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Research Article

Existence and H-U Stability of Solution for Coupled System of Fractional-Order with Integral Conditions Involving Caputo-Hadamard Derivatives, Hadamard Integrals

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In this article, the primary focus of our study is to investigate the existence, uniqueness, and Ulam-Hyers stability results for coupled fractional differential equations of the Caputo-Hadamard type that are supplemented with Hadamard integral boundary conditions. We employ adequate conditions to achieve existence and uniqueness results for the presented problems by utilizing the Banach contraction principle and the Leray-Schauder fixed point theorem. We also show Ulam-Hyers stability using the standard functional analysis technique. Finally, examples are used to validate the results.

1. Introduction

In the past two decades, scientists and researchers have published results on fractional calculus analysis and concluded that integer-order derivatives are not always reliable. The study of turbulent fluid flows, control theory, blood flow through biological tissues, porous media, and signal and image processing, among other fields, have all benefited greatly from the use of fractional calculus. The recent study on fractional calculus, including theory and applications, can be found in [1–13]. Their research is especially pertinent since coupled systems with fractional differential equations are used to address a wide range of real-world problems. Over the past few decades, FDEs have also been the topic of substantial research in the field of stability analysis. Different types of stability, such as Mittag-Leffler and Lyapunov, have been researched in the literature. Very few researches have investigated the Ulam-Hyers stability of a linked system of FDEs. Ulam and Hyers [14, 15] identified the novel type of stability known as Ulam-Hyers stability. Understanding biological processes, fluid motion, semiconductors, population dynamics, heat conduction, and elasticity can all be helped by this kind of research. Meanwhile, the researchers have focused on the differences and results of mathematical models created by these operators and have used a variety of fractional derivation operators in their studies as a result of the diversity of fractional operators described by mathematicians. Different forms of fractional mathematical models, in which the effects of the order of fractional derivatives on the dynamic behavior of the solutions of the assumed systems are rigorously simulated, are some of the well-known works on this topic. The following is just one illustration: Caputo derivatives are used in [16, 17], Caputo-conformable derivatives in [18, 19], generalized derivatives in [20, 21], quantum Caputo derivatives in [22], nonsingular Caputo-Fabrizio derivatives in [23], and nonsingular Mittag-Leffler kernel-type derivatives in [24, 25]. The features of the Caputo and Hadamard operations are combined to define the Caputo-Hadamard fractional derivative, one of the fractional derivatives. By using this operator, very few fractional models and problems were produced. Examples can be seen in [26–29]. However, the Hadamard fractional derivative (HFD) is the most frequently used [30]. Butzer et al. [31] investigated a variety of properties of HFD, which are more general than HFDs. In [32], the authors investigated a hybrid fractional Caputo-Hadamard boundary value problem with hybrid Hadamard integral boundary value conditions. The authors in [33] studied topological degree theory and Caputo-Hadamard fractional boundary value problems. In [34], Etemad et al. investigated a fractional Caputo-Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property. In 2020, Etemad et al. [35] discussed a fractional Caputo-Hadamard problem with boundary value conditions via different orders of the Hadamard fractional operators:

$$\begin{cases} \kappa^{\mathscr{CH}}\mathscr{D}^{\varrho}_{1^+}u(t) + (1-\kappa)^{\mathscr{CH}}\mathscr{D}^{\varpi}_{1^+}w(t) = \alpha\psi(t, w(t)) + \beta^{\mathscr{H}}\mathscr{I}^{\mu}_{1^+}\varphi(t, w(t)) \\ w(1) = 0, \ {}^{\mathscr{CH}}\mathfrak{D}^{\delta}_{1^+}w(e) = 0, \ {}^{\mathscr{CH}}\mathfrak{D}^{\delta}_{1^+}w(1) = 0, \\ {}^{\mathscr{H}}\mathfrak{I}^{\vartheta}_{1^+}w(e) = 0, \end{cases} \tag{1}$$

where $t \in [1, e]$, $\rho, \varpi \in (3, 4]$, $\delta \in (1, 2]$, $\kappa \in (0, 1]$, and $\mu, \vartheta > 0$ with $\delta + \vartheta \neq 0$ and also $\alpha, \beta \in \mathbb{R}$. In 2021, Rezapour et al. [36] investigated Caputo-Hadamard fractional boundary value problem via mixed multi-order integro-derivative conditions:

$$\begin{cases} \lambda^{\mathscr{CH}}\mathfrak{D}^{\varsigma}_{1^+}u(t) + {}^{\mathscr{CH}}\mathfrak{D}^{\theta^*}_{1^+}u(t) = \widehat{A}(t, u(t)), \\ u(1) = 0, \ \mu_1^{*\mathscr{CH}}\mathfrak{D}^{\gamma_1^*}_{1^+}u(M) + {}^{*\mathscr{CH}}\mathfrak{D}^{\gamma_2^*}_{1^+}u(\eta) = \delta_1, \\ \mu_2^{*\mathscr{H}}\mathfrak{I}^{q_1^*}_{1^+}u(M) + {}^{*\mathscr{H}}\mathfrak{I}^{q_2^*}_{1^+}u(\eta) = \delta_2, \end{cases} \tag{2}$$

so that $\lambda, \mu_1^*, \mu_2^* \in (0, 1]$, $\gamma_1^*, \gamma_2^* \in (0, \varsigma - \theta^*$ with $2 < \theta^* < \varsigma < 3$, $q_1^*, q_2^* \in \mathbb{R}^+$, $\delta_1, \delta_2 \in \mathbb{R}$, and $t \in [1, M]$. Recently, in [37], the authors derived existence and uniqueness results for a nonlinear coupled system of Caputo-type FDEs equipped with new coupled boundary conditions given by

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}u(t) = f(t, u(t), v(t)), & t \in J := [0, T], T > 0 \\ {}^{C}\mathscr{D}^{\beta}v(t) = g(t, u(t), v(t)), & t \in J := [0, T], \\ (u+v)(0) = -(u+v)(T), \displaystyle\int_{\eta}^{\xi}(u+v)(s)ds = \mathscr{A}, & 0 < \eta < \xi < T, \end{cases} \tag{3}$$

where ${}^{C}\mathscr{D}^{(\cdot)}_{0^+}$ denote the CFDs of order $(\cdot)$, $\alpha, \beta \in (0, 1]$, $f, g : [0, T] \times \mathscr{R}_e^2 \longrightarrow \mathscr{R}_e$ are continuous functions and A is real constant. In 2022, Belbali et.al [38] existence theory and generalized Mittag-Leffler stability for a nonlinear Caputo-Hadamard fractional initial value problem using the Lyapunov method. By using main ideas of the aforementioned articles, we investigate the Caputo-Hadamard coupled system of FDEs with the Hadamard fractional integral conditions and present its existence, uniqueness, and Ulam-Hyers stability results. We study the following system:

$$\begin{cases} {}^{C}\mathscr{D}^{\varsigma}\mathscr{P}(\vartheta) = \mathscr{Y}_1(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), & \vartheta\epsilon\mathscr{S}[1, \mathscr{T}], \\ {}^{C}\mathscr{D}^{\varrho}\mathscr{U}(\vartheta) = \mathscr{Y}_2(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), & \vartheta\epsilon\mathscr{S}[1, \mathscr{T}], \\ (\mathscr{P}+\mathscr{U})(1) = -(\mathscr{P}+\mathscr{U})(T), \dfrac{1}{\Gamma(\rho)}\displaystyle\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\rho-1}(\mathscr{P}+\mathscr{U})(\nu)\frac{d\nu}{\nu} = \phi, & 1 < \varphi < T, \end{cases} \tag{4}$$

where $\mathscr{Y}_1, \mathscr{Y}_2 : \mathscr{S} \times \mathscr{R}^2 \longrightarrow \mathscr{R}$ are continuous functions; ${}^{H}\mathscr{I}^{\rho}$ is the HFI of order ρ defined by

$$\left({}^{H}\mathscr{I}^{\rho}\mathscr{P}\right)(\vartheta) = \frac{1}{\Gamma(\rho)}\int_1^{\vartheta}\left(\log\frac{\vartheta}{\nu}\right)^{\rho-1}\mathscr{P}(\nu)\frac{d\nu}{\nu}, \ \rho > 0, \tag{5}$$

and ${}^{H}\mathscr{D}^{\varsigma}$ denotes HFD of order ς and is defined by

$$\left({}^{H}\mathscr{D}^{\varsigma}\mathscr{P}\right)(\vartheta) = \frac{1}{\Gamma(n-\varsigma)}\left(\vartheta\frac{d}{d\vartheta}\right)^{n}\int_1^{\vartheta}\left(\log\frac{\vartheta}{\nu}\right)^{n-\varsigma-1} \cdot \mathscr{P}(\nu)\frac{d\nu}{\nu}, \ n-1 < \varsigma < n, \ n = [\varsigma] + 1, \tag{6}$$

see [39, 40], where ${}^{C}\mathscr{D}^{(\cdot)}$ denote the Caputo-Hadamard fractional derivatives (CHFDs) of order $(\cdot)$, $0 < \varsigma, \rho \leq 1$, and ϕ is real constant. In this article, authors have extend the aforementioned articles [35–37] to nonlinear coupled system of Caputo-Hadamard fractional differential equations having the value of the sum of unknown functions $\mathscr{P}$ and $\mathscr{U}$ at the interval endpoints $[1, T]$ being zero, whereas the value of the sum of the unknown functions on an arbitrary domain $(1, \varphi)$ of the given interval $[1, T]$ remains constant. The remainder of the paper is as follows: Section 2 introduces some fundamental definitions, lemmas, and theorems that support our results. In Section 3, we prove the existence and uniqueness of solutions to the given system (4) using various conditions and some regular fixed-point theorems. Finally, examples are given to explain the main results.

2. Preliminaries

In this section, we discuss some relevant definitions and lemmas that will be needed later in our proof [11, 39, 40].

Lemma 1. *If $b, \varsigma, \rho > 0$ then*

$$\left({}^{H}\mathscr{I}_{b}^{\varsigma}\left(\log\frac{\vartheta}{b}\right)^{\rho-1}\right)(\mathscr{P}) = \frac{\Gamma(\rho)}{\Gamma(\rho+\varsigma)}\left(\log\frac{p}{b}\right)^{\rho+\varsigma-1}. \quad (7)$$

Definition 2. Let $0 < b < c < \infty$, $\mathscr{R}_e(\varsigma) \geq 0$, $n = [\mathscr{R}_e(\varsigma) + 1]$. The left and right CHFDs of order ς are, respectively, defined by

$$\begin{aligned}\left({}^{C}\mathscr{D}_{b+}^{\varsigma}\mathscr{P}\right)(\vartheta) &= \mathscr{D}_{b+}^{\varsigma}\left[\mathscr{P}(\nu) - \sum_{k=0}^{n-1}\frac{\delta^k\mathscr{P}(b)}{k!}\left(\log\frac{\nu}{b}\right)^k\right](\vartheta),\\ \left({}^{C}\mathscr{D}_{c-}^{\varsigma}\mathscr{P}\right)(\vartheta) &= \mathscr{D}_{c-}^{\varsigma}\left[h(\nu) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k\mathscr{P}(c)}{k!}\left(\log\frac{c}{\nu}\right)^k\right](\vartheta).\end{aligned} \quad (8)$$

Definition 3. Let $\mathscr{R}_e(\varsigma) > 0$, $n = [\mathscr{R}_e(\varsigma)] + 1$ and $p \in \mathscr{C}[b, c]$. If $\mathscr{R}_e(\varsigma) \neq 0$ or $\varsigma \in \mathbb{N}$, then

$${}^{C}\mathscr{D}_{b+}^{\varsigma}(\mathscr{I}_{b+}^{\varsigma}\mathscr{P})(\vartheta) = \mathscr{P}(\vartheta),\ {}^{C}\mathscr{D}_{c-}^{\varsigma}(\mathscr{I}_{c-}^{\varsigma}\mathscr{P})(\vartheta) = \mathscr{P}(\vartheta). \quad (9)$$

Definition 4. Let $p \in \mathscr{AC}_{\delta}^{n}[b, c]$ or $\mathscr{C}_{\delta}^{n}[b, c]$ and $\varsigma \in \mathbb{C}$, then

$$\begin{aligned}\mathscr{I}_{b+}^{\varsigma}\left({}^{C}\mathscr{D}_{b+}^{\varsigma}\mathscr{P}\right)(\vartheta) &= \mathscr{P}(\vartheta) - \sum_{k=0}^{n-1}\frac{\delta^k\mathscr{P}(b)}{k!}\left(\log\frac{\vartheta}{b}\right)^k,\\ \mathscr{I}_{c-}^{\varsigma}\left({}^{C}\mathscr{D}_{c-}^{\varsigma}\mathscr{P}\right)(\vartheta) &= \mathscr{P}(\vartheta) - \sum_{k=0}^{n-1}\frac{\delta^k\mathscr{P}(c)}{k!}\left(\log\frac{c}{\vartheta}\right)^k.\end{aligned} \quad (10)$$

Definition 5. Let $\mathscr{K}, \mathscr{L} \in \mathscr{C}[1, e]$ and $\mathscr{P}, \mathscr{U} \in \mathscr{AC}(\mathscr{S})$. Then, the solution of the following linear coupled system:

$$\begin{cases} {}^{C}\mathscr{D}^{\varsigma}\mathscr{P}(\vartheta) = \mathscr{K}(\vartheta), & 0 < \varsigma < 1, \vartheta \epsilon \mathscr{S},\\ {}^{C}\mathscr{D}^{\varrho}\mathscr{U}(\vartheta) = \mathscr{L}(\vartheta), & 0 < \varrho < 1, \vartheta \epsilon \mathscr{S},\\ (\mathscr{P}+\mathscr{U})(1) = -(\mathscr{P}+\mathscr{U})(e),\ \dfrac{1}{\Gamma(\rho)}\displaystyle\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\rho-1}(\mathscr{P}+\mathscr{U})(\nu)\frac{d\nu}{\nu} = \phi, & 1 < \varphi < T,\end{cases} \quad (11)$$

$$\begin{aligned}\mathscr{P}(\vartheta) = {}& \frac{1}{\Gamma(\varsigma)}\int_1^{\vartheta}\left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1}\mathscr{K}(\nu)\frac{d\nu}{\nu} + \frac{1}{2}\left\{\frac{\phi}{\eta} - \frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathscr{K}(\nu)\frac{d\nu}{\nu} + \frac{1}{\Gamma(\rho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right]\right.\\ & \left. - \frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\mathscr{K}(\nu)\frac{d\nu}{\nu} - \frac{1}{\Gamma(\rho+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right]\right\},\end{aligned} \quad (12)$$

$$\begin{aligned}\mathscr{U}(\vartheta) = {}& \frac{1}{\Gamma(\rho)}\int_1^{\vartheta}\left(\log\frac{\vartheta}{\nu}\right)^{\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu} + \frac{1}{2}\left\{\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\mathscr{K}(\nu)\frac{d\nu}{\nu} - \frac{1}{\Gamma(\rho+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right]\right.\\ & \left. - \frac{\phi}{\eta} - \frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathscr{K}(\nu)\frac{d\nu}{\nu} + \frac{1}{\Gamma(\rho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right]\right\},\end{aligned} \quad (13)$$

where

$$\eta = \frac{\log\varphi}{\Gamma(\rho+1)} \neq 0. \quad (14)$$

Proof. Using Lemma 2.3 and the operators ${}^{H}\mathscr{I}^{\varsigma}$ and ${}^{H}\mathscr{I}^{\rho}$ on both sides of FDEs in (11), we obtain

$$\mathscr{P}(\vartheta) = {}^{H}\mathscr{I}^{\varsigma}\mathscr{K}(\vartheta) + a_0, \quad (15)$$

$$\mathscr{U}(\vartheta) = {}^{H}\mathscr{I}^{\rho}\mathscr{L}(\vartheta) + b_0, \quad (16)$$

where $a_0, b_0 \in \mathscr{R}_e$, are arbitrary constants. Using the boundary Condition (11) in (15) and (16), we obtain

$$\begin{aligned}a_0 + b_0 = {}& \frac{-1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathscr{K}(\nu)\frac{d\nu}{\nu}\right.\\ & \left. + \frac{1}{\Gamma(\rho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right],\end{aligned} \quad (17)$$

$$\begin{aligned}a_0 - b_0 = {}& \frac{1}{\eta}\left[\varphi - \frac{1}{\Gamma(\varsigma)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathscr{K}(\nu)\frac{d\nu}{\nu}\right.\\ & \left. + \frac{1}{\Gamma(\rho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathscr{L}(\nu)\frac{d\nu}{\nu}\right].\end{aligned} \quad (18)$$

□

Solving the system (17)-(18) for a_0, b_0, we get

$$\begin{aligned} a_0 &= \frac{1}{2}\left\{\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathcal{K}(\nu)\frac{d\nu}{\nu}\right.\right.\\ &\quad\left.+\frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathcal{L}(\nu)\frac{d\nu}{\nu}\right]\\ &\quad-\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\mathcal{K}(\nu)\frac{d\nu}{\nu}\right.\\ &\quad\left.\left.-\frac{1}{\Gamma(\rho+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathcal{L}(\nu)\frac{d\nu}{\nu}\right]\right\},\\ b_0 &= \frac{1}{2}\left\{\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\mathcal{K}(\nu)\frac{d\nu}{\nu}\right.\right.\\ &\quad\left.-\frac{1}{\Gamma(\rho+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathcal{L}(\nu)\frac{d\nu}{\nu}\right]\\ &\quad-\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathcal{K}(\nu)\frac{d\nu}{\nu}\right.\\ &\quad\left.\left.+\frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathcal{L}(\nu)\frac{d\nu}{\nu}\right]\right\}, \end{aligned} \tag{19}$$

where η is given by (14). Substituting the values of a_0 and b_0 yields the result (12) and (13).

3. Existence Results for the Problem (4)

Defined by $\Omega=\mathcal{C}(\mathcal{S},\mathcal{R}_e)\times\mathcal{C}(\mathcal{S},\mathcal{R}_e)$ the Banach space endowed with the norm $\|\mathcal{P},\mathcal{U}\|=\sup_{\vartheta\in\mathcal{S}}|\mathcal{P}(\vartheta)|+\sup_{\vartheta\in\mathcal{S}}|\mathcal{U}(\vartheta)|$, for $(\mathcal{P},\mathcal{U})\in\Omega$. In spite of Lemma 2.4, the following operator $\Xi:\Omega\longrightarrow\Omega$ is associated with the problem (4):

$$\Xi(\mathcal{P},\mathcal{U})(\vartheta)=(\Xi_1(\mathcal{P},\mathcal{U})(\vartheta),\Xi_2(\mathcal{P},\mathcal{U})(\vartheta)), \tag{20}$$

$$\begin{aligned} \Xi_1(\mathcal{P},\mathcal{U})(\vartheta) &= \frac{1}{\Gamma(\varsigma)}\int_1^\vartheta\left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1}\mathcal{Y}_1(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\\ &\quad+\frac{1}{2}\left\{\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\right.\right.\\ &\quad\cdot\mathcal{Y}_1(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho)}\\ &\quad\left.\cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathcal{Y}_2(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\right]\\ &\quad-\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\right.\\ &\quad\cdot\mathcal{Y}_1(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}-\frac{1}{\Gamma(\rho+\rho)}\\ &\quad\left.\left.\cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathcal{Y}_2(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\right]\right\} \end{aligned} \tag{21}$$

$$\begin{aligned} \Xi_2(\mathcal{P},\mathcal{U})(\vartheta) &= \frac{1}{\Gamma(\rho)}\int_1^\vartheta\left(\log\frac{\vartheta}{\nu}\right)^{\rho-1}\mathcal{Y}_2(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\\ &\quad+\frac{1}{2}\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\right.\\ &\quad\cdot\mathcal{Y}_1(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}-\frac{1}{\Gamma(\rho+\rho)}\\ &\quad\left.\cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\mathcal{Y}_2(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\right]\\ &\quad-\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\right.\\ &\quad\cdot\mathcal{Y}_1(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho)}\\ &\quad\left.\left.\cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathcal{Y}_2(\nu,\mathcal{P}(\nu),\mathcal{U}(\nu))\frac{d\nu}{\nu}\right]\right\}. \end{aligned} \tag{22}$$

Following that, we introduce the assumptions necessary to construct the paper's primary results.

Let $\mathcal{Y}_1,\mathcal{Y}_2:\mathcal{S}\times\mathcal{R}_e^2\longrightarrow\mathcal{R}_e$ functions be continuous.

$(\mathcal{F}_1)\exists$ continuous positive functions $\omega_i,\widehat{\omega}_i\in\mathcal{C}(\mathcal{S},\mathcal{R}_e^+)$, $(i=1,2,3)$, such that

$$\begin{aligned} |\mathcal{Y}_1(\vartheta,\mathcal{P},\mathcal{U})| &\le \omega_1(\vartheta)+\omega_2(\vartheta)|\mathcal{P}|+\omega_3(\vartheta)|\mathcal{U}|,\forall(\vartheta,\mathcal{P},\mathcal{U})\in\mathcal{S}\times\mathcal{R}_e^2,\\ |\mathcal{Y}_2(\vartheta,\mathcal{P},\mathcal{U})| &\le \widehat{\omega}_1(\vartheta)+\widehat{\omega}_2(\vartheta)|\mathcal{P}|+\widehat{\omega}_3(\vartheta)|\mathcal{U}|,\forall(\vartheta,\mathcal{P},\mathcal{U})\in\mathcal{S}\times\mathcal{R}_e^2. \end{aligned} \tag{23}$$

$(\mathcal{F}_2)\exists$ positive constants $\varpi_i,\widehat{\varpi}_i\ (i=1,2)$ such that

$$\begin{aligned} &|\mathcal{Y}_1(\vartheta,\mathcal{P}_1,\mathcal{U}_1)-\mathcal{Y}_1(\vartheta,\mathcal{P}_2,\mathcal{U}_2)| \le \varpi_1|\mathcal{P}_1-\mathcal{P}_2|+\varpi_2|\mathcal{U}_1-\mathcal{U}_2|,\\ &|\mathcal{Y}_2(\vartheta,\mathcal{P}_1,\mathcal{U}_1)-\mathcal{Y}_2(\vartheta,\mathcal{P}_2,\mathcal{U}_2)| \le \widehat{\varpi}_1|\mathcal{P}_1-\mathcal{P}_2|+\widehat{\varpi}_2|\mathcal{U}_1-\mathcal{U}_2|,\\ &\forall\vartheta\in\mathcal{S},\mathcal{P}_i,\mathcal{U}_i\in\mathcal{R}_e,\ i=1,2. \end{aligned} \tag{24}$$

We use the notation: For computational ease.

$$Y_1=\frac{1}{4\Gamma(\varsigma+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho+1)}, \tag{25}$$

$$Y_2=\frac{1}{4\Gamma(\rho+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\rho+\rho}}{\Gamma(\rho+\rho+1)}, \tag{26}$$

$$\begin{aligned} \Delta=\min\left\{1-\left[\left(2Y_1+\frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)}\right)\|\omega_2\|\right.\right.\\ \left.+\left(2Y_2+\frac{(\log T)^\rho}{\Gamma(\rho+1)}\right)\|\widehat{\omega}_2\|\right], \end{aligned} \tag{27}$$

$$1-\left[\left(2Y_1+\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|\omega_3\|+\left(2Y_2+\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\|\widehat{\omega}_3\|\right]\Bigg\}. \tag{28}$$

Our first existence result for the problem (4) is based on the following fixed point theorem ([41, 42]).

Lemma 6. *Let $\mathscr{G}:\mathscr{H}\longrightarrow\mathscr{H}$ be a completely continuous operator in the Banach space $\mathscr{H}$, and the set $\Pi=\{\mathscr{P}\in\mathscr{H}\mid \mathscr{P}=\lambda\mathscr{G}\mathscr{P},\ 0<\lambda<1\}$ is bounded. Then, $\mathscr{G}$ has a fixed point in $\mathscr{H}$.*

Theorem 7. *Suppose that ($\mathscr{F}_1$) hold. Then, the problem (4) has at least one solution on $\mathscr{S}$ provided that*

$$\left(2Y_1+\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|\omega_2\|+\left(2Y_2+\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\|\widehat{\omega}_2\|<1, \tag{29}$$

$$\left(2Y_1+\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|\omega_3\|+\left(2Y_2+\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\|\widehat{\omega}_3\|<1. \tag{30}$$

where $Y_j(j=1,2)$ are defined by (25)-(26).

Proof. We begin by demonstrating that the operator $\Xi:\Omega\longrightarrow\Omega$ defined by (20) is completely continuous, i.e., that Ξ is continuous and maps any bounded subset of Ω to a relatively compact subset of Ω. Since the functions $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are continuous, the operator $\Xi:\Omega\longrightarrow\Omega$ is also continuous. Now, let $\Psi_{\hat{r}}\subset\Omega$ be bounded. Then, $\exists$ positive constants $\mathscr{T}_{\mathscr{Y}_1}$ and $\mathscr{T}_{\mathscr{Y}_2}$ such that

$$\begin{aligned}|\mathscr{Y}_1(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta))|\le\mathscr{T}_{\mathscr{Y}_1},\ &|\mathscr{Y}_1(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta))|\\ &\le\mathscr{T}_{\mathscr{Y}_2},\forall(\mathscr{P},\mathscr{U})\in\Psi_{\hat{r}}.\end{aligned} \tag{31}$$

So, for any $(\mathscr{P},\mathscr{U})\in\Psi_{\hat{r}}$, $\vartheta\in\mathscr{S}$, we get

$$\begin{aligned}|\Xi_1(\mathscr{P},\mathscr{U})(\vartheta)|&\le\mathscr{T}_{\mathscr{Y}_1}\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+Y_2\mathscr{T}_{\mathscr{Y}_2}+\frac{\phi}{\eta},\\ |\Xi_2(\mathscr{P},\mathscr{U})(\vartheta)|&\le Y_1\mathscr{T}_{\mathscr{Y}_1}+\mathscr{T}_{\mathscr{Y}_2}\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+Y_2\right)+\frac{\phi}{\eta}.\end{aligned} \tag{32}$$

Thus,

$$\begin{aligned}\|\Xi(\mathscr{P},\mathscr{U})\|&=\|\Xi_1(\mathscr{P},\mathscr{U})\|+\|\Xi_2(\mathscr{P},\mathscr{U})\|\\ &\le\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+2Y_1\right)\mathscr{T}_{\mathscr{Y}_1}+\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+2Y_2\right)\\ &\quad\cdot\mathscr{T}_{\mathscr{Y}_2}+\frac{2\phi}{\eta}.\end{aligned} \tag{33}$$

Thus, the operator Ξ is uniformly bounded as a result of the preceding inequality. Let Ξ prove that it determines bounded sets into equicontinuous sets of Ω, let $\vartheta_1,\vartheta_2\in\mathscr{S}$, $\vartheta_1<\vartheta_2$, and $(\mathscr{P},\mathscr{U})\in\Psi_{\hat{r}}$. Then,

$$\begin{aligned}&|\Xi_1(\mathscr{P},\mathscr{U})(\vartheta_2)-\Xi_2(\mathscr{P},\mathscr{U})(\vartheta_2)|\\ &\le\frac{1}{\Gamma(\varsigma)}\left|\int_1^{\vartheta_1}\left[\left(\log\frac{\vartheta_2}{\nu}\right)^{\varsigma-1}-\left(\log\frac{\vartheta_1}{\theta}\right)^{\varsigma-1}\right]\right|\\ &\quad\cdot\mathscr{Y}_1(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))\left|\frac{d\nu}{\nu}+\int_{\vartheta_1}^{\vartheta_2}\left(\log\frac{\vartheta_2}{\nu}\right)^{\varsigma-1}\right|\\ &\quad\cdot\mathscr{Y}_1(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))\left|\frac{d\nu}{\nu}\right|\\ &\le\mathscr{T}_{\mathscr{Y}_1}\left(\frac{2(\log\vartheta_2-\log\vartheta_1)^{\varsigma}+(\log\vartheta_2^{\varsigma}-\log\vartheta_1^{\varsigma})}{\Gamma(\varsigma+1)}\right).\end{aligned} \tag{34}$$

Take note that in the limit $\vartheta_1\longrightarrow\vartheta_2$, the RHS of the preceding inequalities tends to zero independently of $(\mathscr{P},\mathscr{U})\in\Psi_{\hat{r}}$. Then, The Arzela-Ascoli theorem implies that the operator $\Xi:\Omega\longrightarrow\Omega$ is completely continuous. Following that, we consider the set $\Lambda=\{(\mathscr{P},\mathscr{U})\in\Omega\mid(\mathscr{P},\mathscr{U})=\kappa\Xi(\mathscr{P},\mathscr{U}),0<\kappa<1\}$ and demonstrate that it is bounded. Let $(\mathscr{P},\mathscr{U})\in\Lambda$, then $(\mathscr{P},\mathscr{U})=\kappa\Xi(\mathscr{P},\mathscr{U})$, $0<\kappa<1$. For any $\vartheta\in\mathscr{S}$, we have

$$\mathscr{P}(\vartheta)=\kappa\Xi_1(\mathscr{P},\mathscr{U})(\vartheta),\ \mathscr{U}(\vartheta)=\kappa\Xi_2(\mathscr{P},\mathscr{U})(\vartheta). \tag{35}$$

Using $Y_i(i=1,2)$ given by (25)-(26), we find that

$$\begin{aligned}|\mathscr{P}(\vartheta)|&=\kappa|\Xi_1(\mathscr{P},\mathscr{U})(\vartheta)|\le(\|\omega_1\|+\|\omega_2\|\|\mathscr{P}\|+\|\omega_3\|\|\mathscr{U}\|)\\ &\quad\cdot\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+(\|\widehat{\omega}_1\|+\|\widehat{\omega}_2\|\|\mathscr{P}\|\\ &\quad+\|\widehat{\omega}_3\|\|\mathscr{U}\|)Y_2+\frac{\phi}{\eta},\\ |\mathscr{U}(\vartheta)|&=\kappa|\Xi_2(\mathscr{P},\mathscr{U})(\vartheta)|\le(\|\omega_1\|+\|\omega_2\|\|\mathscr{P}\|+\|\omega_3\|\|\mathscr{U}\|)Y_1\\ &\quad+(\|\widehat{\omega}_1\|+\|\widehat{\omega}_2\|\|\mathscr{P}\|+\|\widehat{\omega}_3\|\|\mathscr{U}\|)\\ &\quad\cdot\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+Y_2\right)+\frac{\phi}{\eta}.\end{aligned} \tag{36}$$

In consequence, we get

$$\begin{aligned}&\|\mathscr{P}\|+\|\mathscr{U}\|\\ &\le\|\omega_1\|\left(\frac{2Y_1+(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)+\|\widehat{\omega}_1\|\left(\frac{2Y_2+(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\\ &\quad+\frac{2\phi}{\eta}+\left[\|\omega_2\|\left(\frac{2Y_1+(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)+\|\widehat{\omega}_2\|\right.\\ &\quad\left.\cdot\left(\frac{2Y_2+(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\right]\|\mathscr{P}\|+\left[\|\omega_3\|\left(\frac{2Y_1+(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right)\right.\\ &\quad\left.+\|\widehat{\omega}_3\|\left(\frac{2Y_2+(\log T)^{\rho}}{\Gamma(\rho+1)}\right)\right]\|\mathscr{U}\|,\end{aligned} \tag{37}$$

Thus, in Conditions (29)-(30), we obtain

$$\|\mathscr{P},\mathscr{U}\| \le \frac{\|\omega_1\|((2Y_1+(\log T)^{\varsigma})/(\Gamma(\varsigma+1))) + \|\widehat{\omega}_1\|((2Y_2+(\log T)^{\rho})/(\Gamma(\rho+1))) + (2\phi/\eta)}{\Delta} \tag{38}$$

T demonstrates that $|\mathscr{P},\mathscr{U}|$ is constrained for $\vartheta \in \mathcal{S}$. As a result, the set Λ is bounded. As a result, the inference of Lemma 6 applies, and the operator Ξ has at least one fixed point, corresponding to a solution of the problem (4). □

The existence of a unique solution to the problem (4) is demonstrated using Banach's contraction mapping theorem in the following result.

Theorem 8. *Suppose that* $(\mathscr{F}_2)$ *hold. Then, the problem (4) has a unique solution on* $\mathcal{S}$ *if*

$$\varpi\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+2Y_1\right)+\widehat{\varpi}\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+2Y_2\right)<1, \tag{39}$$

where $\varpi=\max\{\varpi_1,\varpi_2\}$, $\widehat{\varpi}=\max\{\widehat{\varpi}_1,\widehat{\varpi}_2\}$, *and* Y_i, $i=1,2$ *are defined by (25)-(26).*

Proof. Consider the operator $\Xi:\Omega\longrightarrow\Omega$ denoted by (20) and fix

$$\pi > \frac{\mathscr{W}_1((((\log T)^{\varsigma})/(\Gamma(\varsigma+1)))+2Y_1)+\mathscr{W}_2((((\log T)^{\rho})/(\Gamma(\rho+1)))+2Y_2)}{1-\varpi((((\log T)^{\varsigma})/(\Gamma(\varsigma+1)))+2Y_1)+\widehat{\varpi}((((\log T)^{\rho})/(\Gamma(\rho+1)))+2Y_2)}, \tag{40}$$

where $\mathscr{W}_1=\sup_{\vartheta\in\mathcal{S}}|\mathscr{Y}_1(\vartheta,0,0)|$ and $\mathscr{W}_2=\sup_{\vartheta\in\mathcal{S}}|\mathscr{Y}_2(\vartheta,0,0)|$. Then, we show that $\Xi\mathscr{B}_\pi\subset\mathscr{B}_\pi$, where $\mathscr{B}_\pi=\{(\mathscr{P},\mathscr{U})\in\Omega:\|(\mathscr{P},\mathscr{U})\|\le\pi\}$. For $(\mathscr{P},\mathscr{U})\in\mathscr{B}_\pi$, we have

$$\begin{aligned}
&|\Xi_1(\mathscr{P},\mathscr{U})(\vartheta)|\\
&\le \frac{1}{\Gamma(\varsigma)}\int_1^{\vartheta}\left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1}|\mathscr{Y}_1(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))-\mathscr{Y}_1(\nu,0,0)|\\
&\quad+|\mathscr{Y}_1(\nu,0,0)|\frac{d\nu}{\nu}+\frac{1}{2}\left\{\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\right.\right.\\
&\quad\cdot|\mathscr{Y}_1(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))-\mathscr{Y}_1(\nu,0,0)|+|\mathscr{Y}_1(\nu,0,0)|\frac{d\nu}{\nu}\\
&\quad+\frac{1}{\Gamma(\rho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\rho-1}|\mathscr{Y}_2(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))-\mathscr{Y}_2(\nu,0,0)|\\
&\quad\left.+|\mathscr{Y}_2(\nu,0,0)|\frac{d\nu}{\nu}\right]-\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\right.\\
&\quad\cdot|\mathscr{Y}_1(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))-\mathscr{Y}_1(\nu,0,0)|+|\mathscr{Y}_1(\nu,0,0)|\frac{d\nu}{\nu}\\
&\quad-\frac{1}{\Gamma(\rho+\rho)}\int_1^{\varphi}\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}|\mathscr{Y}_2(\nu,\mathscr{P}(\nu),\mathscr{U}(\nu))\\
&\quad\left.\left.-\mathscr{Y}_2(\nu,0,0)|+|\mathscr{Y}_2(\nu,0,0)|\frac{d\nu}{\nu}\right]\right\}\\
&\le\left(\varpi\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+\widehat{\varpi}Y_2\right)(\|\mathscr{P}\|+\|\mathscr{U}\|)\\
&\quad+\mathscr{W}_1\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+\mathscr{W}_2Y_2,
\end{aligned}\tag{41}$$

which leads to

$$\begin{aligned}
\|\Xi_1(\mathscr{P},\mathscr{U})\|&\le\left(\varpi\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+\widehat{\varpi}Y_2\right)(\|\mathscr{P}\|+\|\mathscr{U}\|)\\
&\quad+\mathscr{W}_1\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+Y_1\right)+\mathscr{W}_2Y_2,
\end{aligned}\tag{42}$$

when the norm for $\vartheta\in\mathcal{S}$. Equivalently, for $(\mathscr{P},\mathscr{U})\in\mathscr{B}_\pi$, one can obtain

$$\begin{aligned}
\|\Xi_2(\mathscr{P},\mathscr{U})\|&\le\left(\varpi Y_1+\widehat{\varpi}\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+Y_2\right)\right)(\|\mathscr{P}\|+\|\mathscr{U}\|)\\
&\quad+\mathscr{W}_1Y_1+\mathscr{W}_2\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+Y_2\right).
\end{aligned}\tag{43}$$

□

Therefore, for any $(\mathscr{P},\mathscr{U})\in\mathscr{B}_\pi$, we have

$$\begin{aligned}
\|\Xi(\mathscr{P},\mathscr{U})\|&=\|\Xi_1(\mathscr{P},\mathscr{U})\|+\|\Xi_2(\mathscr{P},\mathscr{U})\|\\
&\le\left(\varpi\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+2Y_1\right)+\widehat{\varpi}\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+2Y_2\right)\right)\\
&\quad\cdot(\|\mathscr{P}\|+\|\mathscr{U}\|),\\
&\mathscr{W}_1\left(\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+2Y_1\right)+\mathscr{W}_2\left(\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}+2Y_2\right).
\end{aligned}\tag{44}$$

which demonstrates that Ξ maps $\mathscr{B}_\pi$ into itself. To demonstrate that the operator Ξ is a contraction, let $(\mathscr{P}_1, \mathscr{U}_1)$, $(\mathscr{P}_2, \mathscr{U}_2) \in \Omega, \vartheta \in \mathscr{S}$. Then, in view of $\mathscr{F}_2$, we obtain

$$\begin{aligned}
&\|\Xi_1(\mathscr{P}_1,\mathscr{U}_1)(\vartheta) - \Xi_1(\mathscr{P}_1,\mathscr{U}_1)(\vartheta)\| \\
&\quad\le \frac{1}{\Gamma(\varsigma)}\int_1^\vartheta \left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1} |\mathscr{Y}_1(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) \\
&\qquad - \mathscr{Y}_1(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} + \frac{1}{2}\left\{\frac{\phi}{\eta}\right. \\
&\qquad - \frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\int_1^e \left(\log\frac{e}{\nu}\right)^{\varsigma-1}|\mathscr{Y}_1(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu))\right. \\
&\qquad - \mathscr{Y}_1(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} + \frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1} \\
&\qquad \left.\cdot|\mathscr{Y}_2(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) - \mathscr{Y}_2(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu}\right] \\
&\qquad + \frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}|\mathscr{Y}_1(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu))\right. \\
&\qquad - \mathscr{Y}_1(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} + \frac{1}{\Gamma(\rho+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1} \\
&\qquad \left.\left.\cdot|\mathscr{Y}_2(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) - \mathscr{Y}_2(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu}\right]\right\} \\
&\quad\le \left\{\varpi\left(\frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)} + Y_1\right) + \widehat{\varpi}Y_2\right\}(\|\mathscr{P}\| + \|\mathscr{U}\|), \\
&\|\Xi_2(\mathscr{P}_1,\mathscr{U}_1)(\vartheta) - \Xi_2(\mathscr{P}_1,\mathscr{U}_1)(\vartheta)\| \\
&\quad\le \frac{1}{\Gamma(\rho)}\int_1^\vartheta \left(\log\frac{\vartheta}{\nu}\right)^{\rho-1} |\mathscr{Y}_2(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) \\
&\qquad - \mathscr{Y}_2(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} + \frac{1}{2}\left\{\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\right.\right. \\
&\qquad \cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}|\mathscr{Y}_1(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) \\
&\qquad - \mathscr{Y}_1(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} - \frac{1}{\Gamma(\rho+\rho)} \\
&\qquad \cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}|\mathscr{Y}_2(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) \\
&\qquad \left.- \mathscr{Y}_2(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu}\right] - \frac{\phi}{\eta} - \frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\right. \\
&\qquad \cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}|\mathscr{Y}_1(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) \\
&\qquad - \mathscr{Y}_1(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu} + \frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1} \\
&\qquad \left.\left.\cdot|\mathscr{Y}_2(\nu,\mathscr{P}_1(\nu),\mathscr{U}_1(\nu)) - \mathscr{Y}_2(\nu,\mathscr{P}_2(\nu),\mathscr{U}_2(\nu))|\frac{d\nu}{\nu}\right]\right\} \\
&\quad\le \left\{\varpi Y_1 + \widehat{\varpi}\left(\frac{(\log T)^\rho}{\Gamma(\rho+1)} + Y_2\right)\right\}(\|\mathscr{P}\| + \|\mathscr{U}\|).
\end{aligned} \tag{45}$$

Clearly, the preceding inequalities imply that

$$\begin{aligned}
&\|\Xi(\mathscr{P}_1,\mathscr{U}_1) - \Xi(\mathscr{P}_2,\mathscr{U}_2)\| \\
&\quad = \|\Xi_1(\mathscr{P}_1,\mathscr{U}_1) - \Xi_1(\mathscr{P}_2,\mathscr{U}_2)\| \\
&\qquad + \|\Xi_2(\mathscr{P}_1,\mathscr{U}_1) - \Xi_2(\mathscr{P}_2,\mathscr{U}_2)\| \\
&\quad \le \left\{\varpi\left(\frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)} + 2Y_1\right) + \widehat{\varpi}\left(\frac{(\log T)^\rho}{\Gamma(\rho+1)} + 2Y_2\right)\right\} \\
&\qquad \cdot \|(\mathscr{P}_1 - \mathscr{P}_2, \mathscr{U}_1 - \mathscr{U}_2)\|,
\end{aligned} \tag{46}$$

which, in view of (40) means that Ξ is a contraction mapping. As a result of Banach's contraction mapping theorem, Π has a unique fixed point. This demonstrates that the problem (4) has a unique solution on $\mathscr{S}$.

4. Ulam-Hyers Stability Results (4)

The U-H stability of the solutions to the BVPs (4) will be discussed in this section using the integral representation of their solutions defined by

$$\mathscr{P}(\vartheta) = \Xi_1(\mathscr{P},\mathscr{U})(\vartheta), \mathscr{U}(\vartheta) = \Xi_2(\mathscr{P},\mathscr{U})(\vartheta), \tag{47}$$

where ϑ_1 and ϑ_2 are given by (21) and (22). Consider the following definitions of nonlinear operators

$$\begin{aligned}
&\mathscr{Q}_1, \mathscr{Q}_2 \in \mathscr{C}(\mathscr{S},\mathscr{R}_e) \times \mathscr{C}(\mathscr{S},\mathscr{R}_e) \longrightarrow \mathscr{C}(\mathscr{S},\mathscr{R}_e), \\
&\begin{cases} {}^C\mathscr{D}^\varsigma\mathscr{P}(\vartheta) - \mathscr{Y}_1(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)) = \mathscr{Q}_2(\mathscr{P},\mathscr{U})(\vartheta), \vartheta\epsilon\mathscr{S}, \\ {}^C\mathscr{D}^\varrho\mathscr{P}(\vartheta) - \mathscr{Y}_2(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)) = \mathscr{Q}_2(\mathscr{P},\mathscr{U})(\vartheta), \vartheta\epsilon\mathscr{S}. \end{cases}
\end{aligned} \tag{48}$$

It considered the following inequalities for some $\widehat{\lambda}_1, \widehat{\lambda}_2$:

$$\|\mathscr{Q}_1(\mathscr{P},\mathscr{U})\| \le \widehat{\lambda}_1, \|\mathscr{Q}_2(\mathscr{P},\mathscr{U})\| \le \widehat{\lambda}_2. \tag{49}$$

Definition 9. The coupled system (4) is said to be U-H stable if $\mathscr{V}_1, \mathscr{V}_2 > 0$, and there exists a unique solution $(\mathscr{P},\mathscr{U}) \in \mathscr{C}(\mathscr{E},\mathscr{R}_e)$ of a problem (4) with

$$\|(\mathscr{P},\mathscr{U}) - (\mathscr{P}^*,\mathscr{U}^*)\| \le \mathscr{V}_1\widehat{\lambda}_1 + \mathscr{V}_2\widehat{\lambda}_2, \tag{50}$$

$\forall(\mathscr{P},\mathscr{U}) \in \mathscr{C}(\mathscr{S},\mathscr{R}_e)$ of inequality (49).

Theorem 10. *Assume that $(\mathscr{F}_2)$ holds. Then, the problem (4) is U-H stable.*

Proof. Let $\mathscr{C}(\mathscr{S},\mathscr{R}_e) \times \mathscr{C}(\mathscr{S},\mathscr{R}_e)$ be the solution to (4) that satisfies (21) and (22). Let $(\mathscr{P},\mathscr{U})$ be any solution that meets Condition (49):

$$\begin{cases} {}^C\mathscr{D}^\varsigma\mathscr{P}(\vartheta) - \mathscr{Y}_1(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)) + \mathscr{Q}_1(\mathscr{P},\mathscr{U})(\vartheta), \vartheta\epsilon\mathscr{S}, \\ {}^C\mathscr{D}^\varrho\mathscr{P}(\vartheta) - \mathscr{Y}_2(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)) + \mathscr{Q}_2(\mathscr{P},\mathscr{U})(\vartheta), \vartheta\epsilon\mathscr{S}, \end{cases} \tag{51}$$

so,

$$\mathscr{P}*(\vartheta)=\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)+\frac{1}{\Gamma(\varsigma)}\int_1^\vartheta\left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1}\cdot\mathscr{Q}_1(\mathscr{P}*,\mathscr{U}*)(\nu)\frac{d\nu}{\nu}+\frac{1}{2}\left\{\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\mathscr{Q}_1(\mathscr{P}*,\mathscr{U}*)(\nu)\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\mathscr{Q}_2(\mathscr{P}*,\mathscr{U}*)(\nu)\frac{d\nu}{\nu}\right]-\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\mathscr{Q}_1(\mathscr{P}*,\mathscr{U}*)\cdot(\nu)\frac{d\nu}{\nu}-\frac{1}{\Gamma(\rho+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\cdot\mathscr{Q}_2(\mathscr{P}*,\mathscr{U}*)(\nu)\frac{d\nu}{\nu}\right]\right\}. \tag{52}$$

It follows that

$$\begin{aligned}
|\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{P}*(\vartheta)|
&\le\frac{1}{\Gamma(\varsigma)}\int_1^\vartheta\left(\log\frac{\vartheta}{\nu}\right)^{\varsigma-1}\widehat{\lambda}_1\frac{d\nu}{\nu}+\frac{1}{2}\left\{\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\widehat{\lambda}_1\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\widehat{\lambda}_2\frac{d\nu}{\nu}\right]\right.\\
&\quad\left.+\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\widehat{\lambda}_1\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho+\rho)}\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\widehat{\lambda}_2\frac{d\nu}{\nu}\right]\right\}\\
&\le\left(\frac{1}{4\Gamma(\varsigma+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho+1)}\right)\widehat{\lambda}_1+\left(\frac{1}{4\Gamma(\rho+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\rho+\rho}}{\Gamma(\rho+\rho+1)}\right)\widehat{\lambda}_2\le Y_1\widehat{\lambda}_1+Y_2\widehat{\lambda}_2.\\
|\Xi_2(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{U}*(\vartheta)|
&\le\frac{1}{\Gamma(\rho)}\int_1^\vartheta\left(\log\frac{\vartheta}{\nu}\right)^{\rho-1}\widehat{\lambda}_2\frac{d\nu}{\nu}+\frac{1}{2}\left\{\frac{1}{\eta}\left[\frac{1}{\Gamma(\varsigma+\rho)}\cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\varsigma+\rho-1}\widehat{\lambda}_1\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho+\rho)}\cdot\int_1^\varphi\left(\log\frac{\varphi}{\nu}\right)^{\rho+\rho-1}\widehat{\lambda}_2\frac{d\nu}{\nu}\right]+\frac{\phi}{\eta}-\frac{1}{2}\left[\frac{1}{\Gamma(\varsigma)}\cdot\int_1^e\left(\log\frac{e}{\nu}\right)^{\varsigma-1}\widehat{\lambda}_1\frac{d\nu}{\nu}+\frac{1}{\Gamma(\rho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\rho-1}\widehat{\lambda}_2\frac{d\nu}{\nu}\right]\right\},\\
&\le\left(\frac{1}{4\Gamma(\varsigma+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho+1)}\right)\widehat{\lambda}_1+\left(\frac{1}{4\Gamma(\rho+1)}+\frac{1}{2\eta}\frac{(\log\varphi)^{\rho+\rho}}{\Gamma(\rho+\rho+1)}\right)\widehat{\lambda}_2\le Y_1\widehat{\lambda}_1+Y_2\widehat{\lambda}_2.
\end{aligned} \tag{53}$$

where Y_1 and Y_2 are defined in (25)-(26), respectively. As an outcome, we deduce from operator Ξ's fixed-point property, which is defined by (21) and (22)

$$\begin{aligned}
|\mathscr{P}(\vartheta)-\mathscr{P}*(\vartheta)|
&=|\mathscr{P}(\vartheta)-\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)+\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{P}*(\vartheta)|\\
&\le|\Xi_1(\mathscr{P},\mathscr{U})(\vartheta)-\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)|+|\Xi_1(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{P}*(\vartheta)|\\
&\le\left(\left(Y_1\phi_1+Y_1\widehat{\phi}_1\right)+\left(Y_1\phi_2+Y_1\widehat{\phi}_2\right)\right)\cdot\|(\mathscr{P},\mathscr{U})-(\mathscr{P}*,\mathscr{U}*)\|+Y_1\widehat{\lambda}_1+Y_1\widehat{\lambda}_2,
\end{aligned} \tag{54}$$

$$\begin{aligned}
|\mathscr{U}(\vartheta)-\mathscr{U}*(\vartheta)|
&=|\mathscr{U}(\vartheta)-\Xi_2(\mathscr{P}*,\mathscr{U}*)(\vartheta)+\Xi_2(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{U}*(\vartheta)|\\
&\le|\Xi_2(\mathscr{P},\mathscr{U})(\vartheta)-\Xi_2(\mathscr{P}*,\mathscr{U}*)(\vartheta)|+|\Xi_2(\mathscr{P}*,\mathscr{U}*)(\vartheta)-\mathscr{U}*(\vartheta)|\\
&\le\left(\left(Y_2\phi_1+Y_2\widehat{\phi}_1\right)+\left(Y_2\phi_2+Y_2\widehat{\phi}_2\right)\right)\cdot\|(\mathscr{P},\mathscr{U})-(\mathscr{P}*,\mathscr{U}*)\|+Y_2\widehat{\lambda}_1+Y_2\widehat{\lambda}_2.
\end{aligned} \tag{55}$$

From the above Equations (54) and (55), it follows that

$$\begin{aligned}
\|(\mathscr{P},\mathscr{U})-(\mathscr{P}*,\mathscr{U}*)\|
&\le\frac{(Y_1+Y_2)\widehat{\lambda}_1+(Y_1+Y_2)\widehat{\lambda}_2}{1-\left((Y_1+Y_2)(\phi_1+\phi_2)+(Y_1+Y_2)\left(\widehat{\phi}_1+\widehat{\phi}_2\right)\right)}\\
&\le\mathscr{V}_1\widehat{\lambda}_1+\mathscr{V}_2\widehat{\lambda}_2,
\end{aligned} \tag{56}$$

with

$$\begin{aligned}
\mathscr{V}_1&=\frac{(Y_1+Y_2)}{1-\left((Y_1+Y_2)(\phi_1+\phi_2)+(Y_1+Y_2)\left(\widehat{\phi}_1+\widehat{\phi}_2\right)\right)}\\
\mathscr{V}_2&=\frac{(Y_1+Y_2)}{1-\left((Y_1+Y_2)(\phi_1+\phi_2)+(Y_1+Y_2)\left(\widehat{\phi}_1+\widehat{\phi}_2\right)\right)}.
\end{aligned} \tag{57}$$

Hence, the problem (4) is U-H stable. □

5. Examples

Example 11. Consider the following problem:

$$\begin{cases}
{}^{C}\mathscr{D}\dfrac{21}{50}\mathscr{P}(\vartheta)=\mathscr{Y}_1(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)),\vartheta\epsilon[1,2],\\
{}^{C}\mathscr{D}\dfrac{16}{25}\mathscr{U}(\vartheta)=\mathscr{Y}_2(\vartheta,\mathscr{P}(\vartheta),\mathscr{U}(\vartheta)),\vartheta\epsilon[1,2],\\
(\mathscr{P}+\mathscr{U})(1)=-(\mathscr{P}+\mathscr{U})(2),\ {}^{H}\mathscr{I}\dfrac{9}{25}(\mathscr{P}+\mathscr{U})\left(\dfrac{79}{50}\right)=4,
\end{cases} \tag{58}$$

where $\varsigma = 21/50$, $\varsigma = 16/25$, $\rho = 9/25$, $\varphi = 79/50$, $\phi = 4$, $T = 2$, $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$, and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ are set later. Using the provided data, we determine that $Y_1 = 0.852825$ and $Y_2 = 0.72329$, where Y_1 and Y_2 are denoted by (25) and (26), respectively. We will use

$$\begin{aligned} \mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U}) &= \frac{1}{(\log \vartheta + 5)^2}\left(\vartheta + 2\cos p + \frac{q}{4}\right) \text{ and} \\ \mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U}) &= \frac{1}{4\sqrt{(\log \vartheta)^2 + 400}}\left(\frac{p}{4} + 2\sin q + \vartheta\right), \end{aligned} \tag{59}$$

to illustrate Theorem 7. $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are continuous and fulfill the condition $(\mathscr{F}_1)$ with $\omega_1(\vartheta) = \vartheta/((\log \vartheta + 5)^2)$, $\omega_2(\vartheta) = 2/((\log \vartheta + 5)^2)$, $\omega_3(\vartheta) = 1/(4(\log \vartheta + 5)^2)$, $\widehat{\omega}_1(\vartheta) = \vartheta/(4\sqrt{(\log \vartheta)^2 + 400})$, $\widehat{\omega}_2(\vartheta) = 1/(16\sqrt{(\log \vartheta)^2 + 400})$, and $\widehat{\omega}_3(\vartheta) = 1/(2\sqrt{(\log \vartheta)^2 + 400})$. Also $(2Y_1 + ((\log T)^{\varsigma}/(\Gamma(\varsigma + 1))))\|\omega_2\| + (2Y_2 + ((\log T)^{\rho}/(\Gamma(\rho + 1))))\|\widehat{\omega}_2\| \approx 0.9527249493$ and $(2Y_1 + ((\log T)^{\varsigma}/(\Gamma(\varsigma + 1))))\|\omega_3\| + (2Y_2 + ((\log T)^{\rho}/(\Gamma(\rho + 1))))\|\widehat{\omega}_3\| \approx 0.9042700804$. As a result, all of Theorem 10 conditions hold, and there exists at least one solution to the problem (60) involving the equations $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$ and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ specified in (59).

Example 12. Consider the following problem:

$$\begin{cases} {}^{C}\mathscr{D}\dfrac{21}{50}\mathscr{P}(\vartheta) = \mathscr{Y}_1(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), \vartheta\epsilon[1, 2], \\ {}^{C}\mathscr{D}\dfrac{16}{25}\mathscr{U}(\vartheta) = \mathscr{Y}_2(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), \vartheta\epsilon[1, 2], \\ (\mathscr{P} + \mathscr{U})(1) = -(\mathscr{P} + \mathscr{U})(2), \ {}^{H}\mathscr{I}\dfrac{9}{25}(\mathscr{P} + \mathscr{U})\left(\dfrac{79}{50}\right) = 4, \end{cases} \tag{60}$$

where $\varsigma = 21/50$, $\rho = 16/25$, $\rho = 9/25$, $\varphi = 79/50$, $\phi = 4$, $T = 2$, $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$, and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ are set later. Using the provided data, we determine that $Y_1 = 0.852825$ and $Y_2 = 0.72329$, where Y_1 and Y_2 are denoted by (25) and (26), respectively. We chose

$$\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U}) = \frac{1}{30(\log \vartheta + 2)^2}\left(\tan^{-1}u + \frac{|\mathscr{U}|}{1 + |\mathscr{U}|}\right), \tag{61}$$

$$\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U}) = \frac{1}{\sqrt{(\log \vartheta)^2 + 400}}\left(2\cos \mathscr{P} + \tan^{-1}\mathscr{U}\right), \tag{62}$$

to illustrate the implementation of Theorem 8. Remember the fact that $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are continuous and satisfy the condition $(\mathscr{F}_2)$ with $\varpi_1 = \varpi_2 = 1/120 = \varpi$ and $\widehat{\varpi}_1 = \widehat{\varpi}_2 = 1/20 = \widehat{\varpi}$. Also $\varpi((((\log T)^{\varsigma})/(\Gamma(\varsigma + 1))) + 2Y_1) + \widehat{\varpi}((((\log T)^{\rho})/(\Gamma(\rho + 1))) + 2Y_2) \approx 0.9246609812$. As a result, all of Theorem 3.2 conditions are met, and its inference applies to the problem (60) with $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$ and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ given by (61).

Example 13. Consider the following problem:

$$\begin{cases} {}^{C}\mathscr{D}\dfrac{21}{50}\mathscr{P}(\vartheta) = \dfrac{\sqrt{\vartheta}}{2} + \dfrac{1}{5(\vartheta + 25)}\dfrac{|\mathscr{P}(\vartheta)|}{1 + |\mathscr{P}(\vartheta)|} + \dfrac{3}{80}\cos(\mathscr{U}(\vartheta)), \vartheta\epsilon[1, 2], \\ {}^{C}\mathscr{D}\dfrac{16}{25}\mathscr{U}(\vartheta) = \dfrac{\vartheta}{5} + \dfrac{17}{300}\cos(\mathscr{P}(\vartheta)) + \dfrac{1}{70}\dfrac{|\mathscr{U}(\vartheta)|}{1 + |\mathscr{U}(\vartheta)|}, \vartheta\epsilon[1, 2], \\ (\mathscr{P} + \mathscr{U})(1) = -(\mathscr{P} + \mathscr{U})(2), \ {}^{H}\mathscr{I}\dfrac{9}{25}(\mathscr{P} + \mathscr{U})\left(\dfrac{79}{50}\right) = 4, \end{cases} \tag{63}$$

We choose

$$\begin{aligned} &|f(\vartheta, \mathscr{P}_1(\vartheta), \mathscr{U}_1(\vartheta)) - f(\vartheta, \mathscr{P}_2(\vartheta), \mathscr{U}_2(\vartheta))| \\ &\quad = \frac{1}{125}|\mathscr{P}_1(\vartheta) - \mathscr{P}_2(\vartheta)| + \frac{3}{80}|\mathscr{U}_1(\vartheta) - \mathscr{U}_2(\vartheta)|, \end{aligned} \tag{64}$$

$$\begin{aligned} &|g(\vartheta, \mathscr{P}_1(\vartheta), \mathscr{U}_1(\vartheta)) - g(\vartheta, \mathscr{P}_2(\vartheta), \mathscr{U}_2(\vartheta))| \\ &\quad = \frac{17}{300}|\mathscr{P}_1(\vartheta) - \mathscr{P}_2(\vartheta)| + \frac{1}{70}|\mathscr{U}_1(\vartheta) - \mathscr{U}_2(\vartheta)|. \end{aligned} \tag{65}$$

With $\phi_1 = 1/125$, $\phi_2 = 3/80$, $\widehat{\phi}_1 = 17/300$, and $\widehat{\phi}_2 = 1/70$, the functions $\mathscr{P}$ and $\mathscr{U}$ clearly satisfy the $(\mathscr{F}_2)$ condition. Next, we find that $Y_1 = 0.852825$ and $Y_2 = 0.72329$, where Y_1 and Y_2 are, respectively, given by (25) and (26), based on the data available. Thus, $((Y_1 + Y_2)(\phi_1 + \phi_2) + (Y_1 + Y_2)(\widehat{\phi}_1 + \widehat{\phi}_2)) \cong 0.183542 < 1$, all the conditions of Theorem 10 are satisfied, and there is a unique solution for problem (3) on $[0, 1]$, which is stable for Ulam-Hyers, with $\mathscr{P}$ and $\mathscr{U}$ given by (4) and (5), respectively.

Example 14. Consider the following problem:

$$\begin{cases} {}^{C}\mathscr{D}\dfrac{19}{45}\mathscr{P}(\vartheta) = \mathscr{Y}_1(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), \vartheta\epsilon[1, 2], \\ {}^{C}\mathscr{D}\dfrac{13}{20}\mathscr{U}(\vartheta) = \mathscr{Y}_2(\vartheta, \mathscr{P}(\vartheta), \mathscr{U}(\vartheta)), \vartheta\epsilon[1, 2], \\ (\mathscr{P} + \mathscr{U})(1) = -(\mathscr{P} + \mathscr{U})(2), \ {}^{H}\mathscr{I}\dfrac{23}{50}(\mathscr{P} + \mathscr{U})\left(\dfrac{8}{5}\right) = 5, \end{cases} \tag{66}$$

where $\varsigma = 19/45$, $\rho = 13/20$, $\rho = 23/50$, $\varphi = 8/5$, $\phi = 5$, $T = 2$, $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$, and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ are set later. Using the provided data, we determine that $Y_1 = 0.5870324536104914$ and $Y_2 = 0.492344074388438$, where Y_1 and Y_2 are denoted by (25) and (26), respectively. We will use

$$\begin{aligned} \mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U}) &= \frac{\vartheta + \sin p + (q/3)}{(4 + \log \vartheta)^2}, \\ \mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U}) &= \frac{\cos q + (p/2) + \vartheta}{5\sqrt{625 + (\log \vartheta)^2}}, \end{aligned} \tag{67}$$

to illustrate Theorem 7. $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are continuous and fulfill the condition $(\mathscr{F}_1)$ with $\omega_1(\vartheta) = \vartheta/((4+\log\vartheta)^2)$, $\omega_2(\vartheta) = 1/((4+\log\vartheta)^2)$, $\omega_3(\vartheta) = 1/(3(4+\log\vartheta)^2)$, $\widehat{\omega}_1(\vartheta) = \vartheta/(5\sqrt{625+(\log\vartheta)^2})$, $\widehat{\omega}_2(\vartheta) = 1/(2(5\sqrt{625+(\log\vartheta)^2}))$, and $\widehat{\omega}_3(\vartheta) = 1/(5\sqrt{625+(\log\vartheta)^2})$. Also $(2Y_1 + ((\log T)^\varsigma/\Gamma(\varsigma+1)))\|\omega_3\| + (2Y_2 + ((\log T)^\rho/\Gamma(\rho+1)))\|\widehat{\omega}_2\| \approx 0.13130829302659$ and $(2Y_1 + ((\log T)^\varsigma/\Gamma(\varsigma+1)))\|\omega_3\| + (2Y_2 + ((\log T)^\rho/\Gamma(\rho+1)))\|\widehat{\omega}_3\| \approx 0.05486795746034246$. As a result, all of Theorem 3.1 conditions hold, and there exists at least one solution to the problem (66) involving the equations $\mathscr{Y}_1(\vartheta, \mathscr{P}, \mathscr{U})$ and $\mathscr{Y}_2(\vartheta, \mathscr{P}, \mathscr{U})$ specified in (67).

6. Conclusion

This article established and discussed the existence, uniqueness, and Ulam-Hyers stability of solutions for a coupled system of fractional-order nonlinear Caputo-Hadamard fractional differential equations with boundary conditions involving Hadamard fractional integrals. In addition, three examples are provided to demonstrate the applicability of the acquired results. This paper's approach is novel and adds to the field of theory on boundary value problems for nonlinear fractional differential equations. Future research can expand the given fractional boundary value problem to include more complex structures, such as the boundary value requirements for finitely point multi-strip integrals provided by recently developed generalized fractional operators with non-singular kernels.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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