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Retraction

Retracted: A Novel Fuzzy Structure: Infra-Fuzzy Topological Spaces

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Z. A. Ameen, T. M. Al-shami, A. A. Azzam, and A. Mhemdi, "A Novel Fuzzy Structure: Infra-Fuzzy Topological Spaces," *Journal of Function Spaces*, vol. 2022, Article ID 9778069, 11 pages, 2022.

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Research Article

A Novel Fuzzy Structure: Infra-Fuzzy Topological Spaces

Zanyar A. Ameen,[1] **Tareq M. Al-shami**,[2] **A. A. Azzam**,[3,4] **and Abdelwaheb Mhemdi**[5]

[1]*Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq*
[2]*Department of Mathematics, Sana'a University, P.O.Box 1247 Sana'a, Yemen*
[3]*Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*
[4]*Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt*
[5]*Department of Mathematics, College of Sciences and Humanities in Aflaj, Prince Sattam bin Abdulaziz University, Riyadh, Saudi Arabia*

Correspondence should be addressed to Tareq M. Al-shami; tareqalshami83@gmail.com

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Obtaining a weaker condition that preserves some inspired topological properties is always desirable. As a result, we introduce the concept of infra-fuzzy topology, which is a subset family that degrades the concept of fuzzy topology by omitting the condition of closedness under arbitrary unions. Fundamental properties of infra-fuzzy topological spaces are investigated, including infra-fuzzy open and infra-fuzzy closed sets, infra-fuzzy interior and infra-fuzzy closure operators, and the infra-fuzzy boundary of a fuzzy set. It is not possible to expect the latter concepts to have properties identical to those in ordinary fuzzy topological spaces. More precisely, the infra-fuzzy interior of a set need not be infra-fuzzy open, and the infra-fuzzy closure and boundary of a set may not be infra-fuzzy closed. Then, employing infra-fuzzy neighborhood systems, infra-fuzzy Q-neighborhood systems, the basis of infra-fuzzy topology, and infra-fuzzy relative topology, we propose several approaches for generating infra-fuzzy topologies. Finally, we define the notions of continuity, openness, closedness, and homeomorphism of mappings in the context of infra fuzziness and investigate some of their properties and characterizations. We show that the usual characterization of earlier notions in the infra-fuzzy structure is incorrect. We demonstrate that the family of all infra-fuzzy homeomorphisms on an infra-fuzzy topological space forms a group under mappings composition. We finish this work by proving that each infra-fuzzy homeomorphism between two infra-fuzzy topological spaces produces an isomorphism on groups of infra-fuzzy homeomorphisms of the corresponding spaces.

1. Introduction

General topology, it is also named point-set topology, is the branch of topology that deals with the fundamental set-theoretic notions and constructions. It is the basis for most other branches of topology, including differential, geometric, and algebraic topology. The growth of topology has been supported by the variety of classes of topological spaces, examples and their aspects and relationships. The conception of topological space was enlarged.

In the 80s, Mashhour et al. [1] initiated a supra topological space that is not closed under finite intersections. This research has been supported by numerous authors from around the world (see [2–5]). In the following decade, Császár [6] started a comprehensive investigation of families that are closed solely by arbitrary unions under the name of a generalized topological space. This area of research is essential in computer science and its applications. In formal concept analysis and data clustering, Soldano [7] considered the generalized topological space and called it extensional abstraction. It was also used in Banach games and the entropy problem in [8, 9]. Additional topological space generalizations appeared after that. These structures include minimal structure [10, 11], weak structure [12], and generalized weak structure [13] and so on. In fact, the entire study field has developed during the last two decades.

Fuzzy topology [14] and soft topology [15] have a role no less than the general topology and do not lack any of the abovementioned generalizations. Fuzzy topology was generalized to supra fuzzy topology in [16]. The relation between fuzzy soft and soft topological spaces has been recently investigated by [17].

The concept of an infra topological space was explicitly established by Al-Odhari [18] in 2015, although it has attracted little attention from researchers. On the other hand, infra topological spaces are useful to examine, as Witczak demonstrated in [19]. Recently, Al-shami [20] has studied the main properties of infra soft topological spaces. Also, he with coauthors explored the main topological concepts in these structures such as continuity [21], compactness [22], connectedness [23], and separation axioms [24]. The words of Witczak and the work of the Al-shami motivate us to study the infra topological structure in fuzzy settings. We begin by discussing the main properties of infra-fuzzy topological spaces, proceeded by the definition of various operators using infra-fuzzy open and closed sets. Finally, we examine at infra-fuzzy continuous, infra-fuzzy open, and infra-fuzzy closed mappings and infra-fuzzy homeomorphisms.

2. Preliminaries

Let Z be a universe (domain), and let $\mathscr{F}(Z)$ be the class of all fuzzy sets in Z. The closed interval $[0, 1]$ is denoted by I. All undefined terminologies used in the manuscript can be found in [14, 25, 26].

Definition 1 (see [25]). A mapping μ from Z to the unit interval I is named a fuzzy set in Z. The value $\mu(z)$ is called the degree of the membership of z in μ for each $z \in Z$. The support of μ is the set $\{z \in Z : \mu(z) > 0\}$. The complement of μ is, symbolized by μ^c (or $1 - \mu$ if there is no confusion), given by $\mu^c(z) = 1 - \mu(z)$ for all $z \in Z$.

Definition 2 (see [25]). Let $\{\mu_j : j \in J\} \subseteq \mathscr{F}(Z)$, where J is any index set. Then,

(i) $\vee\mu_j(z) = \sup\{\mu_j(z) \colon j \in J\}$, for each $z \in Z$.

(ii) $\wedge\mu_j(z) = \inf\{\mu_j(z) \colon j \in J\}$, for each $z \in Z$.

Definition 3 (see [26]). Let μ be a fuzzy set in Z. Then, μ is said to be a fuzzy point, denoted by z_p, with the support $z \in Z$ and the value $p \in (0, 1]$ if $z_p : Z \longrightarrow I$ is the function defined as follows: for each $y \in Z$,

$$z_p(y) = \begin{cases} p & \text{if } y = z; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A fuzzy point z_p said to be in μ, denoted by $z_p \in \mu$, if $p \le \mu(z)$. We may write $p \in \mu$ if no confusion causes or by a letter p we mean a fuzzy point. We write $\mathscr{F}_p(Z)$ for the set of all fuzzy points in Z.

Definition 4 (see [26]). Let $Z \neq \varnothing$, and let $\mu, \nu \in \mathscr{F}(Z)$. Then, μ, ν intersect if there is $z \in Z$ such that $\mu(z) \wedge \nu(z) \neq 0$.

Definition 5 (see [26]). Let $Z \neq \varnothing$, and let $\mu, \nu \in \mathscr{F}(Z)$. Then,

(i) $z_p \in \mathscr{F}_p(Z)$ is called quasi-coincident with μ if $p + \mu(z) > 1$ and is denoted $z_p \prec \mu$.

(ii) $z_p \in \mathscr{F}_p(Z)$ is called not quasi-coincident with μ if $p + \mu(z) \le 1$ and is denoted $z_p \nprec \mu$.

(iii) μ is called quasi-coincident with ν (at z) if there exists $z \in Z$ such that $\mu(z) + \nu(z) > 1$ and is denoted $\mu \prec \nu$.

Lemma 6 (see [26, Proposition 17]). *Let $Z \neq \varnothing$, $\mu, \nu \in \mathscr{F}(Z)$, $z_p \in \mathscr{F}_p(Z)$. Then,*

(i) if $\mu \prec \nu$ at z, then $\mu(z) \wedge \nu(z) \neq 0$;

(ii) $z_p \in \mu$ iff $z_p \nprec 1 - \mu$; and

(iii) $\mu \le \nu$ iff $\mu \nprec 1 - \nu$.

Definition 7 (see [14]). Let Y, Z be nonempty sets, let $\mu \in \mathscr{F}(Z)$ and $\lambda \in \mathscr{F}(Y)$, and let $\pi : Z \longrightarrow Y$. The image of μ under π is defined by

$$\pi(\mu)(z) = \begin{cases} \sup\limits_{z \in \pi^{-1}(y)} \mu(z), & \text{if } \pi^{-1}(y) \neq \varnothing, \\ 0, & \text{if } \pi^{-1}(y) = \varnothing, \end{cases} \tag{2}$$

for each $y \in Y$, and the inverse image of λ under π is defined by $\pi^{-1}(\lambda)(z) = \lambda(\pi(z))$, for each $z \in Z$.

3. Infra-Fuzzy Topology

In this section, we familiarize the concept of infra-fuzzy topology and present some methods to produce this concept such as the basis of infra-fuzzy topology, infra-fuzzy neighborhood systems, and infra-fuzzy Q-neighborhood systems. To elucidate the obtained results and relationships, we display some examples.

Definition 8. A subcollection T of $\mathscr{F}(Z)$ is said to be an infra-fuzzy topology on Z if

(i) $0, 1 \in T$; and

(ii) $\mu_1 \wedge \mu_2 \in T$ whenever $\mu_1, \mu_2 \in T$.

The pair (Z, T) is called an infra-fuzzy topological space, and the set of all infra-fuzzy topologies on Z is denoted by $\mathrm{IFT}(Z)$. The members of T are called infra-fuzzy open (or IF-open) subsets of Z, and their complements are called infra-fuzzy closed (or IF-closed) sets. The members of T^c are also called IF-closed sets.

Remark 9. Evidently, each fuzzy topology is an infra-fuzzy topology, but not conversely.

Lemma 10. *Let $\{T_j : j \in J\}$ be a subclass of $IFT(Z)$, where J is any index set. Then, $T = \wedge_{j\in J} T_j \in IFT(Z)$.*

Proof. Straightforward. □

Lemma 11. *Let $\mathscr{C}$ be a subclass of $\mathscr{F}(Z)$. There exists a unique $T \in IFT(Z)$ containing $\mathscr{C}$, and if $T^* \in IFT(Z)$ that includes $\mathscr{C}$, then $T \leq T^*$.*

Proof. Note that such an infra-fuzzy topology always exists because $\mathscr{F}(Z)$ is the infra-fuzzy topology on Z which includes $\mathscr{C}$. Consider T, the intersection of all those infra-fuzzy topologies on Z which include $\mathscr{C}$. Then, it follows from Lemma 10 that T is the required infra-fuzzy topology. □

Definition 12. Let $\mathscr{C}$ be a subclass of $\mathscr{F}(Z)$. The unique $T \in \mathrm{IFT}(Z)$ obtained in the above lemma is called the infra-fuzzy topology on Z generated by the collection $\mathscr{C}$ and is denoted by $T(\mathscr{C})$, which is the smallest infra-fuzzy topology on Z containing $\mathscr{C}$.

If $T^*, T' \in \mathrm{IFT}(Z)$, then $T^* \vee T' \in \mathrm{IFT}(Z)$ is false in general.

Example 1. Let μ_1, μ_2 and μ_3 in $\mathscr{F}(I)$ be defined by

$$\begin{aligned}
\mu_1(x) &= \begin{cases} 1, & \text{if } 0 \leq x \leq 1/2 \\ -4x+3, & \text{if } 1/2 \leq x \leq 3/4, \\ 0, & \text{if } 3/4 \leq x \leq 1 \end{cases} \\
\mu_2(x) &= \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4 \\ -1+4x, & \text{if } 1/4 \leq x \leq 1/2, \\ 1, & \text{if } 1/2 \leq x \leq 1 \end{cases} \\
\mu_3(x) &= \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4 \\ -1+4x, & \text{if } 1/4 \leq x \leq 1/2 \\ -4x+3, & \text{if } 1/2 \leq x \leq 3/4 \\ 0, & \text{if } 3/4 \leq x \leq 1 \end{cases}.
\end{aligned} \tag{3}$$

Then, $T^* = \{0, \mu_1, 1\}$, $T' = \{0, \mu_2, 1\} \in \mathrm{IFT}(I)$, but $T^* \vee T' = \{0, \mu_1, \mu_2, 1\} \notin \mathrm{IFT}(I)$, since $\mu_1 \wedge \mu_2 = \mu_3 \notin T^* \vee T'$.

Definition 13. Let $T \in \mathrm{IFT}(Z)$. A subcollection $\mathscr{B}$ of T is called a base for T, if for each $\mu \in T$, there exists a finite $\mathscr{B}_0 \leq \mathscr{B}$ such that $\mu = \wedge \mathscr{B}_0$.

Remark 14. Each subcollection $\mathscr{B}$ of $\mathscr{F}(Z)$ containing 0, 1 can be a base for some $T \in \mathrm{IFT}(Z)$.

Definition 15. Let $T \in \mathrm{IFT}(Z)$. Then, $\eta \in \mathscr{F}(Z)$ is called an infra-fuzzy neighborhood of a fuzzy point $p \in \mathscr{F}_p(Z)$ if there is $\mu \in T$ such that $p \in \mu \leq \eta$. The set of all infra-fuzzy neighborhoods of $p \in \mathscr{F}_p(Z)$ is called an infra-fuzzy neighborhood system of p and is denoted by IFN_p.

Lemma 16. *Let $T \in IFT(Z)$. If $\mu \in T$, then for each $p \in \mu$, $\mu \in IFN_p$.*

Proof. Standard. □

The converse of the above result is generally false as shown below.

Example 2 (see [27, Example 4]). Consider $\mu_1, \mu_2, \mu_3 \in \mathscr{F}(I)$ defined below:

$$\begin{aligned}
\mu_1(x) &= \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2 \\ 2x-1, & \text{if } 1/2 \leq x \leq 1 \end{cases}, \\
\mu_2(x) &= \begin{cases} 1, & \text{if } 0 \leq x \leq 1/4 \\ -4x+2, & \text{if } 1/4 \leq x \leq 1/2, \\ 0, & \text{if } 1/2 \leq x \leq 1 \end{cases} \\
\mu_3(x) &= \begin{cases} 1, & \text{if } 0 \leq x \leq 1/4 \\ -4x+2, & \text{if } 1/4 \leq x \leq 1/2. \\ 2x-1, & \text{if } 1/2 \leq x \leq 1 \end{cases}
\end{aligned} \tag{4}$$

Then, $T = \{0, \mu_1, \mu_2, 1\} \in \mathrm{IFT}(I)$. Note that $\forall p \in \mu_3$, $\mu_3 \in \mathrm{IFN}_p$, but $\mu_3 \notin T$.

Proposition 17. *Let $T \in IFT(Z)$ and let $p \in \mathscr{F}_p(Z)$. Then, IFN_p satisfies the following properties:*

(i) $IFN_p \neq \varnothing$;

(ii) $p \in \eta$ for each $\eta \in IFN_p$;

(iii) if $\eta \in IFN_p$ and $\mu \in \mathscr{F}(Z)$ with $\eta \leq \mu$, then $\mu \in IFN_p$;

(iv) if $\mu, \eta \in IFN_p$, then $\mu \wedge \eta \in IFN_p$; and

(v) if $\eta \in IFN_p$, then there exists $\mu \in IFN_p$ such that $\eta \in IFN_q$ for each $q \in \mu$.

Proof.

(i) Since $1 \in T$ and 1 contains all $p \in \mathscr{F}_p(Z)$, then $1 \in \mathrm{IFN}_p$ for each p.

(ii) It is clear from the definition.

(iii) Let $\eta \in \mathrm{IFN}_p$ and $\mu \in \mathscr{F}(Z)$. There is $\nu \in T$ such that $p \in \nu \leq \eta$. Since $\eta \leq \mu$, so $p \in \nu \leq \mu$. Hence $\mu \in \mathrm{IFN}_p$.

(iv) Let $\mu, \eta \in \mathrm{IFN}_p$. There exists $\nu, \sigma \in T$ such that $p \in \nu \leq \eta$ and $p \in \sigma \leq \mu$. Therefore, $p \in \sigma \wedge \nu \leq \eta \wedge \mu$. Since $\sigma \wedge \nu \in T$, hence $\mu \wedge \eta \in \mathrm{IFN}_p$.

(v) Let $\eta \in \mathrm{IFN}_p$. There is $\nu \in T$ such that $p \in \nu \leq \eta$. Since $\nu \in T$, by Lemma 16, $\nu \in \mathrm{IFN}_q$ for each $q \in \nu$. Therefore, for each $q \in \nu$, there is $\sigma \in T$ such that $q \in \sigma \leq \nu \leq \eta$. Thus, $\eta \in \mathrm{IFN}_q$.

□

Theorem 18. *Let $Z \neq \varnothing$ be a universe. If, for each $p \in \mathscr{F}_p(Z)$, $\mathscr{C}_p$ is a subclass of $\mathscr{F}(Z)$ that fulfills the properties given in Proposition 17, then there exists a unique $T^* \in IFT(Z)$ such that $\mathscr{C}_p = IFN_p(T^*)$, where $IFN_p(T^*)$ is the set of all infra-fuzzy neighborhoods of p in (Z, T^*).*

Proof. Suppose $T^* = \{\mu : \mu \in \mathscr{C}_p, \forall p \in \mu\}$, where $\mu \in \mathscr{F}(Z)$. We first show that $T^* \in \mathrm{IFT}(Z)$. Evidently, $0 \in T^*$. By (III), $1 \in T^*$. Let $\mu, \nu \in T^*$ and let $p \in \mu \wedge \nu$. Then, $p \in \mu$, $p \in \nu$, and so $\mu, \nu \in \mathscr{C}_p$. By (IV), $\mu \wedge \nu \in \mathscr{C}_p$. Therefore, $\mu \wedge \nu \in T^*$. Thus, $T^* \in \mathrm{IFT}(Z)$.

We now prove that $\mathscr{C}_p = \mathrm{IFN}_p(T^*)$. Let $\eta \in \mathrm{IFN}_p(T^*)$. Then, there exists $\mu \in T^*$ such that $p \in \mu \leq \eta$. By definition of T^*, $\mu \in \mathscr{C}_p$ for each $p \in \mu$. By (III), $\eta \in \mathscr{C}_p$. Thus, $\mathrm{IFN}_p(T^*) \leq \mathscr{C}_p$.

Suppose $\eta \in \mathscr{C}_p$. Set $\mu \coloneqq \vee\{q \in \mathscr{F}_p(Z) \colon \eta \in \mathrm{IFN}_q(T^*)\}$. Obviously, $\mu \in \mathscr{F}(Z)$ with $p \in \mu$. By (II), $\mu \leq \eta$. It is enough to prove that $\mu \in T^*$. That is $\mu \in C_q$ for each $q \in \mu$. Assume $q \in \mu$. By definition of μ, $\eta \in \mathrm{IFN}_q(T^*)$. Since $\mathrm{IFN}_q(T^*) \leq C_q$, so $\eta \in C_q$. By (V), there exists $\sigma \in C_q$ such that $\eta \in C_r$ for each $r \in \sigma$. Again, by definition of μ, $r \in \mu$, and so $\sigma \leq \mu$. By (III), $\mu \in C_q$. Hence, $\mathscr{C}_p \leq \mathrm{IFN}_p(T^*)$. □

Definition 19. Let $T \in \mathrm{IFT}(Z)$. Then, $\eta \in \mathscr{F}(Z)$ is called an infra-fuzzy Q-neighborhood of a fuzzy point $p \in \mathscr{F}_p(Z)$ if there is $\mu \in T$ such that $p \prec \mu \leq \eta$. The set of all infra-fuzzy Q-neighborhoods of $p \in \mathscr{F}_p(Z)$ is called an infra-fuzzy Q-neighborhood system of p and is denoted by IFQN_p.

Proposition 20. *Let $T \in IFT(Z)$ and let $p \in \mathscr{F}_p(Z)$. Then, $IFQN_p$ satisfies the following properties:*

(i) $IFQN_p \neq \varnothing$;

(ii) $p \prec \eta$ for each $\eta \in IFQN_p$;

(iii) if $\eta \in IFQN_p$ and $\mu \in \mathscr{F}(Z)$ with $\eta \leq \mu$, then $\mu \in IFQN_p$;

(iv) if $\mu, \eta \in IFQN_p$, then $\mu \wedge \eta \in IFQN_p$; and

(v) if $\eta \in IFQN_p$, then there exists $\mu \in IFQN_p$ such that $\eta \in IFQN_q$ for each $q \prec \mu$.

Proof. Similar to Proposition 17. □

Theorem 21. *Let $Z \neq \varnothing$ be a universe. If, for each $p \in \mathscr{F}_p(Z)$, $\mathscr{C}_p$ is a subclass of $\mathscr{F}(Z)$ that fulfills the properties given in Proposition 20, then there exists a unique $T^* \in IFT(Z)$ such that $\mathscr{C}_p = IFQN_p(T^*)$, where $IFQN_p(T^*)$ is the set of all infra-fuzzy Q-neighborhoods of p in (Z, T^*).*

Proof. Similar to Theorem 18. □

Definition 22. Let (Z, T) be an infra-fuzzy topological space and $Y \in \mathscr{F}(Z)$. Then, $T_Y = \{\mu \wedge Y : \mu \in T\}$ is said to be an infra-fuzzy topological space related to Y. The pair (Y, T_Y) is called an infra-fuzzy topological subspace of (Z, T).

The proofs of results given below are the same as in the case of ordinary fuzzy topology; thus, we leave them out.

Proposition 23. *Let (Y, T_Y) be an infra-fuzzy topological subspace of (Z, T). Then, λ is an IF-closed subset of (Y, T_Y) iff there exists an IF-closed subset ν of (Z, T) such that $\lambda = \nu \wedge Y$.*

Proposition 24. *Let Y be an IF-open subset of (Z, T). Then, μ is an IF-open set in (Y, T_Y) iff it is an IF-open in (Z, T).*

Proposition 25. *Let Y be an IF-closed subset of (Z, T). Then, μ is an IF-closed set in (Y, T_Y) iff it is an IF-closed in (Z, T).*

4. Infra-Fuzzy Operators

In this part, we define infra-fuzzy open and closed operators and scrutinize their main characterizations. Also, we formulate the concept of infra-fuzzy boundary of a fuzzy set and deduce some formulations to calculate it. With the help of examples, we demonstrate that some properties of these concepts which exist in ordinary fuzzy topology evaporate in the frame of infra-fuzzy topology.

Definition 26. Let $T \in \mathrm{IFT}(Z)$ and let $\eta \in \mathscr{F}(Z)$. The infra-fuzzy interior of η is defined as

$$i(\eta) \coloneqq \sup \{\mu : \mu \leq \eta, \mu \in T\}. \quad (5)$$

Lemma 27. *Let $T \in IFT(Z)$, $\eta \in \mathscr{F}(Z)$, and $p \in \eta$. Then, $p \in i(\eta)$ iff there is $\mu \in T$ such that $p \in \mu \leq \eta$.*

Proof. Apply the above definition. □

Theorem 28. *Let $T \in IFT(Z)$, and let $\mu, \nu \in \mathscr{F}(Z)$. Then, the following properties hold:*

(i) $i(\mu) \leq \mu$;

(ii) if $\mu \leq \nu$, then $i(\mu) \leq i(\nu)$;

(iii) if $\mu \in T$, then $i(\mu) = \mu$;

(iv) $i(\mu) = i(i(\nu))$; and

(v) $i(\mu \wedge \nu) = i(\mu) \wedge i(\nu)$.

Proof. We only prove (iv) and (v); the other parts can be followed from the definition:

(i) By (i), we have that $i(i(\mu)) \leq i(\mu)$. Let $p \in i(\mu)$. There exists $\sigma \in T$ such that $p \in \sigma \leq \mu$. By (ii) and (iii), $p \in i(\sigma) = \sigma \leq i(\mu)$. Therefore, $p \in i(i(\mu))$. Hence, $i(\mu) = i(i(\nu))$.

(ii) Since $\mu \wedge \nu \leq \mu$ and $\mu \wedge \nu \leq \nu$, so by (ii), $i(\mu \wedge \nu) \leq i(\mu)$ and $i(\mu \wedge \nu) \leq i(\nu)$. Thus, $i(\mu \wedge \nu) \leq i(\mu) \wedge i(\nu)$. On the other hand, we let $p \in i(\mu) \wedge i(\nu)$. Then, $p \in i(\mu)$, $p \in i(\nu)$. Therefore, there exists $\sigma, \eta \in T$ such that $p \in \sigma \leq \mu$ and $p \in \eta \leq \nu$. Since $\sigma \wedge \eta \in T$ and $p \in \sigma \wedge \eta \leq \mu \wedge \nu$. Hence, $p \in i(\mu \wedge \nu)$ and so $i(\mu \wedge \nu) = i(\mu) \wedge i(\nu)$.

□

It is worth remarking that the converse of (iii) in the above need not be true. On the other hand, it is true in ordinary fuzzy topological spaces.

Example 3. Consider μ_3 and T given in Example 2. One can easily check that $i(\mu_3) = \mu_3$, but $\mu_3 \in T$.

The above example asserts that the infra-fuzzy interior of a fuzzy set need not be an infra-fuzzy open set. However, the following result still true.

Theorem 29. *Let $T \in IFT(Z)$, $p \in \mathscr{F}_p(Z)$, and $\mu \in \mathscr{F}(Z)$. Then $p \in i(\mu)$ iff there exists $\eta \in IFN_p$ such that $\eta \leq \mu$.*

Proof. Apply the definition of $i(\mu)$ and Proposition 17. □

Definition 30. Let $T \in \mathrm{IFT}(Z)$ and let $\eta \in \mathscr{F}(Z)$. The infra-fuzzy closure of η is defined as $c(\eta) \coloneqq \inf \{\mu : \eta \leq \mu, 1 - \mu \in T\}$.

Theorem 31. *Let $T \in IFT(Z)$, $\mu \in \mathscr{F}(Z)$, and $a = z_p \in \mathscr{F}_p(Z)$. Then, $a \in c(\mu)$ iff for each $\eta \in IFQN_a$, $\eta \prec \mu$.*

Proof. If $a \in c(\mu)$, then for all $\kappa \in T^c$ with $\mu \leq \kappa$, $a \in \kappa$ (i.e. $p < \kappa(z)$). By taking the complement, if $a \in c(\mu)$, then for all $\sigma = \kappa^c \in T$ with $\sigma \leq 1 - \mu$, $\sigma(z) \leq 1 - \mu(z)$. Equivalently, for all $\sigma = \kappa^c \in T$ with $\sigma(z) \leq 1 - \mu(z)$, $\sigma \leq 1 - \mu$. This implies that $\sigma \prec \mu$ from Lemma 6. Hence, for each $\eta \in \mathrm{IFQN}_p$, we have $\eta \prec \mu$. □

By reversing the above steps, one can prove the converse.

Theorem 32. *Let $T \in IFT(Z)$ and let $\mu, \nu \in \mathscr{F}(Z)$. Then the following properties hold:*

(i) $\mu \leq c(\mu)$;

(ii) if $\mu \leq \nu$, then $c(\mu) \leq c(\nu)$;

(iii) if $\mu \in T^c$, then $\mu = c(\mu)$;

(iv) $c(\mu) = c(c(\nu))$; and

(v) $c(\mu \vee \nu) = c(\mu) \vee c(\nu)$.

Proof. (i) and (ii) Follow from the definition.

(iii) Use Theorem 31.

(iv) By (i), $c(\mu) \leq c(c(\mu))$. On the other hand, let $p \in c(c(\mu))$. By Theorem 31, for each $\eta \in \mathrm{IFQN}_p$, $c(\eta) \prec \mu$, but $\eta \leq c(\eta)$. Therefore, $\eta \prec \mu$ for each $\eta \in \mathrm{IFQN}_p$. Hence $p \in c(\mu)$. This shows that $c(\mu) = c(c(\nu))$.

(v) The first direction is simple as $\mu \leq \mu \vee \nu$ and $\nu \leq \mu \vee \nu$, then, by (ii), $c(\mu) \leq c(\mu \vee \nu)$ and $c(\nu) \leq c(\mu \vee \nu)$. Thus $c(\mu) \vee c(\nu) \leq c(\mu \vee \nu)$. For other direction, we have that $\mu \leq c(\mu)$ and $\nu \leq c(\nu)$ and so $\mu \vee \nu \leq c(\mu) \vee c(\nu)$. By the definition of infra-fuzzy closure, we must have $c(\mu \vee \nu) \leq c(\mu) \vee c(\nu)$. Hence the result. □

Remark 33. We shall state that the infra-fuzzy closure of a fuzzy set need not be IF-closed (c.f. Theorem 32 (iii)). The example showing this can be concluded from Example 3.

Corollary 34. *Let $T \in IFT(Z)$ and let $\{\mu_j : j \in J\} \subseteq \mathscr{F}(Z)$, where J is any index set. Then,*

(i) $\vee_{j \in J}(c(\mu_j)) \leq c(\vee_{j \in J}\mu_j)$;

(ii) $\vee_{j \in J}(i(\mu_j)) \leq i(\vee_{j \in J}\mu_j)$;

(iii) $c(\wedge_{j \in J}\mu_j) \leq \wedge_{j \in J}(c(\mu_j))$; and

(iv) $i(\wedge_{j \in J}\mu_j) \leq \wedge_{j \in J}(i(\mu_j))$;

Proposition 35. *Let $T \in IFT(Z)$ and let $\mu \in \mathscr{F}(Z)$. Then*

(i) $i(\mu) = 1 - (c(1 - \mu))$;

(ii) $c(\mu) = 1 - (i(1 - \mu))$;

(iii) $1 - c(\mu) = i(1 - \mu)$; and

(iv) $c(1 - \mu) = 1 - i(\mu)$.

Proof. We know that $i(\mu) = \sup \vartheta$, where $\vartheta = \{\nu : \nu \in T, \nu \leq \mu\}$. Obviously, $1 - \vartheta = \{1 - \nu : \nu \in \vartheta\} = \{\kappa : \kappa \in T^c, 1 - \mu \leq \kappa\}$, where $\kappa = 1 - \nu$. Therefore, $c(1 - \mu) = \inf\,(1 - \vartheta)$. Now, we have $1 - c(1 - \mu) = 1 - \inf\,(1 - \vartheta) = \sup\,(1 - (1 - \vartheta)) = \sup \vartheta = i(\mu)$.

The other parts can be deduced by a similar technique. □

Proposition 36. *Let (Y, T_Y) be an infra-fuzzy subspace of (Z, T) and let $q \in \mathscr{F}_p(Y)$. Then, $\eta \in IFN_q(T_Y)$ iff there exists $\lambda \in IFN_q(T)$ such that $\eta = \lambda \wedge Y$.*

Proof. Apply Definition 22. □

Theorem 37. *Let (Y, T_Y) be an infra-fuzzy subspace of (Z, T) and let $\mu \in \mathscr{F}(Y)$. Then*

(i) $i(\mu) = i_Y(\mu) \wedge i(Y)$;

(ii) $i(\mu) \leq i_Y(\mu)$; and

(iii) $c_Y(\mu) = c(\mu) \wedge Y$,

where $i_Y(\mu), c_Y(\mu)$ *are, respectively, the infra-fuzzy interior and the infra-fuzzy closure of* μ *in the subspace* (Y, T_Y).

Proof.

(i) Let $p \in i(\mu)$. There exists $\sigma \in T$ such that $p \in \sigma \leq \mu \leq Y$. Therefore, $p \in \sigma \wedge Y \leq \mu$ and so $p \in i_Y(\mu)$ and $p \in i(Y)$. Thus, $i(\mu) \leq i_Y(\mu) \wedge i(Y)$. For another direction, we let $p \in i_Y(\mu) \wedge i(Y)$. There are $\alpha, \beta \in T$ such that $p \in \alpha \wedge Y \leq \mu$ and $p \in \beta \leq Y$. But $\alpha \wedge \beta \in T$ with $p \in \alpha \wedge \beta \leq \mu$. Therefore, $p \in i(\mu)$, and hence, $i_Y(\mu) \wedge i(Y) \leq i(\mu)$. Hence, (i).

(i) From (i) and Proposition 36.

(ii) By the definition,

$$
\begin{aligned}
c_Y(\mu) &= \inf \{\kappa : 1 - \kappa \in T_Y, \mu \leq \kappa\} \\
&= \inf \{\lambda \wedge Y : 1 - \lambda \in T, \mu \leq \lambda \wedge Y\} \quad (\text{by Proposition 23}) \\
&= \inf \{\lambda \wedge Y : 1 - \lambda \in T, \mu \leq \lambda\} \\
&= (\inf \{\lambda : 1 - \lambda \in T, \mu \leq \lambda\}) \wedge Y \\
&= c(\mu) \wedge Y.
\end{aligned} \tag{6}
$$

□

Definition 38. Let $T \in \mathrm{IFT}(Z)$ and let $\eta \in \mathscr{F}(Z)$. The infra-fuzzy boundary of η is defined by $b(\eta) = c(\eta) - i(\eta)$.

From the achieved results on infra-fuzzy interior and closure, one can prove the following result:

Theorem 39. *Let* $T \in IFT(Z)$ *and let* $\mu \in \mathscr{F}(Z)$. *Then,*

(i) $b(\mu) = c(\mu) \wedge c(1 - \mu)$;

(ii) $b(\mu) = i(\mu) \vee i(1 - \mu)$;

(iii) $b(\mu) = b(1 - \mu)$;

(iv) $b(\mu) \vee \mu \leq c(\mu)$;

(v) $b(\mu)$ *may not be in* T^c;

(vi) if $\mu \in T$, *then* $b(\mu) \wedge \mu = 0$;

(vii) if $1 - \mu \in T$, *then* $b(\mu) \leq \mu$; *and*

(viii) if $\mu, 1 - \mu \in T$, *then* $b(\mu) = 0$.

5. Infra-Fuzzy Continuous Mappings

We dedicate this section to introducing the notion of continuity between infra-fuzzy topological structures. We discuss some descriptions of continuity as well as establish some properties that hold true under continuity.

Definition 40. Let π be a mapping from an infra-fuzzy topological space (Y, R) into another infra-fuzzy topological space (Z, T). The mapping π is called infra-fuzzy continuous at $r \in \mathscr{F}_p(Y)$ if $r \in \pi^{-1}(\beta) \in T$ for each $\beta \in T$ with $\pi(r) \in \beta$. It is said that π is infra-fuzzy continuous if it is infra-fuzzy continuous at each $r \in \mathscr{F}_p(Y)$.

Remark 41. We shall note that the above definition is not equivalent to the usual definition of continuity at a single fuzzy point. Namely, a mapping π is infra-fuzzy continuous at $r \in \mathscr{F}_p(Y)$ if for each $\beta \in T$ with $\pi(r) \in \beta$, there exists $\alpha \in R$ such that $r \in \alpha$ and $\pi(\alpha) \leq \beta$. Example 4 justifies our claim.

Theorem 42. *For a mapping* $\pi : (Y, R) \longrightarrow (Z, T)$, *the following statements are equivalent:*

(i) π *is infra-fuzzy continuous;*

(ii) if $\pi^{-1}(\beta) \in R$ *for each* $\beta \in T$; *and*

(iii) if $\pi^{-1}(\eta) \in R^c$ *for each* $\eta \in T^c$.

Proof. (i) ⇔(ii). From Definitions 8–40.

(ii) ⇒(iii) Let $\eta \in T^c$. Then $1 - \eta \in T$. By (ii), $\pi^{-1}(1 - \eta) \in R$. By Theorem 4.1 in [14], $1 - \pi^{-1}(\eta) = \pi^{-1}(1 - \eta) \in R$. Hence, $\pi^{-1}(\eta) \in R^c$.

(iii) ⇒(ii) By reversing the above steps, the proof will be accomplished. □

Theorem 43. *If* $\pi : (Y, R) \longrightarrow (Z, T)$ *is an infra-fuzzy continuous mapping, then* $\pi^{-1}(i(\mu)) \leq i(\pi^{-1}(\mu))$ *for each* $\mu \in \mathscr{F}(Z)$.

Proof. Let $\mu \in \mathscr{F}(Z)$ and let $p \in \pi^{-1}(i(\mu))$. Then, $\pi(p) \in \pi((\pi^{-1}(i(\mu))) \leq i(\mu)$. There exists $\sigma \in T$ such that $\pi(p) \in \sigma \leq \mu$. Therefore, $p \in \pi^{-1}(\sigma) \leq \pi^{-1}(\mu)$. Since π is infra-fuzzy continuous, then $\pi^{-1}(\sigma) \in R$, and so $p \in i(\pi^{-1}(\mu))$. Thus, $\pi^{-1}(i(\mu)) \leq i(\pi^{-1}(\mu))$. □

The converse of the above theorem is generally false, as shown below:

Example 4. Let μ_1, μ_2, μ_3 and μ_4 in $\mathscr{F}(I)$ be defined by $\mu_1(x) = x$, $\mu_2(x) = 1 - x$,

$$
\begin{aligned}
\mu_3(x) &= \begin{cases} x, & \text{if } 0 \leq x \leq 1/2 \\ 1 - x, & \text{if } 1/2 \leq x \leq 1, \end{cases} \\
\mu_4(x) &= \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1/2 \\ x, & \text{if } 1/2 \leq x \leq 1 \end{cases},
\end{aligned} \tag{7}
$$

Let $\pi : (I, T) \longrightarrow (I, R)$ be the identity mapping, where $T = \{0, \mu_1, \mu_2, \mu_3, 1\}$ and $R = \{0, \mu_1, \mu_2, \mu_3, \mu_4, 1\}$. By some computation, we can check that for each $\mu \in \mathscr{F}(Z)$, we have $\pi^{-1}(i(\mu)) \leq i(\pi^{-1}(\mu))$. On the other hand, $\mu_4 \in R$, but $\pi^{-1}(\mu_4) \notin T$.

Theorem 44. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy continuous injection, then $i(\pi(\mu)) \leq \pi(i(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. Let $q \in \mathscr{F}_p(Z)$. If $q \in i(\pi(\mu))$. Then there is $\sigma \in T$ such that $p \in \sigma \leq \pi(\mu)$. Therefore, $\pi^{-1}(q) \in \pi^{-1}(\sigma) \leq \pi^{-1}(\pi(\mu))$. Since π is infra-fuzzy continuous injective, so $\pi^{-1}(\sigma) \in R$, and hence $\pi^{-1}(q) \in i(\mu)$. Thus, $q \in \pi(i(\mu))$. This shows that $i(\pi(\mu)) \leq \pi(i(\mu))$. □

Lemma 45. *For a mapping $\pi : (Y, R) \longrightarrow (Z, T)$, the following statements are equivalent:*

(i) $\pi^{-1}(i(\mu)) \leq i(\pi^{-1}(\mu))$ for each $\mu \in \mathscr{F}(Z)$;

(ii) $c(\pi^{-1}(\mu)) \leq \pi^{-1}(c(\mu))$ for each $\mu \in \mathscr{F}(Z)$; and

(iii) $\pi(c(\nu)) \leq c(\pi(\nu))$ for each $\nu \in \mathscr{F}(Y)$.

Proof. (i)⇔(ii). Let $\mu \in \mathscr{F}(Z)$. Then, $1 - \mu \in \mathscr{F}(Z)$. By (i), $\pi^{-1}(i(1-\mu)) \leq i(\pi^{-1}(1-\mu))$. Proposition 35 and Theorem 4.1 in [14] guarantee that $\pi^{-1}(1 - c(\mu)) \leq i(1 - \pi^{-1}(\mu)) \Leftrightarrow 1 - \pi^{-1}(c(\mu)) \leq 1 - c(\pi^{-1}(\mu)) \Leftrightarrow c(\pi^{-1}(\mu)) \leq \pi^{-1}(c(\mu))$.

(ii)⇒(iii) Let $\nu \in \mathscr{F}(Y)$. Then $\pi(\nu) \in \mathscr{F}(Z)$, so by (ii), $c(\pi^{-1}(\pi(\nu))) \leq \pi^{-1}(c(\pi(\nu)))$. Therefore, $c(\nu) \leq \pi^{-1}(c(\pi(\nu)))$. This implies that $\pi(c(\nu)) \leq \pi(\pi^{-1}(c(\pi(\nu)))) \leq c(\pi(\nu))$. Thus, we get (iii).

(iii)⇒(ii) This part can be completed using the same technique as the previous one. □

From Theorem 43 and Lemma 45, we have

Corollary 46. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy continuous mapping, then $c(\pi^{-1}(\mu)) \leq \pi^{-1}(c(\mu))$ for each $\mu \in \mathscr{F}(Z)$.*

Corollary 47. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy continuous mapping, then $\pi(c(\mu)) \leq c(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Theorem 48. *Let $(X, S), (Y, R), (Z, T)$ be infra-fuzzy topological spaces. If $A \subseteq Y$ and $B \subseteq Z$, then the following statements hold: [(i)]*

(i) the inclusion mapping $j : (A, R_A) \longrightarrow (Y, R)$ is infra-fuzzy continuous.

(ii) if a mapping $\pi : (Y, R) \longrightarrow (Z, T)$ is infra-fuzzy continuous, then $\pi|_A : (A, R_A) \longrightarrow (Z, T)$ is infra-fuzzy continuous.

(iii) let $\pi : (Y, R) \longrightarrow (Z, T)$ be infra-fuzzy continuous and let $\pi(Y) \subseteq B$. If $\phi : (Y, R) \longrightarrow (B, T_B)$ is a mapping given by $\phi(p) = \pi(p)$ for all $p \in \mathscr{F}_p(Y)$, then ϕ is infra-fuzzy continuous.

(iv) let $\pi : (Y, R) \longrightarrow (B, T_B)$ be infra-fuzzy continuous. If $\phi : (Y, R) \longrightarrow (Z, T)$ is a mapping given by $\phi(p) = \pi(p)$ for all $p \in \mathscr{F}_p(Y)$, then ϕ is infra-fuzzy continuous.

(v) if $\pi : (Y, R) \longrightarrow (Z, T)$ and $\phi : (X, S) \longrightarrow (Y, R)$ are infra-fuzzy continuous mappings, then $\pi \circ \phi : (X, S) \longrightarrow (Z, T)$ is infra-fuzzy continuous.

Proof.

(i) For each $\mu \in R$, we have $j^{-1}(\mu) = \mu \wedge A \in R_A$ by the definition of relative infra-fuzzy topology on A. Hence, $j : (A, R_A) \longrightarrow (Y, R)$ is infra-fuzzy continuous.

(ii) Assume that $\mu \in T$. Then $(\pi|_A)^{-1}(\mu) = \pi^{-1}(\mu) \wedge A$. By infra-fuzzy continuity of π, $\pi^{-1}(\mu) \in R$. Therefore, $(\pi|_A)^{-1}(\mu) \in R_A$. Thus $\pi|_A$ is infra-fuzzy continuous.

(iii) Let $\mu \in T_B$. There exists $\nu \in T$ such that $\mu = \nu \wedge B$. Since $\pi : (Y, R) \longrightarrow (Z, T)$ is infra-fuzzy continuous, $\pi^{-1}(\nu) \in R$. Since $\pi(Y) \subseteq B$, so we have that

$$\phi^{-1}(\mu) = \phi^{-1}(\nu \wedge B) = \phi^{-1}(\nu) \wedge \phi^{-1}(B) = \pi^{-1}(\nu) \wedge Y = \pi^{-1}(\nu). \tag{8}$$

Thus, $\pi^{-1}(\mu) \in R$ and hence ϕ is infra-fuzzy continuous.

(iv) Let $\mu \in T$. Obviously, $\mu \wedge B \in T_B$. By infra-fuzzy continuity of π and by definition of ϕ, we have $\pi^{-1}(\mu \wedge B) = \pi^{-1}(\mu) \wedge \pi^{-1}(B) = \pi^{-1}(\mu) \wedge Y = \pi^{-1}(\mu) \in R$, and $\pi^{-1}(\mu) = \phi^{-1}(\mu)$. Thus ϕ is infra-fuzzy continuous.

(v) Let $\mu \in T$. Then $(\pi \circ \phi)^{-1}(\mu) = \phi^{-1}(\pi^{-1}(\mu))$. Since π is infra-fuzzy continuous, then $\pi^{-1}(\mu) \in R$. Therefore, by infra-fuzzy continuity of ϕ, we get $\phi^{-1}(\pi^{-1}(\mu)) \in S$. Thus $\pi \circ \phi$ is infra-fuzzy continuous.

□

Theorem 49. *Let $\pi : (Y, R) \longrightarrow (Z, T)$ be a mapping and let $Y = \cup_{i=1}^{n} F_i$, where each $F_i \in R^c$. Then π is infra-fuzzy continuous iff each $\pi|_{F_i} : (F_i, R_{F_i}) \longrightarrow (Z, T)$ is infra-fuzzy continuous.*

Proof. The first direction is proved in Theorem 48 (ii).

Conversely, let $\lambda \in T^c$. Since, for each $i = 1, 2, \cdots, n$, $\pi|_{F_i}$ is an infra-fuzzy continuous mapping, $(\pi|_{F_i})^{-1}(\lambda) \in R^c_{F_i}$ and so $(\pi|_{F_i})^{-1}(\lambda) \in R^c$, by Proposition 25, since each $F_i \in R^c$ Therefore,

$$\pi^{-1}(\lambda) = \left(\pi|_{F_1}\right)^{-1}(\lambda) \cup \left(\pi|_{F_2}\right)^{-1}(\lambda) \cup \cdots \cup \left(\pi|_{F_n}\right)^{-1}(\lambda), \tag{9}$$

is IFIF-closed as it is a finite union of IF-closed sets. Hence, π is infra-fuzzy continuous. □

6. Infra-Fuzzy Homeomorphisms

This section studies new types of mappings in the context of infra fuzziness called open, closed, and homomorphism mappings. Their main features and characterizations are explored. Also, it is proven that the family of all infra-fuzzy homeomorphisms on an infra-fuzzy topological space forms a group under mappings composition.

Definition 50. Let π be a mapping from an infra-fuzzy topological space (Y, R) into another infra-fuzzy topological space (Z, T). The mapping π is called

(i) infra-fuzzy open if $\pi(\beta) \in T$ for each $\beta \in R$;

(ii) infra-fuzzy closed if $\pi(\eta) \in T^c$ for each $\eta \in R^c$.

Proposition 51. *Let $(Y, R), (Z, T)$ be infra-fuzzy topological spaces and let $A \subseteq Y$. The following statements hold:*

(i) the inclusion mapping $j : (A, R_A) \longrightarrow (Y, R)$ is infra-fuzzy open (resp.,closed) iff $A \in R$(resp.,$A \in R^c$).

(ii) if a mapping $\pi : (Y, R) \longrightarrow (Z, T)$ is infra-fuzzy open (resp.,closed) and if $A \in R$(resp.,$A \in R^c$), then $\pi|_A : (A, R_A) \longrightarrow (Z, T)$ is infra-fuzzy open (resp.,closed).

Proof.

(i) if $A \in R$ and $\mu \in R_A$, then by Proposition 24, $j(\mu) = \mu \in T$. Thus i is infra-fuzzy open. On the other hand, if j is infra-fuzzy open and since $A \in R_A$, then $j(A) = A \in T$. Hence, the proof.

(ii) Assume that $\nu \in R_A$. By Proposition 24, $\nu \in R$. Since π infra-fuzzy open, so $\pi(\nu) \in T$. But $\pi|_A(\nu) = \pi(\nu)$; thus, $\pi|_A$ is infra-fuzzy open.

The proofs for π is infra-fuzzy closed are analogous. □

Proposition 52. *Let $\pi : (Y, R) \longrightarrow (Z, T)$ be an infra-fuzzy open bijection. Then, π is infra-fuzzy open iff π is infra-fuzzy closed.*

Proof. It can be followed from the fact that $\pi(1 - \mu) = 1 - \pi(\mu)$ for each $\mu \in \mathscr{F}(Y)$ whenever π is bijective. □

Theorem 53. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy open mapping, then $\pi(i(\mu)) \leq i(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. Let $q \in \mathscr{F}_p(Z)$ such that $q \in \pi(i(\mu))$. Then, there is some $p \in \mathscr{F}_p(Y)$ with $p \in i(\mu)$ such that $\pi(p) = q$. Therefore, there is $\sigma \in R$ such that $p \in \sigma \leq \mu$. Evidently, $q = \pi(p) \in \pi(\sigma) \leq \pi(\mu)$. Since π is infra-fuzzy open, so $\pi(\sigma) \in T$ and hence $q \in i(\pi(\mu))$. Thus, $\pi(i(\mu)) \leq i(\pi(\mu))$. □

Remark 54. The converse of the preceding observation may not be true, and we can use the inverse of the mapping in Example 4 as a counterexample.

Lemma 55. *For a mapping $\pi : (Y, R) \longrightarrow (Z, T)$, the following statements are equivalent:*

(i) $\pi(i(\mu)) \leq i(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$; and

(ii) $\pi^{-1}(c(\nu)) \leq c(\pi^{-1}(\nu))$ for each $\nu \in \mathscr{F}(Z)$.

Proof. By a similar technique used as in Lemma 45. □

The following are the immediate consequences of Proposition 17, Theorem 53, and Lemma 55.

Corollary 56. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy open mapping, then $\pi^{-1}(c(\nu)) \leq c(\pi^{-1}(\nu))$ for each $\nu \in \mathscr{F}(Z)$.*

Corollary 57. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy open mapping, then $\pi(\eta) \in IFN_{\pi(p)}$ if $\eta \in IFN_p$ for each $p \in \mathscr{F}_p(Y)$.*

Theorem 58. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy closed mapping, then $c(\pi(\mu)) \leq \pi(c(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. Let $q \in \mathscr{F}_p(Z)$. If $q \notin c(\pi(\mu))$, then $\{\pi^{-1}(q)\} \nleq c(\mu)$. For each $p \in \{\pi^{-1}(q)\}$, there exists $\eta \in \mathrm{IFQN}_{\pi(p)}$ such that $\eta \not\!\!\!\;\! q\, \mu$. From Lemma 6, we obtain that $\eta + \mu \leq 1$ and so $\mu \leq 1 - \eta$. Since π is infra-fuzzy closed, then $\pi(1 - \eta) = 1 - \pi(\eta) \in T^c$. Therefore, $c(\pi(\mu)) \leq 1 - \pi(\eta)$. But $q = \pi(p) \not\!\!\!\;\! q\, 1 - \pi(\eta)$, hence $q \notin c(\pi(\mu))$. Thus $c(\pi(\mu)) \leq \pi(c(\mu))$. □

The opposite of the preceding theorem is false. The counterexample showing this can be concluded from Remark 54 and Proposition 52.

Corollary 59. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy closed mapping, then $\pi(\eta) \in IFQN_{\pi(p)}$ if $\eta \in IFQN_p$ for each $p \in \mathscr{F}_p(Y)$.*

Proof. It follows from Theorems 58–31. □

Corollary 60. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy closed continuous mapping, then $\pi(c(\mu)) = c(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. It is a direct consequence of Theorem 58 and Corollary 47. □

Corollary 61. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy open continuous injection, then $\pi(i(\mu) = i(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. It follows from Theorems 44–53. □

Theorem 62. *Let $\phi : (X, S) \longrightarrow (Y, R)$ and $\pi : (Y, R) \longrightarrow (Z, T)$ be mappings. The following statements are true: [(i)]*

(i) if ϕ and π are infra-fuzzy open, then $\pi \circ \phi$ is infra-fuzzy open;

(ii) *if $\pi \circ \phi$ is infra-fuzzy open and ϕ is infra-fuzzy continuous surjective, then π is infra-fuzzy open; and*

(iii) *if $\pi \circ \phi$ is infra-fuzzy open and π is infra-fuzzy continuous injective, then ϕ is infra-fuzzy open.*

Proof.

(i) Let $\sigma \in S$. Since ϕ is infra-fuzzy open, then $\phi(\sigma) \in R$. By infra-fuzzy openness of π, $\pi(\phi(\sigma)) \in T$. But $\pi \circ \phi = \pi(\phi(\sigma))$. Hence, $\pi \circ \phi$ is infra-fuzzy open.

(ii) Let $\mu \in R$. Since ϕ is infra-fuzzy continuous, then $\phi^{-1}(\mu) \in S$. Therefore, $(\pi \circ \phi)(\phi^{-1}(\mu)) \in T$. By surjectivity of ϕ, we have that $(\pi \circ \phi)(\phi^{-1}(\mu)) = \pi(\phi(\phi^{-1}(\mu))) = \pi(\mu)$ and hence π is infra-fuzzy open.

(iii) Let $\nu \in S$. Since $\pi \circ \phi$ is infra-fuzzy open, then $\pi \circ \phi(\nu) \in T$. Since π is infra-fuzzy continuous, $\pi^{-1}(\pi \circ \phi(\nu)) \in R$. By injectivity of π, we get that $\pi^{-1}(\pi \circ \phi(\nu)) = (\pi^{-1}\pi)(\phi(\nu)) = \phi(\nu)$. This proves that ϕ is infra-fuzzy open.

□

One can prove the next result by using the same arguments as in the above proof.

Theorem 63. *Let $\phi : (X, S) \longrightarrow (Y, R)$ and $\pi : (Y, R) \longrightarrow (Z, T)$ be mappings. The following statements are true:*

(i) *if ϕ and π are infra-fuzzy closed, then $\pi \circ \phi$ is infra-fuzzy closed;*

(ii) *if $\pi \circ \phi$ is infra-fuzzy closed and ϕ is infra-fuzzy continuous surjective, then π is infra-fuzzy closed; and*

(iii) *if $\pi \circ \phi$ is infra-fuzzy closed and π is infra-fuzzy continuous injective, then ϕ is infra-fuzzy closed.*

Definition 64. A mapping $\pi : (Y, R) \longrightarrow (Z, T)$ is called infra-fuzzy homeomorphism if it is an infra-fuzzy open and infra-fuzzy continuous bijection. We say that (Y, R) and (Z, T) are infra-fuzzy homeomorphic if there exists an infra-fuzzy homeomorphism between them and write $(Y, R) \cong (Z, T)$.

Theorem 65. *For an infra-fuzzy bijection $\pi : (Y, R) \longrightarrow (Z, T)$, the following properties are equivalent:*

(i) *π is an infra-fuzzy homeomorphism;*

(ii) *π is infra-fuzzy continuous and π^{-1} is infra-fuzzy continuous;*

(iii) *π is infra-fuzzy continuous and infra-fuzzy open; and*

(iv) *π is infra-fuzzy continuous and infra-fuzzy closed.*

Proof. Apply Definition 64 and Proposition 52. □

Corollary 66. *If $\pi : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy homeomorphism, then for each $\mu \in \mathscr{F}(Y)$ the following properties are true:*

(i) *$\mu \in R$ iff $\pi(\mu) \in T$;*

(ii) *$1 - \mu \in R$ iff $1 - \pi(\mu) \in T$; and*

(iii) *for $p \in \mathscr{F}_p(Y)$, $\mu \in IFN_p$ iff $\pi(\mu) \in IFN_{\pi(p)}$.*

Proof. Apply Definition 64 and Proposition 52. □

Theorem 67. *For an infra-fuzzy homeomorphism $\pi : (Y, R) \longrightarrow (Z, T)$, the following properties hold:*

(i) *$\pi(i(\mu) = i(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$;*

(ii) *$\pi(c(\mu) = c(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$; and*

(iii) *$\pi(b(\mu) = b(\pi(\mu))$ for each $\mu \in \mathscr{F}(Y)$.*

Proof. Both proofs are the immediate consequences of Corollaries 60–61. □

Proposition 68. *If $\phi : (X, S) \longrightarrow (Y, R)$ and $\pi : (Y, R) \longrightarrow (Z, T)$ are infra-fuzzy homeomorphisms, then $\pi \circ \phi$ is an infra-fuzzy homeomorphism.*

Proof. It is a consequence of Theorems 48–62. □

Remark 69. One can use Definition 64 and Proposition 68 to show that being a homeomorphism is an equivalence relation on the class of all infra-fuzzy topological spaces.

Theorem 70. *The set of all infra-fuzzy homeomorphisms $\mathbb{H}(Z, T)$ of an infra-fuzzy space (Z, T) forms a group under the composition of mappings.*

Proof. Let $* : \mathbb{H}(Z, T) \times \mathbb{H}(Z, T) \longrightarrow \mathbb{H}(Z, T)$ be a binary operation defined by $\phi * \pi = \pi \circ \phi$ for $\pi, \phi \in \mathbb{H}(Z, T)$. By Proposition 68, $\mathbb{H}(Z, T)$ is closed under the operation $\circ$. It is not difficult to show that $\circ$ is associative. The identity element exists in $\mathbb{H}(Z, T)$ which is the identity mapping $i : (Z, T) \longrightarrow (Z, T)$. For each $\pi \in \mathbb{H}(Z, T)$, by Theorem 65, $\pi^{-1} \in \mathbb{H}(Z, T)$, and $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = i$. This shows that each element in $\mathbb{H}(Z, T)$ possesses an inverse, and hence $(\mathbb{H}(Z, T), \circ)$ is a group. □

Theorem 71. *If $h : (Y, R) \longrightarrow (Z, T)$ is an infra-fuzzy homeomorphism, then h generates an isomorphism between the groups $\mathbb{H}(Y, R)$ and $\mathbb{H}(Z, T)$.*

Proof. Let $h : (Y, R) \longrightarrow (Z, T)$ be an infra-fuzzy homeomorphism. Define a mapping $\phi_h : \mathbb{H}(Y, R) \longrightarrow \mathbb{H}(Z, T)$ by $\phi_h(\pi) = h \circ \pi \circ h^{-1}$ for $\pi \in \mathbb{H}(Y, R)$. Since h is an infra-fuzzy homeomorphism, then ϕ_h is bijective. We now show that ϕ_h is a homomorphism. Let $\kappa, \pi \in \mathbb{H}(Y, R)$. Then

$$\begin{aligned}\phi_h(\pi \circ \kappa) &= h \circ (\pi \circ \kappa) \circ h^{-1} = \left(h \circ \pi \circ h^{-1}\right) \circ \left(h \circ \kappa \circ h^{-1}\right) \\ &= \phi_h(\pi) \circ \phi_h(\kappa).\end{aligned} \tag{10}$$

Thus, ϕ_h is a homomorphism, and hence ϕ_h is an isomorphism generated by h. □

7. Conclusion

The concept of infra-fuzzy topology was developed in this paper as a novel structure that is weaker than fuzzy topology. The main purpose of studying this notion is to preserve some fuzzy topological properties under requirements that are weaker than the conditions of topology. This part of the research has worked to improve knowledge in this field in the following ways: We have established the fundamental properties of infra-fuzzy topological spaces first. The majority of interior and closure properties are true in the setting of infra-fuzzy topological spaces, but others are not. The interior of each infra-fuzzy open set is the open set itself, but the interior of a fuzzy set does not have to be infra-fuzzy open. The infra-fuzzy closure would be in a similar condition. We have also mentioned that the boundary of a fuzzy set may not be infra-fuzzy closed. Second, we have suggested the infra-fuzzy bases, infra-fuzzy subspaces, infra-fuzzy neighborhood systems, and infra-fuzzy Q-neighborhood systems as ways to generate infra-fuzzy topologies. The infra-fuzzy topologies formed by the third and fourth notions are not the same. Finally, we have focused at the notion of mappings between infra-fuzzy topological spaces and their continuity, openness, closedness, and homomorphism. We have demonstrated that in the context of infra-fuzzy topologies, a full characterization of those notions as in fuzzy topologies is impossible. We have shown that the set of all homeomorphisms of an infra-fuzzy topological space is a group under the composition operation. Furthermore, we have proven that if (Y, R) and (Z, T) are infra-fuzzy homeomorphic by h, then it generates a mapping ϕ_h under which $\mathbb{H}(Y, R)$ and $\mathbb{H}(Z, T)$ are isomorphic.

In the upcoming work, we plan to investigate the concepts of separation axioms, compactness, and connectedness. Also, we study the concepts given herein with respect to some recent types of fuzzy sets such as (3, 2)-fuzzy sets [28] and SR-fuzzy sets [29]. Moreover, we will integrate rough set theory and infra-fuzzy topology to form a new frame aiming to handle uncertain situations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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