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Research Article

Oscillation Criteria of Fourth-Order Differential Equations with Delay Terms

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The aim of this paper is to derive oscillation criteria of the following fourth-order differential equation with delay term $(r(x)(z'''(x))^{\gamma})' + \sum_{i=1}^{n} q_i(x) f(z(\eta_i(x))) = 0$, under the assumption $\int_{x_0}^{\infty} r^{-1/\gamma}(s) \mathrm{d}s = \infty$. The results are based on comparison with the oscillatory behaviour of second-order delay equations and the generalised Riccati transformation. Not only do the provided theorems provide an entirely new technique but also they vastly improve on a number of previously published conclusions. We give three examples to illustrate our findings.

1. Introduction

Higher-order neutral differential equations have recently been recognized as being sufficient to describe a variety of real applications [1–4]. As a result, many researchers have studied the qualitative behaviour of solutions of these equations (see [5–8]). The research of oscillation and oscillatory behaviour of these equations, which has been investigated using multiple approaches and techniques, has received special attention (see [9–11]). The attempt to improve the work and obtain a generalised platform that covers all special cases inspires the investigation of fourth- and higher-order equations.

In this work, we are concerned with oscillation of fourth-order delay differential equations of the form

$$\left(r(x)\left(z'''(x)\right)^{\gamma}\right)' + \sum_{i=1}^{n} q_i(x) f(z(\eta_i(x))) = 0, \tag{1}$$

where $x \geq x_0$. Throughout this work, we suppose the following:

(i) $r \in C^1([x_0,\infty), R)$ and γ is a quotient of odd positive integers

(ii) The following condition holds:

$$\int_{x_0}^{\infty} \frac{1}{r^{1/\gamma}(s)} ds = \infty, \tag{2}$$

for $r(x) > 0$, $r'(x) > 0$, and

(iii) $q_i, \eta_i \in C([x_0,\infty), R)$, $q_i(x) \geq 0$, $\eta_i(x) \leq x$, $\lim_{x \longrightarrow \infty} \eta_i(x) = \infty$ $(i = 1, 2, \cdots)$, and $f \in C(R, R)$ such that

$$\frac{f(x)}{x^{\gamma}} \geq \ell > 0, \quad \text{for } x \neq 0. \tag{3}$$

By a solution of (1), we mean a function $z \in C^3[x_z,\infty)$, $x_z \geq x_0$, that has the property $r(x)(z'''(x))^{\gamma} \in C^1[x_z,\infty)$ and fulfills (1) on $[x_z, \infty)$. If a solution of (1) has arbitrarily large zeros on $[x_z, \infty)$, then it is considered oscillatory; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Next, we give some previous findings in the literature that are relevant to the present work. Grace [12] has studied the equation

$$\left(r(x)z^{(n-\nu)}(x)\right)^{(\nu)}+q(x)f(z[g(x)])=0, \tag{4}$$

in addition to Agarwal et al. [13] and Xu and Xia [14] who have studied the equation

$$\left(z^{(n-1)}(x)\left|z^{(n-1)}(x)\right|^{\gamma-1}\right)'+f(x,z(\eta(x)))=0, \tag{5}$$

subject to condition (2). Zhang et al. [15] obtained oscillatory criteria of the equation

$$\left(r(x)\left(z'''(x)\right)^{\gamma}\right)'+q(x)z^{\gamma}(\eta(x))=0, \tag{6}$$

with the condition

$$\int_{x_0}^{\infty}\frac{1}{r^{1/\gamma}(u)}\,\mathrm{d}u<\infty. \tag{7}$$

Baculikova et al. [16] used the comparison theory to prove that if

$$y'(x)+q(x)f\left(\frac{\delta\eta^{n-1}(x)}{(n-1)!r^{1/\gamma}(\eta(x))}\right)f\left(y^{1/\gamma}(\eta(x))\right)=0 \tag{8}$$

is oscillatory, then

$$\left(r(x)\left(z^{(n-1)}(x)\right)^{\gamma}\right)'+q(x)f(z(\eta(x)))=0 \tag{9}$$

is oscillatory for even n. Grace et al. [7] presented oscillation criteria for fourth-order delay differential equations of the form

$$\left(r_3\left(r_2\left(r_1z'\right)'\right)'\right)'(x)+q(x)z(\eta(x))=0, \tag{10}$$

under the assumption

$$\int_{x_0}^{\infty}\frac{dt}{r_i(x)}<\infty,\quad i=1,2,3. \tag{11}$$

Using the Riccati transformation, an oscillation criterion for fourth-order neutral delay differential equation of the form

$$\left[r(x)\left([z(x)+p(x)z((\eta(x)))]'''\right)^{\alpha}\right]'+\int_a^b q(x,\xi)f(z(g(x,\xi)))\,d\xi=0 \tag{12}$$

was obtained by Chatzarakis et al. [17]. By using the technique of the Riccati transformation and the theory of comparison with first-order delay equations, Bazighifan and Abdeljawad [18] established some new oscillation criteria for fourth-order advanced differential equations with p-Laplacian-like operator of the form

$$\left(b(x)\left|z'''(x)\right|^{p-2}z'''(x)\right)'+\sum_{i=1}^{j}q_i(x)g(z(\eta_i(x)))=0. \tag{13}$$

Very recently, Bazighifan et al. [19] established new criteria for the oscillatory behaviour of the following fourth-order differential equations with middle term

$$\left(r(x)\left|z'''(x)\right|^{p_1-2}z'''(x)\right)'+\sigma(x)\left|z'''(x)\right|^{p_1-2}z'''(x)+q(x)|z(\tau(x))|^{p_2-2}z(\tau(x))=0, \tag{14}$$

by the comparison technique and employing the Riccati transformation under the condition

$$\int_{x_0}^{\infty}\left[\frac{1}{r(s)}\exp\left(-\int_{x_0}^{s}\frac{\sigma(\eta)}{r(\eta)}\mathrm{d}\eta\right)\right]^{1/p_1-1}ds=\infty. \tag{15}$$

For convenience, in the present work, we denote

$$\begin{aligned}
\delta(x)&=\int_x^{\infty}\frac{1}{r^{1/\gamma}(s)}ds,\\
\psi(x)&=\pi(x)\left(\sum_{i=1}^{n}\ell q_i(x)\left(\frac{\eta_i^3(x)}{x^3}\right)^{\gamma}+\frac{\mu x^2-2\gamma}{2r^{1/\gamma}(x)\delta^{\gamma+1}(x)}\right),\\
\varphi(x)&=\frac{\pi'(x)}{\pi(x)}+\frac{(\gamma+1)\mu x^2}{2r^{1/\gamma}(x)\delta(x)},\\
\varphi^*(x)&=\frac{\tau'(x)}{\tau(x)}+\frac{2}{\delta(x)},\\
\psi^*&=\tau(x)\left(\int_x^{\infty}\left(\frac{\ell}{r(x)}\int_x^{\infty}\sum_{i=1}^{n}q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\,\mathrm{d}s\right)^{1/\gamma}\mathrm{d}x+\frac{1-r^{-1/\gamma}(x)}{\delta^2(x)}\right),
\end{aligned} \tag{16}$$

where $\pi,\tau\in C^1((x_0,\infty),(0,\infty))$. The generalised Riccati transformation is defined as

$$\omega(x):=\pi(x)\left(\frac{r(x)\left(z'''\right)^{\gamma}(x)}{z^{\gamma}(x)}+\frac{1}{\delta^{\gamma}(x)}\right), \tag{17}$$

$$\vartheta(x):=\tau(x)\left(\frac{z'(x)}{z(x)}+\frac{1}{\delta(x)}\right). \tag{18}$$

We remark that in the study of the asymptotic behaviour of the positive solutions of (1), there are only two cases:

Case 1. $z^{(j)}(x)>0$ for $j=1,2,3$,

Case 2. $z^{(j)}(x)>0$ for $j=1,3$ and $z''(x)<0$.

In this work, using the Riccati approach and a comparison with a second-order equation, we shall obtain oscillation criteria for (1).

2. Some Significant Auxiliary Lemmas

The following lemmas serve as a basis for our findings.

Lemma 1 (see [20]). *Let α be a ratio of two odd numbers; $H>0$ and K are constants. Then,*

$$P^{(\alpha+1)/\alpha}-(P-Q)^{(\alpha+1)/\alpha}\le\frac{P}{\alpha}Q^{1/\alpha}+Q^{1/\alpha}P-\frac{1}{\alpha}Q^{(1+\alpha)/\alpha},\quad PQ\ge0,\alpha\ge1,$$
$$\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{K^{\alpha+1}}{H^{\alpha}}\ge Km-Hm^{(\alpha+1)/\alpha},\quad H>0. \tag{19}$$

Lemma 2 (see [17]). *Let $f^{(j)})>0$ and $f^{(n+1)}<0$ for all $j=0,1,\cdots,n$. Then,*

$$\frac{n!}{x^{n}}f(x)\ge\frac{(n-1)!}{x^{n-1}}\frac{d}{dx}f(x). \tag{20}$$

Lemma 3 (see [21]). *The equation*

$$\left(a(x)\left(m'(x)^{\gamma}\right)\right)'+q(x)m^{\gamma}(x)=0, \tag{21}$$

where $a\in C[x_0,\infty)$, $a(x)>0$, and $q(x)>0$, is nonoscillatory if and only if there exist $x\ge x_0$ and $\sigma\in C^1[x,\infty)$ such that

$$\sigma'(x)+\frac{\gamma}{r^{1/\gamma}(x)}\sigma^{1+1/\gamma}(x)+q(x)\le0, \tag{22}$$

for $x\ge x_0$.

Lemma 4 (see [22]). *Suppose that $h\in C^{n}([x_0,\infty),(0,\infty))$; then,*

$$h^{(n-1)}(x)h^{(n)}(x)\le0, \tag{23}$$

for every $\lambda\in(0,1)$ and $x\ge x_\lambda$.

3. Oscillation Criteria

In this section, we shall obtain some oscillation criteria for equation (1).

Lemma 5. *Suppose that z is a solution of (1) such that $z>0$ and $z^{(j)}>0$ for all $j=1,2,3$. If we have the function $\omega\in C^1[x,\infty)$ defined in (17), where $\pi\in C^1([x_0,\infty),(0,\infty))$, then*

$$\omega'(x)\le-\psi(x)+\varphi(x)\omega(x)-\frac{\gamma\mu x^2}{2(r(x)\pi(x))^{1/\gamma}}\omega^{(\gamma+1)/\gamma}(x), \tag{24}$$

for all $x>x_1$, where x_1 is large enough.

Proof. Let z be a solution of (1) where $z>0$ and $z^{(j)}(x)>0$ for all $j=1,2,3$. Thus, from Lemma 4, we get

$$z'(x)\ge\frac{\mu}{2}x^2z'''(x), \tag{25}$$

for all $\mu\in(0,1)$ and for every large x. From (17), we have that $\omega(x)>0$ for $x\ge x_1$, and

$$\begin{aligned}\omega'(x)&=\pi'(x)\left(\frac{r(x)\left(z'''(x)\right)^{\gamma}(x)}{z^{\gamma}(x)}+\frac{1}{\delta^{\gamma}(x)}\right)+\pi(x)\frac{\left(r(x)\left(z'''(x)\right)^{\gamma}\right)'}{z^{\gamma}(x)}\\&\quad-\gamma\pi(x)\frac{z^{\gamma-1}(x)z'(x)r(x)\left(z'''(x)\right)^{\gamma}}{z^{2\gamma}(x)}+\frac{\gamma\pi(x)}{r^{1/\gamma}(x)\delta^{\gamma+1}(x)}.\end{aligned} \tag{26}$$

Using (25) and (17), we acquire

$$\begin{aligned}\omega'(x)&\le\frac{\pi'(x)}{\pi(x)}\omega(x)+\pi(x)\frac{\left(r(x)\left(z'''(x)\right)^{\gamma}\right)'}{z^{\gamma}(x)}\\&\quad-\gamma\pi(x)\frac{\mu}{2}x^2\frac{r(x)\left(z'''(x)\right)^{\gamma+1}}{z^{\gamma+1}(x)}+\frac{\gamma\pi(x)}{r^{1/\gamma}(x)\delta^{\gamma+1}(x)}\\&\le\frac{\pi'(x)}{\pi(x)}\omega(x)+\pi(x)\frac{\left(r(x)\left(z'''(x)\right)^{\gamma}\right)'}{z^{\gamma}(x)}\\&\quad-\gamma\pi(x)\frac{\mu}{2}x^2r(x)\left(\frac{\omega(x)}{\pi(x)r(x)}-\frac{1}{r(x)\delta^{\gamma}(x)}\right)^{(\gamma+1)/\gamma}+\frac{\gamma\pi(x)}{r^{1/\gamma}(x)\delta^{\gamma+1}(x)}.\end{aligned} \tag{27}$$

Letting $P=\omega(x)/(\pi(x)r(x))$, $Q=1/(r(x)\delta^{\gamma}(x))$, and $\alpha=\gamma$ and by using Lemma 1, we get

$$\begin{aligned}&\left(\frac{\omega(x)}{\pi(x)r(x)}-\frac{1}{r(x)\delta^{\gamma}(x)}\right)^{(\gamma+1)/\gamma}\\&\quad\ge\left(\frac{\omega(x)}{\pi(x)r(x)}\right)^{(\gamma+1)/\gamma}\\&\quad-\frac{1}{\gamma r^{1/\gamma}(x)\delta(x)}\left((\gamma+1)\frac{\omega(x)}{\pi(x)r(x)}-\frac{1}{r(x)\delta^{\gamma}(x)}\right).\end{aligned} \tag{28}$$

From Lemma 2, we obtain $z(x)\ge(x/3)z'(x)$, and hence,

$$\frac{z(\eta_i(x))}{z(x)}\ge\frac{\eta_i^3(x)}{x^3}. \tag{29}$$

From (1), (27), and (28), we obtain

$$\begin{aligned}\omega'(x)&\le\frac{\pi'(x)}{\pi(x)}\omega(x)-\ell\pi(x)\sum_{i=1}^{n}q_i(x)\left(\frac{\eta_i^3(x)}{x^3}\right)^{\gamma}\\&\quad-\gamma\pi(x)\frac{\mu}{2}x^2r(x)\left(\frac{\omega(x)}{\pi(x)r(x)}\right)^{(\gamma+1)/\gamma}\\&\quad-\gamma\pi(x)\frac{\mu}{2}x^2r(x)\left(\frac{-1}{\gamma r^{1/\gamma}(x)\delta(x)}\left((\gamma+1)\frac{\omega(x)}{\pi(x)r(x)}-\frac{1}{r(x)\delta^{\gamma}(x)}\right)\right)\\&\quad+\frac{\gamma\pi(x)}{r^{1/\gamma}(x)\delta^{\gamma+1}(x)}.\end{aligned} \tag{30}$$

This implies that

$$\omega'(x) \leq \left(\frac{\pi'(x)}{\pi(x)} + \frac{(\gamma+1)\mu x^2}{2r^{1/\gamma}(x)\delta(x)}\right)\omega(x) - \frac{\gamma\mu x^2}{2r^{1/\gamma}(x)\pi^{1/\gamma}(x)}\omega^{(\gamma+1)/\gamma}(x) - \pi(x)\left(\sum_{i=1}^{n} \ell q_i(x)\left(\frac{\eta_i^3(x)}{x^3}\right)^{\gamma} + \frac{\mu x^2 - 2\gamma}{2r^{1/\gamma}(x)\delta^{\gamma+1}(x)}\right). \quad (31)$$

Thus,

$$\omega'(x) \leq -\psi(x) + \varphi(x)\omega(x) - \frac{\gamma\mu x^2}{2(r(x)\pi(x))^{1/\gamma}}\omega^{(\gamma+1)/\gamma}(x). \quad (32)$$

The proof is completed. □

Lemma 6. *Let z be a solution of (1) such that $z > 0$ and $z^{(j)}(x) > 0$ for $j = 1, 3$ and $z''(x) < 0$. If the function $\vartheta \in C^1[x,\infty)$ is defined in (18) such that $\tau \in C^1((x_0,\infty),(0,\infty))$, then*

$$\vartheta'(x) \leq \varphi^*(x)\vartheta(x) - \psi^*(x) - \frac{1}{\tau(x)}\vartheta^2(x), \quad (33)$$

for all $x > x_1$, where x_1 is large enough.

Proof. Let z be a solution of (1) where $z > 0$ and $z^{(j)}(x) > 0$ for $j = 1, 3$ and $z''(x) < 0$. From Lemma 2, we have that $z(x) \geq xz'(x)$. Integrating this inequality from $\eta(x)$ to x, we obtain

$$z(\eta_i(x)) \geq \frac{\eta_i(x)}{x}z(x). \quad (34)$$

Hence, from (3) we have

$$f(z(\eta_i(x))) \geq \ell\frac{\eta_i^{\gamma}(x)}{x^{\gamma}}z^{\gamma}(x). \quad (35)$$

By integrating (1) from x to u and since $z'(x) > 0$, we get

$$r(u)\left(z'''(u)\right)^{\gamma} - r(x)\left(z'''(x)\right)^{\gamma} = -\int_x^u \sum_{i=1}^{n} q_i(s) f(z(\eta_i(s)))ds \leq -\ell z^{\gamma}(x)\int_x^u \sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}ds. \quad (36)$$

Now letting $u \longrightarrow \infty$ yields

$$r(x)\left(z'''(x)\right)^{\gamma} \geq \ell z^{\gamma}(x)\int_x^{\infty}\sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\mathrm{d}s, \quad (37)$$

and so

$$z'''(x) \geq z(x)\left(\frac{\ell}{r(x)}\int_x^{\infty}\sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}ds\right)^{1/\gamma}. \quad (38)$$

Integrating this from x to ∞ gives

$$z''(x) \leq -z(x)\int_x^{\infty}\left(\frac{\ell}{r(x)}\int_x^{\infty}\sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\mathrm{d}s\right)^{1/\gamma}dx. \quad (39)$$

From (18), we have that $\vartheta(x) > 0$ for $x \geq x_1$ and by differentiating, we get

$$\vartheta'(x) = \frac{\tau'(x)}{\tau(x)}\vartheta(x) + \tau(x)\frac{z''(x)}{z(x)} - \tau(x)\left(\frac{\vartheta(x)}{\tau(x)} - \frac{1}{\delta(x)}\right)^2 + \frac{\tau(x)}{r^{1/r}(x)\delta^2(x)}. \quad (40)$$

Now, using Lemma 1 with $P = \vartheta(x)/\tau(x)$, $Q = 1/\delta(x)$, and $\alpha = 1$ yields

$$\left(\frac{\theta(x)}{\tau(x)} - \frac{1}{\delta(x)}\right)^2 \geq \left(\frac{\theta(x)}{\tau(x)}\right)^2 - \frac{1}{\delta(x)}\left(\frac{2\vartheta(x)}{\tau(x)} - \frac{1}{\delta(x)}\right). \quad (41)$$

From (1), (40), and (41), we have the following:

$$\vartheta'(x) \leq \frac{\tau'(x)}{\tau(x)}\vartheta(x) - \tau(x)\int_x^{\infty}\left(\frac{\ell}{r(x)}\int_x^{\infty}\sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\mathrm{d}s\right)^{1/\gamma}dx - \tau(x)\left(\left(\frac{\vartheta(x)}{\tau(x)}\right)^2 - \frac{1}{\delta(x)}\left(\frac{2\vartheta(x)}{\tau(x)} - \frac{1}{\delta(x)}\right)\right) + \frac{\tau(x)}{r^{1/\gamma}(x)\delta^2(x)}. \quad (42)$$

This implies that

$$\vartheta'(x) \leq \left(\frac{\tau'(x)}{\tau(x)} + \frac{2}{\delta(x)}\right)\vartheta(x) - \frac{1}{\tau(x)}\vartheta^2(x) - \tau(x)\left(\int_x^{\infty}\left(\frac{\ell}{r(x)}\int_x^{\infty}\sum_{i=1}^{n} q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\mathrm{d}s\right)^{1/\gamma}dx + \frac{1 - r^{-1/\gamma}(x)}{\delta^2(x)}\right). \quad (43)$$

Thus,

$$\vartheta'(x) \leq \varphi^*(x)\vartheta(x) - \psi^*(x) - \frac{1}{\tau(x)}\vartheta^2(x). \quad (44)$$

The proof is completed. □

Lemma 7. *Let z be a solution of (1) with $z > 0$. If $\pi \in C(x_0,\infty)$ such that*

$$\int_{x_0}^{\infty}\left(\psi(s)-\left(\frac{2}{\mu s^2}\right)^{\gamma}\frac{r(s)\pi(s)(\varphi(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\right)ds=\infty, \tag{45}$$

for some $\mu \in (0,1)$, then z does not fulfill Case 1.

Proof. Let z be a solution of (1) such that $z > 0$. From Lemma 5, we obtain that (24) holds. Using Lemma 1 with

$$K=\varphi(x), H=\frac{\gamma\mu x^2}{\left(2(r(x)\pi(x))^{1/\gamma}\right)}, \tag{46}$$

and $m=\omega$, we get

$$\omega'(x)\leq-\psi(x)+\left(\frac{2}{\mu x^2}\right)^{\gamma}\frac{r(x)\pi(x)(\varphi(x))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}. \tag{47}$$

Now, integrating from x_1 to x yields

$$\int_{x_1}^{\infty}\left(\psi(s)-\left(\frac{2}{\mu s^2}\right)^{\gamma}\frac{r(s)\pi(s)(\varphi(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\right)ds\leq\omega(x_1), \tag{48}$$

which contradicts (45). So, the proof is complete. □

Lemma 8. *Let z be a solution of (1) with $z > 0$ and $z^{(j)}(x) > 0$ for $j = 1, 3$ and $z''(x) < 0$. If $\tau \in C([x_0,\infty))$ such that*

$$\int_{x_0}^{\infty}\left(\psi^*(s)-\frac{1}{4}\tau(s)(\varphi^*(s))^2\right)ds=\infty, \tag{49}$$

then z does not fulfill Case 2.

Proof. Let z be a solution of (1) such that $z > 0$. From Lemma 6, we get that (33) holds. Using Lemma 1 with

$$H=\varphi^*(x), K=\frac{1}{\tau(x)}, \gamma=1, m=\vartheta, \tag{50}$$

we obtain

$$\omega'(x)\leq-\psi^*(x)+\frac{1}{4}\tau(x)(\varphi^*(x))^2. \tag{51}$$

Integrating from x_1 to x gives

$$\int_{x_1}^{\infty}\left(\psi^*(s)-\frac{1}{4}\tau(s)(\varphi^*(s))^2\right)ds\leq\omega(x_1), \tag{52}$$

which contradicts (49). This completes the proof. □

Theorem 9. *Let $\pi, \tau \in C[x_0,\infty)$ such that (45) and (49) hold for some $\mu \in (0,1)$. Then, equation (1) is oscillatory.*

Proof. The proof is very similar to the proofs of Lemmas 7 and 8.

Now, by using the comparison method, we develop additional oscillation results for (1) in the following theorem: □

Theorem 10. *Let (2) hold and assume that*

$$\left[\frac{r(x)}{x^{2\gamma}}\left(z'(x)\right)^{\gamma}\right]'+\psi(x)z^{\gamma}(x)=0, \tag{53}$$

$$z''(x)+z(x)\int_{x}^{\infty}\left(\frac{\ell}{r(x)}\int_{x}^{\infty}\sum_{i=1}^{n}q_i(s)\frac{\eta_i^{\gamma}(s)}{s^{\gamma}}\,ds\right)^{1/\gamma}dx=0, \tag{54}$$

are both oscillatory; then, (1) is oscillatory.

Proof. Assume the contrary that (1) has a positive solution z, and by virtue of Lemma 3 and if we set $\pi(x) = 1$ in (24), then we get

$$\omega'(x)+\frac{\gamma\mu x^2}{2r^{1/\gamma}(x)}\omega^{(\gamma+1)/\gamma}+\psi(x)\leq0. \tag{55}$$

Hence, we have that (53) is nonoscillatory, which is a contradiction. If we set $\tau(x) = 1$ in (33), then we obtain

$$\vartheta'(x)+\psi^*(x)+\vartheta^2(x)\leq0. \tag{56}$$

Thus, equation (54) is nonoscillatory, which is a contradiction. The proof is now complete. □

It is well known (see Řehák [23])) that if

$$\int_{x_0}^{\infty}\frac{1}{r(x)}\,\mathrm{d}x=\infty, \lim_{x\longrightarrow\infty}\inf\left(\int_{x_0}^{x}\frac{1}{r(s)}\,\mathrm{d}s\right)\int_{x}^{\infty}q(s)\mathrm{d}s>\frac{1}{4}, \tag{57}$$

then equation (21) with $\gamma = 1$ is oscillatory.

Theorem 11. *Let (2) hold. Assume that*

$$\int_{x_0}^{\infty}\frac{x^2}{r(x)}dx=\infty, \tag{58}$$

and

$$\lim_{x\longrightarrow\infty}\inf\left(\int_{x_0}^{x}\frac{s^2}{r(s)}ds\right)\int_{x}^{\infty}\psi(s)ds>\frac{1}{2\lambda_1}, \tag{59}$$

for some constant $\lambda_1 \in (0,1)$ and

$$\lim_{x\longrightarrow\infty} \inf \int_x^\infty \left(\left(\int_x^\infty \left(\frac{\ell}{r(x)} \int_x^\infty \sum_{i=1}^n q_i(s) \frac{\eta_i^\gamma(s)}{s^\gamma} ds \right) dx \right) ds > \frac{1}{4}, \tag{60}$$

then every solution of (1) is oscillatory.
The proof is obvious.

4. Examples

In this section, we provide some examples to prove that the results of Section 3 are valid.

Example 1. Consider

$$z^{(4)}(x) + \left(\frac{q_0 - x^2}{x^4} + \frac{1}{x^2} \right) z(x) = 0, \quad x \geq 1, \tag{61}$$

where $q_0 > 0$.

Let $\gamma = 1$, $r(x) = 1$, $q(x) = ((q_0 - x^2)/x^4 + (1/x^2)) = q_0/x$, and $\eta(x) = x$.

Hence, we have

$$\delta(x_0) = \infty, \psi(x) = \frac{q_0}{x}, \varphi(x) = \frac{3}{x}, \varphi^*(x) = \frac{1}{x}, \psi^*(x) = \frac{q_0}{6x}. \tag{62}$$

If we set $\pi(x) = x^3$, $\tau(x) = x$, and $\ell = 1$, then condition (45) becomes

$$\begin{aligned} &\int_{x_0}^\infty \left(\psi(s) - \left(\frac{2}{\mu s^2} \right)^\gamma \frac{r(s)\pi(s)(\varphi(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \right) ds \\ &= \int_{x_0}^\infty \left(\frac{q_0}{s} - \frac{9}{2\mu s} \right) ds = \left(q_0 - \frac{9}{2\mu} \right) \int_{x_0}^\infty \frac{1}{s} ds \\ &= \infty, \quad \text{if } q_0 > \frac{9}{2\mu}. \end{aligned} \tag{63}$$

Therefore, from Lemma 7, if $q_0 > 9/2\mu$, then (61) has no positive solution z satisfying $z''(x) > 0$. Also, condition (49) becomes

$$\int_{x_0}^\infty \left(\psi^*(s) - \frac{1}{4}\tau(s)(\varphi^*(s))^2 \right) ds = \int_{x_0}^\infty \left(\frac{q_0}{6s} - \frac{1}{4s} \right) ds = \infty, \quad \text{if } q_0 > \frac{3}{2}. \tag{64}$$

From Lemma 8, if $q_0 > 3/2$, then (61) has no positive solution z satisfying $z''(x) < 0$. Thus, from Theorem 9 every solution of (61) is oscillatory if $q_0 > \max\{9/2\mu, 3/2\}$.

Example 2. Consider the following differential equation representing equation (1),

$$\left(x^3 \left(z'''(x) \right)^3 \right)' + \left(\frac{c - x^4}{x^7} + \frac{1}{x^3} \right) z^3(\varepsilon x) = 0, \quad x \geq 1, \tag{65}$$

where $c > 0$ and $0 < \varepsilon < 1$ are constants.

Here, $\gamma = 3$, $r(x) = x^3$, $q(x) = ((c - x^4)/x^7) + (1/x^3) = c/x^7$, and $\eta(x) = \varepsilon x$. Hence,

$$\delta(x) = \infty, \psi(x) = \frac{c\varepsilon^9}{x}, \varphi(x) = \frac{6}{x}, \varphi^*(x) = \frac{1}{x}, \psi^*(x) = \left(\frac{c\varepsilon^3}{48} \right)^{1/3} \frac{1}{x}. \tag{66}$$

If we set $\pi(x) = x^6$, $\tau(x) = x$, and $\ell = 1$, then condition (45) yields

$$\begin{aligned} &\int_{x_0}^\infty \left(\psi(s) - \left(\frac{2}{\mu s^2} \right)^\gamma \frac{r(s)\pi(s)(\varphi(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \right) ds \\ &= \int_{x_0}^\infty \left(\frac{c\varepsilon^9}{s} - \frac{81}{2\mu^3 s} \right) ds = \left(c\varepsilon^9 - \frac{81}{2\mu^3} \right) \int_{x_0}^\infty \frac{1}{s} ds. \end{aligned} \tag{67}$$

Therefore, from Lemma 7, if $c > 3/4\varepsilon^3$, then (65) has a solution $z > 0$ satisfying $z''(x) > 0$. Also, from condition (49) we have

$$\begin{aligned} &\int_{x_0}^\infty \left(\psi^*(s) - \frac{1}{4}\tau(s)(\varphi^*(s))^2 \right) ds \\ &= \int_{x_0}^\infty \left(\left(\frac{c\varepsilon^3}{48} \right)^{1/3} \frac{1}{s} - \frac{1}{4s} \right) ds = \left(\left(\frac{c\varepsilon^3}{48} \right)^{1/3} - \frac{1}{4} \right) \int_{x_0}^\infty \frac{1}{s} ds. \end{aligned} \tag{68}$$

Thus, from Theorem 9, every solution of (65) is oscillatory if

$$c > \max\left\{ \frac{3}{4\varepsilon^3}, \frac{81}{2\varepsilon^9\mu^3} \right\}. \tag{69}$$

Example 3. Consider

$$z^{(4)}(x) + \left(\frac{q_0 x - x^2}{x^5} + \frac{1}{x^3} \right) z\left(\frac{x}{2} \right) = 0, \quad x \geq 1, \tag{70}$$

where $q_0 > 0$.

Let $\gamma = 1$, $r(x) = 1$, $q(x) = (((q_0 x - x^2)/x^5) + (1/x^3)) = q_0/x^4$, and $\eta(x) = x/2$. When $\ell = 1$ is used, condition (59) becomes

$$\begin{aligned}&\lim_{x\longrightarrow\infty}\inf\left(\int_{x_0}^{x}\frac{s^2}{r(s)}\,ds\right)\int_{x}^{\infty}\frac{q_0}{s^4}ds\\&=\lim_{x\longrightarrow\infty}\inf\left(\frac{x^3}{3}\right)\int_{x}^{\infty}\frac{q_0}{s^4}\mathrm{d}s=\frac{q_0}{9}>\frac{1}{4},\end{aligned}\tag{71}$$

and condition (60) gives

$$\begin{aligned}&\lim_{x\longrightarrow\infty}\inf\int_{x}^{\infty}\left(\int_{x}^{\infty}\left(\frac{\ell}{r(x)}\int_{x}^{\infty}q(s)\frac{\eta^{\gamma}(s)}{s^{\gamma}}ds\right)dx\right)ds\\&=\lim_{x\longrightarrow\infty}\inf\int_{x}^{\infty}\left(\int_{x}^{\infty}\left(\int_{x}^{\infty}\frac{q_0}{2s^4}ds\right)dx\right)ds=\frac{q_0}{6}>\frac{1}{4}.\end{aligned}\tag{72}$$

Therefore, from Theorem 11, all solutions of (70) are oscillatory if $q_0 > 2.25$.

5. Conclusion

In this paper, we have established some new sufficient criteria which ensure that every solution of the fourth-order differential equations (1) is oscillatory. The approach we used was based on comparisons with the oscillatory behaviour of second-order delay equations and the Riccati transformation. Several illustrative examples have also been presented.

Data Availability

No data were used to support this study.

Disclosure

This work is part of UKM's research # DIP-2021-018.

Conflicts of Interest

All authors have declared they do not have any competing interests.

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