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Research Article

General Decay of a Nonlinear Viscoelastic Wave Equation with Balakrishnân-Taylor Damping and a Delay Involving Variable Exponents

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This paper was aimed at investigating the stability of the following viscoelastic problem with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term $u_{tt}-\mathscr{M}(\|\nabla u\|_2^2)\Delta u+\alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1|u_t|^{p(.)-2}u_t+\mu_2|u_t(t-\tau)|^{p(.)-2}u_t(t-\tau)=0$ in $\Omega\times\mathbb{R}^+$, where Ω is a bounded domain of $\mathbb{R}^n$, $p(.)$: $\bar{\Omega}\longrightarrow\mathbb{R}$ is a measurable function, $g>0$ is a memory kernel that decays exponentially, $\alpha\geq 0$ is the potential, and $\mathscr{M}(\|\nabla u\|_2^2)=a+b\|\nabla u(t)\|_2^2+\sigma\int_\Omega \nabla u\nabla u_t dx$ for some constants $a>0$, $b\geq 0$, and $\sigma>0$. Under some assumptions on the relaxation function, we use some suitable Lyapunov functionals to derive the general decay estimate for the energy. The problem considered is novel and meaningful because of the presence of the flutter panel equation and the spillover problem including memory and variable-exponent time delay control. Our result generalizes and improves previous conclusion in the literature.

1. Introduction

In recent years, much attention has been paid to the study systems with variable exponents of nonlinearities which are models of hyperbolic, parabolic, and elliptic equations. These models may be nonlinear over the gradient of unknown solutions and have nonlinear variable exponents. Researches of these systems usually use the imbedding of Lebesgue and Sobolev spaces with variable exponents (see, e.g., [1, 2]). Or see [3–14] and the references therein for more details of relevant problems.

In this paper, we concentrate on the asymptotic behavior of weak solutions for the following weakly damped viscoelastic wave equation with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term

$$\begin{cases} u_{tt}-\mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u+\alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1|u_t|^{p(x)-2}u_t+\mu_2|u_t(t-\tau)|^{p(x)-2}u_t(t-\tau)=0, & \text{in } \Omega\times(0,\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{in } \Omega,\\ u_t(x,t)=j_0(x,t-\tau), & \text{in } \Omega\times(0,\tau),\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1}$$

where $u:\bar{\Omega}\times[0,\infty)\longrightarrow\mathbb{R}$ is unknown function, $\mu_1\ge 0$,μ_2 is a real number, $\tau>0$ is the time delay, $g>0$ is a memory kernel, and $\alpha\ge 0$ is the potential.

Much attention has been paid to the simulation of phenomena such as the vibration of elastic strings and elastic plates, when $g=0$, and $\mu_1=\mu_2=0$; equation (1)) degrades into the Kirchhoff's original equation

$$\rho h\frac{\partial^2 u}{\partial t^2}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}+f, 0\le x\le L, t\ge 0, \tag{2}$$

which was first introduced to study the oscillations of stretched strings and plates in [15]. In addition, equation (2) is also said to be the wave equation of Kirchhoff type, where the unknown function $u=u(x,t)$ represents lateral deflection and E,ρ,h,L,p_0, and f, respectively, denote Young's modulus, mass density, cross-section area, length, initial axial tension, and external force. The Kirchhoff equation has been investigated in a lot of articles due to its abundant physical background. At the present paper, we try to mention some considerable efforts on this topic.

There are many important results, such as the local solutions in time, well-posedness, and solvability; for the Kirchhoff type, equation (2) in general dimensions and domains has been obtained in lots of articles (see, e.g., [16–24] and the references therein).

When $p>1$ identically equals to a constant, problem (1) with the Balakrishnân-Taylor damping term $(\sigma>0)$ is related to the flutter panel equation and the spillover problem involving time delay term. Balakrishnân and Taylor in [25] and Bass and Zes in [26] introduced Balakrishnân-Taylor damping, which arises from a wind tunnel experiment at supersonic speeds (see, e.g., [22, 27–32]).

On damping terms, we point out several excellent works: Lian and Xu in [33] studied a class of nonlinear wave equations with weak and strong damping terms, and they established the existence of weak solutions and related blow-up results under three different initial energy levels and different conditions. Yang et al. [34] investigated the exponential stability of a system with locally distributed damping. Lian et al. [35] were interested in a fourth-order wave equation with strong and weak damping terms; they obtained the local solution, the global existence, asymptotic behavior, and blow-up of solutions under some condition.

Time delays are common phenomena in many physical, chemical, biological, thermal, and so on (see [36–38] for more details). Several authors have investigated existence and stability of the solutions to the viscoelastic wave equation involving delay term under some appropriate conditions on μ_1,μ_2, and g (see, e.g., [39]). For other related problems, one can also refer to [40–44]. The terminology variable exponents mean that $p(.)$ is a measurable function and not a constant. This term $\mu_1|u_t|^{p(.)-2}u_t+\mu_2|u_t(t-\tau)|^{p(.)-2}u_t(t-\tau)$ is a generalization of $\mu_1 u_t+\mu_2 u_t(t-\tau)$, which corresponds to $p(.)>1$. In fact, (1) is also an extension of the second-order viscoelastic wave equation under variable growth conditions

$$\begin{aligned} &u_{tt}-\mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u\\ &\quad+\alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1 u_t+\mu_2 u_t(t-\tau)\\ &\quad=0 \text{ in } \Omega\times\mathbb{R}^+, \end{aligned} \tag{3}$$

which is obtained when considering $\mu_1|u_t|^{p(.)-2}u_t+\mu_2|u_t(t-\tau)|^{p(.)-2}u_t(t-\tau)$. Equation (3) is a well-known electrorheological fluid model that appears in fluid dynamic treatment (see in [45]). However, the researches related to the viscoelastic wave equation possessing delay terms, Balakrishnân-Taylor damping, and variable growth conditions are not sufficient, and the results about these equations are relatively rare (see [46]). In particular, in [40], the authors considered this class of equations under some suitable assumptions; they use suitable Lyapunov functionals to derive general energy decay results, and one see similar work in [44]. Mingione and Rădulescu [47] were concerned with the regularity theory of elliptic variational problems under nonstandard growth conditions.

This paper devotes to generalize some previous results. In particular, in this case, we will use the relaxation function, the specified initial data, and a special Lyapunov functional, which depends on the behavior of the relation function and is not necessary to decay in some polynomial or exponential form, to get a general decay estimate of the energy.

In addition to the introduction, this paper is divided into two parts. In Section 2, we review some basic definitions about Lebesgue and Sobolev spaces with variable exponentials and give some related properties. At the end of this section, we present our main results. In Section 3, we prove our results, showing that a solution of (1) possesses a general decay with small initial values (u_0,u_1).

2. Functional Setting and Main Results

In this section, we will give some preliminaries and our main results.

Without loss of generality, hereinafter, we suppose $\Omega\subseteq\mathbb{R}^n$ $(n\ge 1)$ is a bounded domain with smooth boundary Γ. Moreover, let $p:\bar{\Omega}\longrightarrow(1,+\infty)$ be a measurable function and denote

$$\begin{aligned} p^- &:= \operatorname*{essinf}_{x\in\Omega}[p(x)],\\ p^+ &:= \operatorname*{esssup}_{x\in\Omega}[p(x)]. \end{aligned} \tag{4}$$

As in [1, 48, 49], we define the following variable-exponent Lebesgue spaces and Sobolev spaces. The first one is the variable-exponent space $L^{p(.)}(\Omega)$:

$$L^{p(.)}(\Omega)=\left\{\psi:\Omega\longrightarrow\mathbb{R}\text{ measurable}|\varrho_{p(.),\Omega}(\psi):=\int_\Omega|\psi(x)|^{p(x)}\mathrm{d}x<+\infty\right\}, \tag{5}$$

and it is obvious a Banach space with the following

Luxemburg norm

$$\|\psi\|_{p(.),\Omega} := \inf\left\{\nu > 0 | \int_\Omega \left|\frac{u(x)}{\nu}\right|^{p(x)} dx \le 1\right\}. \tag{6}$$

Actually, in many respects, variable-exponent Lebesgue spaces are very similar to classical Lebesgue spaces (see [49]). In particular, from the above definition of the norm, we can directly get the following results:

$$\min\left(\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right) \le \varrho_{p(.),\Omega}(u) \le \max\left(\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right). \tag{7}$$

For any measurable function $p : \bar{\Omega} \longrightarrow [p^-, p^+] \subset (2,\infty)$, where $p^\pm$ are constants, we define the second space and the variable-exponent Lebesgue space

$$L^{p(.)}(\Omega) = \left\{\phi : \Omega \longrightarrow \mathbb{R} : \phi \text{ is measurable function on } \Omega, \int_\Omega |\phi(x)|^{p(x)} dx < \infty\right\}, \tag{8}$$

which is a Banach space with the following Luxemburg norm:

$$\|u\|_{p(.)} = \inf\left\{\nu > 0, \int_\Omega \left|\frac{u}{\nu}\right|^{p(x)} dx \le 1\right\}. \tag{9}$$

We also assume that p satisfies the following Zhikov-Fan condition for the local uniform continuity: there exist a constant $M > 0$ such that for all points x,y in Ω with $|x - y| < 1/2$, we have the inequality

$$|p(x) - p(y)| \le \frac{M}{|\log|x - y||}. \tag{10}$$

In addition, $\|.\|_q$ and $\|.\|_{H^1(\Omega)}$ denote the usual $L^q(\Omega)$ norm and $H^1(\Omega)$ norm.

In order to obtain the main results, we give the following lemma firstly.

Lemma 1 (see [1]).

(1) If

$$2 \le p^- := ess \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := ess \sup_{x\in\Omega} p(x) < \infty, \tag{11}$$

then

$$\min\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \le \int_\Omega |u(.)|^{p(x)} dx \le \max\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \tag{12}$$

for any $u \in L^{p(.)}(\Omega)$

(2) Assume that $m, n, p : \bar{\Omega} \longrightarrow (1,+\infty)$ *are measurable functions satisfying*

$$\frac{1}{m(.)} = \frac{1}{p(.)} + \frac{1}{n(.)} \tag{13}$$

Then, for all functions $u \in L^{p(.)}(\Omega)$ *and* $v \in L^{n(.)}(\Omega)$, *we have* $uv \in L^{m(.)}(\Omega)$ *with*

$$\|uv\|_{m(.)} \le \mathscr{C}\|u\|_{p(.)}\|v\|_{n(.)}. \tag{14}$$

Lemma 2. *Suppose that* $p : \Omega \longrightarrow [p^-, p^+] \subset [1,+\infty)$ *is a measurable function satisfying*

$$ess \sup_{x\in\Omega} p(x) < p_* \le \frac{2n}{n-2} \text{ with } p_* = \frac{np(x)}{ess \sup_{x\in\Omega}(n - p(x))}. \tag{15}$$

Then, the embedding $H_0^1(\Omega) = W_0^{1,2}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact, and there is a constant $c_* = c_*(\Omega, p^\pm)$ such that

$$\|\phi\|_{p(.)} \le c_* \|\nabla\phi\|_2 \text{ for } \phi \in H_0^1(\Omega). \tag{16}$$

We assume that the relaxation function g and the potential α satisfy the following assumptions:

Hypothesis g, α: $g, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are nonincreasing differentiable functions such that

$$g(s) \ge 0, l_0 = \int_0^\infty g(s) ds < \infty, \alpha(t) > 0, a - \alpha(t)\int_0^t g(s) ds \ge l > 0 \tag{17}$$

Hypothesis ξ: there exist a positive differentiable functions ξ satisfying

$$g'(t) \le -\xi(t)g(t), \text{ for } t \ge 0, \lim_{t\longrightarrow\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0 \tag{18}$$

Hypothesis $p(.)$: the function $p(.)$ satisfies

$$p^- \ge 2, \text{ if } n = 1, 2, 2 < p^- \le p(x) \le p^+ < \frac{n+2}{n-2} \text{ if } n \ge 3 \tag{19}$$

Hypothesis μ_1 and μ_2: the constants μ_1 and μ_2 satisfy

$$|\mu_2| < p^- \mu_1 \tag{20}$$

Calculating $(d/dt)\alpha(t)(g \circ u)(t)$ with respect to t, it shows that

$$\alpha(t)\int_0^t g(t-s)\int_\Omega u(s)\mathrm{d}s u_t(t)\mathrm{d}x = -\frac{\alpha(t)}{2}g(t)\|u(t)\|_2^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\alpha(t)}{2}(g\circ u)(t) - \frac{\alpha(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s\right] + \frac{\alpha(t)}{2}\left(g'\circ u\right)(t) + \frac{\alpha'(t)}{2}(g\circ u)(t) - \frac{\alpha'(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s, \tag{21}$$

where

$$(g\circ u)(t) = \int_0^t g(t-s)\|u(t)-u(s)\|_2^2\mathrm{d}s. \tag{22}$$

As in [38, 43], we present a new time-dependent variable to deal with the time delay term:

$$z(x,\rho,t) = u_t(x, t-\tau\rho), x\in\Omega, \rho\in(0,1), t>0. \tag{23}$$

Consequently, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \text{in}\, \Omega\times(0,1)\times(0,\infty). \tag{24}$$

Therefore, problem (1) can be transformed into

$$\begin{gathered} u_{tt} - \mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s + \mu_1|u_t|^{p(x)-2}u_t + \mu_2|z(1,t)|^{p(x)-2}z(1,t) = 0, \text{in}\, \Omega\times\mathbb{R}^+, \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \text{in}\, (0,1)\times(0,\infty), \\ z(0,t) = u_t, \text{in}\, (0,+\infty), \\ z(\rho,0) = j_0(-\rho(\tau+1)), \text{in}\, (0,1), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x\in\Omega. \end{gathered} \tag{25}$$

By the standard methods as in Section 3 of [50], we can easily prove the well-posedness of problem (1) presented as follows.

Theorem 3. *Let (17)–(20) be in force and $(u_0, u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $j_0\in L^2((\Omega)\times(0,1))$. Then, problem (1) possesses a unique local solution u such that*

$$u\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right), u_t\in C\left([0,T];H_0^1(\Omega)\right)\cap L^2([0,T]\times(\Omega)). \tag{26}$$

3. Main Asymptotic Theorem

Next, we will give the proof of Theorem 4.

The functional E of problem (25) is as follows:

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(a - \alpha(t)\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x + \frac{1}{2}\alpha(t)(g\circ\nabla(u))(t), \tag{27}$$

where ξ and λ are positive constants and they satisfy

$$\mu_1 p^- - |\mu_2| > \xi > |\mu_2|p^+\frac{p^+-1}{p^-}, \lambda < \frac{1}{\tau}\left|\ln\frac{\mu_2 p^+(p^+-1)}{\xi p^-}\right|. \tag{28}$$

The most important key to solve problem (1) is to obtain a result that concerns the asymptotic stability of solutions.

The main result is as follows.

Theorem 4. *Suppose (17)–(20) and (28) hold. Then, there exists positive constants C_0, C, and $t_1>0$ such that*

$$E(t)\le C_0 e^{-C\int_{t_1}^t \nu}, \textit{for}\ t\ge t_1. \tag{29}$$

To prove this theorem, the following technical lemmas are necessary.

Lemma 5. *If u is a solution of problem (25). Then,*

$$\begin{aligned}E'(t) \le & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{1}{2}\alpha(t)\left(g'\circ\nabla u\right)(t) \\ & - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 + \frac{1}{2}\alpha'(t)(g\circ\nabla u)(t) \\ & - \left(\mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}dx \\ & - \left(\frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}dx \\ & - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}dsdx.\end{aligned} \tag{30}$$

Proof. Using the same idea as in [50], multiply the first equation in (25) by u_t and then integrate in Ω. Similarly, multiply the second equation in (25) by $\xi z e^{-\lambda\tau\rho}$ and integrate in $(0, 1)\times\Omega$. Summarizing the above, we can obtain

$$\begin{aligned}E'(t) = & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t) \\ & - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s - \frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2 \\ & + \frac{\alpha'(t)}{2}(g\circ\nabla u)(t) - \mu_1\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\ & - \xi\int_\Omega \frac{1}{p(x)}e^{-\lambda\tau}|u_t(x,t-\tau)|^{p(x)}\mathrm{d}x \\ & - \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x \\ & + \xi\int_\Omega \boxtimes\frac{1}{p(x)}|u_t(x,t)|^{p(x)}\mathrm{d}x \\ & - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{31}$$

By $z(1,t) = u_t(t-\tau)$ and the Young inequality, we get

$$\begin{aligned}& -\mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x \\ & \quad \le |\mu_2|\frac{p^+-1}{p^-}\int_\Omega |z(1,t)|^{p(x)}dx + \frac{|\mu_2|}{p^-}\int_\Omega |u_t|^{p(x)}\mathrm{d}x.\end{aligned} \tag{32}$$

From (23), we have

$$\begin{aligned}& -\xi\int_\Omega \frac{1}{p(x)}e^{-\lambda\tau}|u_t(x,t-\tau)|^{p(x)}\mathrm{d}x \\ & \quad \le -\frac{\xi}{p^+}e^{-\lambda\tau}\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x.\end{aligned} \tag{33}$$

Comparing (31) and (32), we obtain

$$\begin{aligned}E'(t) \le & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t) \\ & - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s - \frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2 \\ & + \frac{\alpha'(t)}{2}(g\circ\nabla u)(t) - \left(\mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\ & - \left(\frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x \\ & - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{34}$$

Setting

$$\begin{aligned}c_0 &= \mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}, \\ c_1 &= \frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-},\end{aligned} \tag{35}$$

by condition (28), we derived the desired inequality (30). □

Remark 6. If

$$-\frac{1}{2}\alpha'(t)\|\nabla u(t)\|_2^2\int_0^t g(s)\mathrm{d}s \ge 0 \tag{36}$$

holds, $E(t)$ may not be nonincreasing.

Lemma 7. *Assume that u be a solution of problem (25). Then,*

$$\|\nabla u\|_2^2 \le \frac{2E(0)}{l}e^{(l_0/l)\alpha(0)}, t \ge 0, \tag{37}$$

where l_0 and l as in (17).

Proof. From (27) and (30), we have

$$E'(t) \le -\frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds \le -\frac{1}{2}l_0\alpha'(t)\|\nabla u\|_2^2 \le -\frac{l_0}{l}\alpha'(t)E(t). \tag{38}$$

Integrating the above inequality in $(0, t)$, we get

$$E(t) \le E(0)e^{-(l_0/l)\alpha(t)+(l_0/l)\alpha(0)} \le E(0)e^{(l_0/l)\alpha(0)}. \tag{39}$$

From (27), we see that

$$\|\nabla u\|_2^2 \le \frac{2}{l}E(t). \tag{40}$$

Combining it with (39), it gives (37).

Now, we give a modified functional:

$$\mathrm{L}(t) = NE(t) + \varepsilon_1 \alpha(t)\varphi(t) + \varepsilon_2 \alpha(t)\psi(t), \tag{41}$$

$$\varphi(t) = \int_\Omega u(t)u_t(t)\mathrm{d}x + \frac{\sigma}{4}\|\nabla u\|_2^4, \tag{42}$$

$$\psi(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x, \tag{43}$$

where ε_1, ε_2, and N are positive constants. In fact, L is equivalent to E by the following lemma. □

Lemma 8. *There exists $C_1, C_2 > 0$ such that*

$$C_1 E(t) \leq L(t) \leq C_2 E(t), t \geq 0. \tag{44}$$

Proof. By the Poincaré theorem and Young inequality, we have the following results through integrating by parts:

$$\begin{aligned}
|\mathrm{L}(t) - NE(t)| &= \left|\varepsilon_1\alpha(t)\int_\Omega u(t)u_t(t)\mathrm{d}x + \varepsilon_1\alpha(t)\frac{\sigma}{4}\|\nabla u\|_2^4 + \varepsilon_2\alpha(t)\psi(t)\right| \\
&\leq \varepsilon_1|\alpha(t)|\int_\Omega |u(t)||u_t(t)|\mathrm{d}x + \varepsilon_1\frac{\sigma}{4}|\alpha(t)|\|\nabla u\|_2^4 + \varepsilon_2\frac{1}{2}|\alpha(t)|\|u_t\|_2^2 \\
&+ \varepsilon_2\frac{1}{2}|\alpha(t)|c_*^2(a-l)(g\circ\nabla(u))(t) \leq \varepsilon_1\frac{\alpha(0)}{2}c_*^2\|\nabla u\|_2^2 + \varepsilon_1\frac{\alpha(0)}{2}\|u_t\|_2^2 \\
&+ \varepsilon_1\sigma\frac{\alpha(0)}{4}\|\nabla u\|_2^4 + \varepsilon_2\frac{1}{2}\alpha(0)\|u_t\|_2^2 + \varepsilon_2\frac{1}{2}\alpha(0)c_*^2(a-l)(g\circ\nabla(u))(t) \\
&\leq C(\varepsilon_1+\varepsilon_2)E(t),
\end{aligned} \tag{45}$$

where c_* as in Lemma 1, taking $C_1 = N - C(\varepsilon_1+\varepsilon_2)$ and $C_2 = N + C(\varepsilon_1+\varepsilon_2)$, provided ε_1 and ε_2 are sufficiently small, and the proof is completed. □

Lemma 9. *There exists $c_\varepsilon, C_\varepsilon > 0$ fulfilling*

$$\begin{aligned}
\varphi'(t) \leq \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\frac{a}{2l}(g\circ\nabla u)(t) \\
+ c_\varepsilon\left(\int_\Omega |u_t|^{p(x)}dx + \int_\Omega |z(1,t)|^{p(x)}dx\right) \\
+ C_\varepsilon\int_\Omega |u|^{p(x)}dx.
\end{aligned} \tag{46}$$

Proof. By the first equation of (25), we differentiate (42), and then we have

$$\begin{aligned}
\varphi'(t) &= \|u_t\|_2^2 + \int_\Omega v + \sigma\|\nabla u\|_2^2\int_\Omega \nabla u\nabla u_t\mathrm{d}x \\
&= \|u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u(t)\mathrm{d}x \\
&- \mu_1\int_\Omega |u_t|^{p(x)-2}u_t u\mathrm{d}x - \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u\mathrm{d}x = \|u_t\|_2^2 \\
&- a\|\nabla u\|_2^2 - bv_2^4 + I_1 + I_2 + I_3.
\end{aligned} \tag{47}$$

By the Hölder inequality, Sobolev-Poincaré inequalities, and (17), we estimate the second part of the right-hand side in (47).

$$\begin{aligned}
I_1 &= \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u(t)\mathrm{d}x \\
&\leq \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right|^2\mathrm{d}x\right)^{1/2} \\
&\leq \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega\int_0^t g(s)\mathrm{d}s\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{1/2} \\
&\leq \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s\right)^{1/2}\left(\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{1/2}{}^2\mathrm{d}s\mathrm{d}x. \\
&\leq \frac{\alpha(t)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s + \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x \\
&\leq \frac{\alpha(t)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s + \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s) \\
&-\nabla u(t)+\nabla u(t)|
\end{aligned} \tag{48}$$

For every $\eta > 0$, using the Young inequality and (17), we deduce

$$\begin{aligned}
&\frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)[\nabla u(s)-\nabla u(t)+\nabla u(t)]^2\mathrm{d}s\mathrm{d}x \\
&\quad\leq \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)\left((\nabla u(s)-\nabla u(t))^2 + 2|\nabla u(s)-\nabla u(t)||\nabla u| + |\nabla u|^2\right) \\
&\quad\mathrm{d}s\mathrm{d}x \leq \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\mathrm{d}s\mathrm{d}x \\
&\qquad+ \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u|^2\mathrm{d}s\mathrm{d}x \\
&\qquad+ \alpha(t)\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)||\nabla u|\mathrm{d}s\mathrm{d}x \\
&\quad\leq \frac{\alpha(t)}{2}(g\circ\nabla u)(t) + \frac{\alpha(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x \\
&\qquad+ \eta\frac{\alpha(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2\eta}(g\circ\nabla u)(t) \\
&\leq \frac{\alpha(t)}{2}(1+\eta)\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t) \\
&\leq (1+\eta)\frac{(a-l)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t).
\end{aligned} \tag{49}$$

Summarizing the above estimates, (48) and (49), we obtain

$$\begin{aligned}
\alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x &\leq \frac{(a-l)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x \\
&+ \frac{(a-l)}{2}(1+\eta)\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t) \\
&= (2+\eta)\frac{(a-l)}{2}\|\nabla u\|^2 + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t).
\end{aligned} \tag{50}$$

Setting $\eta = l/(a-l)$, it is easy to obtain

$$\begin{aligned}|I_1| &\le \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x \\ &\le \left(a-\frac{l}{2}\right)\|\nabla u\|_2^2 + \frac{a}{2l}\alpha(t)(g\circ\nabla u)(t),\end{aligned} \tag{51}$$

and by means of the Young inequality, we have

$$\begin{aligned}|I_2| \le c_\varepsilon\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \varepsilon \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x &\coloneqq c_\varepsilon\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\ &+ \varepsilon c_2\int_\Omega |u|^{p(x)}\mathrm{d}x,\end{aligned} \tag{52}$$

$$\begin{aligned}|I_3| \le{}& c_\varepsilon\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x \\ &+ \varepsilon \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x \coloneqq c_\varepsilon\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x \\ &+ \varepsilon c_3\int_\Omega |u|^{p(x)}\mathrm{d}x.\end{aligned} \tag{53}$$

Substituting (51)–(53) into (47), we deduce

$$\begin{aligned}\varphi'(t) \le{}& \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 + C_\varepsilon\int_\Omega |u|^{p(x)}\mathrm{d}x - b\|\nabla u\|_2^4 \\ &+ \frac{a}{2l}\alpha(t)(g\circ\nabla u)(t) + c_\varepsilon\left(\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x\right),\end{aligned} \tag{54}$$

set $C_\varepsilon = \varepsilon(c_2 + c_3) > 0$, for ε sufficiently small. □

Lemma 10. *There exists positive constants δ and c_δ satisfying*

$$\begin{aligned}\psi'(t) \le{}& -\left\{\left(\int_0^t g(s)ds\right\} - \delta\right)\|u_t\|_2^2 + \delta\left\{a + 2(a-l)^2\alpha(t)\right\}\|\nabla u\|_2^2 \\ &+ \delta b\|\nabla u\|_2^4 + \delta\frac{2\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \\ &+ \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right)(a-l)\alpha(t)\right\}(g\circ\nabla u)(t) \\ &+ c_\delta\left(\int_\Omega |u_t|^{p(x)}dx + \int_\Omega |z(1,t)|^{p(x)}dx\right) \\ &- \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t).\end{aligned} \tag{55}$$

Proof. Similar to Lemma,9 by the first equation (25), we differentiate (43), and it yields

$$\begin{aligned}\psi'(t) ={}& -\int_\Omega u_{tt}\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\ ={}& \left(a + b\|\nabla u\|_2^2\right)\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &+ \sigma\int_\Omega \nabla u\nabla u_t\mathrm{d}x\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &- \alpha(t)\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\ &+ \mu_1\int_\Omega |u_t|^{p(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &+ \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\ ={}& \sum_{i=1}^6 I_i - \left(\int_0^t \boxtimes g(s)\mathrm{d}s\right)\|u_t\|_2^2.\end{aligned} \tag{56}$$

By the Hölder inequality, Sobolev-Poincaré inequalities, and (17), we estimate the second part of the right-hand side in (56).

$$\begin{aligned}|I_1| &\le \left(a + b\|\nabla u\|_2^2\right)\left\{\delta\|\nabla u\|_2^2 + \frac{l_0}{4\delta}(g\circ\nabla u)(t)\right\} \\ &\le \delta a\|\nabla u\|_2^2 + \delta b\|\nabla u\|_2^4 + \left\{\frac{al_0}{4\delta} + \frac{bl_0E(0)}{2\delta l}e^{(l_0/l)\alpha(0)}\right\}(g\circ\nabla u)(t),\end{aligned} \tag{57}$$

$$\begin{aligned}|I_2| &\le \delta\sigma\left(\int_\Omega \nabla u\nabla u_t\mathrm{d}x\right)^2\|\nabla u\|_2^2 + \frac{\sigma l_0}{4\delta}(g\circ\nabla u)(t) \\ &\le \delta\frac{2\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 + \frac{\sigma l_0}{4\delta}(g\circ\nabla u)(t),\end{aligned} \tag{58}$$

$$\begin{aligned}|I_3| \le{}& \delta\alpha(t)\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)| + |\nabla u(t)|)\mathrm{d}s\right)^2\mathrm{d}x \\ &+ \frac{1}{4\delta}\alpha(t)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\mathrm{d}s\right)^2\mathrm{d}x \\ \le{}& 2\delta l_0^2\alpha(t)\|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)l_0\alpha(t)(g\circ\nabla u)(t),\end{aligned} \tag{59}$$

$$
\begin{aligned}
|I_4| &\le c_\delta \int_\Omega |u_t|^{p(x)}\mathrm{d}x + \delta \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^{p(x)} \mathrm{d}x \\
&\le c_\delta \int_\Omega |u_t|^{p(x)}\mathrm{d}x + \delta\Big\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left(l_0^{p^+-1}, l_0^{p^--1}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right) \\
&\quad \int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^{p(x)}\mathrm{d}s\Big\} \\
&\le c_\delta \int_\Omega |u_t|^{p(x)}\mathrm{d}x + \delta\Big\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right) \max \\
&\quad \left(\left(\frac{2E(0)}{l} e^{(l_0/l)\alpha(0)}\right)^{(p^+-2)/2}, \left(\frac{2E(0)}{l} e^{(l_0/l)\alpha(0)}\right)^{(p^--2)/2}\right) \\
&\quad (g\circ\nabla u)(t)\Big\} \coloneqq c_\delta \int_\Omega |u_t|^{p(x)}\mathrm{d}x + \delta c_4 (g\circ\nabla u)(t).
\end{aligned}
\tag{60}
$$

Similarly,

$$
\begin{aligned}
|I_5| &\le c_\delta \int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x + \delta c_5 (g\circ\nabla u)(t), \\
|I_6| &\le \delta\|u_t\|_2^2 - \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t).
\end{aligned}
\tag{61}
$$

Comparing these above estimates (57)–(61), we have

$$
\begin{aligned}
\psi'(t) \le &-\left(\int_0^t g(s)\mathrm{d}s - \delta\right)\|u_t\|_2^2 + \delta\left\{a + 2l_0^2\alpha(t)\right\}\|\nabla u\|_2^2 \\
&+ \delta b\|\nabla u\|_2^4 + \delta \frac{2\sigma E(0)}{l} e^{(l_0/l)\alpha(0)} \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 \\
&+ \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right) l_0 \alpha(t)\right\}(g\circ\nabla u)(t) \\
&+ c_\delta\left(\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x\right) \\
&- \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t),
\end{aligned}
\tag{62}
$$

where $C_\delta = \{a l_0/4\delta + (b l_0 E(0)/2\delta l)e^{(l_0/l)\alpha(0)} + \sigma l_0/4\delta + \delta(c_4 + c_5)\}$. □

Lemma 11. *There exists positive constants C_3, C_4, and t_0 satisfying*

$$
L'(t) \le -C_3 \alpha(t) E(t) + C_4 \alpha(t)(g\circ\nabla u)(t), t > t_0. \tag{63}
$$

Proof. Since $g > 0$ and is continuous, then for any $t \ge t_0 > 0$, we get

$$
\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s = g_0 > 0. \tag{64}
$$

Differentiate (41), and using Lemmas 9 and 10, we get

$$
\begin{aligned}
\mathrm{L}'(t) &= NE'(t) + \varepsilon_1\alpha'(t)\varphi(t) + \varepsilon_1\alpha(t)\varphi'(t) + \varepsilon_2\alpha'(t)\psi(t) + \varepsilon_2\alpha(t)\psi'(t) \\
&\le -\alpha(t)\{\varepsilon_2(g_0-\delta) - \varepsilon_1\}\|u_t\|_2^2 \\
&\quad - \alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a + 2l_0^2\right)\alpha(t)\right\}\|\nabla u\|_2^2 \\
&\quad - \alpha(t)(b(\varepsilon_1 - \varepsilon_2\delta))\|u_t\|_2^4 \\
&\quad - \alpha(t)\left\{\sigma - \varepsilon_2\delta\frac{\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\right\}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \\
&\quad + \alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta + \frac{1}{4\delta}\right)l_0\alpha(t)\right\}(g\circ\nabla u)(t) \\
&\quad + \alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t) \\
&\quad - \alpha(t)\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\
&\quad - \alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x \\
&\quad - \frac{N\alpha'(t)}{2}\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \varepsilon_1\alpha'(t)\int_\Omega u u_t\mathrm{d}x \\
&\quad + \varepsilon_2\alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x.
\end{aligned}
\tag{65}
$$

Indeed,

$$
\begin{aligned}
&\alpha'(t)\int_\Omega u u_t\mathrm{d}x + \alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
&\quad \le -\alpha'(t)\frac{c_*^2}{2}\|\nabla u\|_2^2 - \alpha'(t)\|u_t\|_2^2 - \alpha'(t)\frac{c_*^2}{2}\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t).
\end{aligned}
\tag{66}
$$

Thus,

$$
\begin{aligned}
\mathrm{L}'(t) &\le -\alpha(t)\left\{\varepsilon_2(g_0-\delta) - \varepsilon_1 + \frac{\alpha'(t)}{\alpha(t)}\right\}\|u_t\|_2^2 \\
&\quad - \alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a + 2l_0^2\right)\alpha(0) + \frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t \boxtimes g(s)\mathrm{d}s\right) + \frac{c_*^2\alpha'(t)}{2\alpha(t)}\right\} \\
&\|\nabla u\|_2^2 - \alpha(t)b(\varepsilon_1 - \varepsilon_2\delta)\|\nabla u\|_2^4 - \alpha(t)\left\{\sigma\varepsilon_1 - \varepsilon_2\delta\frac{\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\right\} \\
&\quad \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 \\
&+ \alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta + \frac{1}{4\delta}\right)l_0\alpha(t) - \frac{c_*^2\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right)\right\} \\
&(g\circ\nabla u)(t) + \alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t) - \alpha(t) \\
&\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\mathrm{d}x - \alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\} \\
&\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x.
\end{aligned}
\tag{67}
$$

Fix $\delta > 0$ such that

$$
g_0 - \delta > \frac{1}{2}g_0, \frac{\delta}{C_\varepsilon}\left(a + 2l_0^2\right)\alpha(0) < \frac{1}{4}g_0, \tag{68}
$$

and take ε_1 and ε_2 small enough to satisfy

$$\begin{aligned}&\frac{g_0}{4}\varepsilon_2<\varepsilon_1<\varepsilon_2\frac{g_0}{2},\\&c_5=\varepsilon_2(g_0-\delta)-\varepsilon_1>0,\\&c_6=\varepsilon_1C_\varepsilon-\varepsilon_2\delta\left(a+2l_0^2\right)\alpha(0)>0.\end{aligned}\tag{69}$$

Select ε_1 and ε_2 small enough to make (44) and (67) hold, and moreover

$$\begin{aligned}&b(\varepsilon_1-\varepsilon_2\delta)>0,\sigma\varepsilon_1-\varepsilon_2\delta\frac{\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}>0,\frac{N}{2}-\varepsilon_2\frac{g(0)c_*^2}{4\delta}>0\\&\frac{c_0}{\alpha(0)}-\varepsilon_1c_\varepsilon-\varepsilon_2c_\delta>0,\frac{c_1}{\alpha(0)}-\varepsilon_1c_\varepsilon-\varepsilon_2c_\delta>0.\end{aligned}\tag{70}$$

Hence, for a generic positive constant c, (67) is equal to the following results:

$$\begin{aligned}L'(t)\le&-\alpha(t)\left\{c+\frac{\alpha'(t)}{\alpha(t)}\right\}\|u_t\|_2^2\\&-\alpha(t)\left\{c+\frac{\alpha'(t)}{2\alpha(t)}\left(\left(\int_0^t g(s)\mathrm{d}s\right)+\frac{c_*^2}{2}\right)\right\}\|\nabla u\|_2^2\\&+\alpha(t)\left\{c-\frac{c_*^2g_0\alpha'(t)}{2\alpha(t)}\right\}(g\circ\nabla u)(t),\forall t\ge t_0.\end{aligned}\tag{71}$$

Noticing that $\lim_{t\longrightarrow\infty}-\alpha'(t)/\xi(t)\alpha(t)=0$, so choose $t_1>t_0$, we see

$$\begin{aligned}L'(t)&\le-\alpha(t)\left(c\|u_t\|_2^2+C\|\nabla u\|_2^2\right)+c(g\circ\nabla u)(t)\\&\le-C_3\alpha(t)E(t)+C_4\alpha(t)(g\circ\nabla u)(t),\forall t\ge t_1,\end{aligned}\tag{72}$$

where C_3 and C_4 are positive constants. □

Now, we are in the position to prove Theorem 4.

Proof of Theorem 4. According to Lemma 5, Lemma 11, and (17), we have

$$\begin{aligned}\zeta(t)L'(t)&\le-C_3\alpha(t)\zeta(t)E(t)+C_4\alpha(t)\zeta(t)(g\circ\nabla u)(t)\\&\le-C_3\alpha(t)\zeta(t)E(t)-C_4\alpha(t)\left(g'\circ\nabla u\right)(t)\\&\le-C_3\alpha(t)\zeta(t)E(t)\\&\quad-C_4\left(2E'(t)+\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\right).\end{aligned}\tag{73}$$

Since $\zeta(t)$ is nonincreasing, by assumption (17) and the definition of $E(t)$, we get

$$\begin{aligned}&\frac{l}{2}\|\nabla u\|_2^2\le E(t),\\&\frac{\mathrm{d}}{\mathrm{d}t}(\zeta(t)L(t)+2C_4E(t))\le-C_3\alpha(t)\zeta(t)E(t)\\&\qquad-C_4\alpha'(t)\left(\int_0^s g(s)\mathrm{d}s\right)\|\nabla u\|_2^2,\end{aligned}\tag{74}$$

which leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\zeta(t)F(t)+2C_4E(t))\le-C_3\alpha(t)\zeta(t)E(t)\\&-C_4\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\le-C_3\alpha(t)\zeta(t)E(t)\\&-\frac{2C_4E(t)}{l}\alpha'(t)\int_0^t g(s)\mathrm{d}s\le-\alpha(t)\zeta(t)\left(C_3+\frac{2C_4l_0\alpha'(t)}{l\alpha(t)\zeta(t)}\right)E(t).\end{aligned}\tag{75}$$

Since $\lim_{t\longrightarrow\infty}-\alpha'(t)/\alpha(t)\zeta(t)=0$, we can choose $t_1\ge t_0$ such that $C_3+2C_4l_0\alpha'(t)/l\alpha(t)\zeta(t)>0$ for $t\ge t_1$. Hence, if we let

$$\mathscr{L}(t)=\zeta(t)L(t)+2C_4E(t),\tag{76}$$

then it is obvious that $\mathscr{L}(t)$ is equivalent to $E(t)$ and satisfies

$$\mathscr{L}'(t)\le-k\zeta(t)\alpha(t)\mathscr{L}(t)\text{ for }t\ge t_1.\tag{77}$$

Consequently, to integrate (77) over (t_1,t), it yields

$$\mathscr{L}(t)\le\mathscr{L}(t_1)e^{-C\int_{t_1}^t\zeta(s)\alpha(s)\mathrm{d}s}t\ge t_0.\tag{78}$$

Thus, the desired result yields from the equivalence relations of $\mathscr{L}(t)$,$L(t)$, and $E(t)$. □

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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