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Research Article

Fixed-Point Results Related to *b*-Intuitionistic Fuzzy Metric Space

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In this study, we contribute to the terminological debate about various fixed-point results' use of the term intuitionistic fuzzy *b*-metric space in defining the structure based on fuzzy sets. As a predominant result, we give an adequate condition for a sequence to be Cauchy in the intuitionistic fuzzy *b*-metric space. Subsequently, we simplify the proofs of manifold fixed-point theorems in the intuitionistic fuzzy *b*-metric spaces under the prominent contraction conditions. Also, we give a satisfactory condition for a solution to Cauchy in the intuitionistic fuzzy *b*-metric spaces.

1. Introduction

Zadeh [1] was very striking with fuzzy logic in which the truth values of variables may be any real number in the interval [0,1]. In this real world, we all are surrounded by the problems of uncertainty. To counter this problem of uncertainty, Zadeh establishes fuzzy logic. Moreover, fixed-point theory can be explored in fuzzy metric space (briefly, FMS) in several different ways. Fuzzy generalization of Banach contraction principle [2] expressed fuzzy mapping as a notion and manifested a theorem for a fuzzy contraction on a fixed point in linear metric spaces.

Fuzzy metric space was pioneered by Kramosil and Michalek [3] by referring to the concept of the fuzzy set. Subsequently, Grabiec [4] interpreted the concept of completeness in FMS and extended the Banach contraction principle to G-complete FMS. Then, later on, George and Veeramani [5] also played a vital role in the theory of FMS and amended the idea of Cauchy sequence which was established by Grabiec and, meantime, amended the notion of a FMS which was introduced by Kramosil and came up with the new idea of Hausdorff topology on FMS.

On a different note, Atanassov [6] generalized the FS and brought up the idea of IFS. Using this concept, Park [7] discovered the concept of IFMS in 2006. Saadati Park [8] introduced the related results. Furthermore, we refer the readers to [9–16]. In metric space, still there are enough scopes. Bakhtin [17] and Czerwik [18] revealed a weaker condition of metric space to generalize the Banach contraction principle [2]. They labeled it as b-MS. To explore more about these spaces, see [14, 19–21]. The *b*-metric and FMSs are related, which we can see in [22]. The idea of a fuzzy b-MS was established in [23]. Using this idea, we proved a very useful lemma by using the extension of t-norm and t-conorm setting that the sequence $\{\eth_{\mathfrak{y}}\}$ is a Cauchy sequence. This paper comprises various fixed-point results in the b-IFMS. After the criticism presented in [24] on the adoption of the term "intuitionistic" in his original algebraic structures, a defense of his position is given by Atanassov in [25] by an "a posteriori" argument. Indeed, in [24], besides the above discussed argument about the use of "intuitionistic" term attributed to an algebraic negation which does not satisfy the great part of the accepted principles of intuitionistic logic, it is stressed that, in a paper of 1984 [26], the term "intuitionistic fuzzy set theory by Takeuti and Titani is an absolute legitimate approach, in the scope of intuitionistic logic, but it has nothing to do with Atanassov's intuitionistic fuzzy sets."

We show a novel utilization of IFMS in a really difficult space of dynamic (for example, professional decision). An illustration of vocation assurance will be introduced, accepting there is a dataset (for example, a portrayal of a bunch of subjects and a bunch of vocations). We will portray the condition of understudies knowing the after effects of their presentation. The difficult portrayal utilizes the idea of IFS that makes it conceivable to deliver two significant realities. To begin with, the upsides of each subject presentation change for every understudy. Second, in a professional assurance dataset depicting vocation for various understudies, it ought to be considered that, for various understudies focusing on a similar vocation, the upsides of a similar subject exhibition can be extraordinary. We utilize the standardized Euclidean metric strategy surrendered to gauge the metric between every understudy and each profession. The littlest acquired worth calls attention to a legitimate vocation assurance dependent on scholarly execution.

2. Preliminaries

Through this paper, we consider $\mathbb{N}$ be a set of all natural numbers and k be a positive real number.

Definition 1 (see [17]). A function $d\colon \mathfrak{Z}\times\mathfrak{Z}\longrightarrow[0,\infty)$ is b-metric if, for every $\mathfrak{b},\mathfrak{e},w\in\mathfrak{Z}$,

(a) $d(\mathfrak{b},\mathfrak{e})=0\Leftrightarrow\mathfrak{b}=\mathfrak{e}$

(b) $d(\mathfrak{b},\mathfrak{e})=d(\mathfrak{e},\mathfrak{b})$

(c) $d(\mathfrak{b},w)\le s[d(\mathfrak{b},\mathfrak{e})+d(\mathfrak{e},w)]$

The ordered pair $(\mathfrak{Z},d)$ is a b-MS. It is key to note that b-MS are not metrizable, especially b-metric might not be a continuous function of its variable.

Definition 2 (see [27]). A binary operation in such a way $\mathfrak{T}\colon[0,1]^2\longrightarrow[0,1]$ is said to be a continuous triangular norm (t-norm) if

(a) $\mathfrak{T}(\mathfrak{p},\mathfrak{b})=\mathfrak{T}(\mathfrak{b},\mathfrak{p})$ and $\mathfrak{T}(\mathfrak{p},(\mathfrak{b},\mathfrak{r}))=\mathfrak{T}((\mathfrak{p},\mathfrak{b}),\mathfrak{r})$

(b) $\mathfrak{T}$ is continuous

(c) $\mathfrak{T}(\mathfrak{p},1)=\mathfrak{p}$, $\forall\mathfrak{p}\in[0,1]$

(d) $\mathfrak{T}(\mathfrak{p},\mathfrak{b})\le\mathfrak{T}(\mathfrak{r},\mathbb{S})$ whenever $\mathfrak{p}\le\mathfrak{r}\&\mathfrak{b}\le\mathbb{S}$, forall $\mathfrak{p},\mathfrak{b},\mathfrak{r},\mathbb{S}\in[0,1]$

Definition 3 (see [28]). Let $\mathfrak{T}$ be a t-norm. Then, $\mathfrak{T}_{\mathfrak{y}}\colon[0,1]^2\longrightarrow[0,1],\mathfrak{y}\in\mathbb{N}$, is defined by

$$\mathfrak{T}_1(k)=\mathfrak{T}(k,k),\mathfrak{T}_{\mathfrak{y}+1}(k)=\mathfrak{T}\left(\mathfrak{T}_{\mathfrak{y}}(k),k\right),\quad \mathfrak{y}\in\mathbb{N},k\in[0,1]. \tag{1}$$

Let

$$\sup_{0<k<1}\mathfrak{T}(k,k)=1. \tag{2}$$

Then, t-norm $\mathfrak{T}$ is of H-type with the assuming the functions family $\{\mathfrak{T}^m(k)\}_{m=1}^{\infty}$ which is equicontinuous at $k=1$, where

$$\mathfrak{T}^1(k)=k,\mathfrak{T}^{m+1}(k)=\mathfrak{T}(\mathfrak{T}^m(k)),\quad m=1,2,3,\ldots,\text{and}\,k\in[0,1]. \tag{3}$$

The t-norm $\mathfrak{T}_{\min}(\eth,v)=\min(\eth,v)$ is an example of H-type.

Each t-norm $\mathfrak{T}$ and t-conorm $\mathbb{S}$ can be drawn out to an n-ary operation taking $(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}})\in[0,1]^{\mathfrak{y}}$ (see [27]):

$$\mathfrak{T}_{i=1}^1\eth_i=\eth_1,\mathfrak{T}_{i=1}^{\mathfrak{y}}=\mathfrak{T}\left(\mathfrak{T}_{i=1}^{\mathfrak{y}}\eth_i,\eth_{\mathfrak{y}}\right)=\mathfrak{T}\left(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}}\right). \tag{4}$$

Definition 4. $\mathfrak{T}_{\min}$, $\mathfrak{T}_L$, and $\mathfrak{T}_P$ can be extended in n-ary in the following ways:

$$\begin{aligned}\mathfrak{T}_{\min}\left(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}}\right)&=\min\left(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}}\right)\\ \mathfrak{T}_L\left(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}}\right)&=\max\sum_{i=\mathfrak{y}}^{\infty}\eth_i-(\mathfrak{y}-1),0\\ \mathfrak{T}_P\left(\eth_1,\eth_2,\ldots,\eth_{\mathfrak{y}}\right)&=\prod_{i=1}^{\mathfrak{y}}\eth_i.\end{aligned} \tag{5}$$

Definition 5 (see [29], t-conorm). "A binary operation $\mathbb{S}\colon[0,1]\times[0,1]\longrightarrow[0,1]$ is continuous t-conorm if

(a) $\mathbb{S}(a,q)=\mathbb{S}(q,a)\&\mathbb{S}(a,(q,c))=\mathbb{S}((a,q),c)$

(b) $\mathfrak{T}$ is continuous

(c) $\mathbb{S}(a,0)=a,\forall a\in[0,1]$

(d) $\mathbb{S}(a,q)\le\mathbb{S}(c,s)$ whenever $a\le c\,\&\,q\le s$, $\forall a,q,c,s\in[0,1]$

Definition 6. A t-conorm, $\mathbb{S}_{\mathfrak{y}}\colon[0,1]\longrightarrow[0,1],\mathfrak{y}\in\mathbb{N}$ is defined by

$$\mathbb{S}_1(k)=\mathbb{S}(k,k),\mathbb{S}_{\mathfrak{y}+1}(k)=\mathbb{S}\left(\mathbb{S}_{\mathfrak{y}}(k),k\right),\quad \mathfrak{y}\in\mathbb{N},k\in[0,1]. \tag{6}$$

Let

$$\inf_{0<k<1}\mathbb{S}(k,k)=1. \tag{7}$$

Then, t-conorm $\mathbb{S}$ is of H-type with the assuming the functions' family $\{\mathbb{S}^m(k)\}_{m=1}^{\infty}$ which is equicontinuous at $k=0$, where

$$\mathbb{S}^1(k) = k, \mathbb{S}^{m+1}(k) = \mathbb{S}(\mathbb{S}^m(k)), \quad m = 1, 2, 3, \ldots, \text{and}\, k \in [0, 1]. \tag{8}$$

The t-conorm $\mathbb{S}_{\max}(ð, \nu) = \max(ð, \nu)$ is an example of H-type.

The t-conorm can be stretched out by associativity in an n-ary operation taking, for $(ð_1, ð_2, \ldots, ð_\mathfrak{y}) \in [0, 1]^\mathfrak{y}$, the values

$$\mathbb{S}^1_{i=1} ð_i = \mathbb{S}\left(\mathbb{S}^{\mathfrak{y}}_{i=1} ð_i, ð_\mathfrak{y}\right) = \mathbb{S}\left(ð_1, ð_2, \ldots, ð_\mathfrak{y}\right). \tag{9}$$

Example 1. The extensions of the t-conorm in n-ary are $\mathbb{S}_{\min}, \mathbb{S}_L, \text{and} \mathbb{S}_P$ are the following:

$$\begin{aligned} \mathbb{S}_{\max}\left(ð_1, ð_2, \ldots, ð_\mathfrak{y}\right) &= \max\left(ð_1, ð_2, \ldots, ð_\mathfrak{y}\right) \\ \mathbb{S}_L\left(ð_1, ð_2, \ldots, ð_\mathfrak{y}\right) &= \min\left(\sum_{i=\mathfrak{y}}^{\infty} ð_i, 1\right) \\ \mathbb{S}_P\left(ð_1, ð_2, \ldots, ð_\mathfrak{y}\right) &= \sum_{i=\mathfrak{y}}^{\infty} ð_i - \prod_{i=1}^{\mathfrak{y}} ð_i. \end{aligned} \tag{10}$$

For any $\left\{ð_\mathfrak{y}\right\}_{\mathfrak{y}\in\mathbb{N}}$ in [0,1], the t-norm and t-conorm can be stretched out to a countable illimitable operation, that is,

$$\begin{aligned} \mathfrak{T}^{i=1}_{\infty} ð_i &= \lim_{n\longrightarrow\infty} \mathfrak{T}^{i=1}_{\mathfrak{y}} ð_i, \\ \mathbb{S}^{i=1}_{\infty} ð_i &= \lim_{n\longrightarrow\infty} \mathbb{S}^{i=1}_{\mathfrak{y}} ð_i. \end{aligned} \tag{11}$$

The sequence $\mathfrak{T}^{i=1}_{\mathfrak{y}} ð_i$ is increasing, and $\mathbb{S}^{i=1}_{\mathfrak{y}} ð_i$ is decreasing and has an upper bound and lower bound, respectively, so the limit exists.

In this fixed-point theory, see [30, 31], it is very interesting to discuss the classes of t-norms $\mathfrak{T}$ and the classes of t-conorm $\mathbb{S}$ and $\left\{ð_\mathfrak{y}\right\}$ in [0, 1] such that $\lim_{n\longrightarrow\infty} ð_\mathfrak{y} = 1$ and

$$\mathfrak{T}^{i=1}_{\infty} ð_i = \lim_{n\longrightarrow\infty} \mathfrak{T}^{i=1}_{\mathfrak{y}} ð_{\mathfrak{y}+i} = 1, \tag{12}$$

$$\mathbb{S}^{i=1}_{\infty} ð_i = \lim_{n\longrightarrow\infty} \mathbb{S}^{i=1}_{\mathfrak{y}} ð_{\mathfrak{y}+i} = 0. \tag{13}$$

Proposition 1. *Let $\left\{ð_\mathfrak{y}\right\}_{\mathfrak{y}\in\mathbb{N}}$ in [0, 1] be $\lim_{n\longrightarrow\infty} ð_\mathfrak{y} = 1$ and $\mathfrak{T}$ be a t-norm of H-type. Then,*

$$\mathfrak{T}^{i=1}_{\infty} ð_i = \lim_{n\longrightarrow\infty} \mathfrak{T}^{i=1}_{\mathfrak{y}} ð_{\mathfrak{y}+i} = 1. \tag{14}$$

Proposition 2. *Let $\left\{ð_\mathfrak{y}\right\}_{\mathfrak{y}\in\mathbb{N}}$ in [0, 1] ne $\lim_{n\longrightarrow\infty} ð_\mathfrak{y} = 0$ and $\mathbb{S}$ be a t-conorm of H-type. Then,*

$$\mathbb{S}^{i=1}_{\infty} ð_i = \lim_{n\longrightarrow\infty} \mathbb{S}^{i=1}_{\mathfrak{y}} ð_{\mathfrak{y}+i} = 0. \tag{15}$$

Definition 7 (see [23]). A 3-tuple $(\mathfrak{Z}, \mathscr{M}, \mathfrak{T})$ is called a b-FMS. If $\forall\, ð, \nu, z \in \mathfrak{Z}, s, k > 0$,

(a) $\mathscr{M}(ð, \nu, k) > 0$

(b) $\mathscr{M}(ð, \nu, k) = 1$, $\forall k > 0$ iff $ð = \nu$

(c) $\mathscr{M}(ð, \nu, k) = \mathscr{M}(\nu, ð, k)$

(d) $\mathscr{M}(ð, \nu, k + s) \geq \mathfrak{T}(\mathscr{M}(ð, \mathfrak{a}, k/b), \mathscr{M}(\mathfrak{a}, \nu, s/b))$

(e) $\mathscr{M}(ð, \nu, \cdot)$: $[0, \infty) \longrightarrow [0, 1]$ (left continuous)

where $\mathfrak{Z}$ is a nonempty set, $\mathfrak{T}$ is continuous t-norm, and $\mathscr{M}$ is a FS on $\mathfrak{Z}^2 \times (0, \infty)$.

Example 2 (see [9]). Let $\mathscr{M}(ð, \nu, k) = e^{-|ð-\nu|^p}$, where $p > 1$. Then, $\mathscr{M}$ is a b-metric with $b = 2^{p-1}$.

Definition 8. (see [7, 8]). A 5-tuple $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ is called intuitionistic fuzzy rectangular metric space. If, for every $ð, \nu, z \in \mathfrak{Z}$ and $s, k > 0$,

(a) $\mathscr{M}(ð, \nu, k) + \mathscr{N}(ð, \nu, k) \leq 1$

(b) $\mathscr{M}(ð, \nu, 0) = 0$

(c) $\mathscr{M}(ð, \nu, k) = 1, \forall k > 0$ iff $ð = \nu$

(d) $\mathscr{M}(ð, \nu, k) = \mathscr{M}(\nu, ð, k)$

(e) $\mathscr{M}(ð, \nu, k + s) \geq \mathfrak{T}(\mathscr{M}(ð, \mathfrak{a}, k/b), \mathscr{M}(\mathfrak{a}, \nu, s/b))$

(f) $\mathscr{M}(ð, \nu, \cdot)$: $[0, \infty) \longrightarrow [0, 1]$ (left continuous)

(g) $\lim_{k\longrightarrow\infty} \mathscr{M}(ð, \nu, k) = 1, \forall ð, \nu \in \mathfrak{Z}$

(h) $\mathscr{N}(ð, \nu, 0) = 1$

(i) $\mathscr{N}(ð, \nu, k) = 0, \forall k > 0$ iff $ð = \nu$

(j) $\mathscr{N}(ð, \nu, k + s) \geq \mathbb{S}(\mathscr{N}(ð, \mathfrak{a}, k/b), \mathscr{M}(\mathfrak{a}, \nu, s/b))$

(k) $\mathscr{N}(ð, \nu, \cdot)$: $[0, \infty) \longrightarrow [0, 1]$ (right continuous)

(l) $\lim_{k\longrightarrow\infty} \mathscr{N}(ð, \nu, k) = 0, \forall ð, \nu \in \mathfrak{Z}$

where $\mathfrak{Z}$ is a random set, $\mathfrak{T}$ is continuous t-norm, $\mathbb{S}$ is continuous t-conorm, and $\mathscr{M}$and$\mathscr{N}$ are FSs on $X^2 \times (0, \infty)$.

Example 3. Let $\mathscr{M}(ð, \nu, k) = e^{-d(ð,\nu)/k}, \mathscr{N}(\mathfrak{a}, \mathfrak{e}, k) = 1 - e^{-d(\mathfrak{a},\mathfrak{e})/k}$, where $p > 1$. Then, $\mathscr{M}$ and $\mathscr{N}$ are b-metric with $b = 2^{p-1}$.

Definition 9 (see [9]). A function ζ: $\mathfrak{R} \longrightarrow \mathfrak{R}$ is called b-nondecreasing if $ð > b\nu$ implies $\zeta(ð) \geq \zeta(ð), \forall ð, \nu \in \mathfrak{R}$.

Definition 10. A function ζ: $\mathfrak{R} \longrightarrow \mathfrak{R}$ is called b-nonincreasing if $ð < b\nu$ implies $\zeta(ð) \leq \zeta(\nu), \forall ð, \nu \in \mathfrak{R}$.

Lemma 1 (see [23, 32]). *Let $\mathscr{M}(ð, \nu, \mathfrak{T})$ be a b-FMS. Then, $\mathscr{M}(ð, \nu, k)$ is b-increasing with respect to $t, \forall ð, \nu \in \mathfrak{Z}$.*

Lemma 2. *Let $\mathscr{N}(ð, \nu, \mathbb{S})$ be an b-IFMS. Then, $\mathscr{N}(ð, \nu, k)$ is b-decreasing with respect to $t, \forall ð, \nu \in \mathfrak{Z}$.*

Definition 11 (see [23, 32]). Let $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ be an b-IFMS. For $k > 0$, the open ball $\mathscr{B}(\nu, r, k)$ with centre $ð \in \mathfrak{Z}$ and radius $0 < r < 1$ are defined as

$$\begin{aligned} \mathscr{B}(ð, r, k) &= \{\nu \in \mathfrak{Z}\colon \mathscr{M}(ð, \nu, k) > 1 - r\}, \\ \mathscr{B}(ð, r, k) &= \{\nu \in \mathfrak{Z}\colon \mathscr{N}(ð, \nu, k) < 1 - r\}. \end{aligned} \tag{16}$$

A sequence $\left\{ð_\mathfrak{y}\right\}$

(a) Converges to $ð$ if $\mathscr{M}(ð_\mathfrak{y}, ð, k) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $k > 0$. In this case, we write $\lim_{n\longrightarrow\infty} ð_\mathfrak{y} = ð$.

(b) Is a Cauchy sequence if $\forall 0<\varepsilon<1$ & $k>0, \exists \mathfrak{y}_0 \in \mathbb{N}$ in such a way that $\mathscr{M}(\eth_{\mathfrak{y}}, \eth_m, k) > 1-\varepsilon, \forall \mathfrak{y}, m \geq \mathfrak{y}_0$.

(c) Converges to $\eth$ if $\mathscr{N}(\eth_{\mathfrak{y}}, \eth, k) \longrightarrow 0$ as $\mathfrak{y} \longrightarrow \infty$, for each $k>0$. In this case, we write $\lim_{n\longrightarrow\infty} \eth_{\mathfrak{y}} = \eth$.

(d) Is a Cauchy sequence if $\forall 0<\varepsilon<1$ & $k>0, \exists \mathfrak{y}_0 \in \mathbb{N}$ in such a way that $\mathscr{N}(\eth_{\mathfrak{y}}, \eth_m, k) < 1-\varepsilon, \forall \mathfrak{y}, m \geq \mathfrak{y}_0$.

Definition 12. The intuitionistic fuzzy b-MS $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ is complete if and only if every Cauchy sequence is a convergent sequence.

Lemma 3. *In an b-IFMS $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$, we have a sequence $\{\eth_{\mathfrak{y}}\}$ in $\mathfrak{Z}$ converges to $\eth$; then, it is definitely a Cauchy sequence and $\eth$ is unique.*

In a b-IFMS, we possess the successive proposition.

Proposition 3. *Let $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ be a b-IFMS and $\{\eth_{\mathfrak{y}}\}$ converges to x. Then,*

$$\begin{aligned}
&\mathscr{M}\left(\eth, \nu, \frac{k}{b}\right) \leq \lim_{n\longrightarrow\infty} \sup \mathscr{M}\left(\eth_{\mathfrak{y}}, \nu, k\right) \leq \mathscr{M}(\eth, \nu, bk),\\
&\mathscr{M}\left(\eth, \nu, \frac{k}{b}\right) \leq \lim_{n\longrightarrow\infty} \inf \mathscr{M}\left(\eth_{\mathfrak{y}}, \nu, k\right) \leq \mathscr{M}(\eth, \nu, bk),\\
&\mathscr{N}(\eth, \nu, bk) \leq \lim_{n\longrightarrow\infty} \sup \mathscr{N}\left(\eth_{\mathfrak{y}}, \nu, k\right) \leq \mathscr{M}\left(\eth, \nu, \frac{k}{b}\right),\\
&\mathscr{N}(\eth, \nu, bk) \leq \lim_{n\longrightarrow\infty} \inf \mathscr{N}\left(\eth_{\mathfrak{y}}, \nu, k\right) \leq \mathscr{M}\left(\eth, \nu, \frac{k}{b}\right).
\end{aligned} \tag{17}$$

Remark 1. A b-IFM is not continuous generally.

Example 4. Let $\mathfrak{Z} = [0, \infty)$, $\mathscr{M}(\eth, \nu, k) = e^{-d(\eth,\nu)/k}$, and $\mathscr{N}(\mathfrak{a}, \mathfrak{e}, k) = 1 - e^{-d(\mathfrak{a},\mathfrak{e})/k}$, and

$$d(\eth, \nu) = \begin{cases} 0, & \text{if } \eth = \nu \\ 2|\eth - \nu|, & \text{if } \eth, \nu \in [0,1) \\ \dfrac{1}{2}|\eth - \nu|, & \text{otherwise.} \end{cases} \tag{18}$$

Then, $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ be a b-IFMS with $b=6$.

3. Main Results

Furthermore, we will use a b-IFMS in terms of definition with additional condition, $\lim_{k\longrightarrow\infty} \mathscr{M}(\eth, \nu, k) = 1$.

Lemma 4. *Let $\{\eth_{\mathfrak{y}}\}$ in a b-IFMS $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ be*

$$\begin{aligned}
&\mathscr{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, k\right) \geq \mathscr{M}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k>0,\\
&\mathscr{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, k\right) \leq \mathscr{N}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k>0,
\end{aligned} \tag{19}$$

and there exist $\eth_0, \eth_1 \in \mathfrak{Z}$ and $\nu \in (0,1)$ such that

$$\begin{aligned}
&\lim_{i\longrightarrow\infty} \mathfrak{T}_{i=1}^{\infty}\left(\eth_1, \eth_1, \frac{k}{\nu^i}\right) = 1,\\
&\lim_{i\longrightarrow\infty} \mathbb{S}_{i=1}^{\infty}\left(\eth_1, \eth_1, \frac{k}{\nu^i}\right) = 0, \quad k>0.
\end{aligned} \tag{20}$$

Then, $\{\eth_{\mathfrak{y}}\}$ is a Cauchy sequence.

Proof. Let $\omega \in (0,1)$. Then, the sum $\sum_{i=1}^{\infty} \omega^i$ is convergent, and then, there exists $\mathfrak{y}_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} \omega^i < 1$, for every $n > \mathfrak{y}_0$. Let $n > m > \mathfrak{y}_0$. Since $\mathscr{M}$ is b-increasing by Definition 8(e) and $\mathscr{N}$ is b-decreasing by Definition 8(j), for every $k>0$, we obtain

$$\begin{aligned}
\mathscr{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+m}, k\right) &\geq \mathscr{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+m}, \frac{k\sum_{i=\mathfrak{y}}^{\mathfrak{y}+m-1} \omega^i}{b}\right)\\
&\geq \mathfrak{T}\left(\mathscr{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, \frac{k\omega^{\mathfrak{y}}}{b^2}\right), \mathscr{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}+m}, \frac{k\sum_{i=\mathfrak{y}+1}^{\mathfrak{y}+m-1} \omega^i}{b^2}\right)\right)\\
&\geq \mathfrak{T}\left(\mathscr{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, \frac{k\omega^{\mathfrak{y}}}{b^2}\right), \mathfrak{T}\left(\mathscr{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}+2}, \frac{k\omega^{\mathfrak{y}+1}}{b^3}\right), \ldots, \mathscr{M}\left(\eth_{\mathfrak{y}+m-1}, \eth_{\mathfrak{y}+m}, \frac{k\omega^{\mathfrak{y}+m-1}}{b^m}\right)\right)\right).
\end{aligned} \tag{21}$$

By (19), it implies $\mathscr{M}(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, k) \geq \mathscr{M}(\eth_0, \eth_1, k/\mu^{\mathfrak{y}})$, $\mathfrak{y} \in \mathbb{N}, k>0$:

$$\begin{aligned}
\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+m}, k\right) &\leq \mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+m}, \frac{k\sum_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\omega^i}{b}\right)\\
&\leq \mathbb{S}\left(\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \frac{k\omega^{\mathfrak{y}}}{b^2}\right), \mathcal{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}+m}, \frac{k\sum_{i=\mathfrak{y}+1}^{\mathfrak{y}+m-1}\omega^i}{b^2}\right)\right)\\
&\geq \mathbb{S}\left(\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \frac{k\omega^{\mathfrak{y}}}{b^2}\right), \mathbb{S}\left(\mathcal{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}+2}, \frac{k\omega^{\mathfrak{y}+1}}{b^3}\right), \dots, \mathcal{N}\left(ð_{\mathfrak{y}+m-1}, ð_{\mathfrak{y}+m}, \frac{k\omega^{\mathfrak{y}+m-1}}{b^m}\right)\right)\right).
\end{aligned} \tag{22}$$

By (19), it implies $\mathcal{N}(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, k) \leq \mathcal{N}(ð_0, ð_1, k/\mu^{\mathfrak{y}})$, $\mathfrak{y} \in \mathbb{N}, k > 0$.

Since $n > m$ and $b > 1$, we have

$$\begin{aligned}
\mathcal{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+m}, k\right) &\geq \mathfrak{T}\left(\mathcal{M}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}}}{b^2\mu^{\mathfrak{y}}}\right), \mathfrak{T}\left(\mathcal{M}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}+1}}{b^3\mu^{\mathfrak{y}+1}}\right), \dots, \mathcal{M}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}+m-1}}{b^{m+1}\mu^{\mathfrak{y}+m-1}}\right)\right), \dots\right)\\
&\geq \mathfrak{T}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{M}\left(ð_0, ð_1, \frac{k\omega^i}{b^{i-\mathfrak{y}+2}\mu^i}\right)\\
&\geq \mathfrak{T}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{M}\left(ð_0, ð_1, \frac{k\omega^i}{b^i\mu^i}\right)\\
&\geq \mathfrak{T}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{M}\left(ð_0, ð_1, \frac{k}{v^i}\right),\\
\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+m}, k\right) &\geq \mathbb{S}\left(\mathcal{N}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}}}{b^2\mu^{\mathfrak{y}}}\right), \mathbb{S}\left(\mathcal{N}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}+1}}{b^3\mu^{\mathfrak{y}+1}}\right), \dots, \mathcal{N}\left(ð_0, ð_1, \frac{k\omega^{\mathfrak{y}+m-1}}{b^{m+1}\mu^{\mathfrak{y}+m-1}}\right)\right), \dots\right)\\
&\geq \mathbb{S}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{N}\left(ð_0, ð_1, \frac{k\omega^i}{b^{i-\mathfrak{y}+2}\mu^i}\right),\\
\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+m}, k\right) &\geq \mathbb{S}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{N}\left(ð_0, ð_1, \frac{k\omega^i}{b^i\mu^i}\right)\\
&\geq \mathbb{S}_{i=\mathfrak{y}}^{\mathfrak{y}+m-1}\mathcal{N}\left(ð_0, ð_1, \frac{k}{v^i}\right),
\end{aligned} \tag{23}$$

where $v = b\mu/\omega$ is $v \in (0, 1)$. By equation (20), it implies $\{ð_{\mathfrak{y}}\}$ is a Cauchy sequence. □

Corollary 1. *Let $\{ð_{\mathfrak{y}}\}$ be a b-IFMS $(\mathfrak{Z}, \mathcal{M}, \mathcal{N}, \mathfrak{T}, \mathbb{S})$, where $\mathfrak{T}$ and $\mathbb{S}$ are H-type. If there exists $\mu \in (0, 1/b)$ such that*

$$\begin{aligned}
\mathcal{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, k\right) &\geq \mathcal{M}\left(ð_{\mathfrak{y}-1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0,\\
\mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, k\right) &\leq \mathcal{N}\left(ð_{\mathfrak{y}-1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0,
\end{aligned} \tag{24}$$

holds, then $\{ð_{\mathfrak{y}}\}$ is a Cauchy sequence.

Lemma 5. *If there exists $\mu \in (0, 1)$ and $ð, v \in \mathfrak{Z}$ such that*

$$\begin{aligned}
\mathcal{M}(ð, v, k) &\geq \mathcal{M}\left(ð_{\mathfrak{y}-1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0,\\
\mathcal{N}(ð, v, k) &\leq \mathcal{N}\left(ð_{\mathfrak{y}-1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0,
\end{aligned} \tag{25}$$

holds, then $ð = v$.

Proof. Condition (25) implies that

$$\begin{aligned} \mathcal{M}(\eth, v, k) &\geq \mathcal{M}\left(\eth, v, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0 \\ &\geq \lim_{n\longrightarrow\infty} \mathcal{M}\left(\eth, v, \frac{k}{\mu^{\mathfrak{y}}}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0, \\ \mathcal{N}(\eth, v, k) &\leq \mathcal{N}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0 \\ &\leq \lim_{n\longrightarrow\infty} \mathcal{N}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0. \end{aligned} \tag{26}$$

By definition of b-IFMS, we have $\eth = v$. □

Theorem 1. *Let $(\mathfrak{Z}, \mathcal{M}, \mathcal{N}, \mathfrak{T}, \mathbb{S})$ be a b-IFMS, which is complete and $Q: \mathfrak{Z} \longrightarrow \mathfrak{Z}$. Suppose there exists $\mu \in (0, 1/b)$ such that*

$$\begin{aligned} \mathcal{M}(Q\eth, Qv, k) &\geq \mathcal{M}\left(\eth, v, \frac{k}{\mu}\right), \quad \eth, v \in \mathfrak{Z}, k > 0, \\ \mathcal{N}(Q\eth, Qv, k) &\geq \mathcal{N}\left(\eth, v, \frac{k}{\mu}\right), \quad \eth, v \in \mathfrak{Z}, k > 0, \end{aligned} \tag{27}$$

holds. Furthermore, there exists $\eth_0 \in \mathfrak{Z}$ and $v \in (0, 1)$ such that

$$\begin{aligned} \lim_{n\longrightarrow\infty} \mathfrak{T}_{i=\mathfrak{y}}^{\infty} \mathcal{M}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) &= 1, \\ \lim_{n\longrightarrow\infty} \mathbb{S}_{i=\mathfrak{y}}^{\infty} \mathcal{N}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) &= 0, \end{aligned} \tag{28}$$

holds. Then, Q has a unique fixed point in $\mathfrak{Z}$.

Proof. Let $\eth_0 \in \mathfrak{Z}$ and $\eth_{\mathfrak{y}+1} = Q\eth_{\mathfrak{y}}, \mathfrak{y} \in \mathbb{N}$. If we take $\eth = \eth_{\mathfrak{y}}$ and $y = \eth_{\mathfrak{y}-1}$ in (27), then we have

$$\begin{aligned} \mathcal{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, k\right) &\geq \mathcal{M}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0, \\ \mathcal{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}+1}, k\right) &\leq \mathcal{N}\left(\eth_{\mathfrak{y}-1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \quad \mathfrak{y} \in \mathbb{N}, k > 0. \end{aligned} \tag{29}$$

By Lemma 4, it implies $\{\eth_{\mathfrak{y}}\}$ is Cauchy sequence. Since $(\mathfrak{Z}, \mathcal{M}, \mathcal{N}, \mathfrak{T}, \mathbb{S})$ is complete, then $\lim_{n\longrightarrow\infty} \eth_{\mathfrak{y}} = \eth$, $\eth \in \mathfrak{Z}$. Therefore,

$$\begin{aligned} \lim_{n\longrightarrow\infty} \mathcal{M}\left(\eth, \eth_{\mathfrak{y}}, k\right) &= 1, \\ \lim_{n\longrightarrow\infty} \mathcal{N}\left((\eth, \eth_{\mathfrak{y}}, k\right) &= 0, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \mathcal{M}(Q\eth, \eth, k) &\geq \mathfrak{T}\left(\mathcal{M}\left(Q\eth, \eth_{\mathfrak{y}}, \frac{k}{2b}\right), \mathcal{M}\left(\eth_{\mathfrak{y}}, \eth, \frac{k}{2b}\right)\right) \\ &\geq \mathfrak{T}\left(\mathcal{M}\left(\eth, \eth_{\mathfrak{y}-1}, \frac{k}{2b\mu}\right), \mathcal{M}\left(\eth_{\mathfrak{y}}, \eth, \frac{k}{2b}\right)\right), \\ \mathcal{N}(Q\eth, \eth, k) &\geq \mathfrak{T}\left(\mathcal{N}\left(Q\eth, \eth_{\mathfrak{y}}, \frac{k}{2b}\right), \mathcal{N}\left(\eth_{\mathfrak{y}}, \eth, \frac{k}{2b}\right)\right) \\ &\geq \mathfrak{T}\left(\mathcal{N}\left(\eth, \eth_{\mathfrak{y}-1}, \frac{k}{2b\mu}\right), \mathcal{N}\left(\eth_{\mathfrak{y}}, \eth, \frac{k}{2b}\right)\right), \quad \forall k > 0. \end{aligned} \tag{31}$$

By (27), as $n \longrightarrow \infty$, we have

$$\begin{aligned} \mathcal{M}(Q\eth, \eth, k) &\geq \mathfrak{T}(1, 1) = 1, \\ \mathcal{N}(Q\eth, \eth, k) &\leq \mathbb{S}(0, 0) = 0. \end{aligned} \tag{32}$$

Let us assume that x and y are fixed points for Q. By (24), we obtain

$$\begin{aligned} \mathcal{M}(\eth, v, k) &= \mathcal{M}(Q\eth, Q\eth, k) \geq \mathcal{M}\left(\eth, v, \frac{k}{\mu}\right), \\ \mathcal{N}(\eth, v, k) &= \mathcal{M}(Q\eth, Q\eth, k) \leq \mathcal{M}\left(\eth, v, \frac{k}{\mu}\right). \end{aligned} \tag{33}$$

Using Lemma 5, we have $\eth = v$. □

Example 5. Assume $\mathfrak{Z} = [0, 1]$. Using Example 3 for $p = 2$ implies $(\mathfrak{Z}, \mathcal{M}, \mathcal{N}, \mathfrak{T}, \mathbb{S})$ is a b-IFMS with $b = 2$ and b-IFM. We define

$$\mathcal{M}(\eth, v, k) = e^{-(\eth-v)^2/k} \text{ and } \mathcal{N}(\eth, v, k) = 1 - e^{-(x-y)^2/k}. \tag{34}$$

Let $Q(v) = kv$, $k < \sqrt{2}/2, \eth \in \mathfrak{Z}$. Then,

$$\begin{aligned} \mathcal{M}(Q\eth, Qv, k) &= e^{-k^2(\eth-v)^2/k} \geq e^{-\mu(\eth-v)^2/k} = \mathcal{M}\left(\eth, v, \frac{k}{b}\right), \quad \eth, v \in \mathfrak{Z}, t > 0, \\ \mathcal{N}(Q\eth, Qv, k) &= 1 - \mathcal{M}(Q\eth, Qv, k) = 1 - e^{-k^2(\eth-v)^2/k} \leq 1 - e^{-\mu(\eth-v)^2/k} = \mathcal{M}\left(\eth, v, \frac{k}{b}\right), \quad \eth, v \in \mathfrak{Z}, t > 0, \end{aligned} \tag{35}$$

for $1/b > \mu > k^2$; hence, equation (27) will be satisfied, and we can say that Q possess a fixed point in $\mathfrak{Z}$ which is unique.

Theorem 2. *Let $(\mathfrak{Z}, \mathcal{M}, \mathcal{N}, \mathfrak{T}, \mathbb{S})$ be a b-IFMS and $Q: \mathfrak{Z} \longrightarrow \mathfrak{Z}$. Assume that $\mu \in (0, 1/b)$ such that*

$$\mathcal{M}(Q\eth, Qv, k) \geq \min\left\{\mathcal{M}\left(\eth, v, \frac{k}{\mu}\right), \mathcal{M}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathcal{M}\left(Qv, v, \frac{k}{\mu}\right)\right\},$$
$$\mathcal{N}(Q\eth, Qv, k) \geq \max\left\{\mathcal{N}\left(\eth, v, \frac{k}{\mu}\right), \mathcal{N}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathcal{N}\left(Qv, v, \frac{k}{\mu}\right)\right\}, \quad \forall \eth, v \in \mathfrak{Z}, k > 0, \tag{36}$$

and there exists $\eth_0 \in \mathfrak{Z}$, $v \in (0, 1)$ *such that*

$$\lim_{n\longrightarrow\infty} \mathfrak{T}_{i=\mathfrak{y}}^{\infty}\mathcal{M}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) = 1,$$
$$\lim_{n\longrightarrow\infty} \mathbb{S}_{i=\mathfrak{y}}^{\infty}\mathcal{N}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) = 0, \quad \forall k > 0. \tag{37}$$

Then, Q has a unique fixed point in $\mathfrak{Z}$.

Proof. Let $\eth_0 \in \mathfrak{Z}$, $\eth_{\mathfrak{y}+1} = Q\eth_{\mathfrak{y}}$, $\mathfrak{y} \in \mathbb{N}$. By (36), with $\eth = \eth_{\mathfrak{y}}$ and $v = \eth_{\mathfrak{y}-1}$, for every $\mathfrak{y} \in \mathbb{N}$, $k > 0$, we have

$$\mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \geq \min\left\{\mathcal{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \mathcal{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right)\right\}$$
$$\geq \min\left\{\mathcal{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right)\right\},$$
$$\mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \leq \min\left\{\mathcal{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right), \mathcal{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right)\right\}$$
$$\leq \min\left\{\mathcal{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right)\right\}. \tag{38}$$

If $\mathfrak{y} \in \mathbb{N}, k > 0$, then

$$\mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \geq \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right),$$
$$\mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \leq \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, \frac{k}{\mu}\right). \tag{39}$$

Lemma 5 implies $\eth_{\mathfrak{y}} = \eth_{\mathfrak{y}+1}$, $\mathfrak{y} \in \mathbb{N}$. Therefore,

$$\mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \geq \mathcal{M}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right),$$
$$\mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth_{\mathfrak{y}}, k\right) \leq \mathcal{N}\left(\eth_{\mathfrak{y}}, \eth_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \tag{40}$$

and by lemma, we have $\left\{\eth_{\mathfrak{y}}\right\}$ is a Cauchy sequence. Hence, $\lim_{n\longrightarrow\infty}\eth_{\mathfrak{y}} = v$, $\eth \in \mathfrak{Z}$:

$$\lim_{n\longrightarrow\infty} \mathcal{M}\left(\eth, \eth_{\mathfrak{y}}, k\right) = 1, \quad k > 0,$$
$$\lim_{n\longrightarrow\infty} \mathcal{N}\left(\eth, \eth_{\mathfrak{y}}, k\right) = 0, \quad k > 0. \tag{41}$$

Let us prove $\eth$ is a fixed point for Q. Let $\omega_1 \in (\mu b, 1)$ and $\omega_2 = 1 - \omega_1$. By (36), we have

$$\mathcal{M}(Q\eth, \eth, k) \geq \mathfrak{T}\left(\mathcal{M}\left(Q\eth, Q\eth_{\mathfrak{y}}, \frac{t\omega_1}{b}\right), \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b}\right)\right)$$
$$\geq \mathfrak{T}\left(\min\left\{\mathcal{M}\left(\eth, \eth_{\mathfrak{y}}, \frac{t\omega_1}{b\mu}\right), \mathcal{M}\left(\eth, Q\eth, \frac{t\omega_1}{b\mu}\right), \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b\mu}\right)\right\}, \mathcal{M}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b}\right)\right),$$
$$\mathcal{N}(Q\eth, \eth, k) \leq \mathbb{S}\left(\mathcal{N}\left(Q\eth, Q\eth_{\mathfrak{y}}, \frac{t\omega_1}{b}\right), \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b}\right)\right)$$
$$\leq \mathbb{S}\left(\max\left\{\mathcal{N}\left(\eth, \eth_{\mathfrak{y}}, \frac{t\omega_1}{b\mu}\right), \mathcal{N}\left(\eth, Q\eth, \frac{t\omega_1}{b\mu}\right), \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b\mu}\right)\right\}, \mathcal{N}\left(\eth_{\mathfrak{y}+1}, \eth, \frac{t\omega_2}{b}\right)\right). \tag{42}$$

Taking $n \longrightarrow \infty$ and using (41), we have

$$\begin{aligned}
\mathscr{M}(Q\eth,\eth,k) &\geq \mathfrak{T}\left(\min\left\{1,\mathscr{M}\left(\eth,Q\eth,\frac{t\omega_1}{b\mu}\right),1\right\},1\right)\\
&= \mathfrak{T}\left(\mathscr{M}\left(\eth,Q\eth,\frac{t\omega_1}{b\mu}\right),1\right)\\
&= \mathscr{M}\left(\eth,Q\eth,\frac{k}{\nu}\right), \quad k>0,\\
\mathscr{N}(Q\eth,\eth,k) &\leq \mathbb{S}\left(\max\left\{0,\mathscr{N}\left(\eth,Q\eth,\frac{t\omega_1}{b\mu}\right),1\right\},0\right)\\
&= \mathbb{S}\left(\mathscr{N}\left(\eth,Q\eth,\frac{t\omega_1}{b\mu}\right),0\right)\\
&= \mathscr{N}\left(\eth,Q\eth,\frac{k}{\nu}\right), \quad k>0,
\end{aligned} \tag{43}$$

where $\nu = b\mu/\omega_1 \in (0,1)$. Therefore, we obtain

$$\begin{aligned}
\mathscr{M}(Q\eth,\eth,k) &\geq \mathscr{M}\left(\eth,Q\eth,\frac{k}{\nu}\right), \quad k>0,\\
\mathscr{N}(Q\eth,\eth,k) &\geq \mathscr{N}\left(\eth,Q\eth,\frac{k}{\nu}\right), \quad k>0.
\end{aligned} \tag{44}$$

Lemma 5 implies $Q\eth = \eth$.

Suppose that $\eth$ and ν are fixed points for Q (i.e., $Q\eth = \eth \,\&Q\nu = \nu$). By (36), we obtain

$$\begin{aligned}
\mathscr{M}(Q\eth,Q\nu,k) &\geq \min\left\{\mathscr{M}\left(\eth,\nu,\frac{k}{\mu}\right),\mathscr{M}\left(\eth,Q\eth,\frac{k}{\mu}\right),\mathscr{M}\left(y,Q\nu,\frac{k}{\mu}\right)\right\}\\
&= \min\left\{\mathscr{M}\left(\eth,\nu,\frac{k}{\mu}\right),1,1\right\}\\
&= \mathscr{M}\left(\eth,\nu,\frac{k}{\mu}\right)\\
&= \mathscr{M}\left(Q\eth,Q\nu,\frac{k}{\mu}\right),\\
\mathscr{N}(Q\eth,Q\nu,k) &\leq \max\left\{\mathscr{N}\left(\eth,\nu,\frac{k}{\mu}\right),\mathscr{N}\left(\eth,Q\eth,\frac{k}{\mu}\right),\mathscr{N}\left(y,Q\nu,\frac{k}{\mu}\right)\right\}\\
&= \max\left\{\mathscr{N}\left(\eth,\nu,\frac{k}{\mu}\right),1,1\right\}\\
&= \mathscr{N}\left(\eth,\nu,\frac{k}{\mu}\right)\\
&= \mathscr{N}\left(Q\eth,Q\nu,\frac{k}{\mu}\right), \quad \text{for } k>0.
\end{aligned} \tag{45}$$

By Lemma 5, it follows that $Q\eth = Q\nu$, which implies that $\eth = \nu$. □

Example 6. Let $X = (0,2), \mathscr{M}(\eth,\nu,k) = e^{-(\eth-\nu)^2/k}$, and $\mathscr{N}(\eth,\nu,k) = 1 - e^{-(\eth-\nu)^2/k}$. Then, $(\eth,\mathscr{M},\mathscr{N},\mathfrak{T},\mathbb{S})$ is b-IFMS which is complete with $b = 2$. Let

$$Q(x) = \begin{cases} 2-\eth, & \text{if } \eth \in (0,1),\\ 1, & \text{if } \eth \in [1,2). \end{cases} \tag{46}$$

Case (i): if $\eth,\nu \in [1,2)$, then $\mathscr{M}(Q\eth,Q\nu,k) = 1$ and$\mathscr{N}(Q\eth,Q\nu,k) = 0$, $k>0$, and conditions (36) will be trivially satisfied.

Case (ii): if $\eth \in [1,2), \nu \in (0,1)$, then $\mu \in (1/4,1/2)$; we have

$$\begin{aligned}
\mathscr{M}(Q\eth,Q\nu,k) &= e^{-(1-\nu)^2/k} \geq e^{-4\mu(1-\nu)^2/k} = \mathscr{M}\left(Q\nu,\nu,\frac{k}{\lambda}\right), \quad k>0,\\
\mathscr{N}(Q\eth,Q\nu,k) &= 1-\mathscr{M}(Q\eth,Q\nu,k) = 1-e^{-(1-\nu)^2/k} \leq 1-e^{-4\mu(1-\nu)^2/k} = \mathscr{N}\left(Q\nu,\nu,\frac{k}{\lambda}\right), \quad k>0.
\end{aligned} \tag{47}$$

Case (iii): for $\mu \in (1/4,1/2)$, we have

$$\begin{aligned}
\mathscr{M}(Q\eth,Q\nu,k) &= \mathscr{M}\left(Q\nu,\nu,\frac{k}{\lambda}\right), k>0,\\
\mathscr{N}(Q\eth,Q\nu,k) &= \mathscr{N}\left(Q\nu,\nu,\frac{k}{\lambda}\right), k>0.
\end{aligned} \tag{48}$$

Case (iv): if $ð, v \in (0,1)$, then $\mu \in (1/4, 1/2)$; we have

$$\begin{aligned}
&\mathscr{M}(Qð, Qv, k) \geq e^{-(ð-v)^2/k} \geq e^{-(1-v)^2/k} \geq e^{-4\mu(1-v)^2/k} = \mathscr{M}\left(Qv, v, \frac{k}{\mu}\right),\\
&\mathscr{N}(Qð, Qv, k) = 1 - \mathscr{M}(Qð, Qv, k) = 1 - e^{-(1-v)^2/k} \leq 1 - e^{-4\mu(1-v)^2/k} = \mathscr{N}\left(Qv, y, \frac{k}{\mu}\right),\\
&\mathscr{M}(Qð, Qv, k) \geq \mathscr{M}\left(Qv, v, \frac{k}{\mu}\right),\\
&\mathscr{N}(Qð, Qv, k) \leq \mathscr{N}\left(Qv, v, \frac{k}{\mu}\right).
\end{aligned} \tag{49}$$

So, condition (36) is satisfied $\forall\, ð, v \in ð, k > 0$, and by Theorem 2, Q possess a unique fixed point.

Theorem 3. *Let $(ð, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ be a complete b-IFMS and $Q: X \longrightarrow X$. If*

$$\begin{aligned}
&\mathscr{M}(Qð, Qv, k) \geq \min\left\{\mathscr{M}\left(ð, v, \frac{k}{\mu}\right), \mathscr{M}\left(Qð, ð, \frac{k}{\mu}\right), \mathscr{M}\left(Qv, v, \frac{k}{\mu}\right), \mathscr{M}\left(Qð, v, \frac{2t}{\mu}\right), \mathscr{M}\left(ð, fv, \frac{k}{\mu}\right)\right\},\\
&\mathscr{N}(Qð, Qv, k) \geq \min\left\{\mathscr{N}\left(ð, v, \frac{k}{\mu}\right), \mathscr{N}\left(Qð, ð, \frac{k}{\mu}\right), \mathscr{N}\left(Qv, v, \frac{k}{\mu}\right), \mathscr{N}\left(Qð, v, \frac{2t}{\mu}\right), \mathscr{M}\left(ð, fv, \frac{k}{\mu}\right)\right\},
\end{aligned} \tag{50}$$

for $\mu(0, 1/b^3)$, then Q possess a unique fixed point in v.

Proof. Let $ð_0 \in \mathfrak{Z}$ and $ð_{\mathfrak{y}+1} = Qð_{\mathfrak{y}}, \mathfrak{y} \in \mathbb{N}$. By (37), with $ð = ð_{\mathfrak{y}}$ and $v = v_{\mathfrak{y}}$, using Definition 8(d) and $\mathfrak{T} = \mathfrak{T}_{\min}$, we have

$$\begin{aligned}
\mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\geq \min\left\{\begin{Bmatrix} \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \\ \min\left\{\mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right)\right\} \end{Bmatrix}\right\}\\
&\geq \min\left\{\mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right)\right\},\\
\mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\leq \max\left\{\begin{Bmatrix} \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{\mu}\right) \\ \max\left\{\mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}}, \frac{k}{\mu}\right)\right\} \end{Bmatrix}\right\}\\
&\leq \max\left\{\mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right)\right\}.
\end{aligned} \tag{51}$$

Proceeding as in proof of Theorem 2, Lemma 5, and Corollary 1, it follows that

$$\mathcal{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) \geq \mathcal{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right),$$
$$\mathcal{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) \leq \mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \tag{52}$$

and $\{ð_{\mathfrak{y}}\}$ is a Cauchy sequence. So, there exists $ð \in \mathfrak{Z}$ such that $\lim_{n\longrightarrow\infty} ð_{\mathfrak{y}} = ð$ and

$$\lim_{n\longrightarrow\infty} \mathcal{M}\left(ð, ð_{\mathfrak{y}}, k\right) = 1, \quad k > 0,$$
$$\lim_{n\longrightarrow\infty} \mathcal{N}\left(ð, ð_{\mathfrak{y}}, k\right) = 0, \quad k > 0. \tag{53}$$

Let $\omega_1 \in (\mu b^3, 1)$ and $\omega_2 = 1 - \omega_1$. By (50) and Definition 8(e), for $\mathfrak{T} = \mathfrak{T}_{\min}$, we have

$$\begin{aligned}
\mathcal{M}(Qð, ð, k) &\geq \min\left\{\mathcal{M}\left(Qð_{\mathfrak{y}}, Qð_{\mathfrak{y}-1}, \frac{t\omega_1}{b}\right), \mathcal{M}\left(Qð_{\mathfrak{y}}, ð, \frac{t\omega_2}{b}\right)\right\} \\
&\geq \min\left\{\begin{array}{l} \min\left\{\begin{array}{l} \mathcal{M}\left(ð, ð_{\mathfrak{y}}, \frac{t\omega_2}{b\mu}\right), \mathcal{M}\left(ð, Qð, \frac{t\omega_2}{b\mu}\right), \mathcal{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \frac{t\omega_2}{b\mu}\right), \\ \min\left\{\mathcal{M}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), \mathcal{M}\left(ð, ð_{\mathfrak{y}}, \frac{t\omega_2}{b^3\mu}\right), \mathcal{M}\left(ð, ð_{\mathfrak{y}+1}, \frac{t\omega_1}{b\mu}\right)\right\}, \end{array}\right. \\ \mathcal{M}\left(ð_{\mathfrak{y}+1}, ð, \frac{t\omega_2}{b}\right), \end{array}\right. \\
\mathcal{N}(Qð, ð, k) &\leq \max\left\{\mathcal{N}\left(Qð_{\mathfrak{y}}, Qð_{\mathfrak{y}-1}, \frac{t\omega_1}{b}\right), \mathcal{N}\left(Qð_{\mathfrak{y}}, ð, \frac{t\omega_2}{b}\right)\right\} \\
&\leq \max\left\{\begin{array}{l} \max\left\{\begin{array}{l} \mathcal{N}\left(ð, ð_{\mathfrak{y}}, \frac{t\omega_2}{b\mu}\right), \mathcal{N}\left(ð, Qð, \frac{t\omega_2}{b\mu}\right), \mathcal{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \frac{t\omega_2}{b\mu}\right), \\ \max\left\{\mathcal{N}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), \mathcal{N}\left(ð, ð_{\mathfrak{y}}, \frac{t\omega_2}{b^3\mu}\right), \mathcal{N}\left(ð, ð_{\mathfrak{y}+1}, \frac{t\omega_1}{b\mu}\right)\right\}, \end{array}\right. \\ \mathcal{N}\left(ð_{\mathfrak{y}+1}, ð, \frac{t\omega_2}{b}\right), \end{array}\right.
\end{aligned} \tag{54}$$

for all $\mathfrak{y} \in \mathbb{N}$ and $k > 0$. Taking $n \longrightarrow \infty$ and using (41), we obtain

$$\begin{aligned}
\mathcal{M}(Qð, ð, k) &\geq \min\left\{\min\left\{1, \mathcal{M}\left(ð, Qð, \frac{t\omega_1}{b\mu}\right), 1, \min\left\{\mathcal{M}\left(Qð, ð, \frac{t\omega_2}{b^3\mu}\right), 1\right\}\right\}, 1\right\} \\
&= \mathcal{M}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), k > 0, \\
\mathcal{N}(Qð, ð, k) &\leq \max\left\{\max\left\{1, \mathcal{N}\left(ð, Qð, \frac{t\omega_1}{b\mu}\right), 1, \max\left\{\mathcal{N}\left(Qð, ð, \frac{t\omega_2}{b^3\mu}\right), 1\right\}\right\}, 1\right\} \\
&= \mathcal{N}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), k > 0,
\end{aligned} \tag{55}$$

and by Lemma 5, with $\nu = b^2\mu/\omega_1 \in (0, 1)$, it follows that $Qð = ð$.

By condition (50), $ð = Qð$ and $\nu = Q\nu$; we have

$$
\begin{aligned}
\mathscr{M}(Q\eth, Qv, k) &\geq \min\left\{\begin{array}{l}\mathscr{M}\left(\eth, v, \frac{k}{\mu}\right), \mathscr{M}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathscr{M}\left(Qv, v, \frac{k}{\mu}\right), \\ \min\left\{\mathscr{M}\left(Q\eth, \eth, \frac{k}{b\mu}\right), \mathscr{M}\left(\eth, v, \frac{k}{b\mu}\right)\right\}, \mathscr{M}\left(\eth, Qv, \frac{k}{\mu}\right)\end{array}\right. \\
&= \min\left\{\mathscr{M}\left(\eth, v, \frac{k}{\mu}\right), 1, 1, \min\left\{, \mathscr{M}\left(\eth, v, \frac{k}{b\mu}\right)\right\}, \mathscr{M}\left(\eth, v, \frac{k}{\mu}\right)\right\} \\
&= \mathscr{M}\left(\eth, v, \frac{k}{b\mu}\right) \\
&= \mathscr{M}\left(Q\eth, Qv, \frac{k}{b\mu}\right), \\
\mathscr{N}(Q\eth, Qv, k) &\leq \max\left\{\begin{array}{l}\mathscr{N}\left(\eth, v, \frac{k}{\mu}\right), \mathscr{N}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathscr{N}\left(Qv, v, \frac{k}{\mu}\right), \\ \max\left\{\mathscr{N}\left(Q\eth, \eth, \frac{k}{b\mu}\right), \mathscr{N}\left(\eth, v, \frac{k}{b\mu}\right)\right\}, \mathscr{N}\left(\eth, Qv, \frac{k}{\mu}\right)\end{array}\right. \\
&= \max\left\{\mathscr{N}\left(\eth, v, \frac{k}{\mu}\right), 1, 1, \max\left\{, \mathscr{N}\left(\eth, v, \frac{k}{b\mu}\right)\right\}, \mathscr{N}\left(\eth, v, \frac{k}{\mu}\right)\right\} \\
&= \mathscr{N}\left(\eth, v, \frac{k}{b\mu}\right) \\
&= \mathscr{N}\left(Q\eth, Qv, \frac{k}{b\mu}\right),
\end{aligned} \tag{56}
$$

and by Lemma 3, it follows that $\eth = v$. □

Theorem 4. *Let* $(\eth, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S}), \mathfrak{T} \geq \mathfrak{T}_p$, *be a b-IFMS which is complete and* $Q: \mathfrak{Z} \longrightarrow \mathfrak{Z}$, *for some* $\mu \in (0, 1/b^2)$, *such that*

$$
\begin{aligned}
\mathscr{M}(Q\eth, Qv, k) &\geq \min\left\{\mathscr{M}\left(\eth, v, \frac{k}{\mu}\right), \mathscr{M}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathscr{M}\left(Qv, v, \frac{k}{\mu}\right), \sqrt{\mathscr{M}\left(Q\eth, v, \frac{2t}{\mu}\right)}, \mathscr{M}\left(\eth, fv, \frac{k}{\mu}\right)\right\}, \\
\mathscr{N}(Q\eth, Qv, k) &\leq \max\left\{\mathscr{N}\left(\eth, v, \frac{k}{\mu}\right), \mathscr{N}\left(Q\eth, \eth, \frac{k}{\mu}\right), \mathscr{N}\left(Qv, v, \frac{k}{\mu}\right), \sqrt{\mathscr{N}\left(Q\eth, v, \frac{2t}{\mu}\right)}, \mathscr{N}\left(\eth, fv, \frac{k}{\mu}\right)\right\},
\end{aligned} \tag{57}
$$

and there exists $\eth_0 \in \mathfrak{Z}$, $v \in (0, 1)$; *we have*

$$
\begin{aligned}
\lim_{n \longrightarrow \infty} \mathfrak{T}_{i=\mathfrak{y}}^{\infty} \mathscr{M}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) &= 1, \\
\lim_{n \longrightarrow \infty} \mathbb{S}_{i=\mathfrak{y}}^{\infty} \mathscr{N}\left(\eth_0, Q\eth_0, \frac{k}{v^i}\right) &= 0.
\end{aligned} \tag{58}
$$

Then, Q possess a unique fixed point in $\mathfrak{Z}$.

Proof. Let $ð_0 \in \mathfrak{Z}$ & $ð_{\mathfrak{y}+1} = Qð_{\mathfrak{y}}, \mathfrak{y} \in \mathbb{N}$. Taking $ð = ð_{\mathfrak{y}}$ and $v = v_{\mathfrak{y}}$ in condition (57), and $\mathfrak{T} = \mathfrak{T}_p$, we have

$$\begin{aligned} \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\geq \min \left\{ \begin{array}{l} \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \dfrac{k}{\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{\mu}\right), \\ \sqrt{\mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \dfrac{k}{b\mu}\right)}, \sqrt{\mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{b\mu}\right)}, \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}}, \dfrac{k}{b\mu}\right), \end{array} \right. \\ \mathscr{N}\left(\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) \right. &\leq \max \left\{ \begin{array}{l} \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \dfrac{k}{\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{\mu}\right), \\ \sqrt{\mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \dfrac{k}{b\mu}\right)}, \sqrt{\mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \dfrac{k}{b\mu}\right)}, \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}}, \dfrac{k}{b\mu}\right). \end{array} \right. \end{aligned} \tag{59}$$

Since $\mathscr{M}(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k)$ is b-nondecreasing and $\sqrt{a.b} \geq \min\{a, b\}$; $\mathscr{N}(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k)$ is b-nonincreasing and $\sqrt{a.b} \leq \max\{a, b\}$.

Therefore,

$$\begin{aligned} \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\geq \min\left\{\mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right)\right\}, \\ \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\leq \max\left\{\mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, \frac{k}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right)\right\}, \end{aligned} \tag{60}$$

for $\mathfrak{y} \in \mathbb{N}$, $k > 0$. By lemma,

$$\begin{aligned} \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\geq \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \\ \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð_{\mathfrak{y}}, k\right) &\geq \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}-1}, \frac{k}{b\mu}\right), \end{aligned} \tag{61}$$

and $\{ð_{\mathfrak{y}}\}$ is a Cauchy sequence. Since $(ð, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ is a complete b-IFMS, then $\lim_{n\longrightarrow\infty} ð_{\mathfrak{y}} = v$, $ð \in \mathfrak{Z}$ and

$$\begin{aligned} \lim_{n\longrightarrow\infty} \mathscr{M}\left(ð, ð_{\mathfrak{y}}, k\right) &= 1, \\ \lim_{n\longrightarrow\infty} \mathscr{N}\left(ð, ð_{\mathfrak{y}}, k\right) &= 0, \end{aligned} \tag{62}$$

$\omega_1 \in (b^3\mu, 1)$ and $\omega_1 \in (b^3\mu, 1)$. Also and for $\mathfrak{T} \geq \mathfrak{T}_p$, we have

$$\begin{aligned} \mathscr{M}(Qð, ð, k) &\geq \mathfrak{T}\left(\mathscr{M}\left(Qð, Qð_{\mathfrak{y}}, \frac{t\omega_1}{b}\right), \mathscr{M}\left(Qð_{\mathfrak{y}}, Qð, \frac{t\omega_2}{b}\right)\right) \\ &\geq \mathfrak{T}\left(\min\left\{ \begin{array}{l} \mathscr{M}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{M}\left(ð, Qð, \dfrac{t\omega_1}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right) \\ \sqrt{\mathscr{M}\left(Qð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)}, \sqrt{\mathscr{M}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)}, \mathscr{M}\left(ð, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right) \end{array} \right. , \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð, \frac{t\omega_2}{b}\right) \right) \end{aligned}$$

$$\geq \mathfrak{T}\left(\min\left\{\begin{array}{l}\mathscr{M}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{M}\left(ð, Qð, \dfrac{t\omega_1}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right), \\ \min\left\{\mathscr{M}\left(Qð, ð, \dfrac{t\omega_1}{b^3\mu}\right), \mathscr{M}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)\right\}, \\ \mathscr{M}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{M}\left(ð_{\mathfrak{y}+1}, ð, \dfrac{t\omega_2}{b}\right)\end{array}\right.\right),$$

$$\mathscr{N}(Qð, ð, k) \leq \mathbb{S}\left(\mathscr{N}\left(Qð, Qð_{\mathfrak{y}}, \frac{t\omega_1}{b}\right), \mathscr{N}\left(Qð_{\mathfrak{y}}, Qð, \frac{t\omega_2}{b}\right)\right)$$

$$\leq \mathbb{S}\left(\max\left\{\begin{array}{l}\mathscr{N}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{N}\left(ð, Qð, \dfrac{t\omega_1}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right) \\ \sqrt{\mathscr{N}\left(Qð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)}, \sqrt{\mathscr{N}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)}, \mathscr{N}\left(ð, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right)\end{array}\right., \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð, \frac{t\omega_2}{b}\right)\right)$$

$$\leq \mathbb{S}\left(\min\left\{\begin{array}{l}\mathscr{N}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{N}\left(ð, Qð, \dfrac{t\omega_1}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right), \\ \max\left\{\mathscr{N}\left(Qð, ð, \dfrac{t\omega_1}{b^3\mu}\right), \mathscr{N}\left(ð, ð_{\mathfrak{y}}, \dfrac{t\omega_1}{b^3\mu}\right)\right\}, \\ \mathscr{N}\left(ð_{\mathfrak{y}}, ð_{\mathfrak{y}+1}, \dfrac{t\omega_1}{b\mu}\right), \mathscr{N}\left(ð_{\mathfrak{y}+1}, ð, \dfrac{t\omega_2}{b}\right)\end{array}\right.\right), \tag{63}$$

for every $\mathfrak{y} \in \mathbb{N}$ and $k > 0$. Taking $n \longrightarrow \infty$ and using

$$\begin{aligned}\mathscr{M}(Qð, ð, k) &\geq \mathfrak{T}\left(\min\left\{1, \mathscr{M}\left(ð, Qð, \frac{t\omega_1}{b\mu}\right), 1, \min\left\{\mathscr{M}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), 1\right\}, 1\right\}, 1\right) \\ &\geq \mathscr{M}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), \quad k > 0, \\ \mathscr{N}(Qð, ð, k) &\leq \mathbb{S}\left(\max\left\{0, \mathscr{N}\left(ð, Qð, \frac{t\omega_1}{b\mu}\right), 0, \max\left\{\mathscr{N}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), 0\right\}, 0\right\}, 0\right) \\ &\leq \mathscr{N}\left(Qð, ð, \frac{t\omega_1}{b^3\mu}\right), \quad k > 0,\end{aligned} \tag{64}$$

and by Lemma 5 with $\nu = b^3\mu/\omega_1 \in (0, 1)$, we have $Qð = ð$ and

$$\mathscr{M}(Qð, Q\nu, k) \geq \mathfrak{T}\left\{\begin{array}{l}\mathscr{M}\left(ð, \nu, \dfrac{k}{\mu}\right), \mathscr{M}\left(Qð, ð, \dfrac{k}{\mu}\right), \mathscr{M}\left(Q\nu, \nu, \dfrac{k}{\mu}\right), \\ \sqrt{\mathscr{M}\left(Qð, ð, \dfrac{k}{b\mu}\right)}, \sqrt{\mathscr{M}\left(ð, \nu, \dfrac{k}{b\mu}\right)}, \mathscr{M}\left(ð, Q\nu, \dfrac{k}{\mu}\right)\end{array}\right.$$

$$
\begin{aligned}
&\geq \mathfrak{T}\left(\mathscr{M}\left(ð, \nu, \frac{k}{\mu}\right), 1, 1, \min\left\{1, \mathscr{M}\left(ð, \nu, \frac{k}{b\mu}\right)\right\}, \mathscr{M}\left(ð, \nu, \frac{k}{b\mu}\right)\right)\\
&= \mathscr{M}\left(Qð, Q\nu, \frac{k}{b\mu}\right), \quad k > 0,\\
\mathscr{N}(Qð, Q\nu, k) &\leq \mathbb{S}\begin{cases} \mathscr{N}\left(ð, \nu, \frac{k}{\mu}\right), \mathscr{N}\left(Qð, ð, \frac{k}{\mu}\right), \mathscr{N}\left(Q\nu, \nu, \frac{k}{\mu}\right),\\ \sqrt{\mathscr{M}\left(Qð, ð, \frac{k}{b\mu}\right)}, \sqrt{\mathscr{N}\left(ð, \nu, \frac{k}{b\mu}\right)}, \mathscr{N}\left(ð, Q\nu, \frac{k}{\mu}\right) \end{cases}\\
&\leq \mathbb{S}\left(\mathscr{N}\left(ð, \nu, \frac{k}{\mu}\right), 1, 1, \max\left\{1, \mathscr{N}\left(ð, \nu, \frac{k}{b\mu}\right)\right\}, \mathscr{N}\left(ð, \nu, \frac{k}{b\mu}\right)\right)\\
&= \mathscr{N}\left(Qð, Q\nu, \frac{k}{b\mu}\right), \quad k > 0.
\end{aligned} \tag{65}
$$

Thus, by Lemma 5, we have $ð = \nu$. □

Example 7. Let $\mathfrak{Z} = \{0, 1, 3\}$, $\mathscr{M}(ð, \nu, k) = e^{-(ð,\nu)^2/k}$, $\mathscr{N}(ð, \nu, k) = 1 - e^{-(ð-\nu)^2/k}$, or $\mathscr{N}(ð, \nu, k) = e^{-(ð-\nu)^2/t-(ð-\nu)^2}$, $\mathfrak{T} = \mathfrak{T}_p$, $\mathbb{S} = \mathbb{S}_p$. Then, $(\mathfrak{Z}, \mathscr{M}, \mathscr{N}, \mathfrak{T}, \mathbb{S})$ is a complete b-IFM with $b = 2$.

Define Q: $\mathfrak{Z} \longrightarrow \mathfrak{Z}$ as $f(0) = f(1) = 0, f(3) = 0$.

We observe that if $ð = \nu$ or $ð, \nu \in \{0, 1\}$, then $\mathscr{M}(Qð, Q\nu, k) = 1and\mathscr{N}(Qð, Q\nu, k) = 0$, $k > 0$, and (57) is satisfied.

If $x = 1$ and $y = 3$, then $\mu \in (1/9, 1/4)$ and

$$
\begin{aligned}
\mathscr{M}(fð, f\nu, k) &= e^{-1/k} \geq \min\left\{e^{-4\mu/k}, 1, e^{-9\mu/k}, e^{-\mu/k}, 1\right\},\\
\mathscr{N}(fð, f\nu, k) &= e^{-1/t-1} \leq \max\left\{e^{-4\mu/t-4\mu}, 1, e^{-9\mu/t-9\mu}, \sqrt{e^{-\mu/2t-4\mu}}, e^{-\mu/t-\mu}\right\}.
\end{aligned} \tag{66}
$$

In the same way, if we take $ð = 3$ and $\nu = 1$ and for $\mu \in (1/9, 1/4)$, condition (57) is satisfied, for all $ð, \nu \in \mathfrak{Z}, k > 0$, and Q possess a unique fixed in $\mathfrak{Z}$.

4. Applications of *b*-Intuitionistic Fuzzy Metric Space

The substance of giving satisfactory data to understudies to legitimate vocation decision cannot be overemphasized. This is a principle on the grounds that the various issues of absence of legitimate vocation control looked by understudies are of extraordinary result on their profession decision and effectiveness. Accordingly, it is practical that understudies be given adequate data on vocation assurance or decision to upgrade satisfactory arranging, arrangement, and capability. Among the vocation deciding elements such as scholarly execution, interest, and character make-up; the first-referenced is by all accounts' abrogating. We use b-IFMS as a device since it joins the enrollment degree (i.e., the marks of the questions answered by the student), the nonparticipation degree (i.e., the marks of the questions the student failed), and the dithering degree (which is the mark allocated to the questions the student do not attempt).

5. Conclusions

The authors introduced and discussed several notions of intuitionistic fuzzy b-metric space from different points of view with a suitable notion for the intuitionistic fuzzy metric of a given intuitionistic fuzzy b-metric space. In particular, we explore several properties of the intuitionistic fuzzy b-metric space. We have presented the b-IFMS and identified with fixed-point results about career determination which is of incredible importance since it gives precise and appropriate professional decision dependent on scholastic execution. We give a satisfactory condition for an arrangement to Cauchy in the b-IFMS. Accordingly, we work on the verifications of complex fixed-point hypotheses. Profession decision is a fragile independent direction issue since it has a reverberatory impact on productivity, and capability is appropriately dealt with. In the proposed application, we utilized standardized Euclidean distance to compute the distance of every understudy from each career regarding the subjects, to acquire outcomes. We use b-IFMS as an instrument since it fuses the membership degree, the nonmembership degree, and the hesitation degree.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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