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Research Article

Kakeya Inequalities by Maximal Functions in Hardy Spaces

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In this paper, we will introduce and study several types of Kakeya inequalities by the maximal functions in Hardy spaces in $\mathbb{R}^n$, $(n \geq 2)$, and we could obtain several inequalities associated with the Kakeya inequalities. We will show that $\|M^t_{\delta S_{\alpha,\beta}} f\|_p \lesssim_{p,n,\varphi,\varepsilon} (1/\delta)^{5((n/r)+2)\varepsilon}\|f\|_p$, when $f(x) \in L^p(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f}(\xi) \subseteq B(0,1)$.

1. Introduction

In 1917, Kakeya [1] proposed a problem to determine the minimal area needed to continuously rotate a unit line segment in the plane by 180 degrees. In 1928, Besicovithch [2] proved the measure of such sets could be arbitrary small. Such sets are called Besicovitch sets or Kakeya sets. The Kakeya conjectures state that the Hausdorff dimension of any Besicovitch sets in $\mathbb{R}^n$ is n. The case for $n \geq 3$ is still an open problem. The so-called maximal Kakeya conjecture (or maximal Nikodym conjecture) is actually a stronger one that involves the following Kakeya maximal function (or Nikodym maximal function):

$$f^*_\delta(\xi) = \sup_{a\in\mathbb{R}^n} \frac{1}{\left|T^\delta_\xi(a)\right|} \int_{T^\delta_\xi(a)} |f(y)|dy, \tag{1}$$

where $T^\delta_\xi(a)$ is a $1 \times \delta$ tube centered at $a \in \mathbb{R}^n$ with the direction $\xi \in S^{n-1}$.

$$f^{**}_\delta(x) = \sup_{x\in T} \frac{1}{|T|} \int_T |f(y)|dy, \tag{2}$$

where the supremum is taken over all $1 \times \delta$ tubes T that contain $x \in \mathbb{R}^n$. Formula (1) is Kakeya maximal function, and Formula (2) is Nikodym maximal function. When $n = 2$, in [3], Cordoba proved that for any $\varepsilon > 0$,

$$\|f^*_\delta\|_{L^2(S^1)} \lesssim_\varepsilon \delta^{-\varepsilon}\|f\|_{L^2(\mathbb{R}^2)}. \tag{3}$$

The Kakeya maximal function conjecture is formulated by Bourgain [4] that

$$\|f^*_\delta\|_{L^p(S^{n-1})} \lesssim_\varepsilon \delta^{-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{4}$$

holds for $p \geq n$ and $n \in \mathbb{N}$, and

$$\|f^*_\delta\|_{L^q(S^{n-1})} \lesssim_\varepsilon \delta^{-n/p+1-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{5}$$

holds for $1 < p \leq n$, $q = (n-1)p'$, and $n \in \mathbb{N}$. In 1983, Drury proved Formula (5) for $p = (d+1)/2$, $q = n+1$, in [5]. In 1991, Bourgain in [4] improved this result for each $n \geq 3$ to some $p(d) \in ((d+1)/2, (d+2)/2)$. By the interpolation theory, (see [6–8] and reference therein),

$$\|f^*_\delta\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon \delta^{-(n-1)/p-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

holds for $p \geq n$ and $n \in \mathbb{N}$.

1.1. Main Result. Inspired by the Formulas (1), (2), and (4)–(6), we will consider maximal functions like $M_{\delta S_{\alpha,\beta}}f(x)$ and $M^t_{\delta S_{\alpha,\beta}}f(x)$ in this paper. Notice that the classical case is $\delta = 1$, and then $M_{1S_{\alpha,\beta}}f(x)$ and $M^t_{1S_{\alpha,\beta}}f(x)$ are classical maximal functions in Hardy spaces. And

$$\left\|M_{1S_{\alpha,\beta}}f\right\|_p \le C\|(f*\varphi)_\nabla\|_p, \left\|M_{1S_{\alpha,\beta}}f\right\|_p \le C\|f\|_p, \quad (7)$$

for some constant $C > 0$.

We will obtain several inequalities in Proposition 6, Theorem 7, and Theorem 8. In Proposition 6, though the coefficient in Formula (71) is not better than the factor $\delta^{-(n-1)/p-\varepsilon}$ in Formula (6), we use a way different to [3, 4] and [5]. And we could obtain Formula (70) which is different to the classical case $\delta = 1$. In Theorem 7, the coefficient is the same as the factor $\delta^{-\varepsilon}$ in Formula (4) when supp $\widehat{f}(\xi) \subseteq B(0,1)$. In Theorem 8, the coefficient is independent on δ when supp $\widehat{f}(\xi) \subseteq \left(B(0, t^{-1}\delta^{-(1+4\varepsilon)})\right)^c$.

1.2. Notations. As usual, we use n to denote the dimension of $\mathbb{R}^n$. supp $f(x)$ is the support set of $f(x)$. If $x \in \mathbb{R}^n$: $x = (x_1, x_2, \cdots, x_n)$, $|x|_e$ denotes $|x|_e = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For $\alpha \in \mathbb{N}^n$: $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha|$ denotes $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We use $\|.\|_p$ to denote $\|.\|_{L^p(\mathbb{R}^n)}$ and $O(\mathbb{R}^n)$ to denote the $n \times n$ unit orthogonal matrix in $\mathbb{R}^n$: $O(\mathbb{R}^n) = \{A : A^T A = 1. A^T$ is the transposed matrix of $A\}$. $S(\mathbb{R}^n)$ designates the space of C^∞ functions on $\mathbb{R}^n$ rapidly decreasing together with their derivatives. $S_{\alpha,\beta}(\mathbb{R}^n)$ denotes $S_{\alpha,\beta}(\mathbb{R}^n) = \{\phi \in S(\mathbb{R}^n): \|\phi\|_{\alpha',\beta'} \le 1, \forall \alpha', \beta' \in \mathbb{N}^n, |\alpha'| \le |\alpha|, |\beta'| \le |\beta|\}$ and ε a positive fixed number (may be very small): $\varepsilon > 0$.

If X and Y are two quantities, $X \lesssim Y$ or $Y \gtrsim X$ denotes that $X \le CY$ for some absolute constant $C > 0$. More generally, given some parameters $a_1, \cdots, a_k$, we use $X \lesssim_{a_1,\cdots,a_k} Y$ or $Y \gtrsim_{a_1,\cdots,a_k} X$to denote the statement that $X \le C_{a_1,\cdots,a_k} Y$ for some constant $C_{a_1,\cdots,a_k}$ which can depend on the parameter $a_1, \cdots, a_k$. We use $X \sim Y$ to denote the statement $X \lesssim Y \lesssim X$, and similarly, $X \sim_{a_1,\cdots,a_k} Y$ denotes $X \lesssim_{a_1,\cdots,a_k} Y \lesssim_{a_1,\cdots,a_k} X$.

2. Preliminaries

For $t, \xi \in \mathbb{R}^n$, $f \in S(\mathbb{R}^n)$, the Fourier transform of f is given by

$$\widehat{f}(\xi) = \mathfrak{F}f(\xi) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i <\xi, t>} dt, \quad (8)$$

and thus $f(x) = (\widehat{f})^\vee(x)$, where $<\xi, t> = \sum_{k=1}^n \xi_k t_k$ and $\vee$ is the inversion of Fourier transform. For $g \in S(\mathbb{R}^n)$, $g_I(x)$ designates

$$g_I(x) = \left(\frac{1}{\delta}\right)^{n-1} g\left(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \cdots, \frac{x_{n-1}}{\delta}, x_n\right). \quad (9)$$

Let $u = (u_1, u_2, \cdots, u_n) = xA^{-1} = (x_1, x_2, \cdots, x_n)A^{-1}$ where A is a variable (not fixed) matrix with $A \in O(\mathbb{R}^n)$, and then $g_{AI}(x)$ is given by

$$g_{AI}(x) = g_I(xA^{-1}) = g_I(u) = \left(\frac{1}{\delta}\right)^{n-1} g\left(\frac{u_1}{\delta}, \frac{u_2}{\delta}, \cdots, \frac{u_{n-1}}{\delta}, u_n\right). \quad (10)$$

If A is a variable (not fixed) matrix with $A \in O(\mathbb{R}^n)$, let

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix} = A\xi = A \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad (11)$$

and thus

$$\begin{aligned} \mathfrak{F}(g_I)(\xi) &= \mathfrak{F}(g)(\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_{n-1}, \xi_n), \\ \mathfrak{F}(g_{AI})(\xi) &= \mathfrak{F}(g_I)(A\xi) = \mathfrak{F}(g_I)(\zeta) \\ &= \mathfrak{F}(g)(\delta\zeta_1, \delta\zeta_2, \cdots, \delta\zeta_{n-1}, \zeta_n). \end{aligned} \quad (12)$$

In this paper, let $\varphi \in S(\mathbb{R}^n)$ always to be a fixed radial function satisfying the following:

$$\begin{cases} \widehat{\varphi}(\xi) = 1, & \text{for } |\xi|_e \le 1, \\ \widehat{\varphi}(\xi) = 0, & \text{for } |\xi|_e \ge 2, \\ \widehat{\varphi}(A\xi) = \widehat{\varphi}(\xi) & \text{for } A \in O(\mathbb{R}^n). \end{cases} \quad (13)$$

$M_Y f(x)$ and $M_{S_{\alpha,\beta}} f(x)$ are given by $M_Y f(x) = \sup_{t>0} |(f * Y_t)(x)|$, $M_{S_{\alpha,\beta}} f(x) = \sup_{Y \in S_{\alpha,\beta}(\mathbb{R}^n)} M_Y f(x)$. And nontangential maximal functions $(f * Y)_\nabla(x)$ are defined as usual: $(f * Y)_\nabla(x) = \sup_{|x-y| \le t} |(f * Y_t)(y)|$. The even larger tangential variant M_{YN}^{**} depending on a parameter N is given by

$$M_{YN}^{**} f(x) = \sup_{v \in \mathbb{R}^n, t>0} \left| \int_{\mathbb{R}^n} f(u) \frac{1}{t^n} Y\left(\frac{x-u-v}{t}\right) \left(1 + \frac{|v|}{t}\right)^{-N} du \right|. \quad (14)$$

Let f to be a distribution, and Hardy spaces $H^p(\mathbb{R}^n)$ are (c.f. [9]): $\|f\|_{H^p(\mathbb{R}^n)} = \|M_{S_{\alpha,\beta}} f\|_{L^p(\mathbb{R}^n)}$, for $0 < p < \infty$ with appropriate α and β depending on p. It is known that $H^p = L^p$ for $p > 1$: $\|f\|_{H^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$.

In this paper, the Kakeya type maximal function $M_{\delta S_{\alpha,\beta}} f(x)$ is given by

$$M_{\delta S_{\alpha,\beta}} f(x) = \sup_{t>0, A \in O(\mathbb{R}^n), Y \in S_{\alpha,\beta}(\mathbb{R}^n)} \left| \int f(x-y) Y_{I_t}(yA^{-1}) dy \right|, \quad (15)$$

where $Y_{AI_t}(y) = Y_{I_t}(yA^{-1}) = (1/t^n) Y_I(yA^{-1}/t)$. For some fixed $t > 0$, $M^t_{\delta S_{\alpha,\beta}} f(x)$ can be defined by

$$M^t_{\delta S_{\alpha,\beta}} f(x) = \sup_{A \in O(\mathbb{R}^n), Y \in S_{\alpha,\beta}(\mathbb{R}^n)} \left| \int f(x-y) Y_{I_t}\left(yA^{-1}\right) dy \right|. \quad (16)$$

Lemma 1 (see [9]). *For any $\psi \in S(\mathbb{R}^n)$, $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, and $N > n/p$, we could obtain*

$$\left\| M^{**}_{\psi N} f \right\|_p \lesssim_{N,p} \left\| (f * \psi)_\nabla \right\|_p \lesssim_{p,\psi} \|f\|_p. \quad (17)$$

Lemma 2 (see[10]). *Let $0 < C_0 < \infty$ and $0 < r < \infty$. Then, there exist constants C_1 and C_2 (that depend only on n, C_0, and r) such that for all $t > 0$ and for all $C^1(\mathbb{R}^n)$ functions u on $\mathbb{R}^n$ whose Fourier transform is supported in the ball $|\xi|_e \leq C_0 t$ and that satisfies $|u(z)| \leq B(1+|z|_e)^{n/r}$ for some $B > 0$, we have the estimation*

$$\sup_{z \in \mathbb{R}^n} \frac{1}{t} \frac{|\nabla u(x-z)|}{\left(1+t|z|_e\right)^{n/r}} \leq C_1 \sup_{z \in \mathbb{R}^n} \frac{|u(x-z)|}{\left(1+t|z|_e\right)^{n/r}} \leq C_2 (M(|u|^r)(x))^{1/r}, \quad (18)$$

where M denotes the Hardy-Littlewood maximal operator. (The constants C_1 and C_2 are independent of B and u.)

Lemma 3 (Phragmen-Lindelöf Lemma). *Let F be analytic in the open strip $S = \{z \in \mathbb{C} : 0 < Re z < 1\}$, continuous, and bounded on its closure, such that $|F(z)| \leq C_0$ when $Re z = 0$ and $|F(z)| \leq C_1$ when $Re z = 1$. Then, $|F(z)| \leq C_0^{1-\theta} C_1^{\theta}$ when $Re z = \theta$ for any $0 < \theta < 1$.*

3. The Case when $2 \leq \delta^{-\varepsilon}$

When $1 \leq \delta^{-\varepsilon} \leq 2$, the case that $\delta \sim_\varepsilon 1$ is trival for the Kakeya type inequalities; thus, we only want to discuss the case when $0 < \delta \ll 1$. In the following of this paper, we will discuss under the assumption that $2 \leq \delta^{-\varepsilon}$.

3.1. Decomposition of the Phase Space. In this section, we will decompose $\mathbb{R}^n$ into a collection of regions:

$$\left\{\xi \in \mathbb{R}^n : \delta^{-(k-1)\varepsilon} \leq |\xi|_e \leq \delta^{-(k+3)\varepsilon}\right\}_{k \geq 1, k \in \mathbb{Z}} \text{ and } \left\{\xi \in \mathbb{R}^n : |\xi|_e \leq 1\right\}. \quad (19)$$

Then, we will give a decomposition of the region $\{\xi \in \mathbb{R}^n : |\xi|_e \leq 1\} \bigcup \{\xi \in \mathbb{R}^n : 1 \leq |\xi|_e \leq \delta^{-4\varepsilon}\}$ into a collection of smaller ones:

$$\left\{\xi \in \mathbb{R}^n : 2^{k-1} \leq |\xi|_e \leq 2^{k+3}\right\}_{s \geq k \geq 1, k \in \mathbb{Z}} \text{ and } \left\{\xi \in \mathbb{R}^n : |\xi|_e \leq 1\right\}. \quad (20)$$

Let the functions $\{\widehat{\Phi_k}(\xi)\}_k$ for $k \in \mathbb{Z}, k \geq 0$ to be defined as

$$\begin{cases} \widehat{\Phi_0}(\xi) = \widehat{\varphi}(\xi), & \Phi_0(x) = \varphi(x), \\ \widehat{\Phi_k}(\xi) = \widehat{\varphi}\left(2^{-k}\xi\right) - \widehat{\varphi}\left(2^{1-k}\xi\right), & \Phi_k(x) = \varphi_{2^{-k}}(x) - \varphi_{2^{-(k-1)}}(x), \text{for } k \geq 1. \end{cases} \quad (21)$$

Then the functions $\{\widehat{\mathbf{\Phi}_k}(\xi)\}_k$ for $k \in \mathbb{Z}, k \geq 0$ are given by

$$\begin{cases} \widehat{\mathbf{\Phi}}_0(\xi) = \widehat{\Phi_0}(\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_{n-1}, \xi_n), & \mathbf{\Phi_0}(x) = (\Phi_0)_I(x), \\ \widehat{\mathbf{\Phi}}_k(\xi) = \widehat{\Phi_k}(\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_{n-1}, \xi_n), & \mathbf{\Phi_k}(x) = (\Phi_k)_I(x), \text{for } k \geq 1. \end{cases} \quad (22)$$

Thus, it is clear that supp $\widehat{\Phi_k}(\xi) \subseteq \{\xi \in \mathbb{R}^n : 2^{k-1} \leq |\xi|_e \leq 2^{k+1}\}$, for $k \geq 1$ and supp $\widehat{\mathbf{\Phi}_k}(\xi) \subseteq \{\xi \in \mathbb{R}^n : 2^{k-1} \leq ((\delta\xi_1)^2 + \cdots + (\delta\xi_{n-1})^2 + (\xi_n)^2)^{1/2} \leq 2^{k+1}\}$ for $k \geq 1$. Also, we could deduce that

$$\sum_{k=0}^{\infty} \widehat{\mathbf{\Phi}_k}(\xi) = 1, with\, \mathbf{\Phi_k}(x) = (\Phi_k)_I(x). \quad (23)$$

In the same way, we could define the functions $\{\Psi_k(x)\}_k$ and $\{\mathbf{\Psi}_k(x)\}_k$ for $k \in \mathbb{Z}, k \geq 0$ as

$$\begin{cases} \widehat{\Psi_0}(\xi) = \widehat{\varphi}(\xi), & \Psi_0(x) = \varphi(x), \\ \widehat{\Psi_k}(\xi) = \widehat{\varphi}\left(\delta^{k\varepsilon}\xi\right) - \widehat{\varphi}\left(\delta^{(k-1)\varepsilon}\xi\right), & \Psi_k(x) = \varphi_{\delta^{k\varepsilon}}(x) - \varphi_{\delta^{(k-1)\varepsilon}}(x), \text{for } k \geq 1, \end{cases}$$

$$\begin{cases} \widehat{\mathbf{\Psi}_0}(\xi) = \widehat{\Psi_0}(\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_{n-1}, \xi_n), & \mathbf{\Psi_0}(x) = (\Psi_0)_I(x) \\ \widehat{\mathbf{\Psi}_k}(\xi) = \widehat{\Psi_k}(\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_{n-1}, \xi_n), & \mathbf{\Psi_k}(x) = (\Psi_k)_I(x), \text{for } k \geq 1. \end{cases} \quad (24)$$

Then, we could deduce that supp $\widehat{\Psi_k}(\xi) \subseteq \{\xi \in \mathbb{R}^n : \delta^{-(k-1)\varepsilon} \leq |\xi|_e \leq \delta^{-(k+3)\varepsilon}\}$ for $k \geq 1$ and supp $\widehat{\mathbf{\Psi}_k}(\xi) \subseteq \{\xi \in \mathbb{R}^n : \delta^{-(k-1)\varepsilon} \leq ((\delta\xi_1)^2 + \cdots + (\delta\xi_{n-1})^2 + (\xi_n)^2)^{1/2} \leq \delta^{-(k+3)\varepsilon}\}$ for $k \geq 1$ hold. Thus, we could have

$$\sum_{k=0}^{\infty} \widehat{\mathbf{\Psi}_k}(\xi) = 1 \text{ with } \mathbf{\Psi_k}(x) = (\Psi_k)_I(x). \quad (25)$$

Notice that $\delta^{1+(k+3)\varepsilon}|\xi|_e \leq 1$ for $\xi \in$ supp $\widehat{\Psi_k}(\xi)$. Then, we could obtain

$$\widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right) = 1, \text{for } \xi \in \text{supp } \widehat{\Psi_k}(\xi),$$

$$\mathfrak{F}(Y_I)(\xi) = \sum_{k=0}^{\infty} \frac{\widehat{\mathbf{\Psi}_k}(\xi)}{\widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right)} \mathfrak{F}(Y_I)(\xi) \widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right). \quad (26)$$

We set $\widehat{\eta_1^k}(\xi)$, (for $0 \le k, k \in \mathbb{Z}$) as

$$\widehat{\eta_1^k}(\xi) = \frac{\widehat{\boldsymbol{\Psi}_k}(\xi)}{\widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right)}\mathfrak{F}(Y_I)(\xi) = \widehat{\boldsymbol{\Psi}_k}(\xi)\mathfrak{F}(Y_I)(\xi).\text{for } 0 \le k, k \in \mathbb{Z}. \tag{27}$$

It is easy to see that $\widehat{\eta_1^k}(\xi)$ and $\eta_1^k(x) \in S(\mathbb{R}^n)$. $\exists s \in \mathbb{N}$, such that

$$\delta^{-4\varepsilon} \sim 2^s, \text{ and } \sum_{k=0}^{s} \widehat{\boldsymbol{\Phi}_k}(\xi) = 1 \text{ for } \xi \in \operatorname{supp} \widehat{\boldsymbol{\Psi}_0} \bigcup \operatorname{supp} \widehat{\boldsymbol{\Psi}_1}. \tag{28}$$

We set $\widehat{\eta_0^k}(\xi)$ $(k = 0, 1, 2, \cdots, s)$ as

$$\widehat{\eta_0^k}(\xi) = \frac{\left(\widehat{\boldsymbol{\Psi}_0}(\xi) + \widehat{\boldsymbol{\Psi}_1}(\xi)\right)\widehat{\boldsymbol{\Phi}_k}(\xi)}{\widehat{\varphi}\left(2^{-(k+1)}\delta\xi\right)}\mathfrak{F}(Y_I)(\xi).\text{for } k = 0, 1, 2, \cdots, s. \tag{29}$$

Notice that $2^{-(k+1)}\delta|\xi|_e \le 1$ holds, when $\xi \in \operatorname{supp} \widehat{\eta_0^k}(\xi)$. Thus, we could obtain

$$\widehat{\varphi}\left(2^{-(k+1)}\delta\xi\right) = 1 \text{ when } \xi \in \operatorname{supp} \widehat{\eta_0^k}(\xi). \tag{30}$$

Thus,

$$\widehat{\eta_0^k}(\xi) = \left(\widehat{\boldsymbol{\Psi}_0}(\xi) + \widehat{\boldsymbol{\Psi}_1}(\xi)\right)\widehat{\boldsymbol{\Phi}_k}(\xi)\mathfrak{F}(Y_I)(\xi) \text{ for } k = 0, 1, 2, \cdots, s. \tag{31}$$

Thus, we could write $\mathfrak{F}(Y_I)(\xi)$ and $\mathfrak{F}(Y_{AI})(\xi)$($\widehat{\varphi}$ as radial, and A is a variable (not fixed) matrix with $A \in O(\mathbb{R}^n$)) as

$$\mathfrak{F}(Y_I)(\xi) = \sum_{k=0}^{s} \widehat{\eta_0^k}(\xi)\widehat{\varphi}\left(2^{-(k+1)}\delta\xi\right) + \sum_{k=2}^{\infty} \widehat{\eta_1^k}(\xi)\widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right), \tag{32}$$

$$\begin{aligned}\mathfrak{F}(Y_{AI})(\xi) = \mathfrak{F}(Y_I)(A\xi) = \sum_{k=0}^{s} \widehat{\eta_0^k}(A\xi)\widehat{\varphi}\left(2^{-(k+1)}\delta\xi\right)\\ + \sum_{k=2}^{\infty} \widehat{\eta_1^k}(A\xi)\widehat{\varphi}\left(\delta^{1+(k+3)\varepsilon}\xi\right),\end{aligned} \tag{33}$$

where $2^s \sim \delta^{-4\varepsilon}$.

3.2. Two Lemmas. In this section, we will estimate the integrals (in Lemmas 4 and 5) associated with $\eta_0^k(x)$ and $\eta_1^k(x)$ given in Formulas (32) and (33).

Lemma 4. *For $N \ge 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, $k \ge 2$, and $Y \in S_{\alpha,\beta}(\mathbb{R}^n)$ with appropriate α, β depending on ε, n, N, we have*

$$\int_{\mathbb{R}^n} \left(1 + \delta^{-(k+3)\varepsilon}\delta^{-1}|x|_e\right)^N \left|\eta_I^k(x)\right| dx \lesssim_{N,n,\varphi,\varepsilon} \delta^{k\varepsilon}. \tag{34}$$

Proof. First, we will prove that for $l \in \mathbb{R}$, $l \ge 0$, $k \in \mathbb{N}$, and $k \ge 2$, the following inequality holds:

$$|x|_e^{l+2n}\left|\eta_1^k(x)\right| \lesssim_{l,n,\varphi,\varepsilon} \delta^{l+k\varepsilon+(k+3)l\varepsilon}. \tag{35}$$

Notice that the following inequality holds for $0 < \delta < 1$, for any $m \in \mathbb{Z}, m \ge 0$:

$$|x|_e^{2m+2n}\left|\eta_1^k(x)\right| \le \left(\left(\frac{x_1}{\delta}\right)^2 + \left(\frac{x_1}{\delta}\right)^2 + \cdots + \left(\frac{x_{n-1}}{\delta}\right)^2 + x_n^2\right)^{m+n}\left|\eta_1^k(x)\right|. \tag{36}$$

Thus, by the formula of integration by parts, we could deduce the following for any $m \in \mathbb{Z}, m \ge 0$:

$$\begin{aligned}&\left(\left(\frac{x_1}{\delta}\right)^2 + \left(\frac{x_1}{\delta}\right)^2 + \cdots + \left(\frac{x_{n-1}}{\delta}\right)^2 + x_n^2\right)^{m+n}\left|\eta_1^k(x)\right| \\ &= \left|\int_{\mathbb{R}^n} C\left(\left(\frac{\partial_{\xi_1}}{\delta}\right)^2 + \left(\frac{\partial_{\xi_2}}{\delta}\right)^2 + \cdots + \left(\frac{\partial_{\xi_{n-1}}}{\delta}\right)^2 + \partial_{\xi_n}^2\right)^{m+n}\widehat{\eta_1^k}(\xi)e^{2\pi i<x,\xi>}d\xi\right|.\end{aligned} \tag{37}$$

Make a variable substitution:

$$(\delta\xi_1, \delta\xi_2 \cdots \delta\xi_{n-1}, \xi_n) \longrightarrow \left(\xi_1', \xi_2' \cdots \xi_{n-1}', \xi_n'\right). \tag{38}$$

We could write Formula (37) as

$$\begin{aligned}&\left(\left(\frac{x_1}{\delta}\right)^2 + \left(\frac{x_1}{\delta}\right)^2 + \cdots + \left(\frac{x_{n-1}}{\delta}\right)^2 + x_n^2\right)^{m+n}\left|\eta_1^k(x)\right| \\ &= \frac{1}{\delta^{n-1}}\left|\int_{\mathbb{R}^n} C\left(\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right)\right)e^{2\pi i<x,\xi>}d\xi'\right|,\end{aligned} \tag{39}$$

where $\Delta_{\xi'}$ is the Laplace operator: $\Delta_{\xi'} = \partial_{\xi'_1}^2 + \partial_{\xi'_2}^2 + \cdots + \partial_{\xi'_n}^2$. We could also deduce that

$$\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right) = \left(\Delta_{\xi'}\right)^{n+m}\left(\widehat{\Psi_k}\left(\xi'\right)\mathfrak{F}(Y)\left(\xi'\right)\right). \tag{40}$$

Thus, $((\Delta_{\xi'})^{n+m}\widehat{\eta_1^k}(\xi')) \in S(\mathbb{R}^n)$, $||\xi'|_e^{|\alpha'|}(\Delta_{\xi'})^{\beta'}\widehat{\eta_1^k}(\xi')| \lesssim_{\alpha',\beta'} 1$, for appropriate α', β', and

$$\begin{aligned}&\operatorname{supp}\left(\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right)\right) \\ &\subseteq \left\{\xi' \in \mathbb{R}^n : \delta^{-(k-1)\varepsilon} \le \left|\xi'\right|_e \le \delta^{-(k+3)\varepsilon}\right\} \text{ for } k \ge 2.\end{aligned} \tag{41}$$

When $\delta^{-(k-1)\varepsilon} \le |\xi'|_e \le \delta^{-(k+3)\varepsilon}$, $k \ge 2$, and $\delta^{-\varepsilon} \ge 2$, we could deduce that

$$\delta^{-(k/2)\varepsilon} \le \delta^{-(k-1)\varepsilon} \le \left|\xi'\right|_e \le \delta^{-(k+3)\varepsilon} \le \delta^{-3k\varepsilon}. \tag{42}$$

That is,

$$\left|\xi'\right|_e^{1/3} \le \delta^{-k\varepsilon} \le \left|\xi'\right|_e^2, \delta^{-\varepsilon} \le \delta^{-k\varepsilon} \le \left|\xi'\right|_e^2. \tag{43}$$

Then, we could deduce that

$$|x|_e^{2m+2n}\left|\eta_1^k(x)\right| \lesssim \frac{1}{\delta^{n-1}}\int_{\mathbb{R}^n}\left|\left(\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right)\right)\right|d\xi' \lesssim \delta^{2m+k\varepsilon+2(k+3)m\varepsilon}\int_{\mathbb{R}^n}\left|\xi'\right|_e^{(2n+4m)/\varepsilon+8m}\left|\left(\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right)\right)\right| d\xi' \lesssim_{m,n,\varphi,\varepsilon}\delta^{2m+k\varepsilon+2(k+3)m\varepsilon}. \tag{44}$$

Thus, similar to Formula (44), we could obtain

$$|x|_e^{2m}\left|\eta_1^k(x)\right| \lesssim \frac{1}{\delta^{n-1}}\int_{\mathbb{R}^n}\left|\left(\left(\Delta_{\xi'}\right)^{n+m}\widehat{\eta_1^k}\left(\xi'\right)\right)\right|d\xi' \lesssim_{m,n,\varphi,\varepsilon}\delta^{2m+k\varepsilon+2(k+3)m\varepsilon}, \tag{45}$$

where $k \geq 2$, $m \in \mathbb{Z}, m \geq 0$. By Lemma 3 and Formula (44), we could deduce Formula (35). Thus, we could obtain the following inequality for $N \geq 0, N \in \mathbb{R}$

$$\left(\frac{\delta^{-(k+3)\varepsilon}}{\delta}|x|_e\right)^N\left|\eta_1^k(x)\right| \lesssim_{N,n,\varphi,\varepsilon}\delta^{k\varepsilon}\frac{1}{|x|_e^{2n}}. \tag{46}$$

By Lemma 3 and Formula (45), for $l \in \mathbb{R}$, $l \geq 0$, $k \in \mathbb{N}$, and $k \geq 2$, the following inequality holds:

$$|x|_e^l\left|\eta_1^k(x)\right| \lesssim_{l,n,\varphi,\varepsilon}\delta^{l+k\varepsilon+(k+3)l\varepsilon}. \tag{47}$$

Then, we could obtain the following inequality for $N \geq 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, and $k \geq 2$,

$$\left(\frac{\delta^{-(k+3)\varepsilon}}{\delta}|x|_e\right)^N\left|\eta_1^k(x)\right| \lesssim_{N,n,\varphi,\varepsilon}\delta^{k\varepsilon}. \tag{48}$$

By Formulas (46) and (48), we could deduce that for $N \geq 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, and $k \geq 2$, the following Formulas (49) and (50) hold:

$$\int_{\mathbb{R}^n}\left|\eta_1^k(x)\right|dx \lesssim_{n,\varphi,\varepsilon}\delta^{k\varepsilon}, \tag{49}$$

$$\int_{\mathbb{R}^n}\left(\delta^{-(k+3)\varepsilon}\delta^{-1}|x|_e\right)^N\left|\eta_1^k(x)\right|dx \lesssim_{N,n,\varphi,\varepsilon}\delta^{k\varepsilon}. \tag{50}$$

Then, we could obtain the Lemma 4 directly from Formulas (49) and (50). This proves the Lemma. □

Lemma 5. *For $N \geq 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, and $0 \leq k \leq s$ where $2^s \sim \delta^{-4\varepsilon}$, $Y \in S_{\alpha,\beta}(\mathbb{R}^n)$, with appropriate α, β depending on ε, n, N, the following two inequalities hold:*

$$\int_{\mathbb{R}^n}\left(1+2^{k+1}\delta^{-1}|x|_e\right)^N\left|\eta_0^k(x)\right|dx \lesssim_{N,n,\varphi,\varepsilon}\delta^{-4(N+1)\varepsilon}\delta^{-N}2^{-k}, \tag{51}$$

$$\int_{\mathbb{R}^n}\left(1+2^{k+1}|x|_e\right)^N\left|\eta_0^k(x)\right|dx \lesssim_{N,n,\varphi,\varepsilon}\delta^{-4(N+1)\varepsilon}2^{-k}. \tag{52}$$

Proof. Notice that $2^s \sim \delta^{-4\varepsilon}$, thus, for any $k \in \{0, 1, \cdots, s\}$ and $N \geq 0, N \in \mathbb{R}$, we have

$$1 \lesssim \left(\frac{1}{2^{k+1}\delta^{4\varepsilon}}\right)^N. \tag{53}$$

Notice that $2^{-(k+1)}\delta|\xi|_e \leq 1$ holds, when $\xi \in \operatorname{supp}\widehat{\eta_0^k}(\xi)$. Thus, we could obtain

$$\widehat{\varphi}\left(2^{-(k+1)}\delta\xi\right) = 1 \, when \, \xi \in \operatorname{supp}\widehat{\eta_0^k}(\xi). \tag{54}$$

Then, we could write $\widehat{\eta_0^k}(\xi)$ as

$$\widehat{\eta_0^k}(\xi) = \left(\left(\widehat{\mathbf{\Psi}_0}(\xi)+\widehat{\mathbf{\Psi}_1}(\xi)\right)\widehat{\mathbf{\Phi}_k}(\xi)\right)\mathfrak{F}(Y_I)(\xi). \tag{55}$$

It is clear that the following Formulas (56)–(60) hold:

$$\left(\partial_\xi^{\alpha_1}\widehat{\mathbf{\Psi}_0}(\xi)\right)^\vee(x) = (-2\pi i x)^{\alpha_1}\varphi_I(x), \tag{56}$$

$$\left(\partial_\xi^{\alpha_1}\widehat{\mathbf{\Psi}_1}(\xi)\right)^\vee(x) = (-2\pi i x)^{\alpha_1}\left(\frac{1}{\delta^\varepsilon}\right)^n\varphi_I\left(\frac{x}{\delta^\varepsilon}\right) - (-2\pi i x)^{\alpha_1}\varphi_I(x), \tag{57}$$

$$\left(\partial_\xi^{\beta_1}\widehat{\mathbf{\Phi}_k}(\xi)\right)^\vee(x) = (-2\pi i x)^{\beta_1}2^{kn}\varphi_I\left(2^k x\right) - (-2\pi i x)^{\beta_1}2^{(k-1)n}\varphi_I\left(2^{(k-1)}x\right) \, (\text{for } k \geq 1), \tag{58}$$

$$\left(\partial_\xi^{\beta_1}\widehat{\mathbf{\Phi}_0}(\xi)\right)^\vee(x) = (-2\pi i x)^{\beta_1}\varphi_I(x), \tag{59}$$

$$\left(\partial_\xi^{\gamma_1}\mathfrak{F}(Y_I)(\xi)\right)^\vee(x) = (-2\pi i x)^{\gamma_1}Y_I(x). \tag{60}$$

By Young inequality, we could have

$$\int\left|\eta_0^k(x)\right|dx \leq \|(\mathbf{\Psi}_0+\mathbf{\Psi}_1)*\mathbf{\Phi}_k*Y_I\|_1 \leq \|\mathbf{\Psi}_0+\mathbf{\Psi}_1\|_1\|\mathbf{\Phi}_k\|_1\|Y_I\|_1 \lesssim_\varphi 1. \tag{61}$$

By the formula of integration by parts, we could deduce the following for any $m \in \mathbb{N}$:

$$|x|_e^{2n+2m}\left|\eta_0^k(x)\right| = \left|\int_{\mathbb{R}^n}C\left(\left(\Delta_\xi\right)^{n+m}\widehat{\eta_0^k}(\xi)\right)e^{2\pi i<x,\xi>}d\xi\right| = \left|\sum_{|\alpha_1|+|\beta_1|+|\gamma_1|=2m+2n}\left(\partial_\xi^{\alpha_1}\widehat{\mathbf{\Psi}_1}(\xi)+\partial_\xi^{\alpha_1}\widehat{\mathbf{\Psi}_0}(\xi)\right)^\vee * \left(\partial_\xi^{\beta_1}\widehat{\mathbf{\Phi}_k}(\xi)\right)^\vee * \left(\partial_\xi^{\gamma_1}\mathfrak{F}(Y_I)(\xi)\right)^\vee(x)\right|. \tag{62}$$

By Young Inequality, Formulas (62) and (56)–(60), we could obtain

$$\int \left| |x|_e^{2n+2m} \eta_0^k(x) \right| dx \le \sum_{|\alpha_1|+|\beta_1|+|\gamma_1|=2m+2n} \left\| \left(\partial_\xi^{\alpha_1} \widehat{\mathbf{\Psi}_1}(\xi) + \partial_\xi^{\alpha_1} \widehat{\mathbf{\Psi}_0}(\xi) \right)^\vee \right\|_1 \left\| \left(\partial_\xi^{\beta_1} \widehat{\mathbf{\Phi}_k}(\xi) \right)^\vee \right\|_1 \left\| \left(\partial_\xi^{\gamma_1} \mathfrak{F}(Y_I)(\xi) \right)^\vee \right\|_1 \lesssim_{\varphi,n,m} 1 \text{ for } m \in \mathbb{N}. \tag{63}$$

By Lemma 3 and Formulas (61) and (63), we could deduce the following Formula (64).

$$\int \left| |x|_e^l \eta_0^k(x) \right| dx \lesssim_{\varphi,l} 1 \text{ for } l \in \mathbb{R}, l \ge 0. \tag{64}$$

By Formulas (61) and (64), the following two inequalities hold for $N \ge 0$, $k \in \{0, 1, \cdots, s\}$:

$$\int_{\mathbb{R}^n} \left(1 + 2^{k+1} \delta^{-1} |x|_e \right)^N \left| \eta_0^k(x) \right| dx \lesssim_{N,n,\varphi,\varepsilon} \delta^{-N} 2^{(k+1)N}, \tag{65}$$

$$\int_{\mathbb{R}^n} \left(1 + 2^{k+1} |x|_e \right)^N \left| \eta_0^k(x) \right| dx \lesssim_{N,n,\varphi,\varepsilon} 2^{(k+1)N}. \tag{66}$$

From Formulas (53), (65), and (66), we could obtain the Formula (51) and (52) together. This proves the Lemma. □

From Lemmas 4 and 5, we could obtain the following inequalities (67)–(69). For $N \ge 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, and $k \ge 2$, we have

$$\int_{\mathbb{R}^n} \left(1 + \delta^{-(k+3)\varepsilon} \delta^{-1} |x|_e \right)^N \left| \eta_1^k \left(xA^{-1} \right) \right| dx \lesssim_{N,n,\varphi,\varepsilon} \delta^{k\varepsilon}, \tag{67}$$

where A is a variable (not fixed) matrix with $A \in O(\mathbb{R}^n)$. For $N \ge 0, N \in \mathbb{R}$, $k \in \mathbb{N}$, and $0 \le k \le s$ where $2^s \sim \delta^{-4\varepsilon}$, we have Formulas (37) and (68)

$$\int_{\mathbb{R}^n} \left(1 + 2^{k+1} \delta^{-1} |x|_e \right)^N \left| \eta_0^k \left(xA^{-1} \right) \right| dx \lesssim_{N,n,\varphi,\varepsilon} \delta^{-4(N+1)\varepsilon} \delta^{-N} 2^{-k}, \tag{68}$$

$$\int_{\mathbb{R}^n} \left(1 + 2^{k+1} |x|_e \right)^N \left| \eta_0^k \left(xA^{-1} \right) \right| dx \lesssim_{N,n,\varphi,\varepsilon} \delta^{-4(N+1)\varepsilon} 2^{-k}, \tag{69}$$

where A is a variable (not fixed) matrix with $A \in O(\mathbb{R}^n)$.

3.3. Main Results. From Formulas (67)–(69), we will obtain our main results in this section:

Proposition 6. *For $p > 1$ with appropriate α, β depending on ε, n, p, we have*

$$\left\| M_{\delta S_{\alpha,\beta}} f \right\|_p \lesssim_{p,n,\varphi,\varepsilon} \left(\frac{1}{\delta} \right)^{4(n/p+2)\varepsilon} \left\| (f * \varphi_\delta)_\nabla \right\|_p, \tag{70}$$

$$\left\| M_{\delta S_{\alpha,\beta}} f \right\|_p \lesssim_{p,n,\varphi,\varepsilon} \left(\frac{1}{\delta} \right)^{n/p+4(n/p+3)\varepsilon} \|f\|_p, \tag{71}$$

where $(f * \varphi_\delta)_\nabla(x) = \sup_{|x-y| \le t} \left| (f * (\varphi_\delta))_t(y) \right|$.

Proof. Notice that $f \in L^p(\mathbb{R}^n)$ is a distribution. By Formula (33), we could obtain

$$\begin{aligned} \left| M_{\delta S_{\alpha,\beta}} f(x) \right| &= \sup_{t>0, A \in O(\mathbb{R}^n), Y \in S_{\alpha,\beta}(\mathbb{R}^n)} \left| \int f(x-y) Y_{I_t}\left(yA^{-1} \right) dy \right| \\ &\le \sum_{k=2}^{\infty} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} f(x-y) \int_{\mathbb{R}^n} t^{-n} \eta_1^k \left(\frac{(uA^{-1})}{t} \right) \varphi_{\delta^{(k+3)\varepsilon}\delta t}(y-u) du dy \right| \\ &\quad + \sum_{k=0}^{s} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} f(x-y) \int_{\mathbb{R}^n} t^{-n} \eta_0^k \left(\frac{(uA^{-1})}{t} \right) \varphi_{2^{-(k+1)}\delta t}(y-u) du dy \right| \\ &\le \sum_{k=2}^{\infty} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} t^{-n} \eta_1^k \left(\frac{(uA^{-1})}{t} \right) f * \varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u) du \right| \\ &\quad + \sum_{k=0}^{s} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} t^{-n} \eta_0^k \left(\frac{(uA^{-1})}{t} \right) f * \varphi_{2^{-(k+1)}\delta t}(x-u) du \right| \\ &\le \sum_{k=2}^{\infty} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| M_{\varphi_\delta N}^{**} f(x) \int_{\mathbb{R}^n} t^{-n} \left| \eta_1^k \left(\frac{(uA^{-1})}{t} \right) \right| \left(1 + \frac{|u|_e}{\delta^{(k+3)\varepsilon} t} \right)^N du \right| \\ &\quad + \sum_{k=0}^{s} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| M_{\varphi_\delta N}^{**} f(x) \int_{\mathbb{R}^n} t^{-n} \left| \eta_0^k \left(\frac{(uA^{-1})}{t} \right) \right| \left(1 + \frac{|u|_e}{2^{-(k+1)} t} \right)^N du \right|, \end{aligned} \tag{72}$$

where $2^s \sim \delta^{-4\varepsilon}$. Lemma 1 and Formulas (67), (69), and (72) yield to

$$\left\| M_{\delta S_{\alpha,\beta}} f \right\|_p \lesssim_{p,N,n,\varphi,\varepsilon} \left(\frac{1}{\delta} \right)^{4(N+2)\varepsilon} \left\| (f * \varphi_\delta)_\nabla \right\|_p \text{ for } p > 1, N > \frac{n}{p}. \tag{73}$$

Similar to Formula (72), we could also obtain

$$\begin{aligned} &\sup_{t>0, A \in O(\mathbb{R}^n)} \left| \int f(x-y) Y_{I_t}\left(yA^{-1} \right) dy \right| \\ &\le \sum_{k=2}^{\infty} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| M_{\varphi N}^{**} f(x) \int_{\mathbb{R}^n} t^{-n} \left| \eta_1^k \left(\frac{(uA^{-1})}{t} \right) \right| \left(1 + \frac{|u|_e}{\delta^{(k+3)\varepsilon} \delta t} \right)^N du \right| \\ &\quad + \sum_{k=0}^{s} \sup_{t>0, A \in O(\mathbb{R}^n)} \left| M_{\varphi N}^{**} f(x) \int_{\mathbb{R}^n} t^{-n} \left| \eta_0^k \left(\frac{(uA^{-1})}{t} \right) \right| \left(1 + \frac{|u|_e}{2^{-(k+1)} \delta t} \right)^N du \right|. \end{aligned} \tag{74}$$

Notice that $2^s \sim \delta^{-4\varepsilon}$; thus, Lemma 1 and Formulas (67), (68), and (74) yield to

$$\left\| M_{\delta S_{\alpha,\beta}} f \right\|_p \lesssim_{p,N,n,\varphi,\varepsilon} \left(\frac{1}{\delta} \right)^N \left(\frac{1}{\delta} \right)^{4(N+2)\varepsilon} \|f\|_p \text{ for } p > 1, N > \frac{n}{p}, N \in \mathbb{R}. \tag{75}$$

Let N be $N = (n/p) + \varepsilon$. From Formulas (73) and (75), we could prove the Proposition 6. □

Theorem 7. *For $\infty > p > r > 1$, $0 < t \le \delta^{-\varepsilon}$, $f(x) \in L^p(\mathbb{R}^n)$, and* $\operatorname{supp} \widehat{f}(\xi) \subseteq B(0,1)$. *Then, with appropriate α, β depending on ε, n, r, we could obtain*

$$\left\| M_{\delta S_{\alpha,\beta}}^t f \right\|_p \lesssim_{p,n,\varphi,\varepsilon} \left(\frac{1}{\delta} \right)^{5\left(\frac{n}{r}+2\right)\varepsilon} \|f\|_p. \tag{76}$$

Proof. By Formula (33), we could write $M^t_{\delta S_{\alpha,\beta}} f$ as

$$\left|M^t_{\delta S_{\alpha,\beta}} f(x)\right| \le \sum_{k=2}^{\infty} \sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)}{\left(1+\left(|u|_e/\delta^{(k+3)\varepsilon}t\right)\right)^{n/r}}\right| \sup_{A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n} t^{-n}\left|\eta_1^k\left(\frac{(uA^{-1})}{t}\right)\right| \left(1+\frac{|u|_e}{\delta^{(k+3)\varepsilon}t}\right)^{n/r} du\right| + \sum_{k=0}^{s} \sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{2^{-(k+1)}\delta t}(x-u)}{\left(1+\left(|u|_e/2^{-(k+1)}t\right)\right)^{n/r}}\right| \sup_{A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n} t^{-n}\left|\eta_0^k\left(\frac{(uA^{-1})}{t}\right)\right| \left(1+\frac{|u|_e}{2^{-(k+1)}t}\right)^{n/r} du\right|, \tag{77}$$

where $2^s \sim \delta^{-4\varepsilon}$.

By Holder inequality, $|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x)|$ and $|f*\varphi_{2^{-(k+1)}\delta t}(x)|$ are both bounded functions for any $x\in\mathbb{R}$, $\delta>0$, $k\in\mathbb{N}$, and $t>0$. Thus, for some $B>0$, we could have

$$|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x)| \le B\left(1+|x|_e\right)^{n/r} \text{ and, } |f*\varphi_{2^{-(k+1)}\delta t}(x)| \le B\left(1+|x|_e\right)^{n/r}. \tag{78}$$

It is also clear that supp $\mathfrak{F}(f*\varphi_{\delta^{(k+3)\varepsilon}\delta t})(\xi)\subseteq B(0,1)$, and supp $\mathfrak{F}(f*\varphi_{2^{-(k+1)}\delta t})(\xi)\subseteq B(0,1)$. We could also deduce that $f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x)\in C^1(\mathbb{R}^n)$, $f*\varphi_{2^{-(k+1)}\delta t}(x)\in C^1(\mathbb{R}^n)$. Thus, by Lemma 2, we could obtain

$$\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)}{\left(1+\left(|u|_e/\delta^{(k+3)\varepsilon}t\right)\right)^{n/r}}\right| \le \sup_{u\in\mathbb{R}^n}\frac{|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)|}{\left(1+|u|_e\right)^{n/r}} \le C_2\left(M\left(|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}|^r\right)(x)\right)^{1/r}, \tag{79}$$

$$\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{2^{-(k+1)}\delta t}(x-u)}{\left(1+\left(|u|_e/2^{-(k+1)}t\right)\right)^{n/r}}\right| \le \sup_{u\in\mathbb{R}^n}\frac{|f*\varphi_{2^{-(k+1)}\delta t}(x-u)|}{\left(1+|u|_e\right)^{n/r}} \le C_2\left(M\left(|f*\varphi_{2^{-(k+1)}\delta t}|^r\right)(x)\right)^{1/r}. \tag{80}$$

Notice that $2^s \sim \delta^{-4\varepsilon}$, by Formulas (67), (69), (77), (79), and (80), we could deduce that for $\infty > p > r > 1$, $0 < t \le 1$

$$\left\|M^t_{\delta S_{\alpha,\beta}} f\right\|_p \lesssim_{p,n,\varphi,\varepsilon} \sum_{k=2}^{\infty}\delta^{k\varepsilon}\left|\int_{\mathbb{R}^n}|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}|^p(x)dx\right|^{1/p} + \sum_{k=0}^{s}\delta^{-4(n/r+1)\varepsilon}2^{-k}\left|\int_{\mathbb{R}^n}|f*\varphi_{2^{-(k+1)}\delta t}|^p(x)dx\right|^{1/p} \lesssim_{p,n,\varphi,\varepsilon}\left(\frac{1}{\delta}\right)^{4(n/r+2)\varepsilon}\|f\|_p. \tag{81}$$

When $1 < t \le \delta^{-\varepsilon}$, notice that

$$\begin{aligned}\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{2^{-(k+1)}\delta t}(x-u)}{\left(1+\left(|u|_e/2^{-(k+1)}t\right)\right)^{n/r}}\right| &\le t^{n/r}\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{2^{-(k+1)}\delta t}(x-u)}{\left(1+\left(|u|_e/2^{-(k+1)}\right)\right)^{1/r}}\right| \\ &\le t^{n/r}\sup_{u\in\mathbb{R}^n}\frac{|f*\varphi_{2^{-(k+1)}\delta t}(x-u)|}{\left(1+|u|_e\right)^{n/r}} \\ &\le C_2\delta^{-(n/r)\varepsilon}\left(M\left(|f*\varphi_{2^{-(k+1)}\delta t}|^r\right)(x)\right)^{1/r},\end{aligned} \tag{82}$$

$$\begin{aligned}\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)}{\left(1+\left(|u|_e/\delta^{(k+3)\varepsilon}t\right)\right)^{n/r}}\right| &\le t^{n/r}\sup_{u\in\mathbb{R}^n}\left|\frac{f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)}{\left(1+\left(|u|_e/\delta^{(k+3)\varepsilon}\right)\right)^{n/r}}\right| \\ &\le t^{n/r}\sup_{u\in\mathbb{R}^n}\frac{|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)|}{\left(1+|u|_e\right)^{n/r}} \\ &\le C_2\delta^{-(n/r)\varepsilon}\left(M\left(|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}|^r\right)(x)\right)^{1/r}\end{aligned} \tag{83}$$

hold. From Formulas (67), (69), (77), (79), (80), (82), and (83), we could deduce that for $\infty > p > r > 1$, $1 < t \le \delta^{-\varepsilon}$

$$\begin{aligned}\left\|M^t_{\delta S_{\alpha,\beta}} f\right\|_p &\lesssim_{p,n,\varphi,\varepsilon}\sum_{k=2}^{\infty}\delta^{-(n/r)\varepsilon}\delta^{k\varepsilon}\left|\int_{\mathbb{R}^n}|f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}|^p(x)dx\right|^{1/p} \\ &\quad+\sum_{k=0}^{s}\delta^{-(n/r)\varepsilon}\delta^{-4\left(\frac{n}{r}+1\right)\varepsilon}2^{-k}\left|\int_{\mathbb{R}^n}|f*\varphi_{2^{-(k+1)}\delta t}|^p(x)dx\right|^{1/p} \\ &\lesssim_{p,n,\varphi,\varepsilon}\left(\frac{1}{\delta}\right)^{5\left(\frac{n}{r}+2\right)\varepsilon}\|f\|_p.\end{aligned} \tag{84}$$

This proves the Theorem. □

Theorem 8. *For $\infty > p > 1$, $0 < t \le \delta^{-\varepsilon}$, $f(x)\in L^p(\mathbb{R}^n)$, and supp $\widehat{f}(\xi)\subseteq\left(B(0,t^{-1}\delta^{-(1+4\varepsilon)})\right)^c$, then with appropriate α, β depending on ε, n, p, we could obtain*

$$\left\|M^t_{\delta S_{\alpha,\beta}} f\right\|_p \lesssim_{p,n,\varphi,\varepsilon}\|f\|_p. \tag{85}$$

Proof. By Formula (33), we could write $M^t_{\delta S_{\alpha,\beta}} f$ as the following:

$$\begin{aligned}\left|M^t_{\delta S_{\alpha,\beta}} f(x)\right| \le &\sum_{k=2}^{\infty}\sup_{A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n}t^{-n}\eta_1^k\left(\frac{(uA^{-1})}{t}\right)f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)du\right| \\ &+\sum_{k=0}^{s}\sup_{A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n}t^{-n}\eta_0^k\left(\frac{(uA^{-1})}{t}\right)f*\varphi_{2^{-(k+1)}\delta t}(x-u)du\right|.\end{aligned} \tag{86}$$

Notice that supp $\widehat{f}(\xi)\subseteq\left(B(0,t^{-1}\delta^{-(1+4\varepsilon)})\right)^c$ and supp $\mathfrak{F}(\varphi_{2^{-(k+1)}\delta t})(\xi)\subseteq B(0,t^{-1}\delta^{-(1+4\varepsilon)})$ for $0\le k\le s, k\in\mathbb{Z}$; thus, we could deduce that $f*\varphi_{2^{-(k+1)}\delta t}=0$. From Formula (86), we could obtain

$$\left|M^t_{\delta S_{\alpha,\beta}} f(x)\right| \leq \sum_{k=2}^{\infty} \sup_{A\in O(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} t^{-n} \eta_1^k \left(\frac{(uA^{-1})}{t} \right) f * \varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)du \right|. \tag{87}$$

By Formula (67) and Lemma 1, we could have

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n}\left(\sum_{k=2}^{\infty}\sup_{A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n} t^{-n}\eta_1^k\left(\frac{(uA^{-1})}{t}\right) f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)du\right|\right)^p dx\right]^{1/p} \\
&\leq \left[\int_{\mathbb{R}^n}\left(\sum_{k=2}^{\infty}\sup_{t>0,A\in O(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n} t^{-n}\eta_1^k\left(\frac{(uA^{-1})}{t}\right) f*\varphi_{\delta^{(k+3)\varepsilon}\delta t}(x-u)du\right|\right)^p dx\right]^{1/p} \\
&\leq \left[\int_{\mathbb{R}^n}\left(\sum_{k=2}^{\infty}\sup_{t>0,A\in O(\mathbb{R}^n)}\left|M^{**}_{\varphi N} f(x)\int_{\mathbb{R}^n} t^{-n}\left|\eta_1^k\left(\frac{(uA^{-1})}{t}\right)\right|\left(1+\frac{|u|_e}{\delta^{(k+3)\varepsilon}\delta t}\right)^N du\right|\right)^p dx\right]^{1/p} \\
&\lesssim_{p,N,n,\varphi,\varepsilon} \|f\|_p \text{ for } p>1, N>\frac{n}{p}, N\in\mathbb{R}.
\end{aligned} \tag{88}$$

Let N be $N = (n/p) + \varepsilon$, from Formulas (86)–(88), and we could prove the Theorem 8. □

Data Availability

There is no data in my paper; thus, the date is available.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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