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Corrigendum

Corrigendum to "Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay"

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In the article titled "Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay" [1], there are a number of minor typographical errors introduced to the equations during the typesetting of the article. The corrected article is as follows.

Abstract
A linearized numerical scheme is proposed to solve the nonlinear time fractional parabolic problems with time delay. The scheme is based on the standard Galerkin finite element method in the spatial direction, the fractional Crank-Nicolson method, and extrapolation methods in the temporal direction. A novel discrete fractional Grönwall inequality is established. Thanks to the inequality, the error estimate of fully discrete scheme is obtained. Several numerical examples are provided to verify the effectiveness of the fully discrete numerical method.

1. Introduction

In this paper, we consider the linearized fractional Crank-Nicolson-Galerkin finite element method for solving the nonlinear time fractional parabolic problems with time delay

$$\begin{cases} {}^{R}D_t^{\alpha}u - \Delta u = f(t, u(x,t), u(x,t-\tau)), & \text{in } \Omega \times (0,T], \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\tau, 0], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1}$$

where Ω is a bounded convex and convex polygon in R^2 (or polyhedron in R^3) and τ is the delay term. ${}^{R}D_t^{\alpha}u$ denotes the Riemann-Liouville fractional derivative, defined by

$${}^{R}D_t^{\alpha}u(\cdot,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}u(\cdot,s)ds, \quad 0<\alpha<1. \tag{2}$$

The nonlinear fractional parabolic problems with time delay have attracted significant attention because of their widely range of applications in various fields, such as biology, physics, and engineering [1–9]. Recently, plenty of numerical methods were presented for solving the linear time fractional diffusion equations. For instance, Chen et al. [10] used finite difference methods and the Kansa method to approximate time and space derivatives, respectively. Dehghan et al. [11] presented a full discrete scheme based on the finite difference methods in time direction and the meshless Galerkin method in space direction and proved that the scheme was unconditionally stable and convergent. Murio [12] and Zhuang [13] proposed a fully implicit finite difference numerical scheme and obtained unconditionally stability. Jin et al. [14] derived the time fractional Crank-Nicolson scheme to approximate Riemann-Liouville fractional derivative. Li et al. [15] used a transformation to develop some new schemes for solving the time-fractional problems. The new schemes admit some advantages for both capturing the initial layer and solving the models with small parameter α. More studies can be found in [16–32].

Recently, it has been one of the hot spots in the investigations of different numerical methods for the nonlinear time fractional problems. For the analysis of the L1-type methods, we refer readers to the paper [33–40]. For the analysis of the convolution quadrature methods or the fractional Crank-Nicolson scheme, we refer to the recent papers [41–46]. The key role in the convergence analysis of the schemes is the fractional Grönwall type inequations. However, as pointed out in [47–49], the similar fractional Grönwall type inequations can not be directly applied to study the convergence of numerical schemes for the nonlinear time fractional problems with delay.

In this paper, we present a linearized numerical scheme for solving the nonlinear fractional parabolic problems with time delay. The time Riemann-Liouville fractional derivative is approximated by fractional Crank-Nicolson type time-stepping scheme, the spatial derivative is approximated by using the standard Galerkin finite element method, and the nonlinear term is approximated by the extrapolation method. To study the numerical behavior of the fully discrete scheme, we construct a novel discrete fractional type Grönwall inequality. With the inequality, we consider the convergence of the numerical methods for the nonlinear fractional parabolic problems with time delay.

The rest of this article is organized as follows. In Section 2, we present a linearized numerical scheme for the nonlinear time fractional parabolic problems with delay and main convergence results. In Section 3, we present a detailed proof of the main results. In Section 4, numerical examples are given to confirm the theoretical results. Finally, the conclusions are presented in Section 5.

2. Fractional Crank-Nicolson-Galerkin FEMs

Denote $\mathcal{T}_h$ is a shape regular, quasi-uniform triangulation of the Ω into d-simplexes. Let $h = \max_{K \in \mathcal{T}_h}\{\text{diam } K\}$. Let X_h be the finite-dimensional subspace of $H_0^1(\Omega)$ consisting of continuous piecewise function on $\mathcal{T}_h$. Let $\Delta t = \tau / m_\tau$ be the time step size, where m_τ is a positive integer. Denote $N = \lceil T/\Delta t \rceil$, $t_j = j\Delta t$, $j = -m_\tau, -m_\tau + 1, \cdots, 0, 1, 2, \cdots, N$.

The approximation to the Riemann-Liouville fractional derivative at point $t = t_{n-(\alpha/2)}$ is given by [14]:

$$\begin{aligned} {}^R D^\alpha_{t_{n-(\alpha/2)}} u(x,t) &= \Delta t^{-\alpha} \sum_{i=0}^{n} \omega_{n-i}^{(\alpha)} u(x, t_i) + \mathcal{O}\left(\Delta t^2\right) \\ &:= {}^R D^\alpha_{\Delta t} u^n + \mathcal{O}\left(\Delta t^2\right), \end{aligned} \tag{3}$$

where

$$\omega_i^{(\alpha)} = (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha - i + 1)}. \tag{4}$$

For simplicity, denote $\|v\| = \left(\int_\Omega |v(x)|^2 dx\right)^{1/2}$, $\eta^{n,\alpha} = (1 - (\alpha/2))\eta^n + (\alpha/2)\eta^{n-1}$, $\widehat{\eta}^{n,\alpha} = (2 - (\alpha/2))\eta^{n-1} - (1 - (\alpha/2))\eta^{n-2}$, $t_n^\alpha = (n\Delta t)^\alpha$.

With the notation, the fully discrete scheme is to find $U_h^n \in X_h$ such that

$$\begin{aligned} &\left\langle {}^R D^\alpha_{\Delta t} U_h^n, v \right\rangle + \langle \nabla U_h^{n,\alpha}, \nabla v \rangle \\ &= \left\langle f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right), v \right\rangle, \forall v \in X_h, n = 1, 2, \cdots, N, \end{aligned} \tag{5}$$

and the initial condition

$$U_h^n = R_h \varphi(x, t_n), \quad n = -m_\tau, -m_\tau + 1, \cdots, 0, \tag{6}$$

where $R_h : H_0^1(\Omega) \longrightarrow X_h$ is Ritz projection operator which satisfies following equality [50]

$$\langle \nabla R_h u, \nabla v \rangle = \langle \nabla u, \nabla v \rangle, \forall u \in H_0^1(\Omega) \cap H^2(\Omega), v \in X_h. \tag{7}$$

We present the main convergence results here and leave its proof in the next section.

Theorem 1. Suppose the system (1) has a unique solution u satisfying

$$\begin{aligned} &\|u_0\|_{H^{r+1}} + \|u\|_{C\left([0,T];H^{r+1}\right)} + \|u_t\|_{C\left([0,T];H^{r+1}\right)} \\ &+ \|u_{tt}\|_{C\left([0,T];H^2\right)} + \left\| {}^R D^\alpha_{\Delta t} u \right\|_{C\left([0,T];H^{r+1}\right)} \le K, \end{aligned} \tag{8}$$

and the source term $f(t, u(x,t), u(x, t-\tau))$ satisfies the Lipschitz condition

$$\begin{aligned} &|f(t, u(x,t), u(x,t-\tau)) - f(t, v(x,t), v(x,t-\tau))| \\ &\le L_1 |u(x,t) - v(x,t)| + L_2 |u(x,t,\tau) - v(x,t,\tau)|, \end{aligned} \tag{9}$$

where K is a constant independent of n, h and Δt, L_1 and L_2 are given positive constants. Then, there exists a positive constant Δt^* such that for $\Delta t \le \Delta t^*$, the following estimate holds that

$$\|u^n - U_h^n\| \le C_1^*\left(\Delta t^2 + h^{r+1}\right), \quad n = 1, 2, \cdots, N, \tag{10}$$

where C_1^* is a positive constant independent of h and Δt.

Remark 2. The main contribution of the present study is that we obtain a discrete fractional Grönwall’s Grönwall's inequality. Thanks to the inequality, the convergence of the fully discrete scheme for the nonlinear time fractional parabolic problems with delay can be obtained.

Remark 3. At present, the convergence of the proposed scheme is proved without considering the weak singularity of the solutions. In fact, if the initial layer of the problem is taken into account, some corrected terms are added at the beginning. Then, the scheme can be of order two in the temporal direction for nonsmooth initial data and some incompatible source term. However, we still have the difficulties to get the similar discrete fractional Grönwall's inequality. We hope to leave the challenging problems in future.

3. Proof of the Main Results

In this section, we will present a detailed proof of the main result.

3.1. Preliminaries and Discrete Fractional Grönwall Inequality

Firstly, we review the definition of weights $\omega_i^{(\alpha)}$, denote $g_n^{(\alpha)} = \sum_{i=0}^{n}\omega_i^{(\alpha)}$. Then, we can get

$$\begin{cases} \omega_0^{(\alpha)} = g_0^{(\alpha)}, \\ \omega_i^{(\alpha)} = g_i^{(\alpha)} - g_{i-1}^{(\alpha)}, \quad 1 \le i \le n. \end{cases} \tag{11}$$

Actually, it has been shown [51] that $\omega_i^{(\alpha)}$ and $g_n^{(\alpha)}$ process following properties.

(1) The weights $\omega_i^{(\alpha)}$ can be evaluated recursively, $\omega_i^{(\alpha)} = (1-((\alpha+1)/i))\omega_{i-1}^{(\alpha)}, i \ge 1, \omega_0^{(\alpha)} = 1$

(2) The sequence $\{\omega_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone increasing $-1 < \omega_i^{(\alpha)} < \omega_{i+1}^{(\alpha)} < 0, i \ge 1$

(3) The sequence $\{g_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone decreasing, $g_i^{(\alpha)} > g_{i+1}^{(\alpha)}$ for $i \ge 0$ and $g_0^{(\alpha)} = 1$

Noticing the definition of $g_i^{(\alpha)}$, ${}^{R}D_{\Delta t}^{\alpha}u^n$ can be rewritten as

$${}^{R}D_{\Delta t}^{\alpha}u^n = \Delta t^{-\alpha}\sum_{i=1}^{n}\left(g_i^{(\alpha)} - g_{i-1}^{(\alpha)}\right)u^{n-i} + \Delta t^{-\alpha}g_0^{(\alpha)}u^n. \tag{12}$$

In fact, rearranging this identity yields

$${}^{R}D_{\Delta t}^{\alpha}u^n = \Delta t^{-\alpha}\sum_{i=1}^{n}g_{n-i}^{(\alpha)}\delta_t u^i + \Delta t^{-\alpha}g_n^{(\alpha)}u^0, \tag{13}$$

where $\delta_t u^i = u^i - u^{i-1}$.

Lemma 4 (see [51]). Consider the sequence $\{\phi_n\}$ given by

$$\phi_0 = 1, \phi_n = \sum_{i=1}^{n}\left(g_{i-1}^{(\alpha)} - g_i^{(\alpha)}\right)\phi_{n-i}, \quad n \ge 1. \tag{14}$$

Then, $\{\phi_n\}$ satisfies the following properties

(i) $0 < \phi_n < 1, \sum_{i=j}^{n}\phi_{n-i}g_{i-j}^{(\alpha)} = 1, 1 \le j \le n$

(ii) $(1/\Gamma(\alpha))\sum_{i=1}^{n}\phi_{n-i} \le (n^{\alpha}/\Gamma(1+\alpha))$

(iii) $(1/(\Gamma(\alpha)\Gamma(1+(k-1)\alpha)))\sum_{i=1}^{n-1}\phi_{n-i}i^{(k-1)\alpha} \le (n^{k\alpha}/\Gamma(1+\alpha)), k = 1, 2 \cdots$.

Lemma 5 (see [51]). Consider the matrix

$$W = 2\mu(\Delta t)^{\alpha}\begin{pmatrix} 0 & \phi_1 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{15}$$

Then, W satisfies the following properties

(i) $W^l = 0, l \ge n$

(ii) $W^k\vec{e} \le (1/\Gamma(1+k\alpha))[(2\Gamma(\alpha)\mu t_n^{\alpha})^k, (2\Gamma(\alpha)\mu t_{n-1}^{\alpha})^k, \cdots, (2\Gamma(\alpha)\mu t_1^{\alpha})^k]', k = 0, 1, 2, \cdots$

(iii) $\sum_{k=0}^{l}W^k\vec{e} = \sum_{k=0}^{n-1}W^k\vec{e} \le [E_\alpha(2\Gamma(\alpha)\mu t_n^{\alpha}), E_\alpha(2\Gamma(\alpha)\mu t_{n-1}^{\alpha}), \cdots, E_\alpha(2\Gamma(\alpha)\mu t_1^{\alpha})]', l \ge n$

where $\vec{e} = [1, 1, \cdots, 1]' \in \mathbb{R}^n$, μ is a constant.

Theorem 6. Assuming $\{u^n | n = -m, -m+1, \cdots, 0, 1, 2, \cdots\}$ and $\{f^n | n = 0, 1, 2, \cdots\}$ are nonnegative sequence, for $\lambda_i > 0$, $i = 1, 2, 3, 4, 5$, if

$${}^{R}D_{\Delta t}^{\alpha}u^j \le \lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1} + f^j, \quad j = 1, 2 \cdots, \tag{16}$$

then there exists a positive constant Δt^*, for $\Delta t < \Delta t^*$, the following holds

$$u^n \le 2\left(\lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \max_{1\le j\le n}f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)} + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}\right)E_\alpha(2\Gamma(\alpha)\lambda t_n^{\alpha}), \quad 1 \le n \le N, \tag{17}$$

where $\lambda = \lambda_1 + (1/(g_0^{(\alpha)} - g_1^{(\alpha)}))\lambda_2 + (1/(g_1^{(\alpha)} - g_2^{(\alpha)}))\lambda_3 + (1/(g_{m-1}^{(\alpha)} - g_m^{(\alpha)}))\lambda_4 + (1/(g_m^{(\alpha)} - g_{m+1}^{(\alpha)}))\lambda_5$, $E_\alpha(z) = \sum_{k=0}^{\infty}(z^k/\Gamma(1+k\alpha))$ is the Mittag-Leffler function, and $M = \max\{u^{-m}, u^{-m+1}, \cdots, u^0\}$.

Proof. By using the definition of ${}^{R}D^{\alpha}_{\Delta t}u^n$ in (13), we have

$$\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k + g_j^{(\alpha)}u^0 \le \Delta t^{\alpha}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + \Delta t^{\alpha} f^j. \quad (18)$$

Multiplying equation (18) by ϕ_{n-j} and summing the index j from 1 to n, we get

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k \le \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}f^j - \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0. \quad (19)$$

We change the order of summation and make use of the definition of ϕ_{n-j} to obtain

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k = \sum_{k=1}^{n}\delta_t u^k\sum_{j=1}^{k}\phi_{n-j}g_{j-k}^{(\alpha)} = \sum_{k=1}^{n}\delta_t u^k = u^n - u^0, \quad (20)$$

and using Lemma 4, we have

$$\Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}f^j \le \Delta t^{\alpha}\max_{1\le j\le n} f^j\sum_{j=1}^{n}\phi_{n-j} \le \Delta t^{\alpha}\max_{1\le j\le n} f^j\frac{\Gamma(\alpha)n^{\alpha}}{\Gamma(1+\alpha)} = \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}. \quad (21)$$

Noticing $g_j^{(\alpha)}$ is monotone decreasing, and using Lemma 4, we have

$$-\sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le u^0\sum_{j=1}^{n}\phi_{n-j}g_{j-1}^{(\alpha)} = u^0. \quad (22)$$

Substituting (20), (21), and (22) into (19), we can obtain

$$u^n \le \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + 2u^0 + \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}. \quad (23)$$

Applying Lemma 4, we have

$$\Delta t^{\alpha}\sum_{j=1}^{m}\phi_{n-j}u^{j-m} \le \frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M, \Delta t^{\alpha}\sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} \le \frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M. \quad (24)$$

Therefore

$$\lambda_4\Delta t^{\alpha}\sum_{j=1}^{m}\phi_{n-j}u^{j-m} + \lambda_5\Delta t^{\alpha}\sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} + 2u^0 + \lambda_2\Delta t^{\alpha}\phi_{n-1}u^0 + \lambda_3\Delta t^{\alpha}\left(\phi_{n-1}u^{-1} + \phi_{n-2}u^0\right) \le \lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}. \quad (25)$$

Denote

$$\Psi_n = \lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)} + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}. \quad (26)$$

(23) can be rewritten as

$$(1-\lambda_1\Delta t^{\alpha})u^n \le \lambda_1\Delta t^{\alpha}\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\Delta t^{\alpha}\sum_{j=2}^{n}\phi_{n-j}u^{j-1} + \lambda_3\Delta t^{\alpha}\sum_{j=3}^{n}\phi_{n-j}u^{j-2} + \lambda_4\Delta t^{\alpha}\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} + \lambda_5\Delta t^{\alpha}\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1} + \Psi_n. \quad (27)$$

Let $\Delta t^* = \sqrt[\alpha]{1/(2\lambda_1)}$, when $\Delta t \le \Delta t^*$, we have

$$u^n \le 2\Psi_n + 2\Delta t^{\alpha}\left[\lambda_1\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\sum_{j=2}^{n}\phi_{n-j}u^{j-1} + \lambda_3\sum_{j=3}^{n}\phi_{n-j}u^{j-2} + \lambda_4\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} + \lambda_5\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1}\right]. \quad (28)$$

Let $V = (u^n, u^{n-1}, \cdots, u^1)^T$, then (28) can be rewritten in the following matrix form

$$V \le 2\Psi_n\vec{e} + (\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3 + \lambda_4 W_4 + \lambda_5 W_5)V, \quad (29)$$

where

$$W_1 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & \phi_1 & \phi_2 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_1 & \phi_2 \\ 0 & 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_2 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_3 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ 0 & 0 & 0 & \cdots & \phi_{n-5} & \phi_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_4 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & \cdots & 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-m-2} & \phi_{n-m-1} \\ 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_5 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_{n-m-4} & \phi_{n-m-3} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \quad (30)$$

Since the definition of ϕ_n, we have

$$\phi_{n-j} \le \frac{1}{g_{j-1}^{(\alpha)} - g_j^{(\alpha)}} \phi_n. \quad (31)$$

Then,

$$\begin{aligned} W_2 V &\le \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} W_1 V, \\ W_3 V &\le \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} W_1 V, \\ W_4 V &\le \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} W_1 V, \\ W_5 V &\le \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} W_1 V. \end{aligned} \quad (32)$$

Hence, (29) can be shown as follows

$$\begin{aligned} V &\le \left(\lambda_1 + \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} \lambda_2 + \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} \lambda_3 \right. \\ &\quad \left. + \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} \lambda_4 + \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} \lambda_5 \right) W_1 V + 2\Psi_n \vec{e} \\ &= WV + 2\Psi_n \vec{e}, \end{aligned} \quad (33)$$

where $W = \lambda W_1$.
Therefore,

$$\begin{aligned} V &\le WV + 2\Psi_n \vec{e} \le W\left(WV + 2\Psi_n \vec{e} \right) + 2\Psi_n \vec{e} \\ &= W^2 V + 2\Psi_n \sum_{j=0}^{1} W^j \vec{e} \le \cdots \le W^n V + 2\Psi_n \sum_{j=0}^{n-1} W^j \vec{e}. \end{aligned} \quad (34)$$

According to Lemma 5, the result can be proved.

Lemma 7 (see [51]). For any sequence $\{e^k\}_{k=0}^N \subset X_h$, the following inequality holds

$$\left\langle {}^R D_{\Delta t}^\alpha e^k, \left(1 - \frac{\alpha}{2}\right) e^k + \frac{\alpha}{2} e^{k-1} \right\rangle \ge \frac{1}{2} \, {}^R D_{\Delta t}^\alpha \left\| e^k \right\|^2, \quad 1 \le k \le N. \quad (35)$$

Lemma 8 (see [52]). There exists a positive constant C_Ω, independent of h, for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$, such that

$$\|v - R_h v\|_{L^2} + h \|\nabla (v - R_h v)\|_{L^2} \le C_\Omega h^s \|v\|_{H^s}, \quad 1 \le s \le r+1. \quad (36)$$

3.2. Proof of Theorem 1

Now, we are ready to prove our main results.

Proof. Taking $t = t_{n-(\alpha/2)}$ in the first equation (1), we can find that u^n satisfies the following equation

$$\left\langle {}^R D^{\alpha}_{\Delta t} u^n, v \right\rangle + \langle \nabla u^{n,\alpha}, \nabla v \rangle = \left\langle f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right), v \right\rangle + \langle P^n, v \rangle, \tag{37}$$

for $n = 1, 2, 3, \cdots, N$ and $\forall v \in X_h$, where

$$\begin{aligned} P^n = {} & {}^R D^{\alpha}_{\Delta t} u^n - {}^R D^{\alpha}_{t_{n-(\alpha/2)}} u + \Delta u^{n-(\alpha/2)} - \Delta u^{n,\alpha} \\ & + f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) \\ & - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \end{aligned} \tag{38}$$

Now, we estimate the error of $\|P^n\|$. Actually, from the definition of $u^{n,\alpha}$ and $\widehat{u}^{n,\alpha}$ and the regularity of the exact solution (8), we can obtain that

$$\begin{aligned} & \left\| u^{n-(\alpha/2)} - u^{n,\alpha} \right\| \\ & \quad = \left\| \left(1 - \frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)} - u^n\right) + \frac{\alpha}{2}\left(u^{n-(\alpha/2)} - u^{n-1}\right) \right\| \\ & \quad = \left\| -\left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t u'(\xi_1) + \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t u'(\xi_2) \right\| \\ & \quad = \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t \left\| \left(u'(\xi_2) - u'(\xi_1)\right) \right\| \\ & \quad \le \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\| ds \le C_1 \Delta t^2, \end{aligned} \tag{39}$$

$$\begin{aligned} & \left\| u^{n-(\alpha/2)} - \widehat{u}^{n,\alpha} \right\| \\ & = \left\| u^{n-(\alpha/2)} - \left(2 - \frac{\alpha}{2}\right)u^{n-1} + \left(1 - \frac{\alpha}{2}\right)u^{n-2} \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)u^{n-(\alpha/2)} - \left(2 - \frac{\alpha}{2}\right)u^{n-1} \right. \\ & \quad \left. + \left(1 - \frac{\alpha}{2}\right)u^{n-2} - \left(1 - \frac{\alpha}{2}\right)u^{n-(\alpha/2)} \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)} - u^{n-1}\right) + \left(1 - \frac{\alpha}{2}\right)\left(u^{n-2} - u^{n-(\alpha/2)}\right) \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t u'(\xi_3) - \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t u'(\xi_4) \right\| \\ & = \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t \left\| u'(\xi_3) - u'(\xi_4) \right\| \\ & \le \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t \int_{t_{n-2}}^{t_{n-1}} \|u_{tt}(s)\| ds \le C_2 \Delta t^2, \end{aligned} \tag{40}$$

where $\xi_1 \in (t_{n-(\alpha/2)}, t_n)$, $\xi_2 \in (t_{n-1}, t_{n-(\alpha/2)})$, $\xi_3 \in (t_{n-(\alpha/2)}, t_{n-1})$, $\xi_4 \in (t_{n-2}, t_{n-(\alpha/2)})$, $C_1 = (1 - (\alpha/2))(\alpha/2)K$, $C_2 = (2 - (\alpha/2))(1 - (\alpha/2))K$ are constants.

Applying (39) and (40) and the Lipschitz condition

$$\begin{aligned} & \left\| f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right) \right\| \\ & \quad \le (L_1 C_1 + L_2 C_2)\Delta t^2, \end{aligned} \tag{41}$$

$$\left\| \Delta\left(u^{n,\alpha} - u^{n-(\alpha/2)}\right) \right\| \le C_1 \Delta t^2, \tag{42}$$

which further implies that

$$\|P^n\| \le C_K (\Delta t)^2, n = 1, 2, 3, \cdots, N, \tag{43}$$

here $C_K = L_1 C_1 + L_2 C_2$.

Denote $\theta_h^n = R_h u^n - U_h^n$, $n = 0, 1, \cdots, N$.

Substituting fully scheme (5) from equation (37) and using the property in (7), we can get that

$$\left\langle {}^R D^{\alpha}_{\Delta t} \theta_h^n, v \right\rangle + \langle \nabla \theta_h^{n,\alpha}, v \rangle = \langle R_1^n, v \rangle + \langle P^n, v \rangle - \left\langle {}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n), v \right\rangle, \tag{44}$$

where

$$R_1^n = f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right) - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \tag{45}$$

Setting $v = \theta_h^{n,\alpha}$ and applying Cauchy-Schwarz inequality, it holds that

$$\begin{aligned} & \left\langle {}^R D^{\alpha}_{\Delta t} \theta_h^n, \theta_h^{n,\alpha} \right\rangle + \|\nabla \theta_h^{n,\alpha}\|^2 \\ & \quad \le \|R_1^n\| \|\theta_h^{n,\alpha}\| + \|P^n\| \|\theta_h^{n,\alpha}\| + \left\| {}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n) \right\| \|\theta_h^{n,\alpha}\|. \end{aligned} \tag{46}$$

Noticing the fact $ab \le 1/2(a^2 + b^2)$ and $\|\nabla \theta_h^{n,\alpha}\|^2 \ge 0$,

$$\begin{aligned} \left\langle {}^R D^{\alpha}_{\Delta t} \theta_h^n, \theta_h^{n,\alpha} \right\rangle \le {} & \frac{1}{2}\left(\|R_1^n\|^2 + \|P^n\|^2 + \left\| {}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n) \right\|^2 \right) \\ & + \frac{3}{2}\|\theta_h^{n,\alpha}\|^2. \end{aligned} \tag{47}$$

Together with (9) and (36), we can arrive that

$$\left\| {}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n) \right\| \le C_\Omega h^{r+1} \left\| {}^R D^{\alpha}_{\Delta t} u^n \right\|_{H^{r+1}} \le C_\Omega K h^{r+1},$$

$$\begin{aligned} & \|\widehat{u}^{n,\alpha} - R_h \widehat{u}^{n,\alpha}\| \\ & \quad = \left\| \left(2 - \frac{\alpha}{2}\right)u^{n-1} - \left(1 - \frac{\alpha}{2}\right)u^{n-2} \right. \\ & \qquad \left. - \left(2 - \frac{\alpha}{2}\right)R_h u^{n-1} + \left(1 - \frac{\alpha}{2}\right)R_h u^{n-2} \right\| \\ & \quad \le \left(2 - \frac{\alpha}{2}\right)\left\| u^{n-1} - R_h u^{n-1} \right\| + \left(1 - \frac{\alpha}{2}\right)\left\| u^{n-2} - R_h u^{n-2} \right\| \\ & \quad \le \left(2 - \frac{\alpha}{2}\right)C_\Omega h^{r+1}\left\| u^{n-1} \right\|_{H^{r+1}} + \left(1 - \frac{\alpha}{2}\right)C_\Omega h^{r+1}\left\| u^{n-2} \right\|_{H^{r+1}} \\ & \quad \le \left(2 - \frac{\alpha}{2}\right)C_\Omega K h^{r+1} + \left(1 - \frac{\alpha}{2}\right)C_\Omega K h^{r+1} \le C_3 h^{r+1}. \end{aligned} \tag{48}$$

Similarly, we have

$$\begin{aligned}
&\|u^{n-m_\tau,\alpha} - R_h u^{n-m_\tau,\alpha}\| \\
&\quad = \left\| \left(1-\frac{\alpha}{2}\right) u^{n-m_\tau} + \frac{\alpha}{2} u^{n-m_\tau-1} \right. \\
&\qquad \left. - \left(1-\frac{\alpha}{2}\right) R_h u^{n-m_\tau} - \frac{\alpha}{2} R_h u^{n-m_\tau-1} \right\| \\
&\quad \le \left(1-\frac{\alpha}{2}\right) C_\Omega K h^{r+1} + \frac{\alpha}{2} C_\Omega K h^{r+1} \le C_4 h^{r+1},
\end{aligned} \tag{49}$$

where $C_3 = 2(2-(\alpha/2))C_\Omega K$, $C_4 = 2\max\{(1-(\alpha/2)), \alpha/2\}C_\Omega K$.

Therefore

$$\begin{aligned}
\|R_1^n\| &= \left\| f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right) - f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right) \right\| \\
&\le L_1 \left\| \widehat{u}^{n,\alpha} - \widehat{U}_h^{n,\alpha} \right\| + L_2 \| u^{n-m_\tau,\alpha} - U_h^{n-m_\tau,\alpha} \| \\
&\le L_1 \left\| \widehat{\theta}_h^{n,\alpha} \right\| + L_2 \|\theta_h^{n-m_\tau,\alpha}\| + L_1 \|\widehat{u}^{n,\alpha} - R_h \widehat{u}^{n,\alpha}\| \\
&\quad + L_2 \|u^{n-m_\tau,\alpha} - R_h u^{n-m_\tau,\alpha}\| \\
&\le L_1 \left\| \widehat{\theta}_h^{n,\alpha} \right\| + L_2 \|\theta_h^{n-m_\tau,\alpha}\| + (L_1 C_3 + L_2 C_4) h^{r+1}.
\end{aligned} \tag{50}$$

Substituting (43), (48), and (50) into (47) and the fact $(a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$, we can get

$$\begin{aligned}
\left\langle {}^R D_{\Delta t}^\alpha \theta_h^n, \theta_h^{n,\alpha} \right\rangle &\le \frac{3}{2} \|\theta_h^{n,\alpha}\|^2 + \frac{3{L_1}^2}{2} \left\| \widehat{\theta}_h^{n,\alpha} \right\|^2 \\
&\quad + \frac{3{L_2}^2}{2} \left\| \theta_h^{n-m_\tau,\alpha} \right\|^2 + \frac{{C_K}^2}{2} (\Delta t)^4 \\
&\quad + \frac{1}{2} \left[3\left({L_1}^2 {C_3}^2 + {L_2}^2 {C_4}^2\right) + (C_K K)^2 \right] h^{2(r+1)} \\
&\le \frac{3}{2} \|\theta_h^{n,\alpha}\|^2 + \frac{3{L_1}^2}{2} \left\| \widehat{\theta}_h^{n,\alpha} \right\|^2 \\
&\quad + \frac{3{L_2}^2}{2} \left\| \theta_h^{n-m_\tau,\alpha} \right\|^2 + \frac{C_4}{2} \left(\Delta t^2 + h^{r+1}\right)^2,
\end{aligned} \tag{51}$$

where $C_4 = \max\{{C_K}^2, 3({L_1}^2 {C_3}^2 + {L_2}^2 {C_4}^2) + (C_K K)^2\}$.

Table 1: The errors and convergence orders in temporal direction by using Q-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	1.6856e-03	*	5.3999e-03	*
10	2.9420e-04	2.5184	1.2503e-03	2.1106
20	5.9619e-05	2.3030	3.0266e-04	2.0465
40	1.3851e-05	2.1058	7.4700e-05	2.0185

Applying Lemma 7, we have

$$\begin{aligned}
{}^R D_{\Delta t}^\alpha \|\theta_h^n\|^2 &\le 3\|\theta_h^{n,\alpha}\|^2 + 3{L_1}^2 \left\| \widehat{\theta}_h^{n,\alpha} \right\|^2 \\
&\quad + 3{L_2}^2 \left\| \theta_h^{n-m_\tau,\alpha} \right\|^2 + C_4 \left(\Delta t^2 + h^{r+1}\right)^2.
\end{aligned} \tag{52}$$

In terms of the definition of $\|\theta_h^{n,\alpha}\|$ and $\widehat{\theta}_h^{n,\alpha}$, we obtain

$$\begin{aligned}
{}^R D_{\Delta t}^\alpha \|\theta_h^n\|^2 &\le 3\left(1-\frac{\alpha}{2}\right)^2 \|\theta_h^n\|^2 + \left(3\left(\frac{\alpha}{2}\right)^2 + 3{L_1}^2\left(2-\frac{\alpha}{2}\right)^2\right) \|\theta_h^{n-1}\|^2 \\
&\quad + 3{L_1}^2 \left(1-\frac{\alpha}{2}\right)^2 \|\theta_h^{n-2}\|^2 + 3{L_2}^2 \left(1-\frac{\alpha}{2}\right)^2 \|\theta_h^{n-m_\tau}\|^2 \\
&\quad + 3{L_2}^2 \left(\frac{\alpha}{2}\right)^2 \left\| \theta_h^{n-m_\tau-1} \right\|^2 + C_4 \left(\Delta t^2 + h^{r+1}\right)^2.
\end{aligned} \tag{53}$$

Using Theorem 6, we can find a positive constant Δt^* such that $\Delta t \le \Delta t^*$, then

$$\|\theta_h^n\|^2 \le C_5 \left(\Delta t^2 + h^{r+1}\right)^2, \tag{54}$$

where C_5 is a nonnegative constant which only dependents on $L_1, L_2, C_4, C_K, C_\Omega$. In terms of the definition of θ_h^n, we have

$$\|u^n - U_h^n\| \le \|u^n - R_h u^n\| + \|R_h u^n - U_h^n\| \le C_1^* \left(\Delta t^2 + h^{r+1}\right). \tag{55}$$

Then, we complete the proof.

4. Numerical Examples

In this section, we give two examples to verify our theoretical results. The errors are all calculated in L2-norm.

Example 1. Consider the nonlinear time fractional Mackey-Glass-type equation

$$\begin{cases}
{}^R D_{\Delta t}^\alpha u(x,y,t) = \Delta u(x,y,t) - 2u(x,y,t) + \dfrac{u(x,y,t-0.1)}{1+u^2(x,y,t-0.1)} + f(x,y,t), (x,y) \in [0,1]^2, t \in [0,1], \\
u(x,y,t) = t^2 \sin(\pi x) \sin(\pi y), \quad (x,y) \in [0,1]^2, t \in [-0.1,0],
\end{cases} \tag{56}$$

Table 2: The errors and convergence orders in spatial direction by using L-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	7.2603e-02	*	7.2065e-02	*
10	1.9449e-02	1.9003	1.9297e-02	1.9009
20	8.7594e-03	1.9673	8.6948e-03	1.9662
40	4.9508e-03	1.9834	4.9180e-03	1.9807

Table 3: The errors and convergence orders in spatial direction by using Q-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	2.0750e-03	*	2.0746e-03	*
10	2.4888e-04	3.0596	2.5148e-04	3.0443
20	7.3251e-05	3.0165	7.5802e-05	2.9577
40	3.0946e-05	2.9952	3.4200e-05	2.7666

where

$$f(x,y,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin(\pi x)\sin(\pi y) + 2t^2\pi^2 \sin(\pi x)\sin(\pi y) - 2t^2 \sin(\pi x)\sin(\pi y) - \frac{(t-0.1)^2 \sin(\pi x)\sin(\pi y)}{1+\left[(t-0.1)^2 \sin(\pi x)\sin(\pi y)\right]^2}. \tag{57}$$

The exact solution is given as

$$u(x,t) = t^2 \sin(\pi x)\sin(\pi y). \tag{58}$$

Table 4: The errors and orders in temporal and spatial direction by using L-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	8.3275e-02	*	8.3375e-02	*
10	2.2615e-02	1.8806	2.2732e-02	1.8749
20	5.8356e-03	1.9543	5.8662e-03	1.9542
40	1.4707e-03	1.9884	1.4784e-03	1.9884

Table 5: The errors and orders in temporal direction and spatial direction by using Q-FEM.

	M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
	8	6.7379e-04	*	6.9141e-04	*
$N = M^{(3/2)}$	10	3.1416e-04	3.0459	3.4945e-04	3.0579
	12	1.9415e-04	3.0968	1.9787e-04	3.1196
	14	1.1891e-04	3.1806	1.1992e-04	3.2485

In order to test the convergence order in temporal direction, we fixed $M = 40$ for $\alpha = 0.4$, $\alpha = 0.6$ and different N. Similarly, to obtain the convergence order in spatial direction, we fixed $N = 100$ for $\alpha = 0.4$, $\alpha = 0.6$ and different M. Table 1 gives the errors and convergence orders in temporal direction by using the Q-FEM. Table 1 shows that the convergence order in temporal direction is 2. Similarly, Tables 2 and 3 give the errors and convergence orders in spatial direction by using the L-FEM and Q-FEM, respectively. These numerical results correspond to our theoretical convergence order.

Example 2. Consider the following nonlinear time fractional Nicholsons blowflies equation

$$\begin{cases} {}^{R}D_{\Delta t}^{\alpha} u(x,y,z,t) = \Delta u(x,y,z,t) - 2u(x,y,z,t) + u(x,y,z,t-0.1)\exp\{-u(x,y,z,t-0.1)\} \\ +f(x,y,z,t), \quad (x,y,z) \in [0,1]^3, \quad t \in [0,1], \\ u(x,y,z,t) = t^2 \sin(\pi x)\sin(\pi y)\sin(\pi z), \quad (x,y,z) \in [0,1]^3, \quad t \in [-0.1,0], \end{cases} \tag{59}$$

where

$$\begin{aligned} f(x,y,z,t) = {} & \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ & + 2t^2\left(\pi^2 - 1\right)\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ & - (t-0.1)^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)\exp \\ & \cdot \left\{-(t-0.1)^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)\right\}, \end{aligned} \tag{60}$$

the exact solution is given as

$$u(x,t) = t^2 \sin(\pi x)\sin(\pi y)\sin(\pi z). \tag{61}$$

In this example, in order to test the convergence order in temporal and spatial direction, we solve this problem by using the L-FEM with $M = N$ and the Q-FEM with $N = M^{(3/2)}$, respectively. Tables 4 and 5 show that the convergence orders in temporal and spatial direction are 2 and 3, respectively. The numerical results confirm our theoretical convergence order.

5. Conclusions

We proposed a linearized fractional Crank-Nicolson-Galerkin FEM for the nonlinear fractional parabolic equations with time delay. A novel fractional Grönwall type inequality is developed. With the help of the inequality, we prove convergence of the numerical scheme. Numerical examples confirm our theoretical results.

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