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Research Article

Univalent Functions by Means of Chebyshev Polynomials

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The primary motivation of the paper is to define a new class $Ch_\delta(\alpha, \beta, \gamma)$ which consists of univalent functions associated with Chebyshev polynomials. For this class, we determine the coefficient bound and convolution preserving property. Furthermore, by using subordination structure, two new subclasses of $Ch_\delta(\alpha, \beta, \gamma)$ are introduced and denoted by $M(\lambda_1, \lambda_2, s)$ and $N(\lambda_1, \lambda_2, s)$, respectively. For these subclasses, we obtain coefficient estimate, extreme points, integral representation, convexity, geometric interpretation, and inclusion results. Moreover, we prove that, under some restrictions on parameters, $Ch_\delta(\alpha, \beta, \gamma) = N(\lambda_1, \lambda_2, s)$.

1. Introduction

Let $\mathscr{A}$ be the class of analytic univalent functions in the open unit disk:

$$\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}, \tag{1}$$

with Taylor expansion series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Also, denote by $\mathscr{S}$ the class of univalent functions which are normalized by $f(0) = f'(0) - 1 = 0$, see [1, 2]. Furthermore, suppose that $\mathscr{N}$ be the subclass of $\mathscr{A}$ consisting of functions with negative coefficients of the type:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{3}$$

The significance of Chebyshev polynomials in numerical analysis is very important in both practical and theoretical points of view. There are four kinds of such polynomials. Many researchers consider orthogonal polynomials of Chebyshev and obtain many interest results.

The Chebyshev polynomials of the first and second kinds are well known and introduced by

$$T_n(t) = \cos\theta \text{ and } U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, (-1 < t < 1), \tag{4}$$

where $t = \cos\theta$ and n is the degree of polynomial. For more details, one may refer to [1–6]. The polynomials in (4) are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t), \quad T_n(t) = U_n(t) - tU_{n-1}(t), \tag{5}$$

$$2T_n(t) = U_n(t) - U_{n-2}(t). \tag{6}$$

We note that if $t = \cos\theta$, where $\theta \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1 - 2\cos\theta z + z^2} = 1 + \sum_{k=2}^{\infty} \frac{\sin(k+1)\theta}{\sin\theta} z^k, \quad (z \in \mathbb{D}). \tag{7}$$

Also, we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots, \quad (z \in \mathbb{D}, \ -1 < t < 1), \tag{8}$$

where

$$U_{n-1}(t) = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}, \quad (n \in \mathbb{N}), \tag{9}$$

are the Chebyshev polynomials of the second kind, see [7–9].

The Hadamard product (convolution) for functions

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{10}$$

is denoted by $f * g$ and defined as follows:

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \mathbb{D}). \tag{11}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$ is given by

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2}, \quad (z \in \mathbb{D}), \tag{12}$$

see [10].

Now, we consider the functions:

$$H_1(z) = 1 + (2\cos\theta + 1)z - H(z,t), \quad H_2(z) = 1 + (1+\cos\theta)z - \frac{1-tz}{1-2tz+z^2}, \tag{13}$$

$$V(z) = (H_1 * H_1) * (H_2 * H_2) * f(z), \tag{14}$$

where $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{N}$ and "$*$" denotes the convolution. By a simple calculation, we conclude that $V(z) \in \mathcal{N}$ and in the form

$$V(z) = z - \sum_{k=2}^{\infty} T_k^2(t) \frac{\sin^2(k+1)\theta}{\sin^2\theta} a_k z^k, \tag{15}$$

where $\theta \in (-\pi/3, \pi/3)$ and $t = \cos\theta$.

Let $Ch_\delta(\alpha,\beta,\gamma)$ denote the subclass of $\mathcal{N}$ consisting of functions of form (15) satisfying the condition:

$$\left| \frac{V'(z) + zV''(z) - 1}{2\gamma V'(z) - \alpha(1+\beta)\gamma} \right| < \delta, \tag{16}$$

where $0 < \alpha \le 1$, $0 < \beta \le 1$, $0 \le \gamma \le 1$, $0 \le \delta < 1$, and $V(z)$ is given by (15), see [11].

2. Main Results

In this section, we introduce a sharp coefficient bound for $V(z) \in Ch_\delta(\alpha,\beta,\gamma)$. Also, convolution preserving property under parameters α and β is proved.

Theorem 1. *$V(z) \in Ch_\delta(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{k(k+2\gamma\delta) T_k^2(t) \sin^2(k+1)\theta}{\sin^2\theta} a_k \le \delta\gamma(2 - \alpha(1+\beta)). \tag{17}$$

Proof. Let inequality (17) hold true, and suppose $z \in \partial\mathbb{D} = \{z \in \mathbb{D}: |z| = 1\}$. Then, we obtain

$$\begin{aligned} &|V'(z) + zV''(z) - 1| - \delta|2\gamma V'(z) - \alpha(1+\beta)\gamma| \\ &= \left| -\sum_{k=2}^{\infty} \left[\frac{kT_k(t)(\sin(k+1)\theta)}{\sin\theta} \right]^2 a_k z^k \right| \\ &\quad - \delta \left| 2\gamma - \sum_{k=2}^{\infty} \frac{2\gamma k T_k^2(t) \sin^2(k+1)\theta}{\sin^2\theta} a_k z^{k-1} - \alpha(1+\beta)\gamma \right| \\ &= \sum_{k=2}^{\infty} \frac{k(k+2\gamma\delta) T_k^2(t) \sin^2(k+1)\theta}{\sin^2\theta} - \delta\gamma(2 - \alpha(1+\beta)) \le 0. \end{aligned} \tag{18}$$

Hence, by maximum modulus theorem, we conclude that $V(z) \in Ch_\delta(\alpha,\beta,\gamma)$.

Conversely, let $V(z)$, defined by (15), be in the class $Ch_\delta(\alpha,\beta,\gamma)$, so condition (16) yields

$$\left| \frac{V'(z) + 2V''(z) - 1}{2\gamma V'(z) - \alpha(1+\beta)\gamma} \right| = \left| \frac{\sum_{k=2}^{\infty} k^2 T_k^2(t) \sin^2(k+1)\theta / \sin^2\theta a_k z^{k-1}}{2\gamma - \sum_{k=2}^{\infty} 2\gamma k T_k^2(t) \sin^2(k+1)\theta / \sin^2\theta a_k z^{k-1} - \alpha(1+\beta)\gamma} \right| < \delta. \tag{19}$$

Since, for any z, $|\mathrm{Re} z| < |z|$, then

$$\mathrm{Re}\left\{ \frac{\sum_{k=2}^{\infty} T_k^2(t) \sin^2(k+1)\theta / \sin^2\theta [k^2] a_k z^{k-1}}{\gamma(2 - \alpha(1+\beta)) - \sum_{k=2}^{\infty} T_k^2(t) \sin^2(k+1)\theta / \sin^2\theta [2\gamma k] a_k z^{k-1}} \right\} < \delta. \tag{20}$$

By letting $z \longrightarrow 1$ through real values, we obtain

$$\sum_{k=2}^{\infty}\frac{T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta}\left[k^2\right]a_k\leq\delta\gamma(2-\alpha(1+\beta))-\sum_{k=2}^{\infty}\frac{T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta}[\alpha\gamma\delta k]a_k, \tag{21}$$

and this completes the proof. □

Remark 1. We note that the function,

$$W(z)=z-\frac{\left(\sin^2\theta\right)\gamma\delta(2-\alpha(1+\beta))}{4(1+\gamma\delta)\cos^2 2\theta\,\sin^2 3\theta}z^2, \tag{22}$$

shows that inequality (17) is sharp.

Theorem 2. *If $f(z)=z-\sum_{k=2}^{\infty}a_kz^k$ and $g(z)=z-\sum_{k=2}^{\infty}b_kz^k$ are in the class $Ch_\delta(\alpha,\beta,\gamma)$, then*

(i) $(f*g)(z)$ belongs to $Ch_\delta(\alpha,\beta,\gamma)[\alpha^*]$, where

$$\alpha^*\leq\frac{2}{1+\beta}-\left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2\frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}. \tag{23}$$

(ii) $(f*g)(z)$ belongs to $Ch_\delta(\alpha,\beta^*,\gamma)$, where

$$\beta^*\leq\frac{2}{\alpha}-\left(1+\left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2\frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}\right). \tag{24}$$

Proof. (i) It is sufficient to show that

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))[\alpha^*]}a_kb_k\leq 1. \tag{25}$$

By using the Cauchy–Schwarz inequality from (31), we obtain

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))[\alpha^*]}\sqrt{a_kb_k}\leq 1. \tag{26}$$

Hence, we find the largest α^* such that

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha^*(1+\beta))}a_kb_k\leq\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))}\sqrt{a_kb_k}\leq 1. \tag{27}$$

This inequality holds if

$$\frac{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))}{k(k+2\gamma\delta)T_k^2(t)\sin^2(k+1)\theta}\leq\frac{2-\alpha^*(1+\beta)}{2-\alpha(1+\beta)}, \tag{28}$$

or equivalently

$$\alpha^*\leq\frac{2}{1+\beta}+\left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2\frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}. \tag{29}$$

(ii) By using the same techniques as in the part (i), we can easily prove the part (ii), so the proof is complete. □

3. Subclass of $Ch_\delta(\alpha,\beta,\gamma)$ and Their Geometric Properties

In this section, we introduce two new subclasses of $Ch_\delta(\alpha,\beta,\gamma)$ and conclude their geometric properties.

For analytic functions $f(z)$ and $F(z)$ in $\mathbb{D}$, we say f is subordinate to F, written $f\prec F$, if there exists a function w analytic in $\mathbb{D}$, with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$, see [3, 12]. If F is univalent, then

$$f\prec F\Leftrightarrow f(0)=F(0),\quad f(\mathbb{D})\subset F(\mathbb{D}). \tag{30}$$

Let $M(\lambda_1,\lambda_2,s)$ consist of all analytic functions $g(z)\in\mathbb{D}$ for which $g(0)=1$ and

$$g(z)\prec\frac{1+(\lambda_2+(\lambda_1-\lambda_2)(1-s))z}{1+\lambda_2z}, \tag{31}$$

where $-1\leq\lambda_1<\lambda_2\leq 1$, $0<\lambda_2\leq 1$, and $0\leq s<1$. Let $N(\lambda_1,\lambda_2,s)$ denote the class of all functions $V(z)\in Ch_\delta(\alpha,\beta,\gamma)$ for which

$$\frac{zV'(z)}{V(z)}\in M(\lambda_1,\lambda_2,s), \tag{32}$$

where $V(z)$ is given by (15).

Theorem 3. *$V(z)\in N(\lambda_1,\lambda_2,s)$ if and only if*

$$\sum_{k=2}^{\infty}\left(1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2a_k<1. \tag{33}$$

Proof. Let $V(z)\in N(\lambda_1,\lambda_2,s)$; then, by (16), (31), and (32), we have

$$\left|\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[(k-1)]a_k z^k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]a_k z^k}\right|<1, \tag{34}$$

which implies that

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k z^k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]a_k z^k}\right\}<1. \tag{35}$$

Now, we choose the values of z on the real axis, and letting $z \longrightarrow 1^-$, we get the required result.

Conversely, assume that condition (33) holds true. We must show that $V(z) \in N(\lambda_1,\lambda_2,s)$ or equivalently

$$|Y| = \left|\frac{V(z)-zV'(z)}{\lambda_2 zV'(z)-(\lambda_2+(\lambda_2-\lambda_1)(1-s))V(s)}\right|<1.$$

However, we have (36)

$$|Y| = \left|\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]}\right|.$$

By using (33), we get $|Y|<1$, so the proof is complete. □

Corollary 1. *Let $V \in N(\lambda_1,\lambda_2,s)$; then,*

$$a_k < \frac{(\lambda_2-\lambda_1)(1-s)(T_k(t)\sin(k+1)\theta/\sin\theta)^2}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}. \tag{37}$$

Theorem 4. *Let $\lambda_2 \neq 1$, $V(z) \in N(\lambda_1,\lambda_2,s)$, and*

$$\frac{zV'(z)}{V(z)} = a+ib = X. \tag{38}$$

Then, the values of X lie in the circle.

Proof. By (31) and (32), we have

$$X = a+ib = \frac{1+(\lambda_2+(\lambda_2-\lambda_1)(1-s))W(z)}{1+\lambda_2 W(z)}, \quad (|W(z)|<1). \tag{39}$$

Then,

$$\begin{aligned}(a+ib)(1+\lambda_2 W(z)) &= 1+(\lambda_2+(\lambda_2-\lambda_1)(1-s))W(z)\\ \text{or } a-1+ib &= [\lambda_2+(\lambda_2-\lambda_1)(1-s)-a\lambda_2-ib\lambda_2]W(z).\end{aligned} \tag{40}$$

After a simple calculation, we obtain

$$\left[a-\frac{1-\lambda_2(\lambda_2+(\lambda_2-\lambda_1)(1-s))}{1-\lambda_2^2}\right]^2+b^2<\left[\frac{(\lambda_2-\lambda_1)(1-s)}{1-\lambda_2^2}\right]^2. \tag{41}$$

Hence, the value of X lies in the circle with center at

$$\left(\frac{1-\lambda_2(\lambda_2+(\lambda_2-\lambda_1)(1-s))}{1-\lambda_2^2}, 0\right) \tag{42}$$

and radius $(\lambda_2-\lambda_1)(1-s)/1-\lambda_2^2$. □

Theorem 5. *If*

$$\frac{\lambda_2+1}{(\lambda_2-\lambda_1)(1-s)} \leq \frac{k^2+2\gamma\delta(k-1)+\gamma\alpha\delta(1+\beta)}{\delta\gamma(2-\alpha(1+\beta))}, \tag{43}$$

then $Ch_\delta(\alpha,\beta,\gamma) = N(\lambda_1,\lambda_2,s)$.

Proof. By equation (32), we have

$$N(\lambda_1,\lambda_2,s) \subseteq Ch_\delta(\alpha,\beta,\gamma). \tag{44}$$

Now, assume that $V \in Ch_\delta(\alpha,\beta,\gamma)$; then, by Theorem 1, we have

$$\sum_{k=2}^{\infty} k(k+2\gamma\delta)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{45}$$

By Theorem 3, it is enough to show that (33) holds true, which is possible when

$$\left[1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}\right] \le \frac{k(k+2\gamma\delta)}{\delta\gamma(2-\alpha(1+\beta))}, \tag{46}$$

or equivalently

$$\frac{(k-1))(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)} \le \frac{k^2+2\gamma\delta(k-1)+\delta\gamma\alpha(1+\beta)}{\delta\gamma(2-\alpha(1+\beta))}. \tag{47}$$

Since k starts from 2, then $k-1\ge 1$, and hence, from the last inequality, we obtain the required result. □

In the next theorems, we prove the inclusion property and convex combination concept. Also, extreme points and integral representation are introduced.

Theorem 6. *Let $0 \le s_2 < s_1 < 1$; then,*

$$N(\lambda_1,\lambda_2,s)[s_1] \subset N(\lambda_1,\lambda_2,s)[s_2]. \tag{48}$$

Proof. Suppose $f \in N(\lambda_1,\lambda_2,s)[s_1]$; then, by Theorem 3, we have

$$\sum_{k=2}^{\infty}\left(1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)[s_1]}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{49}$$

We have to prove

$$\sum_{k=2}^{\infty}\left(1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s))}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{50}$$

However, the last inequality holds true if

$$1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)} \le 1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}, \tag{51}$$

and this inequality by hypothesis $(s_2 < s_1)$ definitely holds true, so the proof is complete. □

Theorem 7. *$N(\lambda_1,\lambda_2,s)$ is a convex set.*

Proof. We have to prove that if

$$V_j(z) = z - \sum_{k=2}^{\infty}\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_{k,j} z^k, \quad (j=1,2,\dots,m), \tag{52}$$

is in the class $N(\lambda_1,\lambda_2,s)$, then the function,

$$L(z) = \sum_{j=1}^{m} d_j V_j(z), \tag{53}$$

is also in $N(\lambda_1,\lambda_2,s)$, where $\sum_{j=1}^{m} d_j = 1$. However, we have

$$L(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 d_j a_{k,j}\right) z^k. \tag{54}$$

We have to prove that if $f_j(z)$ $(j=1,2,\dots,m)$ is in the class $N(\lambda_1,\lambda_2,s)$, then the function $L(z)=\sum_{j=1}^{m} d_j f_j(z)$ is also in $N(\lambda_1,\lambda_2,s)$, where $\sum_{j=1}^{m} d_j = 1$. We have

$$L(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} d_j a_{k,j}\right) z^k. \tag{55}$$

Since, by Theorem 3,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(1+\frac{(\lambda_2+1)(k-1)}{ls}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2\left(\sum_{j=1}^{m} d_j a_{k,j}\right) \\
&= \sum_{j=1}^{m}\left[\sum_{k=2}^{\infty}\left(1+\frac{(\lambda_2+1)(k-1)}{(\lambda_2-\lambda_1)(1-s)}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_{k,j}\right] d_j \\
&< \sum_{j=1}^{m} d_j = 1
\end{aligned} \tag{56}$$

so $L \in N(\lambda_1,\lambda_2,s)$ and the proof is complete. □

Theorem 8. *The function $V_1(z) = z$ and*

$$V_k(z) = z - \left(\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\right)\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 a_k z^k, \quad (k\geq 2), \tag{57}$$

are the extreme points of $N(\lambda_1,\lambda_2,s)$.

Proof. We have to prove that $L\in N(\lambda_1,\lambda_2,s)$ if and only if

$$L(z)=\sum_{k=2}^{\infty}[1]d_kV_k(z), \tag{58}$$

where $d_k\geq 0$ $(k\geq 1)$ and $\sum_{k=2}^{\infty}[1]d_k=1$. □

Proof. Let $L\in N(\lambda_1,\lambda_2,s)$. If we set

$$d_k=\frac{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}{(\lambda_2-\lambda_1)(1-s)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 a_k, \quad (k\geq 2), \tag{59}$$

we get $d_k\geq 0$, and if we put $d_1=1-\sum_{k=2}^{\infty}d_k$, then we obtain

$$\begin{aligned} L(z) &= z-\sum_{k=2}^{\infty}\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_kz^k \\ &= z-\sum_{k=2}^{\infty}d_k\left(z-V_k(z)\right) \\ &= \sum_{k=2}^{\infty}d_kV_k(z). \end{aligned} \tag{60}$$

Conversely, suppose

$$L(z)=\sum_{k=2}^{\infty}[1]d_kV_k(z). \tag{61}$$

Then, we have

$$\begin{aligned} L(z) &= d_1V_1(z)+\sum_{k=2}^{\infty}d_kV_k(z) \\ & d_1z+\sum_{k=2}^{\infty}\left[z-\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_kz^k\right] \\ &= z-\sum_{k=2}^{\infty}\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_kz^k. \end{aligned} \tag{62}$$

Since

$$\sum_{k=2}^{\infty} d_k \left(1 + \frac{(\lambda_2 + 1)(k-1)}{(\lambda_2 - \lambda_1)(1-s)}\right)\left(\frac{(\lambda_2 - \lambda_1)(1-s)}{(\lambda_2 - \lambda_1)(1-s) + (\lambda_2 + 1)(k-1)}\right) \times \times \left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 \left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 = \sum_{k=2}^{\infty} d_k = 1 - d_1 < 1, \tag{63}$$

therefore, by Theorem 3, we conclude the result. □

Theorem 9. *Let $f \in N(\lambda_1, \lambda_2, s)$; then,*

$$V(z) = \exp\left(\int_0^z \frac{1 - (\lambda_2 + (\lambda_2 - \lambda_1)(1-s))W(t)}{t(1 - \lambda_2 W(t))} dt\right), \tag{64}$$

where $|W(z)| < 1$.

Proof. By letting $U(z) = zV'/V$, since $f \in N(\lambda_1, \lambda_2, s)$, so

$$U(z) \prec \frac{1 + (\lambda_2 + (\lambda_1 - \lambda_2)(1-s))z}{1 + \lambda_2 z}, \tag{65}$$

or equivalently

$$\left|\frac{U(z) - 1}{U(z)\lambda_2 - (\lambda_2 + (\lambda_1 - \lambda_2)(1-s))}\right| < 1. \tag{66}$$

Therefore,

$$\frac{U(z) - 1}{U(z)\lambda_2 - (\lambda_2 + (\lambda_1 - \lambda_2)(1-s))} = W(z), \quad (|w(z)| < 1). \tag{67}$$

Hence, we can write

$$\frac{V'(z)}{V(z)} = \frac{1 - (\lambda_2 + (\lambda_1 - \lambda_2)(1-s))W(z)}{z(1 - \lambda_2 W(z))}. \tag{68}$$

After integration, we get the required result. □

4. Conclusion

Univalent functions have always been the main interests of many researchers in geometric function theory. Many studies recently related to Chebyshev polynomials revolved around classes of analytic normalized univalent functions. In this particular work, the geometric properties are obtained for functions in more general class denoted by $Ch_\delta(\alpha, \beta, \gamma)$ using the Chebyshev polynomials associated with the convolution structure. Some other geometric results are introduced for the subclasses of $Ch_\delta(\alpha, \beta, \gamma)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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