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Research Article

New Developments on Ostrowski Type Inequalities via q-Fractional Integrals Involving s-Convex Functions

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In the present paper, q-fractional integral operators are used to construct quantum analogue of Ostrowski type inequalities for the class of s-convex functions. The limiting cases include the nonfractional existing cases from literature. Specially, Ostrowski type inequalities for q-integrals and Ostrowski type inequalities for convex functions are deduced.

1. Introduction

In mid of twentieth century, Jackson (1910) has begun a symmetric investigation on q-integrals. The subject of quantum analysis, depending upon q-integrals, has different applications in different branches of mathematics and material sciences like number hypothesis, combinatorics, symmetrical polynomials, essential hypermathematical capacities, quantum hypothesis, mechanics, and in the hypothesis of relativity. The perusers are suggested to Set [1], Gauchman [2], and Kac and Cheung [3] for q-analogues of fractional calculus.

In numerous pragmatic issues, convexity hypothesis has stayed as a significant device in formation of vital imbalances. In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities such as Ostrowski's inequalities are very useful for this purpose. Ostrowski type inequalities are well known to study the upper bounds for approximation of the integral average by the value of function and definition of s-convex function. Some new Ostrowski type inequalities for Riemann-Liouville fractional integral are established. Fractional calculus has been a well-known topic since it was initiated in the seventeenth century and studied by many great mathematicians of the time. Some classical inequalities including Ostrowski's inequality are examples of it.

2. Preliminary Results

In 1938, Ostrowski [4] established the following well-known and useful integral inequality:

Theorem 1. *Suppose $\psi : I \longrightarrow \mathbb{R}$ is the function differentiable in open interval of I, where $I \subseteq \mathbb{R}$ and let $\xi, \eta \in I^{\circ}$ with $\xi < \eta$. If $|\psi'(\theta)| \leq M$ for all $\theta \in [\xi, \eta]$, then the following inequality holds*

$$\left|\psi(\theta) - \frac{1}{\eta - \xi}\int_{\xi}^{\eta}\psi(u)du\right| \leq M(\eta - \xi)\left[\frac{1}{4} + \frac{(\theta - ((\xi + \eta)/2))^2}{(\eta - \xi)^2}\right], \tag{1}$$

for all $\theta \in [\xi, \eta]$. The least value of constant on R.H.S of (1) is 1/4.

Inequality (1) gives an approximate upper bound for the deviation of integral arithmetic mean $1/(\eta - \xi)\int_{\xi}^{\eta}\psi(\theta)d\theta$ to

the function $\psi(\cdot)$ at point $\theta \in [\xi, \eta]$. In recent years, this inequality is studied extensively by different researchers, and its different variants can be seen in number of research papers including [5–11]. Recently, in [12, 13], Ostrowski type inequalities are studied for q-integrals.

The following notion of s-convex function in the second sense is from [14]:

A mapping $\psi : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense if

$$\psi(u\xi + (1-u)\eta) \leq u^s\psi(\xi) + (1-u)^s\psi(\eta), \tag{2}$$

for all $\xi, \eta \in [0,\infty)$, $u \in [0, 1]$, and some static $s \in (0, 1]$.

The following Lemma is established by Alomari et al. (see [8]).

Lemma 2. *If $\psi' \in L[\xi, \eta]$, then we have the equality*

$$\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du = \frac{(\theta-\xi)^2}{\eta-\xi}\int_0^1 uf'(u\theta + (1-u)\xi)du - \frac{(\eta-\theta)^2}{\eta-\xi}\int_0^1 uf'(u\theta + (1-u)\eta)du, \tag{3}$$

for each $\theta \in [\xi, \eta]$.

By using Lemma 2, Alomari et al. in [8] proved the following results of Ostrowski type inequalities:

Theorem 3. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|$ in term of second sense is s-convex on $[\xi, \eta]$, for unique $s \in (0, 1]$ and $|\psi'(\theta)| \leq M, \theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \leq \frac{M}{\eta-\xi}\left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{s+1}\right] \tag{4}$$

holds, for each $\theta \in [\xi, \eta]$.

Theorem 4. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is a s-convex in second sense in $[\xi, \eta]$ for unique $s \in (0, 1], m > 1, n = m/(m-1)$ and $|\psi'(\theta)| \leq M, \theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \leq \frac{M}{(1+n)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m} \cdot \left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{\eta-\xi}\right] \tag{5}$$

holds, for each $\theta \in [\xi, \eta]$.

Theorem 5. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi, \eta]$ for static $s \in (0, 1], m \geq 1$ and $|\psi'(\theta)| \leq M, \theta \in [\xi, \eta]$, then*

$$\left|\psi(\theta) - \frac{1}{\eta-\xi}\int_\xi^\eta \psi(u)du\right| \leq M\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\xi)^2 + (\eta-\theta)^2}{2(\eta-\xi)}\right] \tag{6}$$

holds, for each $\theta \in [\xi, \eta]$.

Further some existing results on s-convex functions can be seen in [11], and some results involving fractional operators can be found in [15–21]. In case of fractional integrals, see the following lemma from [1].

Lemma 6. *If $\psi' \in L[\xi, \eta]$, then for all $\theta \in [\xi, \eta]$ and $\beta > 0$*

$$\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J_{\theta-}^\beta\psi(\xi) + J_{\theta+}^\beta\psi(\eta)\right] = \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\psi'(u\theta + (1-u)\xi)du - \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\psi'(u\theta + (1-u)\eta)du \tag{7}$$

holds, where $\Gamma(\beta) = \int_0^\infty e^{-u}u^{\beta-1}du$ is Euler Gamma function.

By using Lemma 6, Set in [1] proved the following:

Theorem 7. *For $\psi' \in L[\xi, \eta]$. If $|\psi'(\theta)|$ is s-convex in second sense on $[\xi, \eta]$ for fix $s \in (0, 1]$ and $|\psi'(\theta)| \leq M, \theta \in [\xi, \eta]$, then the following fractional integrals inequality, for $\beta > 0$*

$$\left|\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J_{\theta-}^\beta\psi(\xi) + J_{\theta+}^\beta\psi(\eta)\right]\right| \leq \frac{M}{\eta-\xi}\left(1 + \frac{\Gamma(\beta+1)\Gamma(s+1)}{\Gamma(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1} + (\eta-\theta)^{\beta+1}}{\beta+s+1}\right] \tag{8}$$

holds.

Theorem 8. *Suppose $\psi' \in L[\xi, \eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi, \eta]$ for some fixed $s \in (0, 1]$, $n, m > 1$ and $|\psi'(\theta)| \leq M, \theta \in [\xi, \eta]$, then*

$$\left|\left(\frac{(\theta-\xi)^\beta + (\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta) - \frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J_{\theta-}^\beta\psi(\xi) + J_{\theta+}^\beta\psi(\eta)\right]\right| \leq \frac{M}{(1+n\beta)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\xi)^{\beta+1} + (\eta-\theta)^{\beta+1}}{\eta-\xi}\right] \tag{9}$$

holds, where $(1/n) + (1/m) = 1, \beta > 0$.

Theorem 9. *Suppose $\psi' \in L[\xi,\eta]$, and assume that $|\psi'|^m$ is s-convex in second sense on $[\xi,\eta]$ for some fix $s \in (0,1]$, $m \geq 1, |\psi'(\theta)| \leq M$, and $\theta \in [\xi,\eta]$. Then*

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta)-\frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi)+J^\beta_{\theta+}\psi(\eta)\right]\right| \leq M\left(\frac{1}{1+\beta}\right)^{1-(1/m)}\left(\frac{1}{\beta+s+1}\right)^{1/m} \times\left(1+\frac{\Gamma(\beta+1)\Gamma(s+1)}{\Gamma(\beta+s+1)}\right)^{1/m}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right] \tag{10}$$

holds for each $\beta > 0$.

Theorem 10. *Suppose $\psi' \in L[\xi,\eta]$. If $|\psi'|^m$ is s-convex in second sense on $[\xi,\eta]$ for some fixed $s \in (0,1]$ and $n, m > 1$. Then*

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\psi(\theta)-\frac{\Gamma(\beta+1)}{(\eta-\xi)}\left[J^\beta_{\theta-}\psi(\xi)+J^\beta_{\theta+}\psi(\eta)\right]\right| \leq \frac{2^{(s-1)/m}}{(1+n\beta)^{1/n}(\eta-\xi)}\left[(\theta-\xi)^{\beta+1}\left|\psi'\left(\frac{\theta+\xi}{2}\right)\right| + (\eta-\theta)^{\beta+1}\left|\psi'\left(\frac{\eta+\theta}{2}\right)\right|\right] \tag{11}$$

holds, for $(1/n)+(1/m)=1$ and $\beta > 0$.

Note that if $s = 1$, the definition of s-convexity reduces to classical convexity of functions defined on $\mathbb{R}^+$.

3. Preliminaries about q-Integrals and Related Inequalities

The following properties of q-derivatives are recalled from [3].

3.1. q-Derivative. For $\phi \in C[\xi,\eta]$, q-derivative of ϕ at $\theta \in [\xi,\eta]$ is given by

$$_\xi D_q\phi(\theta) = \frac{\phi(\theta)-\phi(q\theta+(1-q)\xi)}{(1-q)(\theta-\xi)}\theta \neq \xi. \tag{12}$$

For $n \geq 1$, we have the following relation:

$$(\theta-\xi)^n_q = \prod_{J=0}^{n}(\theta-q^J\xi),$$
$$(\xi-\theta)^n_q = \prod_{J=0}^{n}(\xi-q^J\theta). \tag{13}$$

Respective derivatives are

$$D_q(\theta-\xi)^n_q = [n](\theta-\xi)^{n-1}_q,$$
$$D_q(\xi-\theta)^n_q = -[n](\xi-q\theta)^{n-1}_q, \tag{14}$$
$$(\xi-q\theta)^n_q = -\frac{1}{[n+1]}D_q(\xi-\theta)^{n+1}_q,$$

where $[n] = (q^n-1)/(q-1)$. The fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of noninteger fractional order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semiderivative has been raised. This question consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, and Letnikov.

3.2. Fractional Integral from [8]. Let $\psi \in L^1[\xi,\eta]$. The Riemann-Liouville integrals $J^\beta_{\xi+}\psi$ and $J^\beta_{\eta-}\psi$ of order $\beta > 0$ for $\xi \geq 0$ are defined by

$$J^\beta_{\xi^+}\psi(\theta) = \frac{1}{\Gamma(\beta)}\int_\xi^\theta(\theta-u)^{\beta-1}\psi(u)du, \quad \theta > \xi,$$
$$J^\beta_{\eta^-}\psi(\theta) = \frac{1}{\Gamma(\beta)}\int_\theta^\eta(u-\theta)^{\beta-1}\psi(u)du, \quad \theta < \eta. \tag{15}$$

3.3. q-Antiderivative. q-Antiderivative along with its properties can be studied in [22]. Suppose that $\psi \in C[\xi,\eta]$. Then q-definite integral for $\theta \in [\xi,\eta]$ is defined as

$$\int_\xi^\theta \psi(u)_\xi d_q u = (1-q)(\theta-\xi)\sum_{n=0}^{\infty} q^n\psi(q^n\theta+(1-q^n)\xi), \tag{16}$$

which gives

$$\int(\xi-\theta)^\beta_q d_q\theta = -\frac{q(\xi-q^{-1}\theta)^{\beta+1}_q}{[\beta+1]} \quad (\beta \neq -1),$$
$$\int_0^1 u^{\beta+s}d_q u = \frac{1}{[\beta+s+1]},$$
$$\int_0^1 u^\beta(1-u)^s d_q u = \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}. \tag{17}$$

Formula for q-integration by parts [23]: Let $\psi, g \in C[\xi,\eta]$, and $\theta \in [\xi,\eta]$, and then

$$\int_c^\theta \psi(u)_\xi D_q g(u) d_q u = \psi(\theta)g(\theta) - \psi(c)g(c) - \int_c^\theta g(qu + (1-q)\xi)_\xi D_q \psi(u) d_q u. \tag{18}$$

The following q-integral inequality is from [23]:

Theorem 11. *Suppose $\psi : [\xi, \eta] \longrightarrow \mathbb{R}$ is a q-differentiable mapping. If $|D_q\psi(\theta)| \le M$ for all $\theta \in [\xi, \eta]$ and $0 < q < 1$, then*

$$\left|\psi(\theta) - \frac{1}{\eta - \xi}\int_\xi^\eta \psi(u)_\xi d_q u\right| \le M\left[\frac{2q}{1+q}\left(\frac{\theta - (((3q-1)\xi + (1+q)\eta)/4q)}{\eta - \xi}\right)^2 + \left(\frac{(-q^2 + 6q - 1)}{8q(1+q)}\right)\right] \tag{19}$$

holds for all $\theta \in [\xi, \eta]$.

The least value of constant on R.H.S of (19) is $(-q^2 + 6q - 1)/8q(1+q)$.

q-Hölder inequality [24]. Let ψ, Φ be q-integrable on $[\xi, \eta]$ and $0 < q < 1$ and $(1/n) + (1/m) = 1$ with $m > 1$, and then we state as

$$\int_\xi^\eta |\psi(u)\Phi(u)| d_q u \le \left\{\int_\xi^\eta |\psi(u)|^n d_q u\right\}^{1/n} \left\{\int_\xi^\eta |\Phi(u)|^m d_q u\right\}^{1/m}. \tag{20}$$

The following inequalities are derived from q-Hölder inequality:

q-Minkowski's inequality. Let $\xi, \eta \in \mathbb{R}$ and $n > 1$, for continuous functions $\psi, \Phi : [\xi, \eta] \longrightarrow \mathbb{R}$, we have stated

$$\left\{\int_\xi^\eta |(\psi(u) + \Phi(u))|^n d_q u\right\}^{1/n} \le \left\{\int_\xi^\eta |\psi(u)|^n d_q u\right\}^{1/n} + \left\{\int_\xi^\eta |\Phi(u)|^n d_q u\right\}^{1/n}. \tag{21}$$

q-Power mean inequality. Let $(1/n) + (1/m) = 1$ with n, $m > 1$, and let $\xi, \eta \in \mathbb{R}$ and for continuous functions $\psi, g : [\xi, \eta] \longrightarrow \mathbb{R}$, and then

$$\int_\xi^\eta |\psi(u)\Phi(u)| d_q u \le \left\{\int_\xi^\eta |\psi(u)| d_q u\right\}^{1-(1/m)} \left\{\int_\xi^\eta |\psi(u)||\Phi(u)|^m d_q u\right\}^{1/m}. \tag{22}$$

Theorem 12 (see [25]). *Let $\psi : [\xi, \eta] \longrightarrow \mathbb{R}$ be a function and $0 < q < 1$. Then*

$$\int_0^1 \psi(u\eta + (1-u)\xi) d_q u = \frac{1}{\eta - \xi}\int_\xi^\eta \psi(t) d_q t. \tag{23}$$

Example 1.

$$\begin{aligned}\int_\xi^\theta u(u-\xi) d_q u &= \frac{1}{q+1}\left(\theta(\theta-\xi)^2 - \int_\xi^\theta (u-\xi)^2 d_q u\right) \\ &= \frac{1}{q+1}\left(\theta(\theta-\xi)^2 - \frac{(\theta-\xi)^3}{1+q+q^2}\right) \\ &= \frac{(\theta-\xi)^2}{1+q}\left(\frac{\xi(1+q) + q^2 a}{1+q+q^2}\right).\end{aligned} \tag{24}$$

Now, we have

$$\begin{aligned}\int_\xi^\theta u(u-\xi) d_q u &= \int_0^{\theta-\xi} (t+\xi) t d_q t \\ &= \frac{1}{q+1}\left(\theta(\theta-\xi)^2 - \int_0^{\theta-\xi} (qt)^2 d_q t\right) \\ &= \frac{(\theta-\xi)^2}{1+q}\left(\frac{\xi(1+q) + q^2 a}{1+q+q^2}\right).\end{aligned} \tag{25}$$

Proposition 13. *For each $k, r \in \mathbb{N}$ (or $\mathbb{Z}\, q \in \mathbb{R}^+$) [2], we have*

$$[k+r]_q = [k]_q + q^k [r]_q. \tag{26}$$

Exponential functions and Taylor series (q-analogues) from [26]:

$$E_q^\theta = \sum_{n=0}^\infty q^{n(n-1)/2} \frac{\theta^n}{[n]_q!} = (1 + (1-q)\theta)_q^\infty. \tag{27}$$

q-Gamma and q-Beta functions:

For any $u > 0$

$$\Gamma_q(u) = \int_0^\infty \theta^{u-1} \mathrm{E}_q^{-q\theta} d_q \theta \tag{28}$$

is called q-Gamma Euler function and for any $u, p > 0$

$$\beta_q(u, p) = \int_0^1 \theta^{u-1} (1 - q\theta)_q^{p-1} d_q \theta \tag{29}$$

is called q-Beta function.

Relation between q-Gamma and q-Beta Function:

$$\beta_q(\theta, \sigma) = \frac{\Gamma_q(\theta)\Gamma_q(\sigma)}{\Gamma_q(\theta + \sigma)}, \tag{30}$$

$$\Gamma_q(\mathrm{m} + 1) = [m]\Gamma_q(\mathrm{m})\ (\mathrm{m} > 0). \tag{31}$$

The following definition is introduced by Agarwal in [27] when $\xi = 0$ and by Rajkovic et al. [28] for $\xi \neq 0$.

q-Fractional integral from [29]. Let $\psi \in L^1[\xi,\eta]$. The Riemann-Liouville q-integrals $J^{\beta}_{\xi+}\psi$ and $J^{\beta}_{\eta-}\psi$ of order $\beta > 0$ for $\xi \geq 0$ are defined by

$$\begin{aligned} J^{\beta}_{q,\xi^+}\psi(\theta) &= \frac{1}{\Gamma_q(\beta)}\int_{\xi}^{\theta}(\theta - qu)^{(\beta-1)}\psi(u)d_qu, \theta > \xi, \\ J^{\beta}_{q,\eta^-}\psi(\theta) &= \frac{1}{\Gamma_q(\beta)}\int_{q\theta}^{\eta}(u - q\theta)^{(\beta-1)}\psi(u)d_qu, \theta < \eta, \end{aligned} \tag{32}$$

respectively, where $\Gamma_q(\beta) = \int_0^{\infty} u^{(\beta-1)}\mathrm{E}_q^{-qu}d_qu$. Here $J^0_{\xi+}\psi(\theta) = J^0_{\eta-}\psi(\theta) = \psi(\theta)$.

4. Ostrowski Type Inequalities via q-Fractional Integrals

In order to prove our main results, we have to prove the following lemma with the help of ([30], Lemma 1), which can be seen in [31]).

Lemma 14. *Suppose that $\psi : [\xi,\eta] \longrightarrow \mathbb{R}$ is q-differentiable mapping. If $D_q\psi \in L[\xi,\eta]$, then for all $\theta \in [\xi,\eta]$ and $\beta > 0$, we have*

$$\begin{aligned} &\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\xi)d_qu \\ &\quad - \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\eta)d_qu \\ &\quad = \left(\frac{(\theta-\beta)^{\beta} + (\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta} + [\beta](1-q)}{q^{\beta}}\right)\psi(\theta) \\ &\quad - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)}\left[J^{\beta}_{q,\theta^-}\psi(\xi) + J^{\beta}_{q,\theta^+}\psi(\eta)\right], \end{aligned} \tag{33}$$

where $\Gamma_q(\beta) = \int_0^{\infty} u^{(\beta-1)}E_q^{-qu}d_qu$.

Proof. By using the formula of q-integration by parts

$$\begin{aligned} &\int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\xi)d_qu \\ &\quad = u^{\beta}\frac{\psi(u\theta + (1-u)\xi)}{\theta-\xi}\bigg|_0^1 - \int_0^1 [\beta]u^{\beta-1}\frac{\psi(qu\theta + (1-qu)\xi)}{\theta-\xi}d_qu \\ &\quad = \frac{\psi(\theta)}{\theta-\xi} - \frac{[\beta]}{\theta-\xi}\int_{\xi}^{q\theta-q\xi+\xi}\left(\frac{(u-q\xi)}{q(\theta-\xi)}\right)^{\beta-1}\frac{\psi(u)}{q(\theta-\xi)}d_qu \\ &\quad = \frac{\psi(\theta)}{\theta-\xi} + \frac{[\beta]}{q^{\beta}(\theta-\xi)^{\beta+1}}\int_{\xi}^{q\theta-q\xi+\xi}(u-\xi)^{\beta-1}\psi(u)d_qu, \end{aligned} \tag{34}$$

where

$$\begin{aligned} \int_{\xi}^{q\theta-q\xi+\xi}(u-\xi)^{\beta-1}\psi(u)d_qu &= (q\theta - q\xi + \xi - \xi)(1-q)\sum_{n=0}^{\infty}q^n \\ &\quad \cdot (q^n(q\theta - q\xi + \xi) + (1-q^n)\xi - \xi)^{\beta-1} \\ &\quad \times \psi(q^n(q\theta - q\xi + \xi) + (1-q^n)\xi) \\ &= \frac{\psi(\theta)}{\theta-\xi} - \frac{[\beta]q(\theta-\xi)(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}q}\sum_{n=1}^{\infty}q^n(q^n\theta - q^n\xi)^{\beta-1} \\ &\quad \cdot \psi(q^n\theta + (1-q^n)\xi) + \frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}} \\ &\quad \cdot \psi(\theta) - \frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\psi(\theta) \\ &= \frac{\psi(\theta)}{\theta-\xi} - \left[\frac{[\beta](\theta-\xi)(1-q)}{q^{q}(\theta-\xi)^{\beta+1}}\sum_{n=1}^{\infty}q^n(q^n\theta - q^n\xi)^{\beta-1}\right. \\ &\quad \left. \cdot \psi(q^n\theta + (1-q^n)\xi) + \frac{[\beta](\theta-\xi)(\theta-\xi)^{\beta-1}(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\psi(\theta)\right] \\ &\quad + \frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta) = \frac{\psi(\theta)}{\theta-\xi} \\ &\quad - \left[\frac{[\beta](\theta-\xi)(1-q)}{q^{\beta}(\theta-\xi)^{\beta+1}}\sum_{n=0}^{\infty}q^n(q^n\theta - q^n\xi)^{\beta-1}\right. \\ &\quad \left. \cdot \psi(q^n\theta + (1-q^n)\xi)\right] + \frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta). \end{aligned} \tag{35}$$

$$\begin{aligned} \int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\xi)d_qu &= \frac{\psi(\theta)}{\theta-\xi} - \frac{[\beta]\Gamma_q(\beta)}{q^{\beta}(\theta-\xi)^{\beta+1}\Gamma_q(\beta)} \\ &\quad \cdot \int_{\xi}^{q\theta}(u-q\xi)^{\beta-1}\psi(u)d_qu + \frac{[\beta](1-q)}{q^{\beta}(\theta-\xi)}\psi(\theta). \end{aligned} \tag{36}$$

Multiply both sides of (36) by $(\theta-\xi)^{\beta+1}/(\eta-\xi)$

$$\begin{aligned} &\frac{(\theta-\xi)^{\beta}}{\eta-\xi}\int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\xi)d_qu \\ &\quad = \frac{(\theta-\xi)^{\beta}}{\eta-\xi}\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)}J^{\beta}_{q,\theta^-}(\xi) \\ &\quad + \frac{[\beta](1-q)(\theta-\xi)^{\beta}}{q^{\beta}(\eta-\xi)}\psi(\theta). \end{aligned} \tag{37}$$

Similar, calculation gives

$$\begin{aligned} \int_0^1 u^{\beta}D_q\psi(u\theta + (1-u)\eta)d_qu &= \frac{\psi(\theta)}{\theta-\eta} + \frac{[\beta]\Gamma_q(\beta)}{q^{\beta}(\eta-\theta)^{\beta+1}\Gamma_q(\beta)} \\ &\quad \cdot \int_{\eta}^{\theta}(\eta-qu)^{\beta-1}\psi(u)d_qu - \frac{[\beta](1-q)}{q^{\beta}(\eta-\theta)}\psi(\theta). \end{aligned} \tag{38}$$

Multiply both sides of (38) by $(\eta-\theta)^{\beta+1}/(\eta-\xi)$ to get

$$\begin{aligned}\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\eta)d_qu &= \frac{\psi(\theta)}{\theta-\eta}\\ &+\frac{[\beta]\Gamma_q(\beta)}{q^\beta(\eta-\theta)^{\beta+1}\Gamma_q(\beta)}\int_\eta^\theta(\eta-qu)^{\beta-1}\psi(u)d_qu\\ &-\frac{[\beta](1-q)}{q^\beta(\eta-\theta)}\psi(\theta).\end{aligned} \tag{39}$$

By combining (37) and (39), we obtain the following desired result.

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\xi)d_qu-\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\\ &\cdot\int_0^1 u^\beta D_q\psi(u\theta+(1-u)\eta)d_qu=\frac{(\theta-\xi)^\beta}{\eta-\xi}\psi(\theta)\\ &-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}J^\beta_{q,\theta^-}(\xi)+\frac{[\beta](1-q)(\theta-\xi)^\beta}{q^\beta(\eta-\xi)}\psi(\theta)\\ &+\frac{(\eta-\theta)^\beta}{\eta-\xi}\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}J^\beta_{q,\theta^+}(\eta)\\ &+\frac{[\beta](1-q)(\eta-\theta)^\beta}{q^\beta(\eta-\xi)}\psi(\theta)\\ &=\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)\\ &-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right].\end{aligned} \tag{40}$$

□

Remark 15.

(a) Taking $q=1$, Lemma 14 becomes Lemma 6

(b) Taking $\beta=1$, Lemma 14 reduces to ([32], Lemma 3.1)

$$\begin{aligned}&\frac{1}{q}\left[\psi(\theta)-\frac{1}{\eta-\xi}\int_\xi^\eta\psi(u)_ad_qu\right]\\ &=\frac{(\theta-\xi)^2}{\eta-\xi}\int_0^1 uD_q\psi(u\theta+(1-u)\xi)_0d_qu\\ &-\frac{(\eta-\theta)^2}{\eta-\zeta}\int_0^1 uD_q\psi(u\theta+(1-u)\eta)_0d_qu.\end{aligned} \tag{41}$$

By using Lemma 14, we established some Ostrowski type q-fractional integral inequalities.

Theorem 16. *Suppose $\psi:[\xi,\eta]\subset\mathbb{R}^+\longrightarrow\mathbb{R}$ is a q-differentiable mapping in such a way that $D_q\psi\in L[\xi,\eta]$. If $|D_q\psi|$ is $s-$convex in second sense on $[\xi,\eta]$ for some static $s,q\in(0,1]$ and $|D_q\psi(\theta)|\le M,\theta\in[\xi,\eta]$, subsequently, the following integral inequality for q-fractional integrals is valid.*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.\\ &\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\le\frac{M}{\eta-\xi}\\ &\cdot\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right].\end{aligned} \tag{42}$$

Proof. Consider Lemma 14, and since $|D_q\psi|$ is $s-$convex mapping on $[\xi,\eta]$, we can write

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.\\ &\left.\cdot\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right]\right|\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\ &\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\\ &\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\ &\cdot\int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|d_qu+u^\beta(1-u)^s\left|D_q\psi(\xi)\right|d_qu\\ &+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|d_qu+u^\beta(1-u)^s\left|D_q\psi(\eta)\right|d_qu\\ &\le\frac{(\theta-\xi)^{\beta+1}M}{\eta-\xi}\left(\int_0^1 u^{\beta+s}+u^\beta(1-u)^s\right)d_qu+\frac{(\eta-\theta)^{\beta+1}M}{\eta-\xi}\\ &\cdot\left(\int_0^1 u^{\beta+s}+u^\beta(1-u)^s\right)d_qu\le\frac{(\theta-\xi)^{\beta+1}M}{\eta-\xi}\\ &\cdot\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]+\frac{(\eta-\theta)^{\beta+1}M}{\eta-\xi}\\ &\cdot\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]\\ &=\frac{M}{\eta-\xi}\left[\frac{1}{[\beta+s+1]}+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+2)}\right]\\ &\cdot\left[(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}\right]=\frac{M}{\eta-\xi}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\\ &\cdot\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right],\end{aligned} \tag{43}$$

where we have used the fact that

$$\int_0^1 u^{\beta+s} d_q u = \frac{1}{[\beta+s+1]},$$
$$\int_0^1 \mathrm{u}^{\beta}(1-\mathrm{u})^{\mathrm{s}} \mathrm{d}_q \mathrm{u} = \frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+2)}. \tag{44}$$

Therefore by applying the reduction formula $\Gamma_q(m+1) = [m]\Gamma_q(m)(n>0)$ for Euler gamma function, it completes the proof. □

Corollary 17. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$, is q-differentiable function in such a way that $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|$ is convex on $[\xi,\eta]$ for some fixed $s, q \in (0,1]$ and $|D_q\psi(\theta)| \le M$, $\theta \in [\xi,\eta]$, then we have the following q-fractional integral inequality*

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}\psi(\xi) + J^{\beta}_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{M}{\eta-\xi}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+1]}\right] \tag{45}$$

is valid for $\beta > 0$.

Proof. Taking $s = 1$ in (42), the required result follows. □

Remark 18.

(a) q-analogue of Theorem 3, (4) is followed by taking $\beta = 1$ and $q = 1$ in (42)

(b) Taking $q = 1$ in (42), it follows the inequality (8) of Theorem 7

Theorem 19. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is q-differentiable function in such a way that $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is s-convex in second sense on $[\xi,\eta]$ for some static $s, q \in (0,1]$, $n, m > 1$ and $|D_q\psi(\theta)| \le M, \theta \in [\xi,\eta]$, then*

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}\psi(\xi) + J^{\beta}_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{M}{(1+n\beta)^{1/n}} \cdot \left[\frac{1+q\left\{1-\left(1-q^{-1}\right)^{s+1}\right\}}{[s+1]}\right]^{1/m} \left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right], \tag{46}$$

where $(1/n) + (1/m) = 1$, and $\beta > 0$.

Proof. From Lemma 14,

$$\left|\left(\frac{(\theta-\xi)^{\beta}+(\eta-\theta)^{\beta}}{\eta-\xi}\right)\left(\frac{q^{\beta}+[\beta](1-q)}{q^{\beta}}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^{\beta}(\eta-\xi)} \cdot \left[J^{\beta}_{q,\theta^-}(\xi) + J^{\beta}_{q,\theta^+}(\eta)\right]\right| \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\xi)\right| d_q u + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\eta)\right| d_q u. \tag{47}$$

By virtue of Hölder's inequality, we have

$$\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\xi)\right| d_q u + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \cdot \int_0^1 u^{\beta}\left|D_q\psi(u\theta+(1-u)\eta)\right| d_q u \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \cdot \left(\int_0^1 u^{\beta n} d_q u\right)^{1/n}\left(\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u\right)^{1/m} + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n} d_q u\right)^{1/n} \cdot \left(\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_q u\right)^{1/m}. \tag{48}$$

Since $|D_q\psi|^m$ is $s-$convex in the second sense on $[\xi,\eta]$ and $D_q\psi(\theta)$ is bounded by number M,

$$\int_0^1 u^{\beta n} d_q u = \frac{1}{[\beta n+1]} \tag{49}$$

$$\begin{aligned}
\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u &\le \int_0^1 u^s \left|D_q\psi(\theta)\right|^m d_q u + \int_0^1 (1-u)^s \left|D_q\psi(\xi)\right|^m d_q u \le \int_0^1 u^s d_q u + \int_0^1 (1-u)^s d_q u \\
&= M^m\left(\left|\frac{u^{s+1}}{[s+1]}\right|_0^1 + \left|\frac{-q\left(1-q^{-1}u\right)^{s+1}}{[s+1]}\right|_0^1\right) \\
&= M^m\left(\frac{1}{[s+1]} - \frac{q\left(1-q^{-1}\right)^{s+1}}{[s+1]} + \frac{q}{[s+1]}\right) \\
&= M^m\left(\frac{1+q-q\left(1-q^{-1}\right)^{s+1}}{[s+1]}\right).
\end{aligned} \tag{50}$$

Similarly, we have

$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu \le \int_0^1 u^s\left|D_q\psi(\theta)\right|^m d_qu + \int_0^1 (1-u)^s\left|D_q\psi(\eta)\right|^m d_qu, \tag{51}$$

$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu \le M^m\left(\frac{1+q\left(1-\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right). \tag{52}$$

Substitute (49), (50), and (52) in (48) to get,

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\left(\frac{M^m\left(1+q-q\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right)^{1/m}\\&\quad+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\left(\frac{1+q-q\left(1-q^{-1}\right)^{s+1}}{[s+1]}\right)^{1/m}\\&=\frac{M}{([\beta n+1])^{1/n}}\left(\frac{1+q\left(1-\left(1-q^{-1}\right)^{s+1}\right)}{[s+1]}\right)^{1/m}\\&\quad\cdot\left(\frac{(\theta-\beta)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).\end{aligned} \tag{53}$$

Hence, this completes the proof. □

Corollary 20. *Suppose $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is convex on $[\xi,\eta]$, for some static $s,q\in(0,1]$, $m,n>1$, and $|D_q\psi(\theta)|\le M, \theta\in[\xi,\eta]$, then subsequently, we have*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.\\&\quad\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\\&\le\frac{M}{([1+n\beta])^{1/n}}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right],\end{aligned} \tag{54}$$

where $(1/n)+(1/m)=1$ and $\beta>0$.

Proof. Taking $s=1$ in (46), the required result follows. □

Remark 21.

(a) Taking $\beta=1$ and $q=1$ in (46), it follows the q-analogue of (5), Theorem 4

(b) Taking $q=1$ in (46), it follows the inequality (9) of Theorem 8

Theorem 22. *Suppose $\psi:[\xi,\eta]\subset\mathbb{R}^+\longrightarrow\mathbb{R}$ is q-differentiable mapping in such a way that $D_q\psi\in L[\xi,\eta]$. If $|D_q\psi|^m$ is $s-$convex in second sense on $[\xi,\eta]$ for some static $s,q\in(0,1]$, $m\ge1$, and $|D_q\psi(\theta)|\le M, \theta\in[\xi,\eta]$, then for $\beta>0$, we have*

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.\\&\quad\left.\cdot\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right]\right|\le M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\\&\quad\cdot\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m}\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).\end{aligned} \tag{55}$$

Proof. From Lemma 14,

$$\begin{aligned}&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.\\&\quad\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\\&\quad\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\\&\quad\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu.\end{aligned} \tag{56}$$

Now applying familiar power mean inequality, we have

$$\begin{aligned}&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\\&\quad\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu\\&\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^\beta d_qu\right)^{1-(1/m)}\\&\quad\cdot\left(\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_qu\right)^{1/m}\\&\quad+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^\beta\right)^{1-(1/m)}\\&\quad\cdot d_qu\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_qu\right)^{1/m}.\end{aligned} \tag{57}$$

Since $|D_q\psi|^m$ is s-convex in the second sense on $[\xi,\eta]$ and $|D_q\psi(\theta)|\le M$, we get

$$\begin{aligned}
&\int_0^1 u^\beta \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u \\
&\le \int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|^m d_q u + \int_0^1 u^\beta (1-u)^s \left|D_q\psi(\xi)\right|^m d_q u \\
&\le M^m \int_0^1 u^{\beta+s} d_q u + M^m \int_0^1 \mathrm{u}^\beta (1-\mathrm{u})^s \mathrm{d}_q \mathrm{u} \\
&= M^m \left(\int_0^1 u^{\beta+s} d_q u + \int_0^1 \mathrm{u}^\beta (1-u)^s d_q u\right),
\end{aligned}$$

$$\int_0^1 u^{\beta+s} d_q u = \frac{1}{[\beta+s+1]},$$

$$\int_0^1 u^{\beta} d_q u = \frac{1}{[\beta+1]},$$

$$\begin{aligned}
\int_0^1 \mathrm{u}^\beta (1-\mathrm{u})^s \mathrm{d}_q \mathrm{u} &= \frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+2)} \\
&= \frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{[\beta+s+1]\Gamma_q(\beta+\mathrm{s}+1)},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 u^\beta \left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u \\
&\le \frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+1)}\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 u^\beta \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_q u \\
&\le \int_0^1 u^{\beta+s}\left|D_q\psi(\theta)\right|^m d_q u + \int_0^1 u^\beta (1-u)^s \left|D_q\psi(\eta)\right|^m d_q u \\
&\le \frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+1)}\right).
\end{aligned} \tag{58}$$

By using (30), we have

$$\begin{aligned}
&\int_0^1 u^\beta \left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_q u \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)} \\
&\cdot \left(\frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+1)}\right)\right)^{1/m} + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \\
&\cdot \left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{M^m}{[\beta+s+1]}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+1)}\right)\right)^{1/m} \\
&= M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(\mathrm{s}+1)}{\Gamma_q(\beta+\mathrm{s}+1)}\right)\right)^{1/m} \\
&\cdot \left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right),
\end{aligned} \tag{59}$$

which completes the proof. □

Corollary 23. *Suppose $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is convex on $[\xi,\eta]$ and for some static $s, q \in (0,1]$, $m \ge 1$, and $|D_q\psi(\theta)| \le M, \theta \in [\xi,\eta]$, then for $\beta > 0$, we have*

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right| \\
&\le M\left(\frac{1}{[\beta+1]}\right)\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right).
\end{aligned} \tag{60}$$

Proof. Taking $s = 1$ in (55), the required result follows. □

Remark 24.

(a) Taking $\beta = 1$ and $q = 1$, in (55), it follows the q-analogue of (6), Theorem 5

(b) Taking $q = 1$, in (55), it follows formula (10) of Theorem 9

Theorem 25. *Suppose that $\psi : [\xi,\eta] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ is q-differentiable mapping and $D_q\psi \in L[\xi,\eta]$. If $|D_q\psi|^m$ is s-convex in second sense on $[\xi,\eta]$ for some static $s \in (0,1]$ and $m, n > 1$, therefore, the following integral inequality for q-fractional integrals is valid.*

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right| \le \frac{2^{((s-1)/m)}}{(1+n\beta)^{1/n}(\eta-\xi)} \\
&\cdot\left[(\theta-\xi)^{\beta+1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right| + (\eta-\theta)^{\beta+1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|\right],
\end{aligned} \tag{61}$$

where $(1/n)+(1/m) = 1$, and $\beta > 0$.

Proof. From Lemma 14 and keeping the familiar Hölder inequality in use, it follows

$$\begin{aligned}
&\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta) - \frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right. \\
&\left.\cdot\left[J^\beta_{q,\theta^-}(\xi)+J^\beta_{q,\theta^+}(\eta)\right]\right| \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi} \\
&\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right| d_q u + \frac{(\eta-\theta)^{\beta+1}}{\eta-\xi} \\
&\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right| d_q u \le \frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n}\right)^{1/n} \\
&\cdot d_q u\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\xi)\right|^m d_q u\right)^{1/m} \\
&+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\int_0^1 u^{\beta n}\right)^{1/n} d_q u\left(\int_0^1\left|D_q\psi(u\theta+(1-u)\eta)\right|^m d_q u\right)^{1/m}.
\end{aligned} \tag{62}$$

Since $|D_q\psi|^m$ is $s-$concave, we have

$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\xi)\right|^m du \le 2^{s-1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|^m,$$
$$\int_0^1 \left|D_q\psi(u\theta+(1-u)\eta)\right|^m du \le 2^{s-1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|^m. \tag{63}$$

Therefore,

$$\begin{aligned}
&\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\xi)\right|d_qu+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\\
&\cdot\int_0^1 u^\beta\left|D_q\psi(u\theta+(1-u)\eta)\right|d_qu\le\frac{(\theta-\xi)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\\
&\cdot\left(2^{s-1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|^m\right)^{1/m}+\frac{(\eta-\theta)^{\beta+1}}{\eta-\xi}\left(\frac{1}{[\beta n+1]}\right)^{1/n}\\
&\cdot\left(2^{s-1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|^m\right)^{1/m}=\frac{2^{(s-1)/m}}{([\beta n+1])^{1/n}(\eta-\xi)}\\
&\cdot\left((\theta-\xi)^{\beta+1}\left|D_q\psi\left(\frac{\theta+\xi}{2}\right)\right|+(\eta-\theta)^{\beta+1}\left|D_q\psi\left(\frac{\eta+\theta}{2}\right)\right|\right),
\end{aligned} \tag{64}$$

which completes the proof. □

Remark 26. Taking $q=1$ in (61), it follows the inequality (11) of Theorem 10.

5. Applications and Examples

Example 2. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0$, $\eta=1$; $\beta=1$; $q=1/2$, and $s=1$, and then we get verification of Theorem 16.

$$\begin{aligned}
&\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\right.\\
&\left.\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right|\\
&\le\frac{1}{1-0}\left(1+\frac{\Gamma_{1/2}(2)\Gamma_{1/2}(2)}{\Gamma_{1/2}(3)}\right)\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{[3]}\right),
\end{aligned}$$

$$\begin{aligned}
&\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu\right)\right|\\
&\le\left(1+\frac{2}{3}\right)\left(\frac{0.5}{1.75}\right),
\end{aligned}$$

$$\left|1-\frac{2}{3}\right|\le(1.6667)(0.2857),$$
$$0.3333\le0.4762. \tag{65}$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 3. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0, \eta=1$; $\beta=1$; $q=1/2$; $m=n=1/2$, and $s=1$, and then we get verification of Theorem 19.

$$\begin{aligned}
&\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\right.\\
&\left.\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right|\\
&\le\left(\frac{1}{[1+(1/2)]}\right)^{1/2}\left(\frac{1+(1/2)\{1-(1-(1/((1/2)))^2_{1/2}\}}{[2]}\right)^{0.5}\\
&\cdot\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{1-0}\right),
\end{aligned}$$

$$\begin{aligned}
&\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu\right.\right.\\
&\left.\left.+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^1(1-u)d_qu\right)\right|\le\left(\frac{1-q}{1-q^{3/2}}\right)^{0.5}\\
&\cdot\left(\frac{1+.5\{1-0\}}{q+1}\right)^{0.5}(0.5),
\end{aligned}$$

$$\left|1-\frac{2}{3}\right|\le\left(\frac{0.5}{1-0.3536}\right)^{0.5}\left(\frac{3/2}{1+(1/2)}\right)^{0.5}(0.5),$$

$$0.3333\le(0.7735)^{0.5}(0.5),$$

$$0.3333\le0.4398. \tag{66}$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 4. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0, \eta=1$; $\beta=1$; $q=1/2$; $m=n=1/2$, and $s=1$, and then we get verification of Theorem 22.

$$\begin{aligned}
&\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\right.\\
&\left.\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right|\\
&\le(1)\left(\frac{1}{[2]}\right)^{1/2}\left(\frac{1}{[3]}\right)^{1/2}\left(1+\frac{1}{\Gamma_{1/2}(3)}\right)^{1/2}\\
&\cdot\left(\frac{((1/2)-0)^2+(1-(1/2))^2}{1-0}\right),
\end{aligned}$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu\right.\right.$$
$$\left.\left.+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^{1}(1-u)d_qu)\right|\leq\left(\frac{1}{(1/2)+1}\right)^{0.5}$$
$$\cdot\left(\frac{1}{1+(1/2)+(1/2)^2}\right)^{0.5}\left(1+\frac{2}{3}\right)^{1/2}(0.5),$$

$$\left|1-\frac{2}{3}\right|\leq\left(\frac{2}{3}\right)^{0.5}\left(\frac{1}{1.75}\right)^{0.5}(1.6667)^{0.5}(0.5),$$

$$0.3333\leq(0.6667)^{0.5}(0.5414)^{0.5}(1.6667)^{0.5}(0.5),$$

$$0.3333\leq 0.3984. \tag{67}$$

For any $q\in(0,1)$ and $s\in(0,1]$ result holds.

Example 5. Let $\psi(u)=1-u$ and $q\in(0,1)$, and we fixed $\theta=1/2$; $\xi=0,\eta=1$; $\beta=1$; $q=1/2$; and $s=1$, and then we get verification of Theorem 25.

$$\left|\left(\frac{((1/2)-0)^1+(1-(1/2))^1}{1-0}\right)\left(\frac{(1/2)^1+[1](1-(1/2))}{1/2}\right)\right.$$
$$\left.\cdot\psi\left(\frac{1}{2}\right)-\frac{\Gamma_{1/2}(2)}{1/2(1)}\left(J^1_{1/2,1/2^-}\psi(0)+J^1_{1/2,1/2^+}\psi(1)\right)\right|$$
$$\leq\left(\frac{1}{[(1/2)+1]}\right)^{0.5}\left(\left(\frac{1}{2}\right)^2|(-1)|\right)+\left(\left(\frac{1}{2}\right)^2|(-1)|\right),$$

$$\left|(0.5+0.5)2(0.5+0.5)\left(\frac{1}{2}\right)-2\left(\frac{1}{\Gamma_{1/2}(1)}\int_0^{1/2}(1-u)d_qu\right.\right.$$
$$\left.\left.+\frac{1}{\Gamma_{1/2}(1)}\int_{1/2}^{1}(1-u)d_qu)\right|\leq\left(\frac{1-q}{1-q^{3/2}}\right)^{0.5}\left(\left(\frac{1}{2}\right)^2|(-1)|\right)$$
$$+\left(\left(\frac{1}{2}\right)^2|(-1)|\right),$$

$$\left|1-\frac{2}{3}\right|\leq\left(\frac{0.5}{1-0.3536}\right)^{0.5}(0.5),$$

$$0.3333\leq 0.4398. \tag{68}$$

Result holds for several example such as every $q\in(0,1)$ and $\psi(u)=(1-u)^n$, $\psi(s)=u^n$, $\psi(u)=u(1-u)^n$, $\psi(u)=u-\xi$, $\psi(u)=(u-\xi)^n$, $\psi(s)=(\xi-u)$, and $\psi(u)=(\xi-u)^n\ n\in\mathbb{R}^+$ have good approximation. At the end, we compare Theorems 16, 19, and 22. We have seen that Theorem 19 has good approximation than Theorem 16, and Theorem 22 has better approximation than Theorems 16 and 19.

$$\left|\left(\frac{(\theta-\xi)^\beta+(\eta-\theta)^\beta}{\eta-\xi}\right)\left(\frac{q^\beta+[\beta](1-q)}{q^\beta}\right)\psi(\theta)-\frac{\Gamma_q(\beta+1)}{q^\beta(\eta-\xi)}\right.$$
$$\left.\cdot\left[J^\beta_{q,\theta^-}\psi(\xi)+J^\beta_{q,\theta^+}\psi(\eta)\right]\right|\leq M\left(\frac{1}{[\beta+1]}\right)^{1-(1/m)}$$
$$\cdot\left(\frac{1}{[\beta+s+1]}\right)^{1/m}\left(\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\right)^{1/m}$$
$$\cdot\left(\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right),\leq\frac{M}{(1+n\beta)^{1/n}}$$
$$\cdot\left[\frac{1+q\left\{1-\left(1-q^{-1}\right)^{s+1}\right\}}{[s+1]}\right]^{1/m}\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{\eta-\xi}\right]$$
$$\leq\frac{M}{\eta-\xi}\left(1+\frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)}\right)\left[\frac{(\theta-\xi)^{\beta+1}+(\eta-\theta)^{\beta+1}}{[\beta+s+1]}\right]. \tag{69}$$

6. Conclusion

The major goal of this paper is to prove fractional quantum integral identities in order to establish some new quantum Ostrowski type inequalities involving q-fractional integrable inequalities. By the virtue of discrete fractional q-calculus, Ostrowski type inequalities are generalized for q-fractional integrals, which provide a method to study some properties of q-fractional integrals via other classes of integral inequalities. Similar method can be applied to other inequalities, like Sipmson's and Newton's type inequalities.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

X. Wang provided the main idea of the article. K. A. Khan wrote the initial draft and investigated the results. A. Ditta contributed to editing of the original draft and methodology. A. Nosheen dealt with the conceptualization and handled the latex work. K. M. Awan performed the validation and formal analysis. R. M. Mabela performed review and editing along with the submission of manuscript. All authors read and approved the final manuscript.

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