\}\Big),\pi\in\Pi\Big\},\text{ if }\delta\in\Theta-\Lambda,\end{aligned}\tag{4}$$

$$\begin{aligned}(\sqsupset,\Delta)=\Big\{\Big(\delta,\Big\{\Big(\pi,&\left[\max\left\{\tau^{+}_{1\Gamma(\delta)}(\pi),\tau^{+}_{1\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{1\Gamma(\delta)}(\pi),1-\eta^{+}_{1\Psi(\delta)}(\pi)\right\}\right]\\&\cdot\left[\max\left\{\tau^{+}_{2\Gamma(\delta)}(\pi),\tau^{+}_{2\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{2\Gamma(\delta)}(\pi),1-\eta^{+}_{2\Psi(\delta)}(\pi)\right\}\right],\cdots,\\&\left[\max\left\{\tau^{+}_{n\Gamma(\delta)}(\pi),\tau^{+}_{n\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{n\Gamma(\delta)}(\pi),1-\eta^{+}_{n\Psi(\delta)}(\pi)\right\}\right],\\&\left[\min\left\{-1-\eta^{-}_{1\Gamma(\delta)}(\pi),-1-\eta^{-}_{1\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{1\Gamma(\delta)}(\pi),\tau^{-}_{1\Psi(\delta)}(\pi)\right\}\right],\\&\left[\min\left\{-1-\eta^{-}_{2\Gamma(\delta)}(\pi),-1-\eta^{-}_{2\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{2\Gamma(\delta)}(\pi),\tau^{-}_{2\Psi(\delta)}(\pi)\right\}\right],\cdots,\\&\left[\min\left\{-1-\eta^{-}_{n\Gamma(\delta)}(\pi),-1-\eta^{-}_{n\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{n\Gamma(\delta)}(\pi),\tau^{-}_{n\Psi(\delta)}(\pi)\right\}\right]\Big)\Big\}\Big),\pi\in\Pi\Big\},\text{ if }\delta\in\Lambda\cap\Theta.\end{aligned}\tag{5}$$

Definition 18. The union of a family $\{(\Gamma_i,\Lambda_i): i\in I\}$ of bipolar-valued multivague soft sets of dimension n over a universal set Π is a bipolar-valued multivague soft set $(\sqsupset,\Delta)$ of dimension n, written as $\sqsupset_{\Delta}=(\sqsupset,\Delta)=\tilde{\cup}(\Gamma_i,\Lambda_i)$, where $\Delta=\cup\Lambda_i$, for all $i\in I$ defined as follows, for all $\delta\in\Delta$:

$$\sqsupset_{\Delta}(\delta)=\begin{cases}\Gamma_i(\delta) & \text{if }\delta\in\Lambda_i-\bigcup_{j\neq i}\Lambda_j,\text{ for all } i\in I,\\ \bigcup_{i\in I}\Gamma_i(\delta), & \text{if }\delta\in\bigcap_{i\in I}\Lambda_i.\end{cases}\tag{6}$$

Definition 19. The restricted union of two bipolar-valued multivague soft sets (Γ,Λ) and (Ψ,Θ) of dimension n on a common universal set Π is a bipolar-valued multivague soft set $(\sqsupset,\Delta)$ of dimension n, written as $(\sqsupset,\Delta)=(\Gamma,\Lambda)\tilde{\cup}_R(\Psi,\Theta)$, where $\Delta=\Lambda\cap\Theta\neq\phi$ and for all $\delta\in\Delta$:

$$\begin{aligned}(\sqsupset,\Delta)=\Big\{\Big(\delta,\Big\{\Big(\pi,&\left[\max\left\{\tau^{+}_{1\Gamma(\delta)}(\pi),\tau^{+}_{1\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{1\Gamma(\delta)}(\pi),1-\eta^{+}_{1\Psi(\delta)}(\pi)\right\}\right],\\&\left[\max\left\{\tau^{+}_{2\Gamma(\delta)}(\pi),\tau^{+}_{2\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{2\Gamma(\delta)}(\pi),1-\eta^{+}_{2\Psi(\delta)}(\pi)\right\}\right],\cdots,\\&\left[\max\left\{\tau^{+}_{n\Gamma(\delta)}(\pi),\tau^{+}_{n\Psi(\delta)}(\pi)\right\},\right.\\&\left.\max\left\{1-\eta^{+}_{n\Gamma(\delta)}(\pi),1-\eta^{+}_{n\Psi(\delta)}(\pi)\right\}\right],\\&\left[\min\left\{-1-\eta^{-}_{1\Gamma(\delta)}(\pi),-1-\eta^{-}_{1\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{1\Gamma(\delta)}(\pi),\tau^{-}_{1\Psi(\delta)}(\pi)\right\}\right],\\&\left[\min\left\{-1-\eta^{-}_{2\Gamma(\delta)}(\pi),-1-\eta^{-}_{2\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{2\Gamma(\delta)}(\pi),\tau^{-}_{2\Psi(\delta)}(\pi)\right\}\right],\cdots,\\&\left[\min\left\{-1-\eta^{-}_{n\Gamma(\delta)}(\pi),-1-\eta^{-}_{n\Psi(\delta)}(\pi)\right\},\right.\\&\left.\min\left\{\tau^{-}_{n\Gamma(\delta)}(\pi),\tau^{-}_{n\Psi(\delta)}(\pi)\right\}\right]\Big)\Big\}\Big):\\&\delta\in\Lambda\cap\Theta,\pi\in\Pi\Big\}.\end{aligned}\tag{7}$$

Definition 20. The restricted union of a family $\{(\Gamma_i,\Lambda_i): i\in I\}$ of bipolar-valued multivague soft sets of dimension n over a universal set Π is a bipolar-valued multivague soft set $(\sqsupset,\Delta)$ of dimension n, written as $\sqsupset_{\Delta}=(\sqsupset,\Delta)=\tilde{\cup}_R(\Gamma_i,\Lambda_i)$, where $\Delta=\cap\Lambda_i$, for all $i\in I$ defined by $\sqsupset_{\Delta}(\delta)=\cup\Gamma_i(\delta)$, $\delta\in\Delta=\cap\Lambda_i$.

Definition 21. The intersection of two bipolar-valued multivague soft sets (Γ,Λ) and (Ψ,Θ) of dimension n over a common universal set Π is a bipolar-valued multivague soft set $(\sqsupset,\Delta)$ of dimension n, written as $(\sqsupset,\Delta)=(\Gamma,\Lambda)\tilde{\cap}(\Psi,\Theta)$, where $\Delta=\Lambda\cup\Theta$ and for all $\delta\in\Delta$:

$$(\sqsupset, \Delta) = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \big[\tau^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Gamma(\delta)}(\pi)\big], \big[\tau^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Gamma(\delta)}(\pi)\big], \cdots, \big[\tau^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Gamma(\delta)}(\pi)\big], \big[-1-\eta^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Gamma(\delta)}(\pi)\big], \big[-1-\eta^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Gamma(\delta)}(\pi)\big], \cdots, \big[-1-\eta^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Gamma(\delta)}(\pi)\big] \Big) \Big\} \Big), \pi \in \Pi \Big\}, \text{if } \delta \in \Lambda - \Theta, \tag{8}$$

$$(\sqsupset, \Delta) = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \big[\tau^+_{1\Psi(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\big], \big[\tau^+_{2\Psi(\delta)}(\pi), 1-\eta^+_{2\Psi(\delta)}(\pi)\big], \cdots, \big[\tau^+_{n\Psi(\delta)}(\pi), 1-\eta^+_{n\Psi(\delta)}(\pi)\big], \big[-1-\eta^-_{1\Psi(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\big], \big[-1-\eta^-_{2\Psi(\delta)}(\pi), \tau^-_{2\Psi(\delta)}(\pi)\big], \cdots, \big[-1-\eta^-_{n\Psi(\delta)}(\pi), \tau^-_{n\Psi(\delta)}(\pi)\big] \Big) \Big\} \Big), \pi \in \Pi \Big\}, \text{if } \delta \in \Theta - \Lambda, \tag{9}$$

$$\begin{aligned}(\sqsupset, \Delta) = \Big\{ \Big(\delta, \Big\{ \Big(\pi, &\Big[\min\big\{\tau^+_{1\Gamma(\delta)}(\pi), \tau^+_{1\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\min\big\{\tau^+_{2\Gamma(\delta)}(\pi), \tau^+_{2\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Psi(\delta)}(\pi)\big\}\Big], \cdots,\\ &\Big[\min\big\{\tau^+_{n\Gamma(\delta)}(\pi), \tau^+_{n\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\max\big\{-1-\eta^-_{1\Gamma(\delta)}(\pi), -1-\eta^-_{1\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\max\big\{-1-\eta^-_{2\Gamma(\delta)}(\pi), -1-\eta^-_{2\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Psi(\delta)}(\pi)\big\}\Big], \cdots,\\ &\Big[\max\big\{-1-\eta^-_{n\Gamma(\delta)}(\pi), -1-\eta^-_{n\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Psi(\delta)}(\pi)\big\}\Big] \Big) \Big\} \Big), \pi \in \Pi \Big\}, \text{if } \delta \in \Lambda \cap \Theta.\end{aligned} \tag{10}$$

Definition 22. The intersection of a family $\{(\Gamma_i, \Lambda_i): i \in I\}$ of bipolar-valued multivague soft sets of dimension n over a universal set Π is a bipolar-valued multivague soft set $(\sqsupset, \Delta)$ of dimension n, written as $\sqsupset_\Delta = (\sqsupset, \Delta) = \cap\,\tilde{}\,(\Gamma_i, \Lambda_i)$, where $\Delta = \cup\Lambda_i$, for all $i \in I$ defined as follows, for all $\delta \in \Delta$:

$$\sqsupset_\Delta(\delta) = \begin{cases} \Gamma_i(\delta), & \text{if } \delta \in \Lambda_i - \bigcup_{j \neq i} \Lambda_j, \text{for all } i \in I,\\ \bigcap_{i \in I} \Gamma_i(\delta), & \text{if } \delta \in \bigcap_{i \in I} \Lambda_i. \end{cases} \tag{11}$$

Definition 23. The restricted intersection of two bipolar-valued multivague soft sets (Γ, Λ) and (Ψ, Θ) of dimension n on a common universal set Π is a bipolar-valued multivague soft set $(\sqsupset, \Delta)$ of dimension n, written as $(\sqsupset, \Delta) = (\Gamma, \Lambda) \cap\,\tilde{}_R (\Psi, \Theta)$, where $\Delta = \Lambda \cap \Theta \neq \phi$ and for all $\delta \in \Delta$:

$$\begin{aligned}(\sqsupset, \Delta) = \Big\{ \Big(\delta, \Big\{ \Big(\pi, &\Big[\min\big\{\tau^+_{1\Gamma(\delta)}(\pi), \tau^+_{1\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\min\big\{\tau^+_{2\Gamma(\delta)}(\pi), \tau^+_{2\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Psi(\delta)}(\pi)\big\}\Big], \cdots,\\ &\Big[\min\big\{\tau^+_{n\Gamma(\delta)}(\pi), \tau^+_{n\Psi(\delta)}(\pi)\big\}, \min\big\{1-\eta^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\max\big\{-1-\eta^-_{1\Gamma(\delta)}(\pi), -1-\eta^-_{1\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\big\}\Big],\\ &\Big[\max\big\{-1-\eta^-_{2\Gamma(\delta)}(\pi), -1-\eta^-_{2\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Psi(\delta)}(\pi)\big\}\Big], \cdots,\\ &\Big[\max\big\{-1-\eta^-_{n\Gamma(\delta)}(\pi), -1-\eta^-_{n\Psi(\delta)}(\pi)\big\}, \max\big\{\tau^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Psi(\delta)}(\pi)\big\}\Big] \Big) \Big\} \Big): \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\}.\end{aligned} \tag{12}$$

Definition 24. The restricted intersection of a family $\{(\Gamma_i, \Lambda_i): i \in I\}$ of bipolar-valued multivague soft sets of dimension n over a universal set Π is a bipolar-valued multivague soft set $(\sqsupset, \Delta)$ of dimension n, written as $\sqsupset_\Delta = (\sqsupset, \Delta) = \cap\,\tilde{}_R(\Gamma_i, \Lambda_i)$, where $\Delta = \cap\,\Lambda_i$, for all $i \in I$ defined by $\sqsupset_\Delta(\delta) = \cap\,\Gamma_i(\delta)$, $\delta \in \Delta = \cap\,\Lambda_i$.

Example 1. Suppose that the bipolar-valued multivague soft set $(\sqsupset, Y)$ of dimension (order) 3 describes some features of five employees nominated for a position in a certain organization, denoted by $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ in the universal set Π, i.e., $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$. Let the two major sets of attributes be $Y = \{\lambda_1, \lambda_2, \lambda_3\}$, $\lambda_i (i = 1, 2, 3)$ and its opposite (counter) set $-Y = \{-\lambda_1, -\lambda_2, -\lambda_3\}$, $-\lambda_i (i = 1, 2, 3)$ stand for the properties and the counter-properties, respectively. These attributes are represented in the following main parameters: mental abilities, leadership or social skills, and professional or organizational talents, respectively. Mental abilities and their counter-abilities are as follows: "intelligence" and "stupidity"; "memory" and "forgetfulness"; and "problem-solving" and "puzzlement." Leadership skills and their counter-skills are as follows: "patience" and "boredom"; "fairness" and "injustice"; and "cooperation" and "competition." Professional talents and their counter-talents are as follows: "flexibility" and "strictness"; "precision" and "negligence"; and "experience" and "rawness." After doing several oral and written tests for them as well as subjecting them to some personal interviews and tests for intelligence and psychological aspects by a selection committee, the following bipolar-valued multivague soft set of dimension 3, namely $(\sqsupset, Y)$ is obtained:

$(\sqsupset, Y) = \{(\lambda_1, \{(\pi_1, [0.8,0.9], [0.2,0.3], [0.4,0.5], [-0.2,-0.1], [-0.8,-0.6], [-0.3,-0.2]), (\pi_2, [0.5,0.7], [0.4,0.6], [0.9,1], [-0.3,-0.1], [-0.5,-0.4], [-1,-0.8]), (\pi_3, [0.4,0.6], [0.7,0.8], [0.2,0.3], [-0.5,-0.4], [-0.8,-0.7], [-0.9,-0.6]), (\pi_4, [0.9,1], [0.1,0.3], [0.5,0.6], [-0.2,-0.1], [-0.6,-0.5], [-0.8,-0.7]), (\pi_5,$

$[0.5,0.6],[0,0.1], [0.6,0.9], [-1,-0.8], [-0.5,-0.4], [-1,-0.9])\}), (\lambda_2,\{(\pi_1, [0.1,0.2], [0.7,0.8], [0.5,0.6], [-0.3,-0.2], [-0.9,-0.8], [-0.2,-0.1]), (\pi_2, [0.8,0.9], [0.6,0.7], [0.5,0.6], [-0.4,-0.3], [-0.8,-0.7], [-0.9,-0.8]), (\pi_3, [0.3,0.4], [0.9,1], [0.2,0.3], [-1, -0.9],[-0.2,-0.1],[-0.5,-0.4]),(\pi_4, [0.8,0.9],[0.2,0.4],[0.7,0.8], [-0.6,-0.5], [-0.7,-0.6], [-0.2,-0.1]), (\pi_5, [0.7,0.8], [0.5,0.6], [0.8,0.9], [-0.9,-0.8], [-0.5,-0.4], [-0.7,-0.6])\}), (\lambda_3, \{(\pi_1, [0.2,0.3],[0.8,0.9], [0.8,1],[-0.8,-0.7],[-0.9,-0.8],[-0.6,-0.5]), (\pi_2, [0.8,1], [0.5,0.7], [0.8,0.9], [-0.2,-0.1], [-0.4,-0.3], [-0.9, -0.7]), (\pi_3, [0.6,0.7], [0.2,0.4], [0.9,1], [-1,-0.9], [-0.2,-0.1], [-0.4,-0.3]),(\pi_4, [0.4,0.5],[0.6,0.8],[0.3,0.4],[-0.7,-0.6], [-0.3, -0.2], [-0.8,-0.7]), (\pi_5, [0.3,0.4], [0.5,0.6], [0.8,0.9], [-1,-0.9], [-0.6,-0.5], [-0.3,-0.2])\})\}$Thus, the complement of $(\sqsupset, Y)$, which is again of dimension 3, is defined as following:

$$
\begin{aligned}
(\sqsupset^c, Y) = \{&(\lambda_1, \{(\pi_1, [0.1,0.2], [0.7,0.8], [0.5,0.6], \\
&[-0.9,-0.8], [-0.4,-0.2], [-0.8,-0.7]), \\
&(\pi_2, [0.3,0.5], [0.4,0.6], [0,0.1], [-0.9,-0.7], \\
&[-0.6,-0.5], [-0.2,0]), (\pi_3, [0.4,0.6], [0.2,0.3], \\
&[0.7,0.8], [-0.6,-0.5], [-0.3,-0.2], [-0.4,-0.1]), \\
&(\pi_4, [0,0.1], [0.7,0.9], [0.4,0.5], [-0.9,-0.8], \\
&[-0.5,-0.4], [-0.3,-0.2]), (\pi_5, [0.4,0.5], [0.9,1], \\
&[0.1,0.4], [-0.2,0], [-0.6,-0.5], [-0.1,0])\}), \\
&(\lambda_2, \{(\pi_1, [0.8,0.9], [0.2,0.3], [0.4,0.5], [-0.8,-0.7], \\
&[-0.2,-0.1], [-0.9,-0.8]), (\pi_2, [0.1,0.2], [0.3,0.4], \\
&[0.4,0.5], [-0.7,-0.6], [-0.3,-0.2], [-0.2,-0.1]), \\
&(\pi_3, [0.6,0.7], [0,0.1], [0.7,0.8], [-0.1,0], [-0.9,-0.8], \\
&[-0.6,-0.5]), (\pi_4, [0.1,0.2], [0.6,0.8], [0.2,0.3], \\
&[-0.5,-0.4], [-0.4,-0.3], [-0.9,-0.8]), \\
&(\pi_5, [0.2,0.3], [0.4,0.5], [0.1,0.2], [-0.2,-0.1], \\
&[-0.6,-0.5], [-0.4,-0.3])\}), (\lambda_3, \{(\pi_1, [0.7,0.8], \\
&[0.1,0.2], [0,0.2], [-0.3,-0.2], [-0.2,-0.1], \\
&[-0.5,-0.4]), (\pi_2, [0,0.2], [0.3,0.5], [0.1,0.2], \\
&[-0.9,-0.8], [-0.7,-0.6], [-0.3,-0.1]), (\pi_3, [0.3,0.4], \\
&[0.6,0.8], [0,0.1], [-0.1,0], [-0.9,-0.8], [-0.7,-0.6]), \\
&(\pi_4, [0.5,0.6], [0.2,0.4], [0.6,0.7], [-0.4,-0.3], \\
&[-0.8,-0.7], [-0.3,-0.2]), (\pi_5, [0.6,0.7], [0.4,0.5], \\
&[0.1,0.2], [-0.1,0], [-0.5,-0.4], [-0.8,-0.7])\})\}.
\end{aligned} \tag{13}
$$

In addition, we can define two bipolar-valued multivague soft sets (Γ, Λ) and (Ψ, Θ) of dimension 3 on Π, respectively, as follows:

$$
\begin{aligned}
(\Gamma, \Lambda) = \{&(\lambda_1, \{(\pi_1, [0,0.1], [0.2,0.3], [0.1,0.2], [-0.4,-0.3], \\
&[-0.2,-0.1], [-0.6,-0.5]), (\pi_2, [0.2,0.3], [0.4,0.6], \\
&[0, 0], [-0.8,-0.7], [-0.4,-0.3], [-0.1,0]), \\
&(\pi_3, [0.3,0.4], [0.1,0.2], [0.6,0.7], [-0.5,-0.4], \\
&[-0.2,-0.1], [-0.2,-0.1]), (\pi_4, [0,0.1], [0.6,0.8], \\
&[0.1,0.2], [-0.5,-0.5], [-0.3,-0.2], [0, 0]), \\
&(\pi_5, [0.1,0.2], [0.3,0.5], [0,0.2], [0, 0], [-0.4,-0.3], \\
&[-0.1,0])\})\},
\end{aligned} \tag{14}
$$

It is clear that $(\Gamma,\Lambda) \subseteq^{\sim} (\Psi, \Theta)$.

Moreover, we have the union and intersection of bipolar-valued multivague soft sets of dimension 3, namely $(\sqsupset, \Delta) = (\Gamma,\Lambda)\cup^{\sim}(\Psi, \Theta)$, and $(\Sigma, \Delta) = (\Gamma,\Lambda) \cap^{\sim}(\Psi, \Theta)$, which are again of dimension 3, where $\Delta = \Lambda \cup \Theta = \{\lambda_1, \lambda_2, \lambda_3\}$, respectively, as follows:

$$
(\sqsupset, \Delta) = \left\{ \begin{array}{c}
\left(\lambda_1, \left\{ \begin{array}{c}
(\pi_1, [0.1,0.2], [0.7,0.8], [0.5,0.6], [-0.9,-0.8], [-0.4,-0.2], [-0.8,-0.7]), \\
(\pi_2, [0.3,0.5], [0.4,0.6], [0,0.1], [-0.9,-0.7], [-0.6,-0.5], [-0.2,0]), \\
(\pi_3, [0.4,0.6], [0.2,0.3], [0.7,0.8], [-0.6,-0.5], [-0.3,-0.2], [-0.4,-0.1]), \\
(\pi_4, [0,0.1], [0.7,0.9], [0.4,0.5], [-0.9,-0.8], [-0.5,-0.4], [-0.3,-0.2]), \\
(\pi_5, [0.4,0.5], [0.9,1], [0.1,0.4], [-0.2,0], [-0.6,-0.5], [-0.1,0])
\end{array} \right\} \right), \\
\left(\lambda_2, \left\{ \begin{array}{c}
(\pi_1, [0.8,0.9], [0.2,0.3], [0.4,0.5], [-0.8,-0.7], [-0.2,-0.1], [-0.9,-0.8]), \\
(\pi_2, [0.1,0.2], [0.3,0.4], [0.4,0.5], [-0.7,-0.6], [-0.3,-0.2], [-0.2,-0.1]), \\
(\pi_3, [0.6,0.7], [0,0.1], [0.7,0.8], [-0.1,0], [-0.9,-0.8], [-0.6,-0.5]), \\
(\pi_4, [0.1,0.2], [0.6,0.8], [0.2,0.3], [-0.5,-0.4], [-0.4,-0.3], [-0.9,-0.8]), \\
(\pi_5, [0.2,0.3], [0.4,0.5], [0.1,0.2], [-0.2,-0.1], [-0.6,-0.5], [-0.4,-0.3])
\end{array} \right\} \right)
\end{array} \right\}, \tag{15}
$$

$$
(\Sigma, \Delta) = \left\{ \begin{array}{c}
\left(\lambda_1, \left\{ \begin{array}{c}
(\pi_1, [0,0.1], [0.2,0.3], [0.1,0.2], [-0.4,-0.3], [-0.2,-0.1], [-0.6,-0.5]), \\
(\pi_2, [0.2,0.3], [0.4,0.6], [0, 0], [-0.8,-0.7], [-0.4,-0.3], [-0.1,0]), \\
(\pi_3, [0.3,0.4], [0.1,0.2], [0.6,0.7], [-0.5,-0.4], [-0.2,-0.1], [-0.2,-0.1]), \\
(\pi_4, [0,0.1], [0.6,0.8], [0.1,0.2], [-0.5,-0.5], [-0.3,-0.2], [0, 0]), \\
(\pi_5, [0.1,0.2], [0.3,0.5], [0,0.2], [0, 0], [-0.4,-0.3], [-0.1,0])
\end{array} \right\} \right) \\
\left(\lambda_2, \left\{ \begin{array}{c}
(\pi_1, [0.8,0.9], [0.2,0.3], [0.4,0.5], [-0.8,-0.7], [-0.2,-0.1], [-0.9,-0.8]), \\
(\pi_2, [0.1,0.2], [0.3,0.4], [0.4,0.5], [-0.7,-0.6], [-0.3,-0.2], [-0.2,-0.1]), \\
(\pi_3, [0.6,0.7], [0,0.1], [0.7,0.8], [-0.1,0], [-0.9,-0.8], [-0.6,-0.5]), \\
(\pi_4, [0.1,0.2], [0.6,0.8], [0.2,0.3], [-0.5,-0.4], [-0.4,-0.3], [-0.9,-0.8]), \\
(\pi_5, [0.2,0.3], [0.4,0.5], [0.1,0.2], [-0.2,-0.1], [-0.6,-0.5], [-0.4,-0.3])
\end{array} \right\} \right)
\end{array} \right\}. \tag{16}
$$

Finally, we have the restricted union and the restricted intersection of bipolar-valued multivague soft sets of dimension 3, namely $(\urcorner, \Omega) = (\Gamma,\Lambda)\cup^{\sim}{}_R(\Psi, \Theta)$ and $(\Phi, \Omega) = (\Gamma, \Lambda) \cap^{\sim}{}_R(\Psi, \Theta)$, which are again of dimension 3, where $\Omega = \Lambda \cap \Theta = \{\lambda_1, \lambda_2\}$, respectively, as follows:

$$
(\urcorner, \Omega) = \left\{ \left(\lambda_1, \left\{ \begin{array}{c}
(\pi_1, [0.1,0.2], [0.7,0.8], [0.5,0.6], [-0.9,-0.8], [-0.4,-0.2], [-0.8,-0.7]), \\
(\pi_2, [0.3,0.5], [0.4,0.6], [0,0.1], [-0.9,-0.7], [-0.6,-0.5], [-0.2,0]), \\
(\pi_3, [0.4,0.6], [0.2,0.3], [0.7,0.8], [-0.6,-0.5], [-0.3,-0.2], [-0.4,-0.1]) \\
(\pi_4, [0,0.1], [0.7,0.9], [0.4,0.5], [-0.9,-0.8], [-0.5,-0.4], [-0.3,-0.2]), \\
(\pi_5, [0.4,0.5], [0.9,1], [0.1,0.4], [-0.2,0], [-0.6,-0.5], [-0.1,0])
\end{array} \right\} \right) \right\},
$$

$$
\begin{aligned}
(\Phi, \Omega) = \{&(\lambda_1, \{(\pi_1, [0,0.1], [0.2,0.3], [0.1,0.2], [-0.4,-0.3], [-0.2,-0.1], [-0.6,-0.5]), \\
&(\pi_2, [0.2,0.3], [0.4,0.6], [0, 0], [-0.8,-0.7], [-0.4,-0.3], [-0.1,0]), (\pi_3, [0.3,0.4], \\
&[0.1,0.2], [0.6,0.7], [-0.5,-0.4], [-0.2,-0.1], [-0.2,-0.1])(\pi_4, [0,0.1], [0.6,0.8], \\
&[0.1,0.2], [-0.5,-0.5], [-0.3,-0.2], [0, 0]), (\pi_5, [0.1,0.2], [0.3,0.5], [0,0.2], [0, 0], \\
&[-0.4,-0.3], [-0.1,0]),)\}
\end{aligned} \tag{17}
$$

Assume the following throughout the remainder of the paper, except otherwise indicated:

$$
\begin{aligned}
(\Gamma,\Lambda)=\Big\{\Big(\lambda,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Gamma(\lambda)}(\pi),1-\eta^{+}_{1\Gamma(\lambda)}(\pi)\right],\\
&\left[\tau^{+}_{2\Gamma(\lambda)}(\pi),1-\eta^{+}_{2\Gamma(\lambda)}(\pi)\right],\cdots,\left[\tau^{+}_{n\Gamma(\lambda)}(\pi),1-\eta^{+}_{n\Gamma(\lambda)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\Gamma(\lambda)}(\pi),\tau^{-}_{1\Gamma(\lambda)}(\pi)\right],\left[-1-\eta^{-}_{2\Gamma(\lambda)}(\pi),\tau^{-}_{2\Gamma(\lambda)}(\pi)\right],\cdots,\\
&\left[-1-\eta^{-}_{n\Gamma(\lambda)}(\pi),\tau^{-}_{n\Gamma(\lambda)}(\pi)\right]\Big)\Big\}\Big):\lambda\in\Lambda,\pi\in\Pi\Big\},\\
(\Psi,\Theta)=\Big\{\Big(\theta,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Psi(\theta)}(\pi),1-\eta^{+}_{1\Psi(\theta)}(\pi)\right],\\
&\left[\tau^{+}_{2\Psi(\theta)}(\pi),1-\eta^{+}_{2\Psi(\theta)}(\pi)\right],\cdots,\left[\tau^{+}_{n\Psi(\theta)}(\pi),1-\eta^{+}_{n\Psi(\theta)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\Psi(\theta)}(\pi),\tau^{-}_{1\Psi(\theta)}(\pi)\right],\left[-1-\eta^{-}_{2\Psi(\theta)}(\pi),\tau^{-}_{2\Psi(\theta)}(\pi)\right],\cdots,\\
&\left[-1-\eta^{-}_{n\Psi(\theta)}(\pi),\tau^{-}_{n\Psi(\theta)}(\pi)\right]\Big)\Big\}\Big):\theta\in\Theta,\pi\in\Pi\Big\},\\
(\Sigma,\Omega)=\Big\{\Big(\omega,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Sigma(\omega)}(\pi),1-\eta^{+}_{1\Sigma(\omega)}(\pi)\right],\\
&\left[\tau^{+}_{2\Sigma(\omega)}(\pi),1-\eta^{+}_{2\Sigma(\omega)}(\pi)\right],\cdots,\left[\tau^{+}_{n\Sigma(\omega)}(\pi),1-\eta^{+}_{n\Sigma(\omega)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\Sigma(\omega)}(\pi),\tau^{-}_{1\Sigma(\omega)}(\pi)\right],\left[-1-\eta^{-}_{2\Sigma(\omega)}(\pi),\tau^{-}_{2\Sigma(\omega)}(\pi)\right],\cdots,\\
&\left[-1-\eta^{-}_{n\Sigma(\omega)}(\pi),\tau^{-}_{n\Sigma(\omega)}(\pi)\right]\Big)\Big\}\Big):\omega\in\Omega,\pi\in\Pi\Big\}.
\end{aligned}
\tag{18}
$$

Theorem 25. *Let (Γ,Λ) be any bipolar-valued multivague soft set of dimension n, $\mathscr{C}^n_\Lambda$ be the complete bipolar-valued multivague soft set of dimension n, and ϕ^n_Λ be the null bipolar-valued multivague soft set of dimension n on a common universal set Π, then:*

$$
\begin{aligned}
&(1)\ (\Gamma,\Lambda)\tilde{\cup}(\Gamma,\Lambda)=(\Gamma,\Lambda),\ 18pt(4)\ (\Gamma,\Lambda)\tilde{\cap}\phi^n_\Lambda=\phi^n_\Lambda,(7)\ \mathscr{C}^n_\Lambda\tilde{\cup}\phi^n_\Lambda=\mathscr{C}^n_\Lambda,\\
&(2)\ (\Gamma,\Lambda)\tilde{\cap}(\Gamma,\Lambda)=(\Gamma,\Lambda),\ 18pt(5)\ (\Gamma,\Lambda)\tilde{\cup}\mathscr{C}^n_\Lambda=\mathscr{C}^n_\Lambda,(8)\ \mathscr{C}^n_\Lambda\tilde{\cap}\phi^n_\Lambda=\phi^n_\Lambda.\\
&(3)\ (\Gamma,\Lambda)\tilde{\cup}\phi^n_\Lambda=(\Gamma,\Lambda),\ (6)(\Gamma,\Lambda)\tilde{\cap}\mathscr{C}^n_\Lambda=(\Gamma,\Lambda).
\end{aligned}
\tag{19}
$$

Proof. We prove (8), and from (1) to (12) follow similarly. From Definitions 12 and 13, $\mathscr{C}^n_\Lambda=\{(\lambda,\{(\pi,[1,1],\overset{n-times}{\cdots},[1,1],[-1,-1],\overset{n-times}{\cdots},[-1,-1])\}):\lambda\in\Lambda,\pi\in\Pi\}$ and $\phi^n_\Lambda=\{(\lambda,\{(\pi,[0,0],\overset{2n-times}{\cdots},[0,0])\}):\lambda\in\Lambda,\pi\in\Pi\}$, respectively. Assume, for $\Delta=\Lambda\cup\Lambda=\Lambda$, that

$$
\begin{aligned}
\mathscr{C}^n_\Lambda\tilde{\cap}\phi^n_\Lambda=(\beth,\Delta)=\Big\{\Big(\delta,\Big\{\Big(\pi,&\left[\tau^{+}_{1\beth(\delta)}(\pi),1-\eta^{+}_{1\beth(\delta)}(\pi)\right],\\
&\left[\tau^{+}_{2\beth(\delta)}(\pi),1-\eta^{+}_{2\beth(\delta)}(\pi)\right],\cdots,\\
&\left[\tau^{+}_{n\beth(\delta)}(\pi),1-\eta^{+}_{n\beth(\delta)}(\pi)\right],\left[-1-\eta^{-}_{1\beth(\delta)}(\pi),\tau^{-}_{1\beth(\delta)}(\pi)\right],\\
&\left[-1-\eta^{-}_{2\beth(\delta)}(\pi),\tau^{-}_{2\beth(\delta)}(\pi)\right],\cdots,\\
&\left[-1-\eta^{-}_{n\beth(\delta)}(\pi),\tau^{-}_{n\beth(\delta)}(\pi)\right]\Big)\Big\}\Big):18pt\delta\in\Lambda,\pi\in\Pi\Big\}\\
=\Big\{\Big(\delta,\Big\{\Big(\pi,&[\min\{1,0\},\min\{1,0\}],\overset{n-times}{\cdots},\\
&[\min\{1,0\},\min\{1,0\}],[\max\{-1,0\},\max\{-1,0\}],\overset{n-times}{\cdots},\\
&[\max\{-1,0\},\max\{-1,0\}]\Big)\Big\}\Big):\delta\in\Lambda,\pi\in\Pi\Big\}\\
=\Big\{\Big(\delta,\Big\{\Big(\pi,&\Big([0,0],\overset{n-times}{\cdots},[0,0]\Big),\Big([0,0],\overset{n-times}{\cdots},[0,0]\Big)\Big)\Big\}\Big)\\
&:\delta\in\Lambda,\pi\in\Pi\Big\}=\Big\{\Big(\delta,\Big\{\Big(\pi,[0,0],\overset{2n-times}{\cdots},[0,0]\Big)\Big\}\Big)\\
&:\delta\in\Lambda,\pi\in\Pi\Big\}=\phi^n_\Lambda.
\end{aligned}
\tag{20}
$$

This result is true for the third case (when $\delta\in\Lambda\cap\Lambda=\Lambda$) of Definition 21 of intersection of two bipolar-valued multivague soft sets of dimension n. For the first and second case, we have no parameters, since $\delta\in\Lambda-\Lambda=\phi$. □

Theorem 26. *Assume that (Γ,Λ) and (Ψ,Θ) are two bipolar-valued multivague soft sets of dimension n on a universal set Π; then we have the absorption properties satisfied for them as follows:*

$$
\begin{aligned}
&(1)\ (\Gamma,\Lambda)\tilde{\cup}((\Gamma,\Lambda)\tilde{\cap}_R(\Psi,\Theta))=(\Gamma,\Lambda),\\
&(2)(\Gamma,\Lambda)\tilde{\cap}_R((\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta))=(\Gamma,\Lambda).
\end{aligned}
\tag{21}
$$

Proof. We just prove (1). The same steps can be followed to prove (5). Let

$$
\begin{aligned}
(\Phi,\Xi)&=(\Gamma,\Lambda)\tilde{\cap}_R(\Psi,\Theta),\Xi=\Lambda\cap\Theta\\
&=\Big\{\Big(\xi,\Big\{\Big(\pi,\left[\tau^{+}_{1\Phi(\xi)}(\pi),1-\eta^{+}_{1\Phi(\xi)}(\pi)\right],\\
&\left[\tau^{+}_{2\Phi(\xi)}(\pi),1-\eta^{+}_{2\Phi(\xi)}(\pi)\right],\cdots,\left[\tau^{+}_{n\Phi(\xi)}(\pi),1-\eta^{+}_{n\Phi(\xi)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\Phi(\xi)}(\pi),\tau^{-}_{1\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{2\Phi(\xi)}(\pi),\tau^{-}_{2\Phi(\xi)}(\pi)\right],\\
&\cdots,\left[-1-\eta^{-}_{n\Phi(\xi)}(\pi),\tau^{-}_{n\Phi(\xi)}(\pi)\right]\Big)\Big\}\Big):\xi\in\Xi,\pi\in\Pi\Big\},\\
(\beth,\Delta)&=(\Gamma,\Lambda)\tilde{\cup}(\Phi,\Xi),\Delta=\Lambda\cup\Xi\\
&=\Big\{\Big(\delta,\Big\{\Big(\pi,\left[\tau^{+}_{1\beth(\delta)}(\pi),1-\eta^{+}_{1\beth(\delta)}(\pi)\right],\\
&\left[\tau^{+}_{2\beth(\delta)}(\pi),1-\eta^{+}_{2\beth(\delta)}(\pi)\right],\cdots,\left[\tau^{+}_{n\beth(\delta)}(\pi),1-\eta^{+}_{n\beth(\delta)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\beth(\delta)}(\pi),\tau^{-}_{1\beth(\delta)}(\pi)\right],\left[-1-\eta^{-}_{2\beth(\delta)}(\pi),\tau^{-}_{2\beth(\delta)}(\pi)\right],\\
&\cdots,\left[-1-\eta^{-}_{n\beth(\delta)}(\pi),\tau^{-}_{n\beth(\delta)}(\pi)\right]\Big)\Big\}\Big):\delta\in\Delta,\pi\in\Pi\Big\}.
\end{aligned}
\tag{22}
$$

According to Definition 17, we have to show that (1) holds for all following three cases:

(i) If $\delta\in\Lambda-\Theta$, then we obtain from Definition 23 that:

$$
\begin{aligned}
(\Phi,\Xi)=(\Gamma,\Lambda)\tilde{\cap}_R(\Psi,\Theta)=\Big\{\Big(\xi,\Big\{\Big(\pi,&\left[\tau^{+}_{1\Phi(\xi)}(\pi),1-\eta^{+}_{1\Phi(\xi)}(\pi)\right],\\
&\left[\tau^{+}_{2\Phi(\xi)}(\pi),1-\eta^{+}_{2\Phi(\xi)}(\pi)\right],\cdots,\left[\tau^{+}_{n\Phi(\xi)}(\pi),1-\eta^{+}_{n\Phi(\xi)}(\pi)\right],\\
&\left[-1-\eta^{-}_{1\Phi(\xi)}(\pi),\tau^{-}_{1\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{2\Phi(\xi)}(\pi),\tau^{-}_{2\Phi(\xi)}(\pi)\right],\\
&\cdots,\left[-1-\eta^{-}_{n\Phi(\xi)}(\pi),\tau^{-}_{n\Phi(\xi)}(\pi)\right]\Big)\Big\}\Big):\xi\in\Lambda-\Theta,\pi\in\Pi\Big\}=\phi.
\end{aligned}
\tag{23}
$$

Then, by using (6) from Theorem 25, we have:

$$
(\beth,\Delta)=(\Gamma,\Lambda)\tilde{\cup}(\Phi,\Xi)=(\Gamma,\Lambda)\tilde{\cup}\phi=(\Gamma,\Lambda).
\tag{24}
$$

(ii) If $\delta\in\Theta-\Lambda$, then we obtain from Definition 23 that:

$$(\Phi, \Xi) = (\Gamma,\Lambda) \tilde{\cap}_R (\Psi, \Theta) = \Big\{ \Big(\xi, \Big\{ \Big(\pi, \big[\tau^+_{1\Phi(\xi)}(\pi), 1-\eta^+_{1\Phi(\xi)}(\pi) \big], \big[\tau^+_{2\Phi(\xi)}(\pi), 1-\eta^+_{2\Phi(\xi)}(\pi) \big], \cdots, \big[\tau^+_{n\Phi(\xi)}(\pi), 1-\eta^+_{n\Phi(\xi)}(\pi) \big], \big[-1-\eta^-_{1\Phi(\xi)}(\pi), \tau^-_{1\Phi(\xi)}(\pi) \big], \big[-1-\eta^-_{2\Phi(\xi)}(\pi), \tau^-_{2\Phi(\xi)}(\pi) \big], \cdots, \big[-1-\eta^-_{n\Phi(\xi)}(\pi), \tau^-_{n\Phi(\xi)}(\pi) \big] \Big) \Big\} \Big) : \xi \in \Theta - \Lambda, \pi \in \Pi \Big\} = \phi. \tag{25}$$

Then, by using (6) from Theorem 25, we have:

$$(\sqsupset, \Delta) = (\Gamma,\Lambda) \tilde{\cup} (\Phi, \Xi) = (\Gamma,\Lambda) \tilde{\cup} \phi = (\Gamma, \Lambda). \tag{26}$$

(iii) If $\delta \in \Lambda \cap \Theta$, then we obtain from Definition 23 that:

$$\begin{aligned}(\Phi, \Xi) &= (\Gamma,\Lambda) \tilde{\cap}_R (\Psi, \Theta) \\ &= \Big\{ \Big(\xi, \Big\{ \Big(\pi, \big[\tau^+_{1\Phi(\xi)}(\pi), 1-\eta^+_{1\Phi(\xi)}(\pi) \big], \big[\tau^+_{2\Phi(\xi)}(\pi), 1-\eta^+_{2\Phi(\xi)}(\pi) \big], \cdots, \big[\tau^+_{n\Phi(\xi)}(\pi), 1-\eta^+_{n\Phi(\xi)}(\pi) \big], \\ &\quad \big[-1-\eta^-_{1\Phi(\xi)}(\pi), \tau^-_{1\Phi(\xi)}(\pi) \big], \big[-1-\eta^-_{2\Phi(\xi)}(\pi), \tau^-_{2\Phi(\xi)}(\pi) \big], \cdots, \big[-1-\eta^-_{n\Phi(\xi)}(\pi), \tau^-_{n\Phi(\xi)}(\pi) \big] \Big) \Big\} \Big) : \xi \in \Lambda \cap \Theta, \pi \in \Pi \Big\} \\ &= \Big\{ \Big(\xi, \Big\{ \Big(\pi, \Big[\min \big\{ \tau^+_{1\Gamma(\xi)}(\pi), \tau^+_{1\Psi(\xi)}(\pi) \big\}, \min \big\{ 1-\eta^+_{1\Gamma(\xi)}(\pi), 1-\eta^+_{1\Psi(\xi)}(\pi) \big\} \Big], \\ &\quad \Big[\min \big\{ \tau^+_{2\Gamma(\xi)}(\pi), \tau^+_{2\Psi(\xi)}(\pi) \big\}, \min \big\{ 1-\eta^+_{2\Gamma(\xi)}(\pi), 1-\eta^+_{2\Psi(\xi)}(\pi) \big\} \Big], \cdots, \\ &\quad \Big[\min \big\{ \tau^+_{n\Gamma(\xi)}(\pi), \tau^+_{n\Psi(\xi)}(\pi) \big\}, \min \big\{ 1-\eta^+_{n\Gamma(\xi)}(\pi), 1-\eta^+_{n\Psi(\xi)}(\pi) \big\} \Big], \\ &\quad \Big[\max \big\{ -1-\eta^-_{1\Gamma(\xi)}(\pi), -1-\eta^-_{1\Psi(\xi)}(\pi) \big\}, \max \big\{ \tau^-_{1\Gamma(\xi)}(\pi), \tau^-_{1\Psi(\xi)}(\pi) \big\} \Big], \\ &\quad \Big[\max \big\{ -1-\eta^-_{2\Gamma(\xi)}(\pi), -1-\eta^-_{2\Psi(\xi)}(\pi) \big\}, \max \big\{ \tau^-_{2\Gamma(\xi)}(\pi), \tau^-_{2\Psi(\xi)}(\pi) \big\} \Big], \cdots, \\ &\quad \Big[\max \big\{ -1-\eta^-_{n\Gamma(\xi)}(\pi), -1-\eta^-_{n\Psi(\xi)}(\pi) \big\}, \max \big\{ \tau^-_{n\Gamma(\xi)}(\pi), \tau^-_{n\Psi(\xi)}(\pi) \big\} \Big] \Big) \Big\} \Big) : \xi \in \Lambda \cap \Theta, \pi \in \Pi \Big\}. \end{aligned} \tag{27}$$

Therefore, we have from Definition 17 that:

$$\begin{aligned}(\sqsupset, \Delta) &= (\Gamma,\Lambda) \tilde{\cup} (\Phi, \Xi) = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \big[\tau^+_{1\sqsupset(\delta)}(\pi), 1-\eta^+_{1\sqsupset(\delta)}(\pi) \big], \big[\tau^+_{2\sqsupset(\delta)}(\pi), 1-\eta^+_{2\sqsupset(\delta)}(\pi) \big], \cdots, \big[\tau^+_{n\sqsupset(\delta)}(\pi), 1-\eta^+_{n\sqsupset(\delta)}(\pi) \big], \\ &\quad \big[-1-\eta^-_{1\sqsupset(\delta)}(\pi), \tau^-_{1\sqsupset(\delta)}(\pi) \big], \big[-1-\eta^-_{2\sqsupset(\delta)}(\pi), \tau^-_{2\sqsupset(\delta)}(\pi) \big], \cdots, \big[-1-\eta^-_{n\sqsupset(\delta)}(\pi), \tau^-_{n\sqsupset(\delta)}(\pi) \big] \Big) \Big\} \Big) : \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\} \\ &= \Big\{ \Big(\delta, \Big\{ \Big(\pi, \Big[\max \Big\{ \tau^+_{1\Gamma(\delta)}(\pi), \min \big\{ \tau^+_{1\Gamma(\delta)}(\pi), \tau^+_{1\Psi(\delta)}(\pi) \big\} \Big\}, \max \Big\{ 1-\eta^+_{1\Gamma(\delta)}(\pi), \min \big\{ 1-\eta^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi) \big\} \Big\} \Big], \\ &\quad \Big[\max \Big\{ \tau^+_{2\Gamma(\delta)}(\pi), \min \big\{ \tau^+_{2\Gamma(\delta)}(\pi), \tau^+_{2\Psi(\delta)}(\pi) \big\} \Big\}, \max \Big\{ 1-\eta^+_{2\Gamma(\delta)}(\pi), \min \big\{ 1-\eta^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Psi(\delta)}(\pi) \big\} \Big\} \Big], \\ &\quad \cdots, \Big[\max \Big\{ \tau^+_{n\Gamma(\delta)}(\pi), \min \big\{ \tau^+_{n\Gamma(\delta)}(\pi), \tau^+_{n\Psi(\delta)}(\pi) \big\} \Big\}, \max \Big\{ 1-\eta^+_{n\Gamma(\delta)}(\pi), \min \big\{ 1-\eta^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Psi(\delta)}(\pi) \big\} \Big\} \Big], \\ &\quad \Big[\min \Big\{ -1-\eta^-_{1\Gamma(\delta)}(\pi), \max \big\{ -1-\eta^-_{1\Gamma(\delta)}(\pi), -1-\eta^-_{1\Psi(\delta)}(\pi) \big\} \Big\}, \min \Big\{ \tau^-_{1\Gamma(\delta)}(\pi), \max \big\{ \tau^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi) \big\} \Big\} \Big], \\ &\quad \Big[\min \Big\{ -1-\eta^-_{2\Gamma(\delta)}(\pi), \max \big\{ -1-\eta^-_{2\Gamma(\delta)}(\pi), -1-\eta^-_{2\Psi(\delta)}(\pi) \big\} \Big\}, \min \Big\{ \tau^-_{2\Gamma(\delta)}(\pi), \max \big\{ \tau^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Psi(\delta)}(\pi) \big\} \Big\} \Big], \cdots, \\ &\quad \Big[\min \Big\{ -1-\eta^-_{n\Gamma(\delta)}(\pi), \max \big\{ -1-\eta^-_{n\Gamma(\delta)}(\pi), -1-\eta^-_{n\Psi(\delta)}(\pi) \big\} \Big\}, \min \Big\{ \tau^-_{n\Gamma(\delta)}(\pi), \max \big\{ \tau^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Psi(\delta)}(\pi) \big\} \Big\} \Big] \Big) \Big\} \Big) \\ &\quad : \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\} = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \big[\tau^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Gamma(\delta)}(\pi) \big], \big[\tau^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Gamma(\delta)}(\pi) \big], \cdots, \big[\tau^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Gamma(\delta)}(\pi) \big], \\ &\quad \big[-1-\eta^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Gamma(\delta)}(\pi) \big], \big[-1-\eta^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Gamma(\delta)}(\pi) \big], \cdots, \big[-1-\eta^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Gamma(\delta)}(\pi) \big] \Big) \Big\} \Big) \\ &\quad : \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\} = (\Gamma, \Lambda). \end{aligned} \tag{28}$$

□

Corollary 27. *For two bipolar-valued multivague soft sets of dimension n on a common universal set* (Γ, Λ) *and* (Ψ, Θ)*:*

$$(\Gamma,\Lambda) \tilde{\cup} ((\Gamma,\Lambda) \tilde{\cap}_R (\Psi, \Theta)) = (\Gamma,\Lambda) \tilde{\cap}_R ((\Gamma,\Lambda) \tilde{\cup} (\Psi, \Theta)) = (\Gamma, \Lambda). \tag{29}$$

Proof. One can prove this result directly with the help of the above Theorem 26. □

Theorem 28. *Suppose that* (Γ, Λ) *and* (Ψ, Θ) *are two bipolar-valued multivague soft sets of dimension n on a universal set* Π*; then, we have the commutative (Abelian) property satisfied for them as follows:*

$$\begin{aligned} &(1)\ (\Gamma,\Lambda) \tilde{\cap} (\Psi, \Theta) = (\Psi,\Theta) \tilde{\cap} (\Gamma, \Lambda), \\ &(2)\ (\Gamma,\Lambda) \tilde{\cup} (\Psi, \Theta) = (\Psi,\Theta) \tilde{\cup} (\Gamma, \Lambda). \end{aligned} \tag{30}$$

Proof. By applying similar techniques used to prove the above theorems with the help of Definitions 21 and 17 of the intersection and union of two bipolar-valued multivague soft sets of dimension n on a universal set Π, respectively, we can show that (1) and (5) hold. □

Proposition 29. *If* (Γ, Λ) *and* (Ψ, Θ) *are two bipolar-valued multivague soft sets of dimension n,* $(\Gamma,\Lambda) \subseteq^{\sim} (\Psi, \Theta)$, *then*

$$(1)\ (\Gamma,\Lambda) \cap^{\sim}_{R} (\Psi, \Theta) = (\Gamma, \Lambda),\ (2)\ (\Gamma,\Lambda) \cup^{\sim} (\Psi, \Theta) = (\Psi, \Theta). \tag{31}$$

Proof. Apply similar methods but by using Definitions 23 and 19 of restricted intersection and restricted union of two bipolar-valued multivague soft sets of dimension n, respectively, to prove (1) and (5). □

Theorem 30. *Suppose that* (Γ, Λ), (Ψ, Θ), *and* (Σ, Ω) *are bipolar-valued multivague soft sets of dimension n on a common universal set* Π; *then associative and distributive laws, respectively, hold for them as follows:*

$$\begin{gathered}
(\Gamma,\Lambda) \cap^{\sim} \left((\Psi,\Theta) \cap^{\sim} (\Sigma, \Omega)\right) = \left((\Gamma,\Lambda) \cap^{\sim} (\Psi,\Theta)\right) \cap^{\sim} (\Sigma, \Omega),\\
(\Gamma,\Lambda) \cup^{\sim} \left((\Psi,\Theta) \cup^{\sim} (\Sigma, \Omega)\right) = \left((\Gamma,\Lambda) \cup^{\sim} (\Psi,\Theta)\right) \cup^{\sim} (\Sigma, \Omega),\\
(\Gamma,\Lambda) \cap^{\sim} \left((\Psi,\Theta) \cup^{\sim} (\Sigma, \Omega)\right) = \left((\Gamma,\Lambda) \cap^{\sim} (\Psi,\Theta)\right) \cup^{\sim} \left((\Gamma,\Lambda) \cap^{\sim} (\Sigma, \Omega)\right),\\
(\Gamma,\Lambda) \cup^{\sim} ((\Psi,\Theta) \cap^{\sim} (\Sigma, \Omega)) = ((\Gamma,\Lambda) \cup^{\sim} (\Psi,\Theta)) \cap^{\sim} ((\Gamma,\Lambda) \cup^{\sim} (\Sigma, \Omega)).
\end{gathered} \tag{32}$$

Proof. Using Definitions 21 and 17 of the intersection and union of two bipolar-valued multivague soft sets of dimension n on a universal set Π, respectively, one can prove the theorem, similarly as the above theorems. □

Theorem 31. *De Morgan's laws are valid for any two bipolar-valued multivague soft sets* (Γ, Λ) *and* (Ψ, Θ) *of dimension n on a common universal set* Π *as follows:*

$$\begin{gathered}
(1)\ \left((\Gamma,\Lambda) \cup^{\sim} (\Psi, \Theta)\right)^c = (\Gamma,\Lambda)^c \cap^{\sim} (\Psi, \Theta)^c,\\
(2)\ \left((\Gamma,\Lambda) \cap^{\sim} (\Psi, \Theta)\right)^c = (\Gamma,\Lambda)^c \cup^{\sim} (\Psi, \Theta)^c.
\end{gathered} \tag{33}$$

Proof. To prove (1), let $\delta \in \Delta = \Lambda \cup \Theta$. We must show that (1) is true for all three cases according to Definitions 17 and 21. Assume that $\delta \in \Lambda \cap \Theta$; therefore, we obtain from Definitions 14, 17, and 21 that:

$$\begin{aligned}
\left((\Gamma,\Lambda) \cup^{\sim} (\Psi, \Theta)\right)^c = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \Big[&\max \left\{\tau^{+}_{1\Gamma(\delta)}(\pi), \tau^{+}_{1\Psi(\delta)}(\pi)\right\},\\
&\max \left\{1-\eta^{+}_{1\Gamma(\delta)}(\pi), 1-\eta^{+}_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[\max \left\{\tau^{+}_{2\Gamma(\delta)}(\pi), \tau^{+}_{2\Psi(\delta)}(\pi)\right\},\\
&\max \left\{1-\eta^{+}_{2\Gamma(\delta)}(\pi), 1-\eta^{+}_{2\Psi(\delta)}(\pi)\right\}\Big], \cdots,\\
&\Big[\max \left\{\tau^{+}_{n\Gamma(\delta)}(\pi), \tau^{+}_{n\Psi(\delta)}(\pi)\right\},\\
&\max \left\{1-\eta^{+}_{n\Gamma(\delta)}(\pi), 1-\eta^{+}_{n\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[\min \left\{-1-\eta^{-}_{1\Gamma(\delta)}(\pi), -1-\eta^{-}_{1\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{1\Gamma(\delta)}(\pi), \tau^{-}_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[\min \left\{-1-\eta^{-}_{2\Gamma(\delta)}(\pi), -1-\eta^{-}_{2\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{2\Gamma(\delta)}(\pi), \tau^{-}_{2\Psi(\delta)}(\pi)\right\}\Big], \cdots,\\
&\Big[\min \left\{-1-\eta^{-}_{n\Gamma(\delta)}(\pi), -1-\eta^{-}_{n\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{n\Gamma(\delta)}(\pi), \tau^{-}_{n\Psi(\delta)}(\pi)\right\}\Big]\Big)\Big\}\Big)\\
&: \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\}^c,
\end{aligned}$$

$$\begin{aligned}
\left((\Gamma,\Lambda) \cup^{\sim} (\Psi, \Theta)\right)^c = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \Big[&\max \left\{\tau^{+}_{1\Gamma(\delta)}(\pi), \tau^{+}_{1\Psi(\delta)}(\pi)\right\},\\
&1-\min \left\{\eta^{+}_{1\Gamma(\delta)}(\pi), \eta^{+}_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[\max \left\{\tau^{+}_{2\Gamma(\delta)}(\pi), \tau^{+}_{2\Psi(\delta)}(\pi)\right\},\\
&1-\min \left\{\eta^{+}_{2\Gamma(\delta)}(\pi), \eta^{+}_{2\Psi(\delta)}(\pi)\right\}], \cdots,\\
&\Big[\max \left\{\tau^{+}_{n\Gamma(\delta)}(\pi), \tau^{+}_{n\Psi(\delta)}(\pi)\right\},\\
&1-\min \left\{\eta^{+}_{n\Gamma(\delta)}(\pi), \eta^{+}_{n\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[-1-\max \left\{\eta^{-}_{1\Gamma(\delta)}(\pi), \eta^{-}_{1\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{1\Gamma(\delta)}(\pi), \tau^{-}_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[-1-\max \left\{\eta^{-}_{2\Gamma(\delta)}(\pi), \eta^{-}_{2\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{2\Gamma(\delta)}(\pi), \tau^{-}_{2\Psi(\delta)}(\pi)\right\}\Big], \cdots,\\
&\Big[-1-\max \left\{\eta^{-}_{n\Gamma(\delta)}(\pi), \eta^{-}_{n\Psi(\delta)}(\pi)\right\},\\
&\min \left\{\tau^{-}_{n\Gamma(\delta)}(\pi), \tau^{-}_{n\Psi(\delta)}(\pi)\right\}\Big]\Big)\Big\}\Big)\\
&: \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\}^c,
\end{aligned}$$

$$\begin{aligned}
\left((\Gamma,\Lambda) \cup^{\sim} (\Psi, \Theta)\right)^c = \Big\{ \Big(\delta, \Big\{ \Big(\pi, \Big[&\min \Big\{\eta^{+}_{1\Gamma(\delta)}(\pi), \eta^{+}_{1\Psi(\delta)}(\pi),\\
&1-\max \left\{\tau^{+}_{1\Gamma(\delta)}(\pi), \tau^{+}_{1\Psi(\delta)}(\pi)\right\}\Big\}\Big],\\
&\Big[\min \Big\{\eta^{+}_{2\Gamma(\delta)}(\pi), \eta^{+}_{2\Psi(\delta)}(\pi),\\
&1-\max \left\{\tau^{+}_{2\Gamma(\delta)}(\pi), \tau^{+}_{2\Psi(\delta)}(\pi)\right\}\Big\}\Big], \cdots,\\
&\Big[\min \Big\{\eta^{+}_{n\Gamma(\delta)}(\pi), \eta^{+}_{n\Psi(\delta)}(\pi),\\
&1-\max \left\{\tau^{+}_{n\Gamma(\delta)}(\pi), \tau^{+}_{n\Psi(\delta)}(\pi)\right\}\Big\}\Big],\\
&\Big[-1-\min \left\{\tau^{-}_{1\Gamma(\delta)}(\pi), \tau^{-}_{1\Psi(\delta)}(\pi)\right\},\\
&\max \left\{\eta^{-}_{1\Gamma(\delta)}(\pi), \eta^{-}_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[-1-\min \left\{\tau^{-}_{2\Gamma(\delta)}(\pi), \tau^{-}_{2\Psi(\delta)}(\pi)\right\},\\
&\max \left\{\eta^{-}_{2\Gamma(\delta)}(\pi), \eta^{-}_{2\Psi(\delta)}(\pi)\right\}\Big], \cdots,\\
&\Big[-1-\min \left\{\tau^{-}_{n\Gamma(\delta)}(\pi), \tau^{-}_{n\Psi(\delta)}(\pi)\right\},\\
&\max \left\{\eta^{-}_{n\Gamma(\delta)}(\pi), \eta^{-}_{n\Psi(\delta)}(\pi)\right\}\Big]\Big)\Big\}\Big)\\
&: \delta \in \Lambda \cap \Theta, \pi \in \Pi \Big\},
\end{aligned}$$

$$\begin{aligned}((\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta))^c = \Big\{\Big(\delta, \Big\{\Big(\pi, \Big[&\min\left\{\eta^+_{1\Gamma(\delta)}(\pi), \eta^+_{1\Psi(\delta)}(\pi),\right.\\
&\left.\min\left\{1-\tau^+_{1\Gamma(\delta)}(\pi), 1-\tau^+_{1\Psi(\delta)}(\pi)\right\}\right\}\Big],\\
&\Big[\min\left\{\eta^+_{2\Gamma(\delta)}(\pi), \eta^+_{2\Psi(\delta)}(\pi),\right.\\
&\left.\min\left\{1-\tau^+_{2\Gamma(\delta)}(\pi), 1-\tau^+_{2\Psi(\delta)}(\pi)\right\}\right\}\Big],\\
&\cdots, \Big[\min\left\{\eta^+_{n\Gamma(\delta)}(\pi), \eta^+_{n\Psi(\delta)}(\pi),\right.\\
&\left.\min\left\{1-\tau^+_{n\Gamma(\delta)}(\pi), 1-\tau^+_{n\Psi(\delta)}(\pi)\right\}\right\}\Big],\\
&\Big[\max\left\{-1-\tau^-_{1\Gamma(\delta)}(\pi), -1-\tau^-_{1\Psi(\delta)}(\pi)\right\},\\
&\max\left\{\eta^-_{1\Gamma(\delta)}(\pi), \eta^-_{1\Psi(\delta)}(\pi)\right\}\Big],\\
&\Big[\max\left\{-1-\tau^-_{2\Gamma(\delta)}(\pi), -1-\tau^-_{2\Psi(\delta)}(\pi)\right\},\\
&\max\left\{\eta^-_{2\Gamma(\delta)}(\pi), \eta^-_{2\Psi(\delta)}(\pi)\right\}\Big], \cdots,\\
&\Big[\max\left\{-1-\tau^-_{n\Gamma(\delta)}(\pi), -1-\tau^-_{n\Psi(\delta)}(\pi)\right\},\\
&\max\left\{\eta^-_{n\Gamma(\delta)}(\pi), \eta^-_{n\Psi(\delta)}(\pi)\right\}\Big]\Big)\Big\}\Big)\\
&: \delta \in \Lambda \cap \Theta, \pi \in \Pi\Big\},\end{aligned}$$

$$\begin{aligned}((\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta))^c = \Big\{\Big(\delta, \Big\{\Big(\pi, &\Big[\eta^+_{1\Gamma(\delta)}(\pi), 1-\tau^+_{1\Gamma(\delta)}(\pi)\Big],\\
&\Big[\eta^+_{2\Gamma(\delta)}(\pi), 1-\tau^+_{2\Gamma(\delta)}(\pi)\Big], \cdots,\\
&\Big[\eta^+_{n\Gamma(\delta)}(\pi), 1-\tau^+_{n\Gamma(\delta)}(\pi)\Big],\\
&\Big[-1-\tau^-_{1\Gamma(\delta)}(\pi), \eta^-_{1\Gamma(\delta)}(\pi)\Big],\\
&\Big[-1-\tau^-_{2\Gamma(\delta)}(\pi), \eta^-_{2\Gamma(\delta)}(\pi)\Big], \cdots,\\
&\Big[-1-\tau^-_{n\Gamma(\delta)}(\pi), \eta^-_{n\Gamma(\delta)}(\pi)\Big]\Big)\Big\}\Big) : \in \Lambda \cap \Theta, \pi \in \Pi\Big\}\\
&\cdot \tilde{\cap}\Big\{\Big(\delta, \Big\{\Big(\pi, \Big[\eta^+_{1\Psi(\delta)}(\pi), 1-\tau^+_{1\Psi(\delta)}(\pi)\Big],\\
&\Big[\eta^+_{2\Psi(\delta)}(\pi), 1-\tau^+_{2\Psi(\delta)}(\pi)\Big], \cdots,\\
&\Big[\eta^+_{n\Psi(\delta)}(\pi), 1-\tau^+_{n\Psi(\delta)}(\pi)\Big],\\
&\Big[-1-\tau^-_{1\Psi(\delta)}(\pi), \eta^-_{1\Psi(\delta)}(\pi)\Big],\\
&\Big[-1-\tau^-_{2\Psi(\delta)}(\pi), \eta^-_{2\Psi(\delta)}(\pi)\Big], \cdots,\\
&\Big[-1-\tau^-_{n\Psi(\delta)}(\pi), \eta^-_{n\Psi(\delta)}(\pi)\Big]\Big)\Big\}\Big)\\
&: \delta \in \Lambda \cap \Theta, \pi \in \Pi\Big\},\end{aligned}$$

$$\begin{aligned}((\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta))^c = \Big\{\Big(\delta, \Big\{\Big(\pi, &\Big[\tau^+_{1\Gamma(\delta)}(\pi), 1-\eta^+_{1\Gamma(\delta)}(\pi)\Big],\\
&\Big[\tau^+_{2\Gamma(\delta)}(\pi), 1-\eta^+_{2\Gamma(\delta)}(\pi)\Big], \cdots,\\
&\Big[\tau^+_{n\Gamma(\delta)}(\pi), 1-\eta^+_{n\Gamma(\delta)}(\pi)\Big],\\
&\Big[-1-\eta^-_{1\Gamma(\delta)}(\pi), \tau^-_{1\Gamma(\delta)}(\pi)\Big],\\
&\Big[-1-\eta^-_{2\Gamma(\delta)}(\pi), \tau^-_{2\Gamma(\delta)}(\pi)\Big], \cdots,\\
&\Big[-1-\eta^-_{n\Gamma(\delta)}(\pi), \tau^-_{n\Gamma(\delta)}(\pi)\Big]\Big)\Big\}\Big) : \delta \in \Lambda, \pi \in \Pi\Big\}^c\\
&\cdot \tilde{\cap}\Big\{\Big(\delta, \Big\{\Big(\pi, \Big[\tau^+_{1\Psi(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\Big],\\
&\Big[\tau^+_{1\Psi(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\Big], \cdots,\\
&\Big[\tau^+_{1\Psi(\delta)}(\pi), 1-\eta^+_{1\Psi(\delta)}(\pi)\Big],\\
&\Big[-1-\eta^-_{1\Psi(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\Big],\\
&\Big[-1-\eta^-_{1\Psi(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\Big], \cdots,\\
&\Big[-1-\eta^-_{1\Psi(\delta)}(\pi), \tau^-_{1\Psi(\delta)}(\pi)\Big]\Big)\Big\}\Big) : \delta \in \Theta, \pi \in \Pi\Big\}^c\\
&= (\Gamma,\Lambda)^c \cap \tilde{}\,(\Psi,\Theta)^c.\end{aligned} \tag{34}$$

This proves the third case. The first and second cases are trivial and follow similarly as above. In addition, (5) can be proved similarly as (1). □

4. Generalized Decision-Making Technique

This section focuses on a practical, real-world example of how to handle a socialist economic decision-making issue. Generalizing the Roy and Maji method [38], one can apply their technique in the bipolar-valued multivague soft set of dimension n environment rather than the fuzzy soft set environment. This makes it possible to pick up the best option for issues with uncertain or ambiguous information more effectively than with the prior models.

Definition 32. (comparison table). The comparison table is a square table; its rows and columns are labeled by the object name of the universe such as $\pi_1, \pi_2, \pi_3, \cdots, \pi_n$, and the entries d_{ij}, where d_{ij} is the number of parameters for which, the value of d_i (the membership value of π_i) exceeds or equal to ($\geq$) the value of d_j (the membership value of π_j).

4.1. Algorithm. Step (1): construct the tabular representations of the n positive poles membership intervals

Step (2): compute the comparison tables, as stated in Definition 32, for the tables obtained in step (1) of the n positive poles membership intervals.

Step (3): for each comparison table, compute the sum of every row $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_n$, and the sum of every column $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n$, then calculate their membership scores $\mathfrak{R}_1$-$\mathfrak{C}_1$, $\mathfrak{R}_2$-$\mathfrak{C}_2$, $\cdots$, $\mathfrak{R}_n$-$\mathfrak{C}_n$, to put them in the n positive poles score tables

Step (4): construct the final positive score table by calculating $\mathfrak{F}^+$ for each alternative π_i obtained by adding its n positive pole scores

Step (5): repeat steps (1), (2), (3), and (4) but for negative information

Step (6): compute the final score $\mathfrak{F}^+ - \mathfrak{F}^-$ for each element π_i by subtracting its final negative information score $\mathfrak{F}^-$ from its final positive information score $\mathfrak{F}^+$, then put these final scores in a table called the final score table

Step (7): determine k under which $\pi_k = \max \pi_i$; therefore, the best alternative is π_k

Step (8): obtain the ranking of alternatives π_i by ranking their final scores in descending order (ordinal ranking)

Table 1: The first positive pole membership intervals.

Π	λ_1'	λ_2'	λ_3'
π_1	[0.8,0.9]	[0.1,0.2]	[0.2,0.3]
π_2	[0.5,0.7]	[0.8,0.9]	[0.8,1]
π_3	[0.4,0.6]	[0.3,0.4]	[0.6,0.7]
π_4	[0.9,1]	[0.8,0.9]	[0.4,0.5]

Table 2: The second positive pole membership intervals.

Π	λ_1''	λ_2''	λ_3''
π_1	[0.2,0.3]	[0.7,0.8]	[0.8,0.9]
π_2	[0.4,0.6]	[0.6,0.7]	[0.5,0.7]
π_3	[0.7,0.8]	[0.9,1]	[0.2,0.4]
π_4	[0.1,0.3]	[0.2,0.4]	[0.6,0.8]
π_5	[0,0.1]	[0.5,0.6]	[0.5,0.6]

Table 3: The third positive pole membership intervals.

Π	λ_1'''	λ_2'''	λ_3'''
π_1	[0.4,0.5]	[0.5,0.6]	[0.8,1]
π_2	[0.9,1]	[0.5,0.6]	[0.8,0.9]
π_3	[0.2,0.3]	[0.2,0.3]	[0.9,1]
π_4	[0.5,0.6]	[0.7,0.8]	[0.3,0.4]
π_5	[0.6,0.9]	[0.8,0.9]	[0.8,0.9]

Table 4: The comparison table of Table 1.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	1	1	0	1
π_2	2	3	3	2	3
π_3	2	0	3	1	1
π_4	3	2	2	3	3
π_5	2	0	2	0	3

Example 2. Consider the same data of Example 1. If the manager of this organization wants to select the most suitable employee among the five nominated employees to become the assistant director of his organization according to the $(\sqsupset, Y)$ description.

4.1.1. Solution 1. Construct the tabular representations of the first, second, and third positive poles membership intervals, putting them in Tables 1–3, respectively.

4.1.2. Solution 2. Construct the comparison Tables 4–6, as stated in Definition 32, of Tables 1–3, respectively.

4.1.3. Solution 3. For Tables 4–6, compute the sum of every row ($\mathfrak{R}_1$, $\mathfrak{R}_2$, and $\mathfrak{R}_3$, respectively) and the sum of every column ($\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_3$, respectively), then calculate their membership scores ($\mathfrak{R}_1$-$\mathfrak{C}_1$, $\mathfrak{R}_2$-$\mathfrak{C}_2$, and $\mathfrak{R}_3$-$\mathfrak{C}_3$, respectively) to put them in the first, second, and third positive pole score tables, namely, Tables 7–9, respectively.

Table 5: The comparison table of Table 2.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	2	1	3	3
π_2	1	3	1	2	3
π_3	2	2	3	2	2
π_4	0	1	1	3	2
π_5	0	0	1	1	3

Table 6: The comparison table of Table 3.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	2	2	1	1
π_2	2	3	2	2	2
π_3	1	1	3	1	1
π_4	2	1	2	3	0
π_5	2	2	2	3	3

Table 7: The first positive pole score table.

Π	$\mathfrak{R}_1$	$\mathfrak{C}_1$	$\mathfrak{R}_1$-$\mathfrak{C}_1$
π_1	6	12	-6
π_2	13	6	7
π_3	7	11	-4
π_4	13	6	7
π_5	7	11	-4

Table 8: The second positive pole score table.

Π	$\mathfrak{R}_2$	$\mathfrak{C}_2$	$\mathfrak{R}_2$-$\mathfrak{C}_2$
π_1	12	6	6
π_2	10	8	2
π_3	11	7	4
π_4	7	11	-4
π_5	5	13	-8

4.1.4. Solution 4. Compute the final positive score ($\mathfrak{F}^+$) for each alternative by adding its first, second, and third positive pole scores, putting them in Table 10.

4.1.5. Solution 5. Repeat steps (1), (2), (3), and (4) for negative information as follows. Construct the tabular representations of the first, second, and third negative poles membership intervals, putting them in Tables 11–13, respectively. Then, construct Tables 14–16 of Tables 11–13, as stated in Definition 32. After that, for Tables 14–16, compute the sum of every row ($\mathfrak{R}_4$, $\mathfrak{R}_5$, and $\mathfrak{R}_6$, respectively)

Table 9: The third positive pole score table.

Π	$\mathfrak{R}_3$	$\mathfrak{C}_3$	$\mathfrak{R}_3$-$\mathfrak{C}_3$
π_1	9	10	-1
π_2	11	9	2
π_3	7	11	-4
π_4	8	10	-2
π_5	12	7	5

Table 10: The final positive score table.

Π	$\mathfrak{R}_1$-$\mathfrak{C}_1$	$\mathfrak{R}_2$-$\mathfrak{C}_2$	$\mathfrak{R}_3$-$\mathfrak{C}_3$	$\mathfrak{F}^+$
π_1	-6	6	-1	-1
π_2	7	2	2	11
π_3	-4	4	-4	-4
π_4	7	-4	-2	1
π_5	-4	-8	5	-7

Table 11: The first negative pole membership intervals.

Π	$-\lambda_1'$	$-\lambda_2'$	$-\lambda_3'$
π_1	[-0.2,-0.1]	[-0.3,-0.2]	[-0.8,-0.7]
π_2	[-0.3,-0.1]	[-0.4,-0.3]	[-0.2,-0.1]
π_3	[-0.5,-0.4]	[-1,-0.9]	[-1,-0.9]
π_4	[-0.2,-0.1]	[-0.6,-0.5]	[-0.7,-0.6]
π_5	[-1,-0.8]	[-0.9,-0.8]	[-1,-0.9]

Table 12: The second negative pole membership intervals.

Π	$-\lambda_1''$	$-\lambda_2''$	$-\lambda_3''$
π_1	[-0.8,-0.6]	[-0.9,-0.8]	[-0.9,-0.8]
π_2	[-0.5,-0.4]	[-0.8,-0.7]	[-0.4,-0.3]
π_3	[-0.8,-0.7]	[-0.2,-0.1]	[-0.2,-0.1]
π_4	[-0.6,-0.5]	[-0.7,-0.6]	[-0.3,-0.2]
π_5	[-0.5,-0.4]	[-0.5,-0.4]	[-0.6,-0.5]

Table 13: The third negative pole membership intervals.

Π	$-\lambda_1'''$	$-\lambda_2'''$	$-\lambda_3'''$
π_1	[-0.3,-0.2]	[-0.2,-0.1]	[-0.6,-0.5]
π_2	[-1,-0.8]	[-0.9,-0.8]	[-0.9,-0.7]
π_3	[-0.9,-0.6]	[-0.5,-0.4]	[-0.4,-0.3]
π_4	[-0.8,-0.7]	[-0.2,-0.1]	[-0.8,-0.7]
π_5	[-1,-0.9]	[-0.7,-0.6]	[-0.3,-0.2]

Table 14: The comparison table of Table 11.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	1	0	2	0
π_2	2	3	0	1	0
π_3	3	3	3	3	2
π_4	2	2	0	3	0
π_5	3	3	3	3	3

Table 15: The comparison table of Table 12.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	3	2	3	3
π_2	0	3	2	1	2
π_3	1	1	3	1	1
π_4	0	1	2	3	2
π_5	0	2	2	1	3

Table 16: The comparison table of Table 13.

Π	π_1	π_2	π_3	π_4	π_5
π_1	3	0	1	1	1
π_2	3	3	3	3	2
π_3	2	0	3	2	1
π_4	3	0	1	3	1
π_5	2	1	2	2	3

Table 17: The first negative pole score table.

Π	$\mathfrak{R}_4$	$\mathfrak{C}_4$	$\mathfrak{R}_4$-$\mathfrak{C}_4$
π_1	6	13	-7
π_2	6	12	-6
π_3	14	6	8
π_4	7	12	-5
π_5	15	5	10

Table 18: The second negative pole score table.

Π	$\mathfrak{R}_5$	$\mathfrak{C}_5$	$\mathfrak{R}_5$-$\mathfrak{C}_5$
π_1	14	4	10
π_2	8	10	-2
π_3	7	11	-4
π_4	8	9	-1
π_5	8	11	-3

and the sum of every column ($\mathfrak{C}_4$, $\mathfrak{C}_5$, and $\mathfrak{C}_6$, respectively), then calculate their membership scores ($\mathfrak{R}_4$-$\mathfrak{C}_4$, $\mathfrak{R}_5$-$\mathfrak{C}_5$, and $\mathfrak{R}_6$-$\mathfrak{C}_6$, respectively) to put them in the first, second, and third negative poles score tables, namely, Tables 17–19, respectively. Finally, compute the final negative score ($\mathfrak{F}^-$) for each alternative by adding its first, second, and third negative pole scores, putting them in Table 20.

Table 19: The third negative pole score table.

Π	$\mathfrak{R}_6$	$\mathfrak{C}_6$	$\mathfrak{R}_6$-$\mathfrak{C}_6$
π_1	6	13	-7
π_2	14	4	10
π_3	8	10	-2
π_4	8	11	-3
π_5	10	8	2

Table 20: The final negative score table.

Π	$\mathfrak{R}_4$-$\mathfrak{C}_4$	$\mathfrak{R}_5$-$\mathfrak{C}_5$	$\mathfrak{R}_6$-$\mathfrak{C}_6$	$\mathfrak{F}^-$
π_1	-7	10	-7	-4
π_2	-6	-2	10	2
π_3	8	-4	-2	2
π_4	-5	-1	-3	-9
π_5	10	-3	2	9

Table 21: The final score table.

Π	$\mathfrak{F}^+$	$\mathfrak{F}^-$	$\mathfrak{F}^+ - \mathfrak{F}^-$
π_1	-1	-4	3
π_2	11	2	9
π_3	-4	2	-6
π_4	1	-9	10
π_5	-7	9	-16

4.1.6. Solution 6. Now, subtract each final negative score (from Table 10) from its opposite final positive score (from Table 20) to calculate entries of the final score table $\mathfrak{F}^+ - \mathfrak{F}^-$, then put all of them in Table 21.

4.1.7. Solution 7. Since the fourth alternative (employee) π_1 has the maximum final score (which is 10) among the others, then the decision is that the most suitable employee to be the manager assistant in this organization among the five employees is the fourth one π_1.

4.1.8. Solution 8. Consequently, we have the ranking of the alternatives (employees) as follows:

$$\pi_4 > \pi_2 > \pi_1 > \pi_3 > \pi_5. \quad (35)$$

5. Concluding Notes and Future Researches

Set theory extensions have been considered a vital issue nowadays because of their importance in decision-making methods. In fact, combining any two basic extensions of the set theory not only is more general and flexible than applying one only of them but also gives more accurate applicable extended results. Some mathematicians have studied a few types of these combined extensions, such as bipolar-valued multifuzzy soft sets, multivague sets, and multivague soft sets. They have presented their characteristics and have used them in various areas as an illustration of actual problem-solving in decision-making. It goes without saying that several of these extensions are extensions for one another, not just for set theory.

In this article, the bipolar-valued multivague soft set of dimension n has been introduced as a novel hybrid set. Its kinds and its related concepts, as well as its operations, have been established, reflected by an example on each. Moreover, absorption properties, commutative properties, associative properties, and distributive laws, as well as De Morgan's laws, are investigated with detailed proof on each. Furthermore, Roy and Maji's technique has been generalized by using the concept of the bipolar-valued multivague soft set of dimension n instead of the fuzzy soft set concept to make more effective decisions to choose the optimal object among others. A useful real-world example has also been provided to further illustrate and clarify the suggested approach. This type of research fills in some gaps in the literature.

In future research work, authors can investigate some necessary comparison analysis between the proposed method and other existing approaches. The analysis will be valuable and interesting. Moreover, several new results with some more generalized measures can be introduced using analogous methods in this article. Furthermore, our future research ideas can be extended to spherical vague soft sets and Pythagorean vague soft sets. By using any of those combinatorial extensions, one can create numerous different applications in a variety of scientific fields.

Data Availability

All the data sets are provided within the main body of the paper.

Conflicts of Interest

The authors declare having no conflicts of interest.

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