Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9991466, 6 pages
https://doi.org/10.1155/2022/9991466

Research Article

The Ball-Relaxed Gradient-Projection Algorithm for Split Feasibility Problem

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Received 16 November 2021; Accepted 14 March 2022; Published 11 April 2022

Academic Editor: Chi Ming Chen

In this paper, we concern with the split feasibility problem (SFP) whenever the convex sets involved are composed of level sets. By applying Gradient-projection algorithm which is used to solve constrained convex minimization problem of a real valued convex function, we construct two new algorithms for the split feasibility problem and prove that both of them are convergent weakly to a solution of the feasibility problem. In the end, as an application, we obtain a new algorithm for solving the split equality problem.

1. Introduction

The split feasibility problem (SFP) was first introduced by Censor and Elfving [1]. And it is formulated as finding a point x satisfying the property:

$$x \in \mathscr{C} \quad \text{and} \quad Ax \in \mathscr{Q}, \tag{1}$$

where $\mathscr{C}$ and $\mathscr{Q}$ are nonempty, closed, and convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and A is a bounded linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$.

Many inverse problems arising from real world can be summarized as SFP. So, the SFP has attracted many scholars to study it. Of course, various algorithms by far have been invented to solve the SFP (see, e.g., [2–9]). And one of the most famous methods for solving the SFP is $\mathscr{C}\mathscr{Q}$ algorithm which is introduced by Byrne. Take an initial guess $x_0 \in \mathscr{H}_1$ and define a recursively equation as

$$x_{n+1} = P_{\mathscr{C}}[I - \alpha A^*(I - P_{\mathscr{Q}})A]x_n, \tag{2}$$

where $0 < \alpha < 2/L$ and L denotes the largest eigenvalue of the matrix $A^T A$. Later, Byrne introduced another recursively equation as

$$x_{n+1} = P_{\mathscr{C}}[I - \alpha_n A^*(I - P_{\mathscr{Q}})A]x_n, \tag{3}$$

where $0 < \alpha_n < 2/\|A\|^2$. Byrne has shown that the sequence $\{x_n\}$ produced by (2) or (3) is convergent weakly to a solution of the SFP (1). From (2) and (3), we know that we have first to compute or estimate the largest eigenvalue of the matrix $A^T A$ and the operator norm $\|A\|^2$ before implementing algorithms. However, computing or estimating the matrix norm is in general not an easy work in practice. So the conditions that Byrne put on his proposed two algorithms seem restrictive. Recently, Lopez et al. [10] proposed a novel way to construct another stepsize which has no connection with eigenvalue and matrix norm. And Lopez et al. suggested the following stepsize:

$$\alpha_n = \frac{\|(I - P_{\mathscr{Q}})Ax_n\|^2}{\|A^*(I - P_{\mathscr{Q}})Ax_n\|^2}, \tag{4}$$

and showed the weak convergence of the $\mathscr{C}\mathscr{Q}$ algorithm (3).

In this paper, we consider the SFP whenever both $\mathscr{C}$ and $\mathscr{Q}$ are level sets for given strongly convex function, that is,

$$\mathscr{C} = \{x \in H_1 | c(x) \le 0\}, \tag{5}$$

$$\mathscr{Q} = \{y \in H_2 | q(y) \le 0\}, \tag{6}$$

where $c : \mathscr{H}_1 \longrightarrow (-\infty, +\infty]$ is an α-strongly convex lower semicontinuous function and $q : \mathscr{H}_2 \longrightarrow (-\infty, +\infty]$ is a β-strongly convex lower semicontinuous function. However, in general, computing the orthogonal projections onto $\mathscr{C}$ and $\mathscr{Q}$ is not an easy task. To overcome this difficulty, Yu et al. [11] have considered the case when $\{\mathscr{C}_n\}$ and $\{\mathscr{Q}_n\}$ are two sequences of balls defined, respectively, by

$$\mathscr{C}_n = \left\{ x \in \mathscr{H}_1 \middle| c(x_n) \le \langle \xi_n, x_n - x \rangle - \frac{\alpha}{2}\|x - x_n\|^2 \right\}, \xi_n \in \partial c(x_n), \tag{7}$$

$$\mathscr{Q}_n = \left\{ y \in \mathscr{H}_2 \middle| q(Ax_n) \le \langle \eta_n, Ax_n - y \rangle - \frac{\beta}{2}\|y - Ax_n\|^2 \right\}, \eta_n \in \partial q(Ax_n), \tag{8}$$

where c and q are strongly convex functions with constant α and β, respectively. Yu et al. [11] have shown that $\mathscr{C}_n$ is a ball whose centre is $x_n - (1/\alpha)\xi_n$ and radius is $\sqrt{(1/\alpha^2)\|\xi_n\|^2 - (2/\alpha)c(x_n)}$. And $\mathscr{Q}_n$ is also a ball whose centre is $Ax_n - (1/\beta)\eta_n$ and radius is $\sqrt{(1/\beta^2)\|\eta_n\|^2 - (2/\beta)q(Ax_n)}$. It is easy to see that $\mathscr{C} \subseteq \mathscr{C}_n$ and $\mathscr{Q} \subseteq \mathscr{Q}_n$ for any $n \ge 0$. Since $\mathscr{C}_n$ and $\mathscr{Q}_n$ are balls, the associated projections can be easily calculated.

The rest of this paper is organized as follows. In Section 2, we review some basic definitions and lemmas that we will use in the remaining sections. In Section 3, we introduce two new ball-relaxed gradient-projection algorithms for solving the SFP and prove the weak convergence of our algorithms. In Section 4, according to the algorithms that we suggest in Section 3, we obtain a new iterative algorithm for solving the split equality problem and establish its weak convergence.

2. Preliminaries

Throughout this paper, we always assume that $\mathscr{H}_i (i = 1, 2)$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote by I the identity operator on $\mathscr{H}_i (i = 1, 2)$ and by $w_w(x_n)$ the set of all cluster points of $\{x_n\}$. The notation "$\longrightarrow$" stands for strong convergence, and "$\rightharpoonup$" stands for weak convergence.

Definition 1 ([12]). Let $\mathscr{D}$ be a nonempty subset of $\mathscr{H}_1$, and let $T : \mathscr{D} \longrightarrow \mathscr{H}_1$. Then, T is

(1) Nonexpansive if it is Lipschitz continuous with constant 1, i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in \mathscr{D}. \tag{9}$$

(2) Firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \le \|x - y\|^2, \quad \forall x \in \mathscr{D}, y \in \mathscr{D}. \tag{10}$$

(3) ν-inverse strongly monotone (ν-ism, $\nu > 0$) if

$$\langle Tx - Ty, x - y \rangle \ge \nu \|Tx - Ty\|^2, \quad \forall x \in \mathscr{D}, y \in \mathscr{D}. \tag{11}$$

Definition 2. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}_1$. Then, an orthogonal projection $P_{\mathscr{C}} : \mathscr{H}_1 \longrightarrow \mathscr{C}$ is defined by

$$P_{\mathscr{C}} x = \arg\min_{y \in \mathscr{C}} \|x - y\|^2, \quad x \in \mathscr{H}_1. \tag{12}$$

Lemma 3. *Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}_1$; then,*

(1) For every x and p in $\mathscr{H}_1$

$$p = P_{\mathscr{C}} x \Leftrightarrow \{p \in \mathscr{C} \quad and \quad (\forall y \in \mathscr{C}) \quad \langle x - p, y - p \rangle \le 0\}. \tag{13}$$

(2) $P_{\mathscr{C}}$ and $I - P_{\mathscr{C}}$ both are nonexpansive

Definition 4 (see [12]). Let $\lambda \in (0, 1)$ and $f : \mathscr{H}_1 \longrightarrow (-\infty, +\infty]$ be a proper function.

(1) f is convex if

$$f(\lambda x + (1 - \lambda) y) \le \lambda f(x) + (1 - \lambda) f(y), \forall x, y \in \mathscr{H}_1. \tag{14}$$

(2) f is strongly convex with constant α where $\alpha > 0$ if

$$f(\lambda x + (1 - \lambda) y) + \frac{\alpha}{2}\lambda(1 - \lambda)\|x - y\|^2 \le \lambda f(x) + (1 - \lambda) f(y), \forall x, y \in \mathscr{H}_1. \tag{15}$$

(3) A vector $u \in \mathscr{H}_1$ is a subgradient of a point x if

$$\langle y - x, u \rangle + f(x) \le f(y), \forall y \in \mathscr{H}_1. \tag{16}$$

(4) The subdifferential of f is the set-valued operator

$$\partial f : \mathscr{H} \longrightarrow 2^{\mathscr{H}} \\ : x \longrightarrow \{u \in \mathscr{H} | (\forall y \in \mathscr{H}) \langle y - x, u\rangle + f(x) \leq f(y)\}. \quad (17)$$

(5) f is subdifferentiable at $x \in \mathscr{H}$ if $\partial f(x) \neq \varnothing$; the elements of $\partial f(x)$ are the subgradients f at x

Definition 5. Let $f : \mathscr{H}_1 \longrightarrow (-\infty, +\infty]$ be a proper function; then, we can obtain the following:

(1) f is lower semicontinuous at a point x if $x_n \longrightarrow x$ implies

$$f(x) \leq \liminf_{n \longrightarrow \infty} f(x_n). \quad (18)$$

(2) f is weakly lower semicontinuous at a point x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \longrightarrow \infty} f(x_n). \quad (19)$$

(3) f is lower semicontinuous on $\mathscr{H}_1$ if it is lower semicontinuous at any point $x \in \mathscr{H}_1$

(4) f is weakly lower semicontinuous on $\mathscr{H}_1$ if it is weakly lower semicontinuous at any point $x \in \mathscr{H}_1$

Lemma 6 (see [12]). *Assume that $f : \mathscr{H}_1 \longrightarrow (-\infty, +\infty]$ is a proper convex function; then, f is a lower semicontinuous function if and only if it is a weakly lower semicontinuous function.*

Definition 7. Let $\mathscr{C}$ be a nonempty closed convex subset in $\mathscr{H}_1$, and $\{x_n\}$ is a sequence in $\mathscr{H}_1$, if for any $n \geq 0$ and $z \in \mathscr{C}$, we have

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad (20)$$

and then, we say that the sequence $\{x_n\}$ is Fejér-monotone with respect to $\mathscr{C}$.

Lemma 8 (see [12]). *Assume that a sequence $\{x_n\}$ in $\mathscr{H}_1$ is Fejér-monotone with respect to $\mathscr{C}$ which is a nonempty closed convex subset of $\mathscr{H}_1$; then, $\{x_n\}$ is weakly convergent to a point of $\mathscr{C}$ if and only if its any weak cluster point belongs to $\mathscr{C}$.*

3. The Algorithm Proposed and Proved

In this section, we still concern with the case the involved subsets are composed of level sets, that is, the case whenever $\mathscr{C}$ and $\mathscr{Q}$ are given by (5) and (6). In this case, we shall assume that problem (1) is consistent, namely, its solution set, denoted by $\mathcal{S}$, is nonempty. Besides, we need to assume that ∂c and ∂q are bounded on bounded sets.

We know that, in $\mathscr{H}_1$, when consider the constrained convex minimization problem of a real valued convex function $f : \mathscr{C} \longrightarrow R$, one of the most famous methods is the gradient projection algorithm (GPA) that generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = P_{\mathscr{C}}(x_n - \alpha_n \nabla f(x_n)), \quad n \geq 0, \quad (21)$$

where the parameter $\{\alpha_n\}$ is a sequence of positive real numbers. And if ∇f is Lipschitz continuous with constant $L \geq 0$ and the parameter sequence $\{\alpha_n\}$ satisfies $0 < \liminf_{n \longrightarrow \infty} \alpha_n \leq \limsup_{n \longrightarrow \infty} \alpha_n < 2/L$, Xu [13] showed that the sequence $\{x_n\}$ generated by the GPA converges weakly to a minimizer of f.

We know that the solution of the SFP amounts to unconstrained minimization of

$$f(x) = \frac{1}{2}\|(I - P_{\mathscr{C}})x\|^2 + \frac{1}{2}\|(I - P_{\mathscr{Q}})Ax\|^2, \quad (22)$$

and $\nabla f = I - P_{\mathscr{C}} + A^*(I - P_{\mathscr{Q}})A$. By a simple calculation, we have ∇f is Lipschitz continuous with $2 + \|A\|^2$. In this case, method (21) is reduced to

$$x_{n+1} = P_{\mathscr{C}}\{x_n - \alpha_n[(I - P_{\mathscr{C}})x_n + A^*(I - P_{\mathscr{Q}})Ax_n]\}. \quad (23)$$

From the above equation, we know that the implementation of the above iterative algorithm needs to calculate the projections onto $\mathscr{C}$ and $\mathscr{Q}$ first. Since $\mathscr{C}$ and $\mathscr{Q}$ are both level sets defined by (5) and (6), it is very difficult to calculate the projection onto the level set at each step. To facilitate the computation of the projection onto $\mathscr{C}$ and $\mathscr{Q}$, we will compute the projections onto $\mathscr{C}_n$ and $\mathscr{Q}_n$ which were defined by (7) and (8) instead of $\mathscr{C}$ and $\mathscr{Q}$. Now, we give our ball-relaxed gradient-projection algorithm for solving the SFP (1).

Algorithm 1. Let x_0 be arbitrary, and generate $\{x_n\}$ according to the following iterative formula.

$$x_{n+1} = P_{\mathscr{C}_n}\left\{x_n - \alpha_n\left[\left(I - P_{\mathscr{C}_n}\right)x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\right]\right\}, \quad (24)$$

where $\mathscr{C}_n$ and $\mathscr{Q}_n$ are defined by (7) and (8), and the parameter sequence $\{\alpha_n\}$ satisfies $0 < a \leq \alpha_n \leq b < 2/L$, where a and b are two positive real numbers and $L = 2 + \|A\|^2$.

Theorem 9. *Let $\{x_n\}$ be the sequence generated by Algorithm 1; then, $\{x_n\}$ converges weakly to a solution of the SFP (1).*

Proof. On the one hand, we will show that the sequence $\{x_n\}$ is Fejér-monotone with respect to S.

Now, denoted $(I - P_{\mathscr{C}_n})x_n + A^*(I - P_{\mathscr{Q}_n})Ax_n$ by u_n; then, (24) is equivalent to $x_{n+1} = P_{\mathscr{C}_n}(x_n - \alpha_n u_n)$. For any $z \in \mathcal{S}$, that is, $z \in \mathscr{C} \subseteq \mathscr{C}_n$, and $Az \in \mathscr{Q} \subseteq \mathscr{Q}_n$, we have

$$\|x_{n+1}-z\|^2 = \|P_{\mathscr{C}_n}(x_n-\alpha_n u_n)-z\|^2 \le \|x_n-z-\alpha_n u_n\|^2 = \|x_n-z\|^2 - 2\alpha_n\langle x_n-z, u_n\rangle + \alpha_n^2\|u_n\|^2. \quad (25)$$

From Lemma 3, we obtain

$$\begin{aligned}\langle x_n-z, u_n\rangle &= \left\langle x_n-z, \left(I-P_{\mathscr{C}_n}\right)x_n + A^*\left(I-P_{\mathscr{Q}_n}\right)Ax_n\right\rangle \\ &= \left\langle x_n-z, x_n-P_{\mathscr{C}_n}x_n\right\rangle + \left\langle Ax_n-Az, Ax_n-P_{\mathscr{Q}_n}Ax_n\right\rangle \\ &\ge \|x_n-P_{\mathscr{C}_n}x_n\|^2+\|Ax_n-P_{\mathscr{Q}_n}Ax_n\|^2.\end{aligned} \quad (26)$$

Besides,

$$\begin{aligned}\|u_n\|^2 &= \|x_n-P_{\mathscr{C}_n}x_n + A^*\left(I-P_{\mathscr{Q}_n}Ax_n\right)\|^2 \\ &\le \left(1+\|A\|^2\right)\|x_n-P_{\mathscr{C}_n}x_n\|^2 + \left(1+\frac{1}{\|A\|^2}\right)\|A^*\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2 \\ &\le \left(1+\|A\|^2\right)\left(\|x_n-P_{\mathscr{C}_n}x_n\|^2+\|\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2\right).\end{aligned} \quad (27)$$

Substituting (26) and (27) into (25), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|x_n-z\|^2 - 2\alpha_n\left(\|x_n-P_{\mathscr{C}_n}x_n\|^2+\|\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2\right) \\ &\quad + \alpha_n^2\left(1+\|A\|^2\right)\left(\|x_n-P_{\mathscr{C}_n}x_n\|^2+\|\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2\right) \\ &= \|x_n-z\|^2 - \alpha_n\left[2-\alpha_n\left(1+\|A\|^2\right)\right] \\ &\quad \cdot \left(\|x_n-P_{\mathscr{C}_n}x_n\|^2+\|\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2\right).\end{aligned} \quad (28)$$

Since $0 < a \le \alpha_n \le b < 2/2+\|A\|^2$, we can obtain $\alpha_n[2-\alpha_n(1+\|A\|^2)] > 0$. The above inequality (28) implies that $\{x_n\}$ is Fejér-monotone with respect to S.

On the other hand, we will show that any weak cluster point of the sequence $\{x_n\}$ belongs to the solution set $\mathscr{S}$, i.e., $w_w(x_n) \subseteq \mathscr{S}$.

The inequality (28) implies that the sequence $\{\|x_n-z\|\}$ is bounded and converges to some finite limit. Passing to the limit in (28), we have

$$\lim_{n\longrightarrow\infty}\left(\|x_n-P_{\mathscr{C}_n}x_n\|^2+\|Ax_n-P_{\mathscr{Q}_n}Ax_n\|^2\right) = 0, \quad (29)$$

that is,

$$\lim_{n\longrightarrow\infty}\|x_n-P_{\mathscr{C}_n}x_n\| = 0, \qquad \lim_{n\longrightarrow\infty}\|Ax_n-P_{\mathscr{Q}_n}Ax_n\| = 0. \quad (30)$$

Besides, since the sequence $\{x_n\}$ is Fejér-monotone with respect to $\mathscr{S}$, it is bounded, and so is the sequence $\{Ax_n\}$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ convergent weakly to $\bar{x}$ ($\bar{x} \in w_w(x_n)$). What is more, due to ∂c and ∂q are bounded on bounded sets, there are two constants $\beta > 0$ and $\delta > 0$, such that $\|\xi_n\| \le \beta$ and $\|\zeta_n\| \le \delta$ for all $\xi_n \in \partial c(x_n)$, $\zeta_n \in \partial q(Ax_n)$ with $n \ge 0$.

From (7) and the fact that $P_{\mathscr{C}_{n_k}}x_{n_k} \in \mathscr{C}_{n_k}$, we obtain

$$\begin{aligned}c\left(x_{n_k}\right) &\le \left\langle \xi_{n_k}, x_{n_k}-P_{\mathscr{C}_{n_k}}x_{n_k}\right\rangle - \frac{\alpha}{2}\|x_{n_k}-P_{\mathscr{C}_{n_k}}x_{n_k}\|^2 \\ &\le \left\langle \xi_{n_k}, x_{n_k}-P_{\mathscr{C}_{n_k}}x_{n_k}\right\rangle \le \|\xi_{n_k}\|\cdot\|x_{n_k}-P_{\mathscr{C}_{n_k}}x_{n_k}\| \\ &\le \beta\|x_{n_k}-P_{\mathscr{C}_{n_k}}x_{n_k}\| \longrightarrow 0, \quad (k\longrightarrow\infty).\end{aligned} \quad (31)$$

Since c is convex and lower semicontinuous, then it is also weakly lower semicontinuous by Lemma 6. This together with (31) implies that

$$c(\bar{x}) \le \liminf_{n\longrightarrow\infty} c\left(x_{n_k}\right) \le 0. \quad (32)$$

It turns out that $\bar{x} \in \mathscr{C}$.

Similarly, from (8) and the fact that $P_{\mathscr{Q}_{n_k}}Ax_{n_k} \in \mathscr{Q}_{n_k}$, we obtain

$$\begin{aligned}q\left(Ax_{n_k}\right) &\le \left\langle \eta_{n_k}, Ax_{n_k}-P_{\mathscr{Q}_{n_k}}Ax_{n_k}\right\rangle - \frac{\beta}{2}\|Ax_{n_k}-P_{\mathscr{Q}_{n_k}}Ax_{n_k}\|^2 \\ &\le \left\langle \eta_{n_k}, Ax_{n_k}-P_{\mathscr{Q}_{n_k}}Ax_{n_k}\right\rangle \le \|\eta_{n_k}\|\cdot\|Ax_{n_k}-P_{\mathscr{Q}_{n_k}}Ax_{n_k}\| \\ &\le \delta\|Ax_{n_k}-P_{\mathscr{Q}_{n_k}}Ax_{n_k}\| \longrightarrow 0, \quad (k\longrightarrow\infty).\end{aligned} \quad (33)$$

Since q is convex and lower semicontinuous, then it is also weakly lower semicontinuous by Lemma 6. This together with (33) implies that

$$q(A\bar{x}) \le \liminf_{n\longrightarrow\infty} q\left(Ax_{n_k}\right) \le 0. \quad (34)$$

It turns out that $A\bar{x} \in \mathscr{Q}$. Then, $\bar{x} \in \mathscr{S}$ which implies that $w_w(x_n) \subseteq \mathscr{S}$. From Lemma 8, we can get $\{x_n\}$ converges weakly to $\bar{x}$ as $n \longrightarrow \infty$. This completes the proof. □

From Algorithm 1, we know that we have first to estimate the operator norm $\|A\|^2$ so that we can select appropriate parameters to implement Algorithm 1. However, computing or estimating the matrix norm $\|A\|^2$ is not an easy work in practice. To overcome this difficulty, we construct a variable stepsize that does not require the matrix norm.

Algorithm 2. Let x_0 be arbitrary, and generate $\{x_n\}$ according to the following iterative formula.

$$x_{n+1} = P_{\mathscr{C}_n}\left\{x_n - \alpha_n\left[\left(I-P_{\mathscr{C}_n}\right)x_n + A^*\left(I-P_{\mathscr{Q}_n}\right)Ax_n\right]\right\}, \quad (35)$$

where $\mathscr{C}_n$ and $\mathscr{Q}_n$ are defined by (7) and (8), and the parameter sequence $\{\alpha_n\}$ is given by

$$\alpha_n = \frac{\lambda_n}{a}\frac{\|\left(I-P_{\mathscr{C}_n}\right)x_n\|^2+\|\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2}{\|\left(I-P_{C_n}\right)x_n + A^*\left(I-P_{\mathscr{Q}_n}\right)Ax_n\|^2}, \quad (36)$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and a is any positive real number.

Theorem 10. *Let $\{x_n\}$ be the sequence generated by Algorithm 2; if $0 < \varepsilon \le \lambda_n \le 2a - \varepsilon$, then $\{x_n\}$ converges weakly to a solution of the SFP (1).*

Proof. For any $z \in S$, that is, $z \in \mathscr{C} \subseteq \mathscr{C}_n$, and $Az \in \mathscr{Q} \subseteq \mathscr{Q}_n$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_{\mathscr{C}_n}(x_n - \alpha_n u_n) - z\|^2 \le \|x_n - z - \alpha_n u_n\|^2 \\ &= \|x_n - z\|^2 - 2\alpha_n \langle x_n - z, u_n\rangle + \alpha_n^2 \|u_n\|^2.\end{aligned} \tag{37}$$

From Lemma 8, we obtain

$$\begin{aligned}\langle x_n - z, u_n\rangle &= \left\langle x_n - z, \left(I - P_{\mathscr{C}_n}\right)x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\right\rangle \\ &= \left\langle x_n - z, x_n - P_{\mathscr{C}_n}x_n\right\rangle + \left\langle Ax_n - Az, Ax_n - P_{\mathscr{Q}_n}Ax_n\right\rangle \\ &\ge \|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{\mathscr{Q}_n}Ax_n\|^2.\end{aligned} \tag{38}$$

Substituting (36) and (38) into (37), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \|x_n - z\|^2 - \frac{\lambda_n}{a}\left(2 - \frac{\lambda_n}{a}\right)\frac{\left(\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{Q_n}Ax_n\|^2\right)^2}{\|x_n - P_{C_n}x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\|^2} \\ &\le \|x_n - z\|^2 - \frac{\varepsilon^2}{a^2}\frac{\left(\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{Q_n}Ax_n\|^2\right)^2}{\|x_n - P_{C_n}x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\|^2}.\end{aligned} \tag{39}$$

The above inequality (39) implies that $\{x_n\}$ is Fejér-monotone with respect to $\mathcal{S}$.

Nextly, we show that any weak cluster point of the sequence $\{x_n\}$ belongs to the solution set S, i.e., $w_w(x_n) \subseteq S$. The inequality (39) implies that the sequence $\{\|x_n - z\|\}$ is bounded and converges to some finite limit. Passing to the limit in (39), we have

$$\lim_{n\longrightarrow\infty} \frac{\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{\mathscr{Q}_n}Ax_n\|^2}{\|x_n - P_{C_n}x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\|} = 0, \tag{40}$$

while

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left(\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{\mathscr{Q}_n}Ax_n\|^2\right) \\ &\quad = \lim_{n\longrightarrow\infty} \frac{\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|Ax_n - P_{\mathscr{Q}_n}Ax_n\|^2}{\|x_n - P_{\mathscr{C}_n}x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\|} \\ &\quad\quad \cdot \|x_n - P_{\mathscr{C}_n}x_n + A^*\left(I - P_{\mathscr{Q}_n}\right)Ax_n\| = 0,\end{aligned} \tag{41}$$

so, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - P_{\mathscr{C}_n}x_n\| = 0, \quad \lim_{n\longrightarrow\infty}\|Ax_n - P_{\mathscr{Q}_n}Ax_n\| = 0. \tag{42}$$

The rest proof is similar to Theorem 9. □

Remark 11. Recently, there are some new results about SFP, such as Shehu and Gibali introduced a relaxed $\mathscr{C}\mathscr{Q}$ method with alternated inertial step for solving SFP in [14]; Shehu et al. introduced some new computational technique for solving proximal SFP by using a modified proximal split feasibility algorithm in [15–17]. Our computational techniques are different from theirs.

4. Application in the Split Equality Problem

The split equality (SEP) is an inverse problem that requests to finding

$$(x, y) \in \mathscr{C} \times \mathscr{Q}, \quad s.t. \quad Ax = By, \tag{43}$$

where $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_3$ and $B : \mathscr{H}_2 \longrightarrow \mathscr{H}_3$ are two linear bounded operators, and $\mathscr{C} \subset \mathscr{H}_1$ and $\mathscr{Q} \subset \mathscr{H}_2$ are two nonempty closed convex subsets.

We know that, for any $\mathscr{X} = (x_1, x_2)$ and $\mathscr{Y} = (y_1, y_2)$ in $\mathscr{H}_1 \times \mathscr{H}_2$, if we defined

$$\begin{aligned}\langle \mathscr{X}, \mathscr{Y}\rangle &= \langle x_1, y_1\rangle + \langle x_2, y_2\rangle, \\ \|\mathscr{X}\|^2 &= \|x_1\|^2 + \|x_2\|^2, \\ A\mathscr{X} &= Ax_1 - Bx_2,\end{aligned} \tag{44}$$

where A and B are defined as the same to (43). Then, it has been proved that A is a bounded linear operator [11], and the SEP can be regarded as a special SFP: find $\mathscr{X} = (x, y) \in \mathscr{H}_1 \times \mathscr{H}_2$, such that

$$\mathscr{X} \in \mathscr{C}, \quad A\mathscr{X} \in \mathscr{Q}, \tag{45}$$

where $\mathscr{C} = \mathscr{C} \times \mathscr{Q}$, $\mathscr{Q} = \{0\}$. Motivated by (35), we can propose a new method for solving the SEP (43).

Algorithm 3. For any arbitrary initial guess (x_0, y_0), define (x_n, y_n) recursively by the following.

$$\begin{aligned}x_{n+1} &= P_{\mathscr{C}_n}\left[x_n - \alpha_n\left(x_n - P_{\mathscr{C}_n}x_n + A^*(Ax_n - By_n)\right)\right], \\ y_{n+1} &= P_{\mathscr{Q}_n}\left[y_n - \alpha_n\left(y_n - P_{\mathscr{Q}_n}y_n - B^*(Ax_n - By_n)\right)\right],\end{aligned} \tag{46}$$

where $\mathscr{C}_n$ and $\mathscr{Q}_n$ are, respectively, given by (7) and (8), and parameter sequence $\{\alpha_n\}$ is chosen as

$$\alpha_n = \frac{\lambda_n}{a}\frac{\|x_n - P_{\mathscr{C}_n}x_n\|^2 + \|y_n - P_{\mathscr{Q}_n}y_n\|^2 + \|Ax_n - By_n\|^2}{\|x_n - P_{\mathscr{C}_n}x_n + A^*(Ax_n - By_n)\|^2 + \|y_n - P_{\mathscr{Q}_n}y_n - B^*(Ax_n - By_n)\|^2}, \tag{47}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers, and a is any positive real number.

Theorem 11. *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 3, and if $0 < \varepsilon \le \lambda_n \le 2a - \varepsilon$, then $\{(x_n, y_n)\}$ converges weakly to a solution of the SEP (43).*

Proof. Let $\mathscr{X}_n = (x_n, y_n)$, and $A : \mathscr{H}_1 \times \mathscr{H}_2 \longrightarrow \mathscr{H}_3$ is a bounded linear operator which is defined by $A\mathscr{X} = Ax - By$; then, Algorithm 3 can be rewritten as

$$\mathcal{X}_{n+1} = P_{\mathcal{C}_n}\left[\mathcal{X}_n - \alpha_n\left(\mathcal{X}_n - P_{\mathcal{C}_n}\mathcal{X}_n + \mathcal{A}^*\left(I - P_{\{0\}}\right)A\mathcal{X}_n\right)\right], \tag{48}$$

where $\mathcal{C}_n = \mathcal{C}_n \times \mathcal{Q}_n$. Hence, by applying Theorem 10, we can conclude that $\mathcal{X}_n$ converges weakly to a solution of the SEP (43). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

Acknowledgments

This research is supported by the Teacher Education Curriculum Reform Project of Henan Province (2021-JSJYYB-019), the Postgraduate Education Reform and Quality Improvement Project of Henan Province (YJS2022ZX33), the higher education teaching reform and practice project (postgraduate education) of Henan Normal University (YJS2019JG01), and the Scientific Research Project for Postgraduates of Henan Normal University (YL202121).

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