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Research Article

A New Version of the Generalized F-Expansion Method for the Fractional Biswas-Arshed Equation and Boussinesq Equation with the Beta-Derivative

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In this article, a new version of the generalized F-expansion method is proposed enabling to obtain the exact solutions of the Biswas-Arshed equation and Boussinesq equation defined by Atangana's beta-derivative. First, the new version generalized F-expansion method is introduced, and then, the exact solutions of the nonlinear fractional differential equations expressed with Atangana's beta-derivative are given. When the results are examined, it is seen that single, combined, and mixed Jacobi elliptic function solutions are obtained. From the point of view, it is understood that the new version generalized F-expansion method can give significant results in finding the exact solutions of equations containing beta-derivatives.

1. Introduction

In recent years, many articles have been published on obtaining numerical and exact solutions of some physical phenomena that can be mathematically modeled using fractional derivatives [1–4]. Many physical phenomena are usually expressed in nonlinear fractional partial differential equations. These equations have application areas such as biology, engineering, dynamics, control theory, signal processing, chemistry, continuum mechanics, and physics, respectively. There are different types of fractional derivative operators defined in the literature. Examples of these derivative operators are Riemann-Liouville derivative [5], Jumarie's modified Riemann-Liouville derivative [6], Caputo derivative [7], Caputo-Fabrizio [8], and Atangana-Baleanu derivative [9]. It is very substantial to find the exact solutions of the nonlinear fractional differential equations. Different methods aiming to find analytical, numerical, and exact solutions of the nonlinear partial differential equations including these derivative operators have been improved as follows: unified method [10], modified trial equation method [11], extended trial equation method [12], fractional local homotopy perturbation transformation method [13], Fourier spectral method [14], variational iteration method [15], Laplace transforms [16], Chebyshev-Tau method [17], finite difference method [18], finite element method [19], etc.

A new definition of the fractional derivative called as conformable derivative has been given, and the exact solutions of the time-heat differential equation created by using this derivative are obtained [20, 21]. In later years, Atangana et al. [22] gave some new features and definitions about the conformable derivative. By using these definitions and properties, some methods have been applied [23, 24]. In the next year, a new definition of fractional derivative called as beta-derivative was given by Atangana et al. [25]. In that article, they obtained the analytical solution of the Hunter-Saxton equation. Exact solutions of the Hunter-Saxton, Sharma-Tasso-Olver, space-time fractional modified Benjamin-Bona-Mahony, and time fractional Schrödinger equations expressed by Atangana's beta-derivative are obtained by using the first integral method [26]. They applied the fractional subequation method to obtain the exact solutions of the space-time conformable generalized Hirota-Satsuma coupled KdV equation, coupled mKdV equation, and space-time resonance nonlinear Schrodinger equations created with Atangana's beta-derivative [27, 28]. Ghanbari and Gomez-Aguilar attained the exact solutions by applying the generalized exponential rational function method to the Radhakrishnan-Kundu-Lakshmanan equation with Atangana's beta-derivative [29]. Like the problems discussed in this article, it is very difficult to find analytical and numerical solutions for nonlinear partial differential equations involving fractional order derivative, especially problems with complex coefficients and absolute value functions. For this reason, the motivation to research the exact solutions of these problems has occurred. From this point of view, it is considered to apply the new version generalized F-expansion method in order to determine solutions such as rational forms of Jacobi elliptic functions that are not in the literature. The double-period Jacobi elliptic functions and their rational combinations, which cannot be found by every method in the literature, can be reached with a new generalized F-expansion method. This method can be successfully applied to a wide variety of equations.

In this article, for the first time, the new version generalized F-expansion method has been investigated in order to find the exact solutions of the differential equations consisting of Atangana's beta-derivative. With this offered method, it is aimed at finding new and several exact solutions of fractional order differential equations that are not actual in the literature. This method, which has been discussed in some studies in the literature, has been applied to various nonlinear partial differential equations [30–32]. There are different F-expansion methods that allow procuring the elliptic function solutions, which are among these exact solutions [33–36].

Firstly we will investigate the exact solutions of the Biswas-Arshed equation with Atangana's beta-derivative:

$$i{}_0^AD_t^\beta\phi + k_1\phi_{xx} + k_2{}_0^AD_t^\beta(\phi_x) + i\left(l_1\phi_{xxx} + l_2{}_0^AD_t^\beta(\phi_{xx})\right) - i\left(\varepsilon\left(|\phi|^2\phi\right)_x + \mu\phi\left(|\phi|^2\right)_x + \theta|\phi|^2\phi_x\right) = 0, \quad (0 < \beta \le 1), \tag{1}$$

where $\phi = \phi(x, t)$ is a complex function [37–40]; k_1 and k_2 are the parameters of the group velocity dispersion and the spatiotemporal dispersion, respectively; l_1 and l_2 are the parameters of the third-order dispersion and the spatiotemporal third-order dispersion, respectively; ε is the parameter of the self-steepening effect; and μ and θ present the parameters of the nonlinear dispersions. Also, we will research the exact solutions of the Boussinesq equation with the beta-derivative [41]

$${}_0^AD_t^\beta\Psi + bD_x^{2\beta}\Psi + cD_x^{2\beta}\left(\Psi^2\right) + \gamma D_x^{4\beta}\Psi = 0, \quad (0 < \beta \le 1), \tag{2}$$

where b, c, and γ are constants. Also, c is the parameter controlling nonlinearity, and γ is the dispersion parameter depending on the rigidity characteristics of the material and compression.

The remaining lines of the article are regulated as follows: in Section 2, Atangana's conformable fractional derivative and its properties are given. In Section 3, the new version generalized F-expansion method is explained in detail. Applications of the method are given in Sections 4 and 5. This article is completed with conclusions in Section 6.

2. The Properties and Definition of Beta-Derivative

There are different definitions of the conformable fractional derivatives in literature. One of them is given by Khalil et al. in the paper [20]. Then, Abdeljawad developed the basic concepts in this conformable fractional calculus [42]. The conformable derivative of the function $g : [0,\infty)$ of the order α from type $t > 0, \alpha \in (0, 1)$ is as follows:

$${}_0D_t^\alpha\{g(t)\} = \lim_{\varepsilon \longrightarrow 0} \frac{g\left(t + \varepsilon t^{1-\alpha}\right) - g(t)}{\varepsilon}. \tag{3}$$

When g which is α-differentiable in the interval of $(0, a)$, $a > 0$ and $\lim_{\varepsilon \longrightarrow 0^+} g^{(\alpha)}(t)$ exists, then it can be defined as $g^{(\alpha)}(0) = \lim_{\varepsilon \longrightarrow 0^+} g^{(\alpha)}(t)$.

The other conformable fractional derivative called as the beta-derivative is defined in [22] as

$${}_0^AD_t^\alpha\{g(t)\} = \lim_{\varepsilon \longrightarrow 0} \frac{g\left(t + \varepsilon(t + (1/\Gamma(\alpha)))^{1-\alpha}\right) - g(t)}{\varepsilon}. \tag{4}$$

The mathematical model considered in the study that depends on Atangana's conformable fractional derivative is selected because it provides some properties of the basic derivative rules. According to all these cases, the various features of Atangana's conformable fractional derivative are as follows:

(i) If $h \neq 0$ and g functions are differentiable according to beta in the range $\beta \in (0, 1]$, then the equation that the functions f and g can satisfy for all the real numbers q and r is as follows:

$${}_0^AD_x^\alpha\{qg(x) + rh(x)\} = q{}_0^AD_x^\alpha\{g(x)\} + r{}_0^AD_x^\alpha\{h(x)\}. \tag{5}$$

(ii) Let us take any constant p. It can be easily seen that it satisfies the following equality:

$${}_0^AD_x^\alpha\{p\} = 0. \tag{6}$$

(iii) ${}_0^AD_x^\alpha\{g(x)h(x)\} = h(x){}_0^AD_x^\alpha\{g(x)\} + g(x){}_0^AD_x^\alpha\{h(x)\}$

(iv) ${}_0^AD_x^\alpha\{g(x)/h(x)\} = (h(x){}_0^AD_x^\alpha\{g(x)\} - g(x){}_0^AD_x^\alpha\{h(x)\})/h^2(x)$

If $\lambda = (x + (1/\Gamma(\alpha)))^{\alpha-1}\nu$ is substituted instead of λ in Equation (4) and $\nu \longrightarrow 0$, when $\lambda \longrightarrow 0$, it is observed as follows:

$$ {}_{0}^{A}D_{x}^{\alpha}\{g(x)\} = \left(x + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} \frac{dg(x)}{dx}, \quad (7) $$

with

$$ \eta = \frac{\delta}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad (8) $$

where δ is any constant. Therefore, the relation between Atangana's conformable fractional derivative and the classical derivative is determined as follows:

$$ {}_{0}^{A}D_{x}^{\alpha}\{g(\eta)\} = \delta \frac{dg(\eta)}{d\eta}. \quad (9) $$

3. Definition of the New Version of Generalized F-Expansion Method

In this section, the application steps of the new version of the generalized F-expansion method to obtain the combined and mixed Jacobi elliptic function solutions of differential equations will be given [30–32]. With this new method, different and new results can be acquired from the results obtained from other methods.

Let us consider the partial differential equation with the Atangana (beta) fractional derivative as

$$ \tilde{S}\left(\phi, {}_{0}^{A}D_{t}^{\beta}\phi, {}_{0}^{A}D_{x}^{\beta}\phi, {}_{0}^{A}D_{t}^{2\beta}\phi, {}_{0}^{A}D_{x}^{2\beta}\phi, \cdots,\right) = 0, \quad (0 < \beta \leq 1), \quad (10) $$

where $\phi(x, t, \cdots)$ is an unknown function, $x, t, \cdots$ is the independent variables, and $\tilde{S}$ is a polynomial of ϕ and its fractional derivatives, in which the highest-order derivatives and the nonlinear terms are contained. When we implemented the wave transform to Equation (10),

$$ \phi(x, t) = \phi(\eta), \quad \eta = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \quad (11) $$

where τ and λ are constants that will be determined later; we can diminish Equation (10) to nonlinear ordinary differential equation

$$ H\left(\phi, \phi', \phi'', \phi''', \cdots\right) = 0, \quad (12) $$

where the prime demonstrates differentiation pursuant to η. Suppose that the solution function of Equation (12) is as follows:

$$ \phi(\eta) = a_0 + \sum_{i=1}^{M}\left(a_i F^i + \frac{b_i}{F^i} + c_i\left(\frac{F'}{F}\right)^i + d_i\left(\frac{F}{F'}\right)^i\right), \quad (13) $$

where $a_0, a_i, b_i, c_i, d_i (i = 1, 2, 3, \cdots, M)$ are constants, $F = F(\eta)$, and $F' = F'(\eta)$. $F(\eta)$ and $F'(\eta)$ functions in Equation (13) provide the following equation:

$$ F'^2(\eta) = PF^4(\eta) + QF^2(\eta) + R, \quad (14) $$

and using Equation (14), the related derivatives are found as follows:

$$ \begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi), \\ F'''(\xi) = \left(6PF^2(\xi) + Q\right)F'(\xi), \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + \left(Q^2 + 12PR\right)F(\xi), \\ F^{(5)}(\xi) = \left(120P^2F^4(\xi) + 60PQF^2(\xi) + Q^2 + 12PR\right)F'(\xi), \\ \cdots \end{cases} \quad (15) $$

where P, Q, and R are all coefficients. To determine the value of M in Equation (13), we use the derivatives in Equation (15). The process of finding the number M is called the balancing process. The number M is a positive number and is determined by balancing the highest-order derivative terms in Equation (12) with the highest-power nonlinear terms. When finding this number in terms of F^M, $1/F^M$, $(F'/F)^M$ and $(F/F')^M$ in the solution function (12) are considered with respect to Equation (14) in conjunction with the degree of derivatives. Therefore, proposed solution function (13) arranged and requisite terms in place of Equation (12) are attached to $(F')^k F^l (k = 0, 1; l = 0, \pm1, \pm2, \cdots)$ function; then, the polynomial is attained. When this polynomial equation is set to zero, then a system of algebraic equations is attained with the coefficients with respect to zero. When the algebraic equation system is solved according to the specified algorithm, the necessary τ, λ, and $a_0, a_i, b_i, c_i, d_i (i = 1, 2, 3, \cdots, M)$ coefficients for the solution function are found. Thence, the new combined and mixed Jacobi elliptic function solutions are gained. If different values of P, Q, and R are taken, diverse Jacobi elliptic function solutions $F(\eta)$ can be attained from Equation (14).

4. Application of the New Version Method to the Biswas-Arshed Equation

In this section, the new version generalized F-expansion method is implemented to the Biswas-Arshed equation with Atangana's beta-derivative. The Biswas-Arshed equation with Atangana's beta-derivative defines pulse propagation through optical fiber. Optical fibers are the main element of data transmission in telecommunications systems. The main aim of the researchers is to improve the quality of transmitted signals, reduce losses, and increase transmission speed. For this reason, it is important to obtain the solutions of such physical equations.

Hosseini et al. found the exact solutions of Equation (1) via the Jacobi and Kudryashov methods [37]. Akbulut and

Islam implemented modified extended auxiliary equation mapping and improved F-expansion methods to acquire the exact solutions of Eq. (1) in [39]. On the other hand, Han et al. utilized the polynomial full discriminant system method to find the exact solutions of Eq. (1) in [40]. Jacobi elliptic function solutions can be found with the methods in the literature, but it is very difficult to find rational function solutions containing the Jacobi elliptic functions we obtained with the method we used in this article, because the method we used includes not only the F function, which expresses the Jacobi elliptic function solutions obtained from the elliptic differential equation, but also the F'/F and F/F' functions. Thus, the rational function solutions or combined containing Jacobi elliptic functions are reached by this way.

Firstly, we acquaint wave transformation for this complex variable equation:

$$\begin{aligned} \phi(x,t) &= \phi(\eta)e^{i\varphi(x,t)}, \\ \eta &= x - \frac{\rho}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \\ \varphi(x,t) &= -kx + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \end{aligned} \tag{16}$$

where ρ, κ, and ω are constants which represents the speed of the wave, frequency, and wave number, respectively. By using the wave transformation in Equation (16), Equation (1) reduces the real and imaginary parts as follows:

$$\begin{aligned} &(2\kappa\rho l_2 - 3\kappa l_1 + \omega l_2 + \rho k_2 - k_1)\phi''(\eta) \\ &+ \left(\kappa^3 l_1 - \kappa^2\omega l_2 + \kappa^2 k_1 - \kappa\omega k_2 + \omega\right)\phi(\eta) \\ &+ (\kappa\varepsilon + \kappa\theta)\phi^3(\eta) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &(\rho l_2 - l_1)\phi'''(\eta) + \left(-\kappa^2\rho l_2 + 3\kappa^2 l_1 - 2\kappa\omega l_2 - \kappa\rho k_2\right. \\ &\quad \left. + 2\kappa k_1 - \omega k_2 + \rho\right)\phi'(\eta) + (3\varepsilon + 2\mu + \theta)\phi^2(\eta)\phi'(\eta) = 0. \end{aligned} \tag{18}$$

From Equation (12), the following equations are easily obtained:

$$\begin{aligned} \rho &= \frac{l_1}{l_2}, \\ \varepsilon &= \frac{-2\mu - \theta}{3}, \\ \omega &\frac{2\kappa^2 l_1 l_2 + 2\kappa k_1 l_2 - \kappa k_2 l_1 + l_1}{l_2(2\kappa l_2 + k_2)}. \end{aligned} \tag{19}$$

When these obtained values are substituted in Equation (17), the following second-order nonlinear ordinary differential equation is found:

$$\begin{aligned} &\left(-\kappa l_1 + \frac{2\kappa^2 l_1 l_2 + 2\kappa k_1 l_2 - \kappa k_2 l_1 + l_1}{2\kappa l_2 + k_2} + \frac{l_1 k_2}{l_2} - k_1\right)\phi''(\eta) \\ &+ \left(\kappa^3 l_1 - \frac{\kappa^2\left(2\kappa^2 l_1 l_2 + 2\kappa k_1 l_2 - \kappa k_2 l_1 + l_1\right)}{2\kappa l_2 + k_2}\right. \\ &- \frac{\kappa k_2\left(2\kappa^2 l_1 l_2 + 2\kappa k_1 l_2 - \kappa k_2 l_1 + l_1\right)}{l_2(2\kappa l_2 + k_2)} \\ &\left. + \frac{2\kappa^2 l_1 l_2 + 2\kappa k_1 l_2 - \kappa k_2 l_1 + l_1}{l_2(2\kappa l_2 + k_2)} + \kappa^2 k_1\right)\phi(\eta) \\ &+ \left(\kappa\left(\frac{-2\mu - \theta}{3}\right) + \kappa\theta\right)\phi^3(\eta) = 0. \end{aligned} \tag{20}$$

According to the balance procedure for the functions $\phi''(\eta)$ and $\phi^3(\eta)$ in Equation (20), we can find $M = 1$, so the solution of Equation (1) is assumed that it provides the following equation:

$$\phi(\eta) = a_0 + a_1 F(\eta) + \frac{b_1}{F(\eta)} + c_1\left(\frac{F'(\eta)}{F(\eta)}\right) + d_1\left(\frac{F(\eta)}{F'(\eta)}\right). \tag{21}$$

When the calculated $\phi''(\eta)$ and $\phi^3(\eta)$ expressions from Equation (21) are replaced in Equation (20), a zero polynomial dependent on $F(\eta)$ and $F'(\eta)$ is obtained. When the algebraic equation system, which is found by equating the coefficients of this zero polynomial to zero, is resolved with the help of the Mathematica package program, the a_0, a_1, b_1, c_1, d_1, and κ coefficients are obtained. While applying the method, since the number of variables is more than the number of equations in the solution of the nonlinear algebraic system of equations, some constants in the partial differential equations are taken as arbitrary parameters and the parametric solutions of the system are reached. When the obtained coefficients and the inverse transformation are substituted to the solution function (21), the following exact solutions are obtained, which depends on the elliptic functions of $F(\eta)$ and $F'(\eta)$. If the elliptic function here is specially chosen as $F(\eta) = sn(\eta)$, where $P = m^2$, $Q = -(1 + m^2)$, and $R = 1$, then the new combined and mixed exact solutions are specified in the following cases.

Case 1.

$$\begin{aligned} &a_0 = b_1 = c_1 = d_1 = 0, \\ &a_1 = a_1, \\ &l_1 = \frac{\kappa(\theta - \mu) l_2 a_1^2\left(k_2\left(\kappa^2 + Q\right) - 2\kappa\right)}{3P}, \end{aligned} \tag{22}$$

$$k_1 = \frac{\kappa(\theta - \mu) a_1^2\left(\left(k_2^2 + l_2\right)\left(\kappa^2 + Q\right) - 2\kappa k_2 + 1\right)}{3P}. \tag{23}$$

Substituting Equation (22) into Equation (21), we attain single Jacobi elliptic function solutions of Equation (1).

$$\phi(\eta_1) = A_1 e^{i\varphi_1} sn(\eta_1), \tag{24}$$

where $\eta_1 = x - (\kappa(\theta-\mu)a_1^2(k_2(\kappa^2-1-m^2)-2\kappa)/3m^2\beta)(t+(1/\Gamma(\beta)))^\beta$, $A_1 = a_1$, and $\varphi_1 = -\kappa x - (\kappa(\theta-\mu)a_1^2(1+m^2+\kappa^2+\kappa k_2(1+m^2-\kappa^2))/3m^2\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 2.

$$\begin{aligned} &a_0 = a_1 = c_1 = d_1 = 0,\\ &b_1 = b_1,\\ &l_1 = \frac{\kappa(\theta-\mu)l_2 b_1^2\left(k_2\left(\kappa^2+Q\right)-2\kappa\right)}{3R}, \end{aligned} \tag{25}$$

$$k_1 = \frac{\kappa(\theta-\mu)b_1^2\left(\left(k_2^2+l_2\right)\left(\kappa^2+Q\right)-2\kappa k_2+1\right)}{3R}. \tag{26}$$

If the obtained coefficients in expression (25) are subrogated in the solution function (21), we find the Jacobi elliptic function solution of Equation (1).

$$\phi(\eta_2) = A_2 e^{i\varphi_2} ns(\eta_2), \tag{27}$$

where $\eta_2 = x - (\kappa(\theta-\mu)b_1^2(k_2(\kappa^2-1-m^2)-2\kappa)/3\beta)(t+(1/\Gamma(\beta)))^\beta$, $A_2 = b_1$, and $\varphi_2 = -\kappa x - (\kappa(\theta-\mu)b_1^2(1+m^2+\kappa^2+\kappa k_2(1+m^2-\kappa^2))/3\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 3.

$$\begin{aligned} &a_0 = a_1 = b_1 = d_1 = 0,\\ &c_1 = c_1,\\ &l_1 = \frac{\kappa(\theta-\mu)l_2 c_1^2\left(k_2\left(\kappa^2-2Q\right)-2\kappa\right)}{3}, \end{aligned} \tag{28}$$

$$k_1 = \frac{\kappa(\theta-\mu)c_1^2\left(\left(k_2^2+l_2\right)\left(\kappa^2-2Q\right)-2\kappa k_2+1\right)}{3}. \tag{29}$$

When the obtained coefficients in Equation (28) are set into Equation (21), we attain new types of the combined Jacobi elliptic function solution as follows:

$$\phi(\eta_3) = A_3 e^{i\varphi_3} cs(\eta_3)dn(\eta_3), \tag{30}$$

where $\eta_3 = x - (\kappa(\theta-\mu)c_1^2(k_2(\kappa^2+2(1+m^2))-2\kappa)/3\beta)(t+(1/\Gamma(\beta)))^\beta$, $A_3 = c_1$, and $\varphi_3 = -\kappa x + (\kappa(\theta-\mu)c_1^2(2+2m^2-\kappa^2+\kappa k_2(2+2m^2+\kappa^2))/3\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 4.

$$\begin{aligned} &a_0 = a_1 = b_1 = c_1 = 0,\\ &d_1 = d_1,\\ &l_1 = \frac{\kappa(\theta-\mu)l_2 d_1^2\left(k_2\left(\kappa^2-2Q\right)-2\kappa\right)}{3\left(Q^2-4PR\right)}, \end{aligned} \tag{31}$$

$$k_1 = \frac{\kappa(\theta-\mu)d_1^2\left(\left(k_2^2+l_2\right)\left(\kappa^2-2Q\right)-2\kappa k_2+1\right)}{3\left(Q^2-4PR\right)}. \tag{32}$$

When achieved coefficients in Equation (31) are replaced into Equation (21), we gain new exact solution called as combined Jacobi elliptic function solutions of Equation (1) as follows:

$$\phi(\eta_4) = A_4 e^{i\varphi_4} sc(\eta_4)nd(\eta_4), \tag{33}$$

where $\eta_4 = x - (\kappa(\theta-\mu)d_1^2(k_2(\kappa^2+2(1+m^2))-2\kappa)/3(1-2m^2+m^4)\beta)(t+(1/\Gamma(\beta)))^\beta$, $A_4 = d_1$, and $\varphi_4 = -\kappa x + (\kappa(\theta-\mu)d_1^2(2+2m^2-\kappa^2+\kappa k_2(2+2m^2+\kappa^2))/3(m^2-1)^2\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 5.

$$\begin{aligned} &a_0 = c_1 = d_1 = 0,\\ &a_1 = a_1,\\ &b_1 = -\sqrt{\frac{R}{P}}a_1,\\ &l_1 = \frac{\kappa(\theta-\mu)l_2 a_1^2\left(k_2\left(\kappa^2+6\sqrt{PR}+Q\right)-2\kappa\right)}{3P}, \end{aligned} \tag{34}$$

$$k_1 = \frac{\kappa(\theta-\mu)a_1^2\left(\left(k_2^2+l_2\right)\left(\kappa^2+6\sqrt{PR}+Q\right)-2\kappa k_2+1\right)}{3P}. \tag{35}$$

Substituting the coefficients in Equation (34) into Equation (21), we get the exact solutions of Equation (1).

$$\phi(\eta_5) = A_5 e^{i\varphi_5}\frac{\left(msn^2(\eta_5)-1\right)}{sn(\eta_5)}, \tag{36}$$

where $\eta_5 = x - (\kappa(\theta-\mu)a_1^2(k_2(\kappa^2-1-m^2+6m)-2\kappa)/3m^2\beta)(t+(1/\Gamma(\beta)))^\beta$, $A_5 = a_1/m$, and $\varphi_5 = -\kappa x - (\kappa(\theta-\mu)a_1^2(1+m^2-6m+\kappa^2+\kappa k_2(1+m^2-6m-\kappa^2))/3m^2\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 6.

$$
\begin{aligned}
&a_0 = a_1 = b_1 = 0,\\
&c_1 = c_1,\\
&d_1 = -c_1\sqrt{Q^2 - 4PR},
\end{aligned}
\tag{37}
$$

$l_1 = \kappa(\theta-\mu)l_2 c_1(6k_2 c_1\sqrt{Q^2-4PR} + c_1(k_2(\kappa^2-2Q)-2\kappa))/3$, $k_1 = \kappa(\theta-\mu)c_1(6c_1(k_2^2+l_2)\sqrt{Q^2-4PR} + c_1((k_2^2+l_2)(\kappa^2-2Q)-2\kappa k_2+1))/3$.

When obtained coefficients in Equation (37) are replaced into Equation (21), we get new exact solution named as mixed Jacobi elliptic function solutions of Equation (1) as follows:

$$
\phi(\eta_6) = A_3 e^{i\varphi_6}\frac{cn^2(\eta_6)\left(dn^2(\eta_6)-1\right)+dn^2(\eta_6)}{cn(\eta_6)dn(\eta_6)sn(\eta_6)},
\tag{38}
$$

where $\eta_6 = x - (\kappa(\theta-\mu)c_1(6k_2c_1\sqrt{m^4-2m^2+1} + c_1(k_2(\kappa^2+2+2m^2)-2\kappa))/3\beta)(t+(1/\Gamma(\beta)))^\beta$ and $\varphi_6 = -\kappa x + (\kappa(\theta-\mu)c_1^2(6(m^2-1)(1+\kappa k_2)+2+2m^2-\kappa^2+\kappa k_2(2+2m^2+\kappa^2))/3\beta)(t+(1/\Gamma(\beta)))^\beta$.

Case 7.

$$
\begin{aligned}
&a_0 = d_1 = 0,\\
&a_1 = -\sqrt{P}c_1 b_1 = -\sqrt{P}c_1,\\
&c_1 = c_1,\\
&l_1 = \frac{2\kappa(\theta-\mu)c_1^2 l_2\left(k_2\left(Q+6\sqrt{PR}-2\kappa^2\right)-4\kappa\right)}{3},
\end{aligned}
\tag{39}
$$

$$
k_1 = \frac{2\kappa(\theta-\mu)c_1^2\left(2-4\kappa k_2-\left(k_2^2+l_2\right)\left(Q+6\sqrt{PR}-2\kappa^2\right)\right)}{3}.
\tag{40}
$$

When acquired coefficients in Equation (39) are put into Equation (21), we attain new exact combined Jacobi elliptic function solution of Equation (1).

$$
\phi(\eta_7) = A_3 e^{i\varphi_7}\frac{\left(cn(\eta_7)dn(\eta_7)-msn^2(\eta_7)-1\right)}{sn(\eta_7)},
\tag{41}
$$

where $\eta_7 = x - (2\kappa(\theta-\mu)c_1^2(k_2(-1-m^2+6m-2\kappa^2)-2\kappa)/3\beta)(t+(1/\Gamma(\beta)))^\beta$ and $\varphi_7 = -\kappa x + (2\kappa(\theta-\mu)c_1^2(1+m^2-6m-2\kappa^2+\kappa k_2(1+m^2-6m+2\kappa^2))/3\beta)(t+(1/\Gamma(\beta)))^\beta$.

Remark 1. When the literature review of the obtained results is made, it is seen that all Jacobi elliptic function solutions obtained by the new version generalized F-expansion method of Equation (1) are new and different wave solutions. Besides, two- and three-dimensional graphics of the attained exact solution functions are shown in Figures 1–7 with appropriate coefficient values.

5. Application of the New Version Method to the Boussinesq Equation with Beta-Derivative

In this section, the implementation of the new version of the generalized F-expansion method to Boussinesq equation with beta-derivative is presented. Firstly, we acquaint wave transformation of Equation (2) as follows:

$$
\begin{aligned}
&\Psi(x,t) = \psi(\vartheta),\\
&\vartheta = \frac{k}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta - \frac{\sigma}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta,
\end{aligned}
\tag{42}
$$

where k and σ are constants. Equation (2) is reduced to a nonlinear 4-order ordinary differential equation in the following form under the transformation (42):

$$
\left(\sigma^2+bk^2\right)\psi'' + ck^2\left(\psi^2\right)'' + \gamma k^4\psi^{(iv)} = 0.
\tag{43}
$$

If Equation (43) is integrated twice according to ϑ and the integration constant is assumed to be zero, then a nonlinear second-order ordinary differential equation is found as follows:

$$
\left(\sigma^2+bk^2\right)\psi + ck^2\psi^2 + \gamma k^4\psi'' = 0.
\tag{44}
$$

According to the proposed new version of generalized F-expansion method, before applying the solution function (13) to Equation (44), the balance operation is performed. The balance procedure is applied between the ψ'' term containing the highest-order derivative and the nonlinear ψ^2 terms of the highest order in Equation (44). Accordingly, as a result of the transactions made between the terms providing balancing $M = 1$ is found, thus, the solution function of Equation (2) is as follows:

$$
\begin{aligned}
\psi(\vartheta) = {} & a_0 + a_1F(\vartheta) + a_2F^2(\vartheta) + \frac{b_1}{F(\vartheta)} + \frac{b_2}{F^2(\vartheta)}\\
& + c_1\left(\frac{F'(\vartheta)}{F(\vartheta)}\right) + c_2\left(\frac{F'(\vartheta)}{F(\vartheta)}\right)^2\\
& + d_1\left(\frac{F(\vartheta)}{F'(\vartheta)}\right) + d_2\left(\frac{F(\vartheta)}{F'(\vartheta)}\right)^2.
\end{aligned}
\tag{45}
$$

When the computed $\psi''(\vartheta)$ and $\psi^2(\vartheta)$ terms from Equation (45) are substituted in Equation (44), a zero polynomial dependent on $F(\eta)$ and $F'(\eta)$ is attained. When the algebraic equation system is solved with the help of the Mathematica package program, the a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, k, and σ coefficients are acquired. When the obtained coefficients and the inverse transformation are substituted to the solution function (45), the following exact solutions are found, which depends on $F(\eta)$ and $F'(\eta)$. If the elliptic function here is specially chosen as $F(\eta) = sn(\eta)$, where P

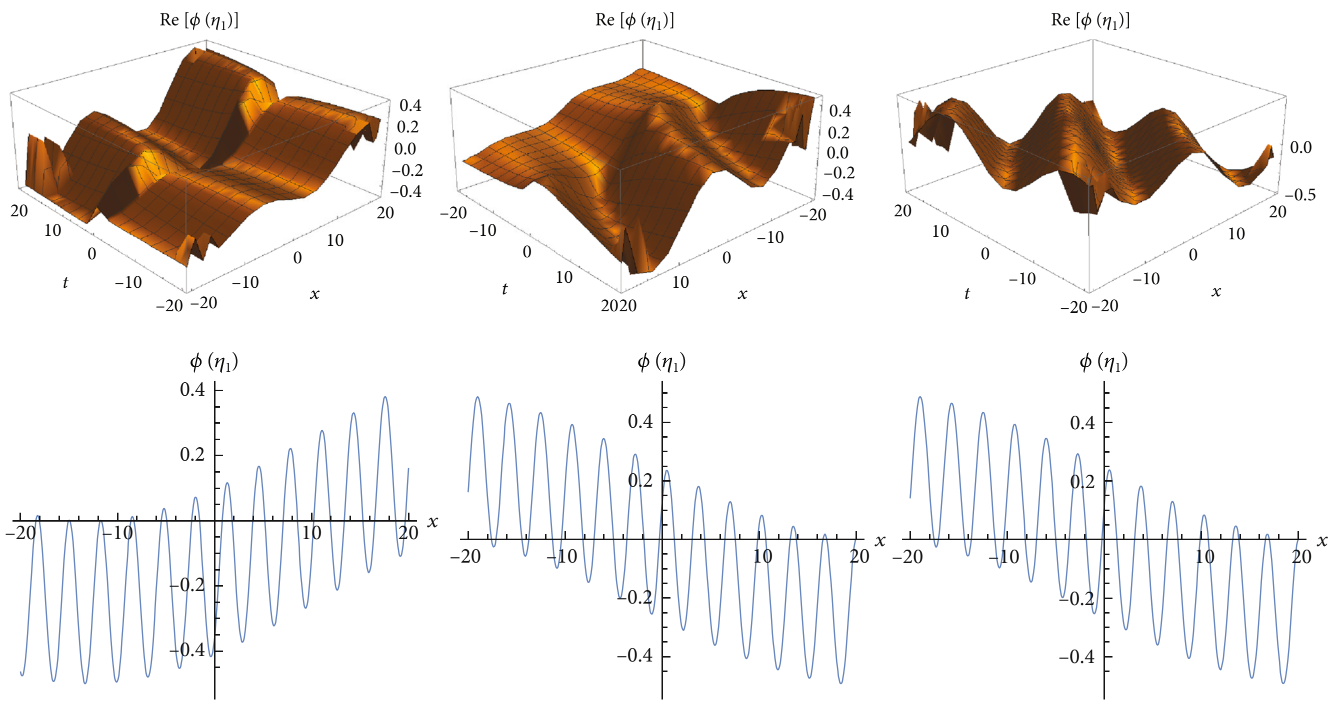

FIGURE 1: Three- and two-dimensional graphs of the solution $\phi(\eta_1)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $a_1 = m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

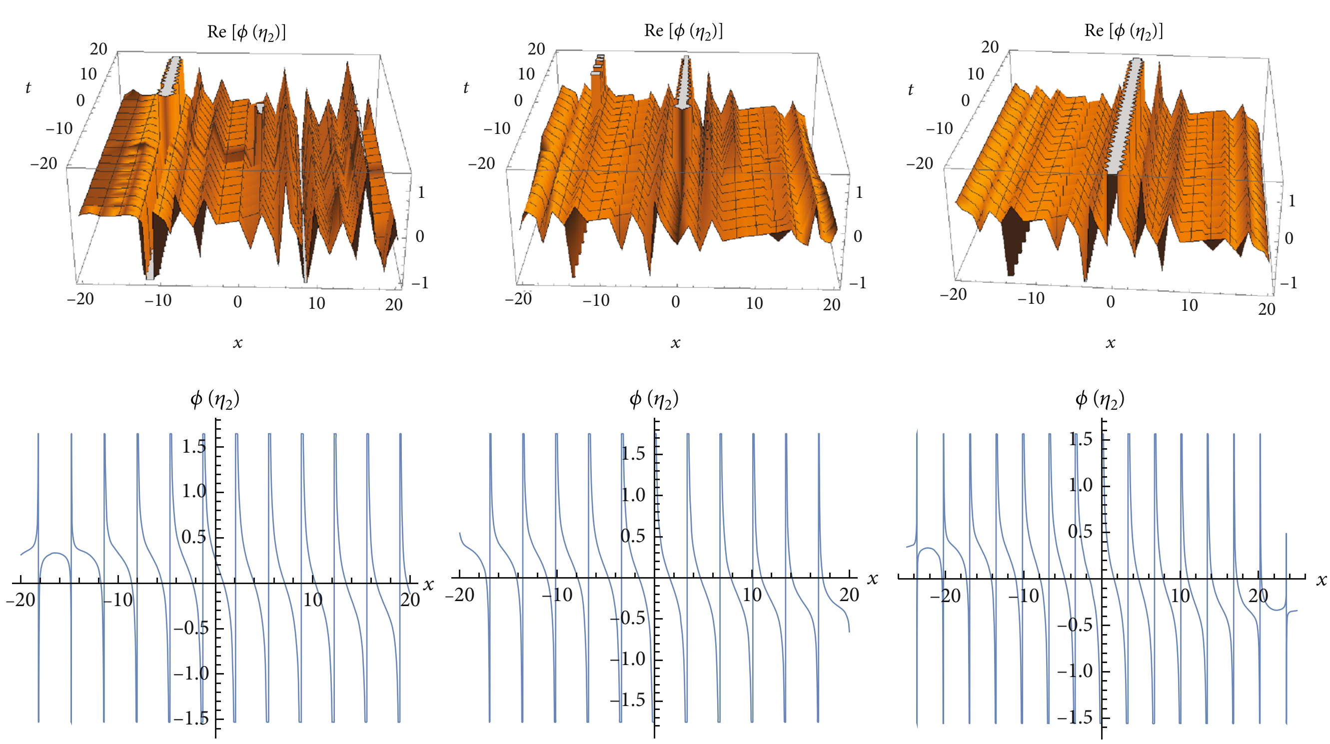

FIGURE 2: Three-dimensional graphs of the solution $\phi(\eta_2)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $b_1 = 1/3$, $m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

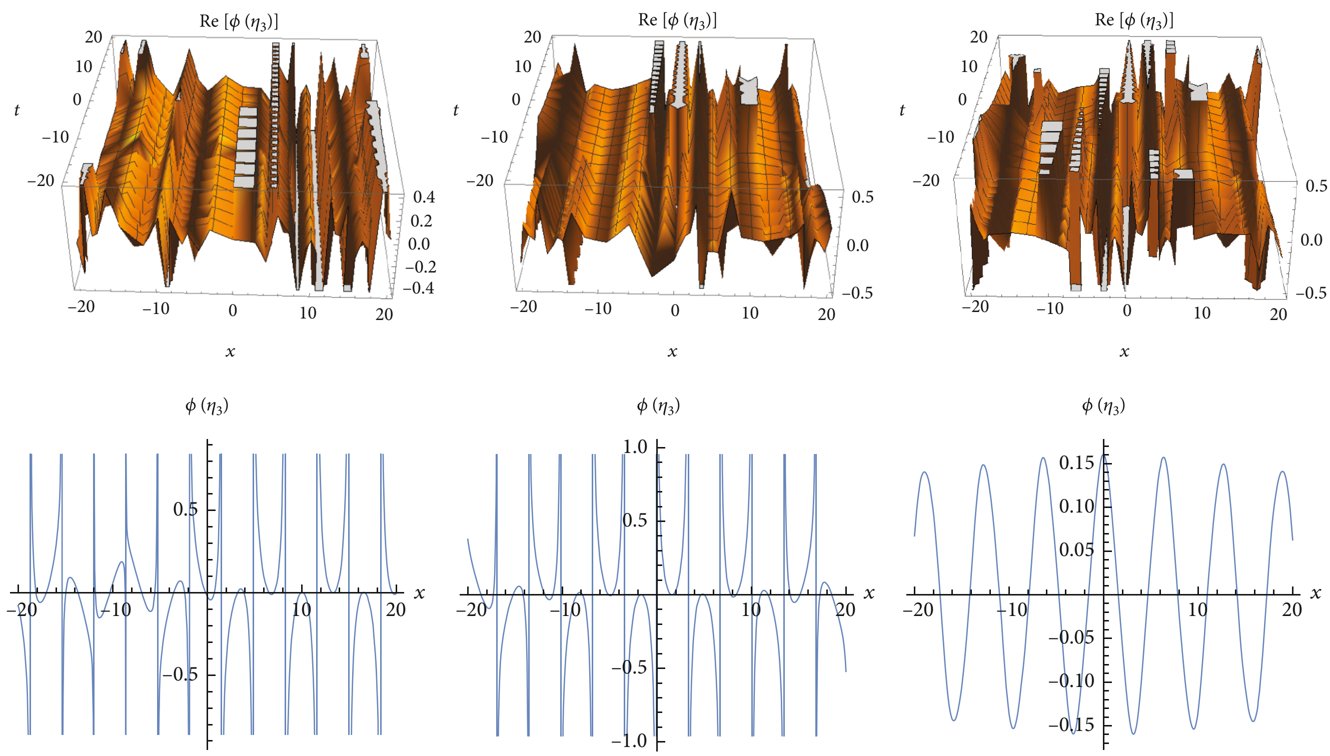

FIGURE 3: Three- and two-dimensional graphs of the solution $\phi(\eta_3)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $c_1 = 1/4$, $m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

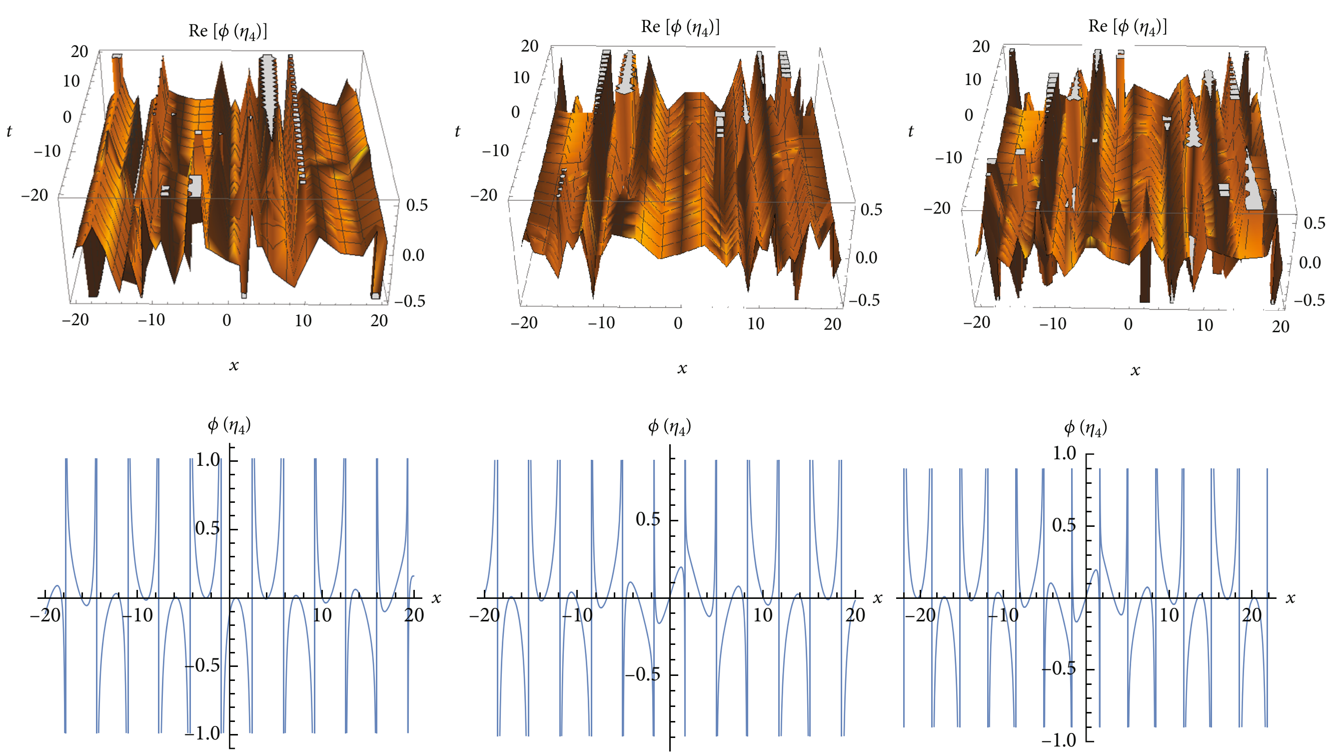

FIGURE 4: Three- and two-dimensional graphs of the solution $\phi(\eta_4)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $d_1 = 1/5$, $m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

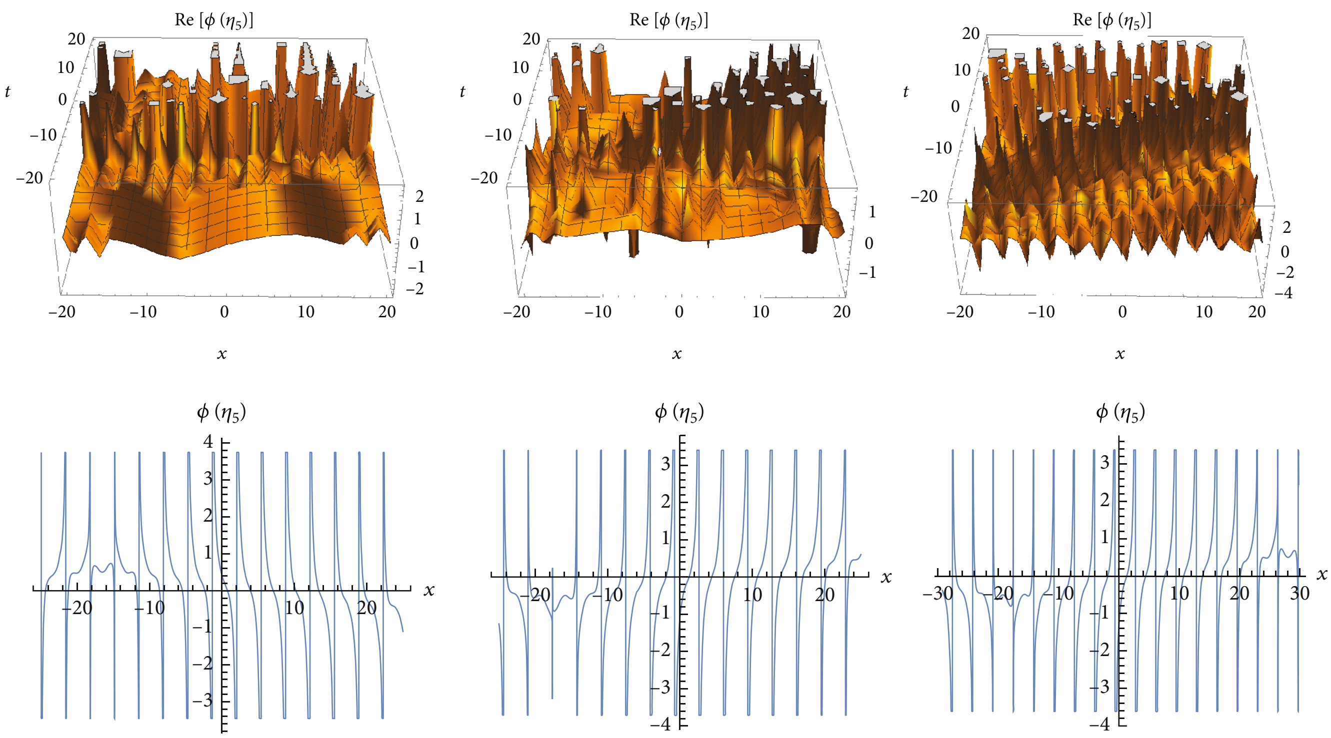

FIGURE 5: Three- and two-dimensional graphs of the solution $\phi(\eta_5)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $a_1 = m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

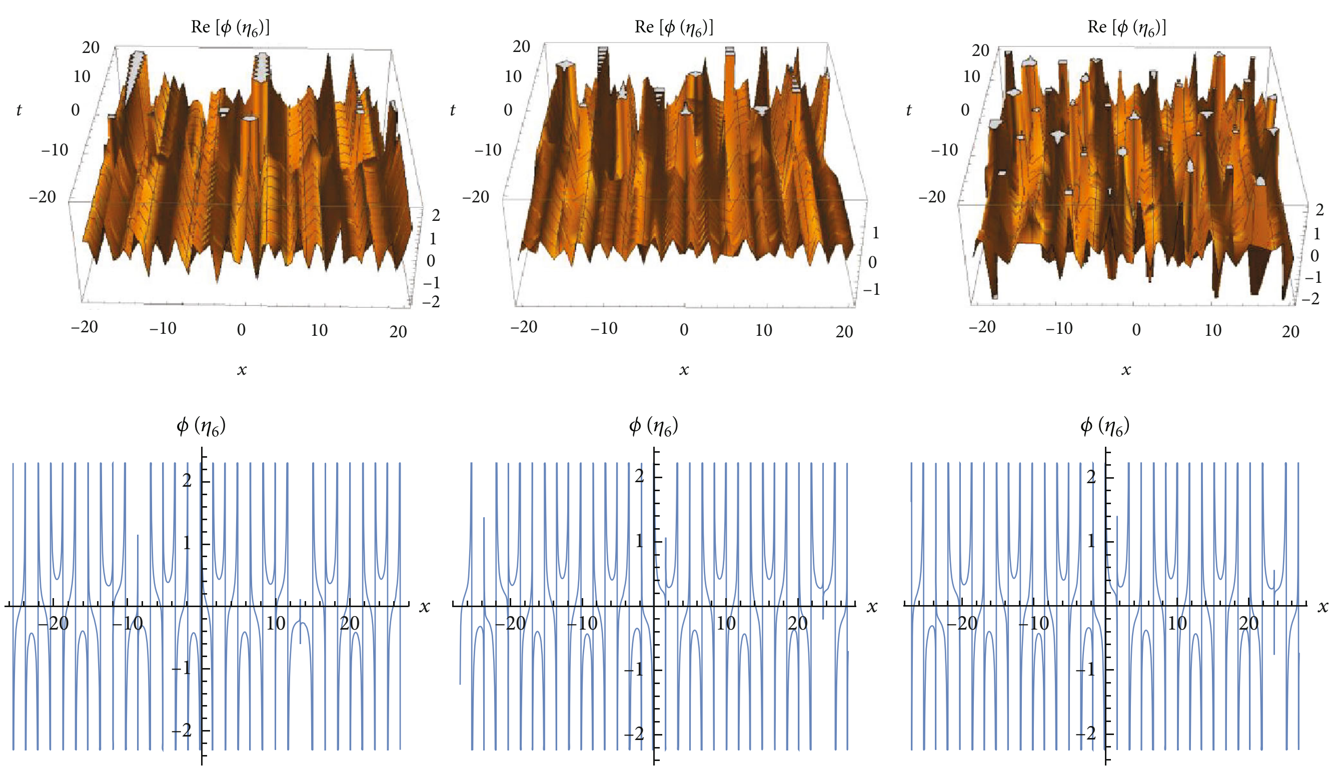

FIGURE 6: Three- and two-dimensional graphs of the solution $\phi(\eta_6)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $c_1 = 1/4$, $m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

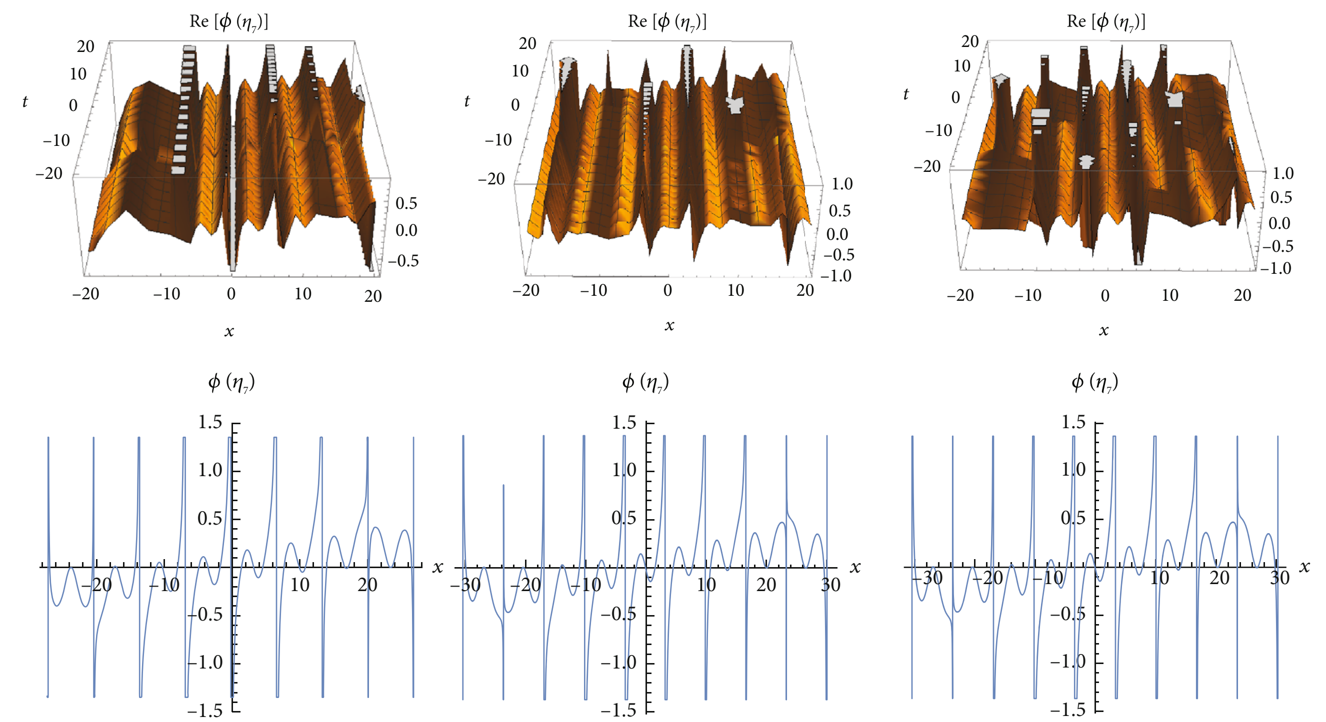

FIGURE 7: Three- and two-dimensional graphs of the solution $\phi(\eta_7)$ for different $\beta = 0.01, 0.55, 0.98$ corresponding to the values $c_1 = 1/4$, $m = 1/2$, $\kappa = -1$, $\theta = 0.3$, $\mu = 0.2$, and $k_2 = 1$.

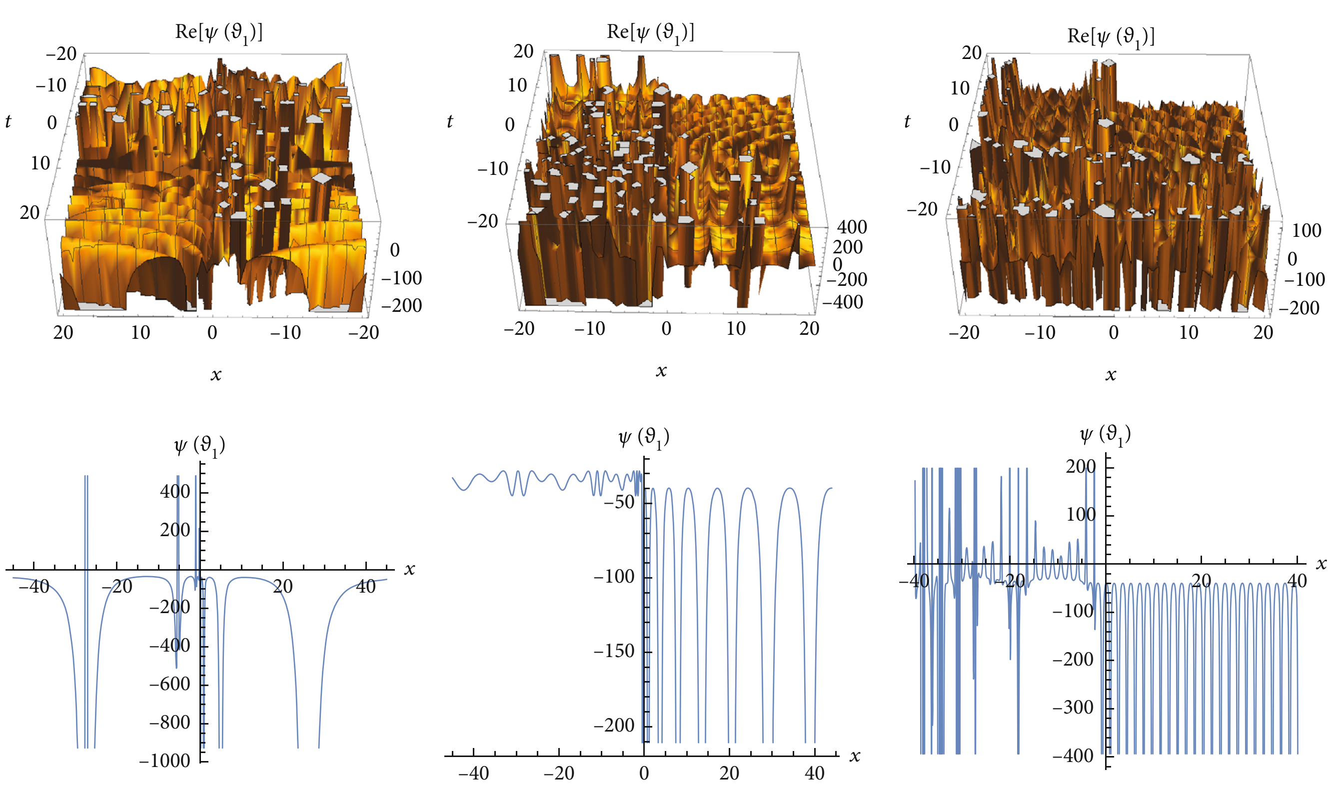

FIGURE 8: Three- and two-dimensional graphs of the solution $\psi(\vartheta_1)$ for different $\beta = 0.01, 0.50, 0.98$ corresponding to the values $c_2 = 1/4$, $m = 1/2$, and $c = k = \gamma = b = 1$.

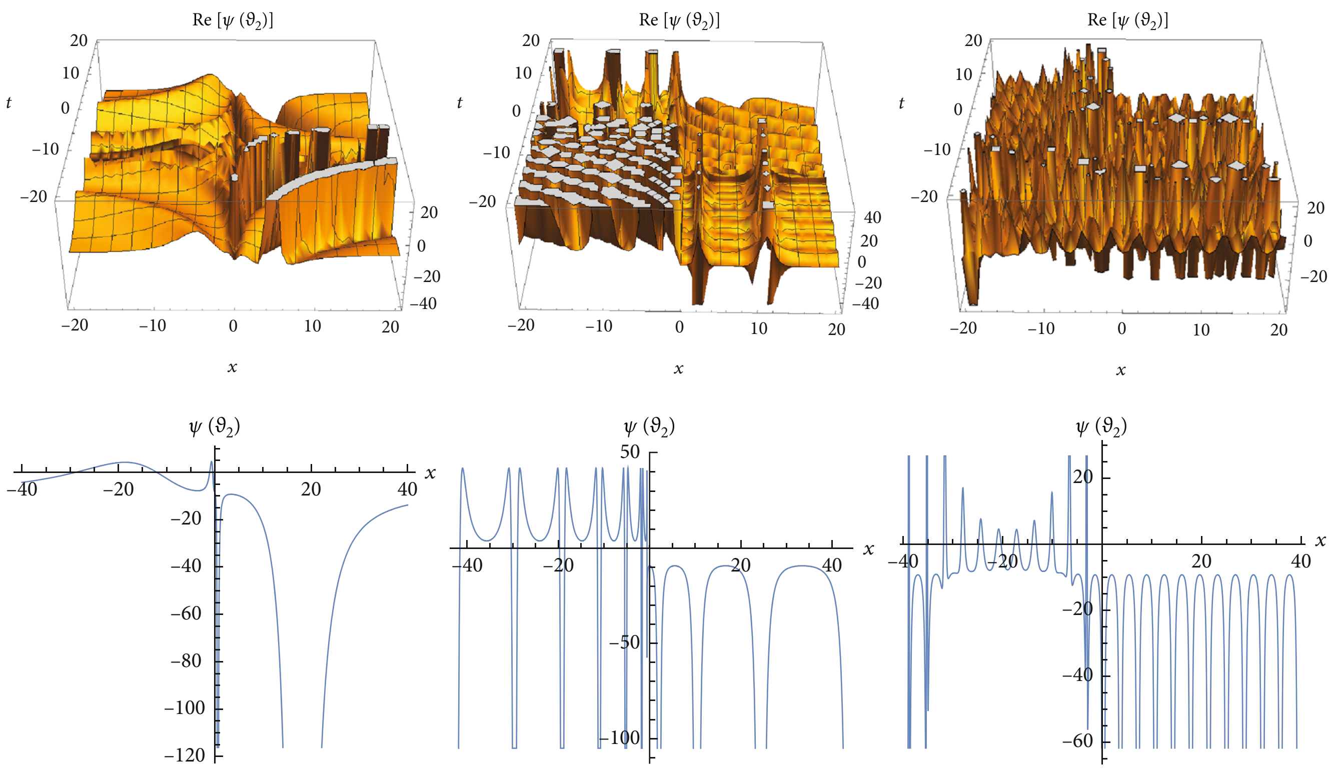

Figure 9: Three- and two-dimensional graphs of the solution $\psi(\vartheta_2)$ for different $\beta = 0.01, 0.50, 0.98$ corresponding to the values $c_2 = 1/4$, $m = 1/2$, and $c = k = \gamma = b = 1$.

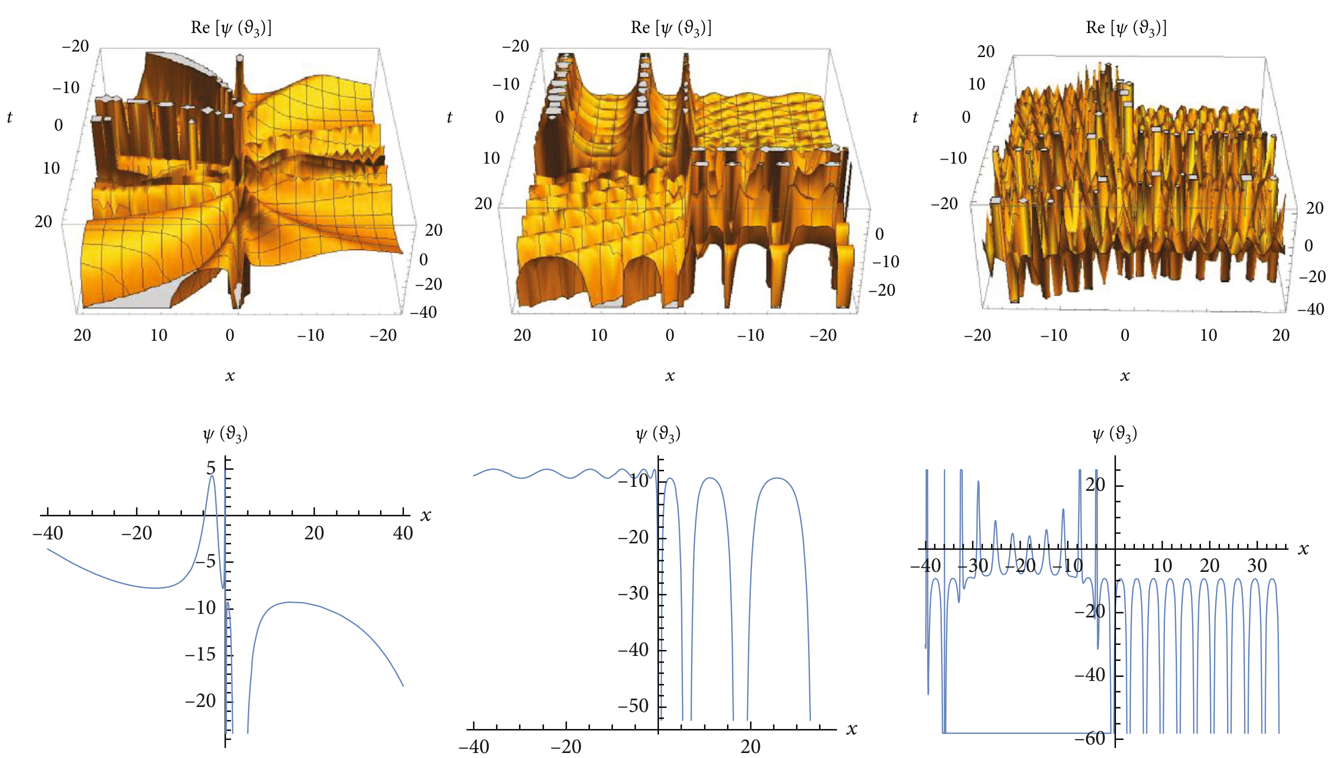

Figure 10: Three- and two-dimensional graphs of the solution $\psi(\vartheta_3)$ for different $\beta = 0.01, 0.50, 0.98$ corresponding to the values $c_2 = 1/4$, $m = 1/2$, and $c = k = \gamma = b = 1$.

$= m^2$, $Q = -(1+m^2)$, and $R = 1$, then the new exact solutions are specified in the following cases.

Case 1.

$$\begin{aligned} a_0 &= \frac{4k^4 Q\gamma - 8k^4\gamma\sqrt{Q^2+12PR}}{ck^2} + 2Qc_2, \\ a_1 &= b_1 = c_1 = d_1 = 0, \\ b_2 &= -\frac{(Q^2-4PR)(6k^2\gamma + cc_2)}{c}, \end{aligned} \tag{46}$$

$$\begin{aligned} a_2 &= -\frac{6k^2\gamma - cc_2}{c}, \\ c_2 &= c_2, \\ d_2 &= \frac{-96k^2PR\gamma}{c}, \\ \sigma &= -\sqrt{16k^4\gamma\sqrt{Q^2+12PR} - bk^2}. \end{aligned} \tag{47}$$

When obtained coefficients in Equation (46) are replaced into Equation (45), we get new exact solution named as mixed Jacobi elliptic function solutions of Equation (2) as follows:

$$\begin{aligned} \psi(\vartheta_1) = {} & B_1 + B_2\frac{cn^2(\vartheta_1)dn^2(\vartheta_1)}{sn^2(\vartheta_1)} + B_3\frac{sn^2(\vartheta_1)}{cn^2(\vartheta_1)dn^2(\vartheta_1)} + B_4\frac{\left(m^2sn^2(\vartheta_1)cn^2(\vartheta_1) + sn^2(\vartheta_1)dn^2(\vartheta_1) + cn^2(\vartheta_1)dn^2(\vartheta_1)\right)^2}{sn^2(\vartheta_1)cn^2(\vartheta_1)dn^2(\vartheta_1)} \\ & + B_5\frac{sn^2(\vartheta_1)cn^2(\vartheta_1)dn^2(\vartheta_1)^2}{\left(m^2sn^2(\vartheta_1)cn^2(\vartheta_1) + sn^2(\vartheta_1)dn^2(\vartheta_1) + cn^2(\vartheta_1)dn^2(\vartheta_1)\right)}, \end{aligned} \tag{48}$$

where $B_1 = (4k^2\gamma(-1-m^2) - 8\gamma k^4\sqrt{1+14m^2+m^4} + 2ck^2c_2(-1-m^2)/k^2c)$, $B_2 = (-6k^2\gamma - cc_2)/c$, $B_3 = (-1+2m^2-m^4)(6k^2\gamma + cc_2)/c$, $B_4 = c_2$, $B_5 = -96k^2m^2\gamma/c$, and $\vartheta_1 = (k/\beta)(x + (1/\Gamma(\beta)))^\beta + (\sqrt{16k^4\gamma\sqrt{1+14m^2+m^4} - bk^2}/\beta)(t + (1/\Gamma(\beta)))^\beta$.

Case 2.

$$\begin{aligned} a_0 &= \frac{4k^4 Q\gamma - 2k^4\gamma\sqrt{Q^2+12PR}}{ck^2} + 2Qc_2, \\ a_1 &= b_1 = c_1 = d_1 = d_2 = 0, \\ b_2 &= (4PR - Q^2)c_2, \end{aligned} \tag{49}$$

$$\begin{aligned} a_2 &= -\frac{6k^2\gamma - cc_2}{c}, \\ c_2 &= c_2, \\ \sigma &= -\sqrt{4k^4\gamma\sqrt{Q^2+12PR} - bk^2}. \end{aligned} \tag{50}$$

Substituting Equation (49) into Equation (45), we find mixed Jacobi elliptic function solutions of Equation (2).

$$\psi(\vartheta_2) = B_6 + B_7\frac{sn^2(\vartheta_2)dn^2(\vartheta_2) - 1}{sn^2(\vartheta_2)}, \tag{51}$$

where $B_6 = 2k^2\gamma(m^2 - 2 - \sqrt{1+14m^2+m^4})/c$, $B_7 = 6k^2\gamma/c$, and $\vartheta_2 = k/\beta(x + (1/\Gamma(\beta)))^\beta + (\sqrt{4k^4\gamma\sqrt{1+14m^2+m^4} - bk^2/\beta})(t + (1/\Gamma(\beta)))^\beta$.

Case 3.

$$\begin{aligned} a_0 &= \frac{4k^4 Q\gamma - 2k^4\gamma\sqrt{Q^2+12PR}}{ck^2} + 2Qc_2, \\ a_1 &= b_1 = c_1 = d_1 = d_2 = 0, \\ b_2 &= \frac{(4PR - Q^2)(6k^2\gamma + cc_2)}{c}, \end{aligned} \tag{52}$$

$$\begin{aligned} a_2 &= -c_2, \\ c_2 &= c_2, \\ \sigma &= \sqrt{4k^4\gamma\sqrt{Q^2+12PR} - bk^2}. \end{aligned} \tag{53}$$

Using Equations (45) and (52), we obtain the following mixed Jacobi elliptic functions of Equation (2).

$$\psi(\vartheta_3) = \frac{B_8\left[B_9 + B_{10}dn^2(\vartheta_3) - 2dn^4(\vartheta_3) - cn^2(\vartheta_3)\left((2+B_{11})dn^2(\vartheta_3) - 3\right)\right]}{ccn^2(\vartheta_3)dn^2(\vartheta_3)}, \tag{54}$$

where $B_8 = 2k^2\gamma$, $B_9 = 3 - 3m^2$, $B_{10} = m^2 - 4$, $B_{11} = \sqrt{1 + 14m^2 + m^4}$, and $\vartheta_3 = k/\beta(x + (1/\Gamma(\beta)))^\beta - (\sqrt{4k^4\gamma\sqrt{1 + 14m^2 + m^4} - bk^2}/\beta)(t + (1/\Gamma(\beta)))^\beta$

Remark 2. When the results of Equation (2), which are found using the new version generalized F-expansion method, are examined, the solutions of single, combined, and mixed Jacobi elliptic functions are new. And these solutions are obtained for the first time in the literature. Furthermore, two- and three-dimensional graphics of the attained exact solutions are drawn in Figures 8–10 according to the selected parameter values.

6. Conclusions

In this paper, the new version generalized F-expansion method is applied for the first time to acquire new exact solutions of the Biswas-Arshed and Boussinesq equations defined by Atangana's beta-derivative. This method makes it possible to get dissimilar states of the new Jacobi elliptic function solutions. The new results for the Biswas-Arshed and Boussinesq equations seem to be very diverse and surprising. These exact solutions consist of single, combined, and mixed Jacobi elliptic function solutions. Thus, none of the solution functions obtained by the various methods in Ref. [37–41] articles contain the solutions found by the method used in this article. Owing to the F'/F and F/F' terms contained in the finite series in the applied method, various rational solution combinations of the double-period Jacobi elliptic functions, which have not yet been found in the literature, have been reached. Also, the graphs (Figures 1–10) drawn for these solution functions help us comprehend the complex wave phenomena of the considered physical problems. It is also shown in the Mathematica package program that all exact solutions obtained in this study provide the fractional Biswas-Arshed equation and Boussinesq equation with the beta-derivative. Also, we would like to mention that all codes were written using Mathematica 11 on an HP Z420 workstation, with an Intel (R) Xeon(R) CPU E5-1620 3.8 GHz processor, 32 GB RAM DDR3, and 1 TB storage. As a result, we can say that the new version generalized F-expansion method gives very effective results in obtaining the exact solutions of the nonlinear differential equations defined by Atangana's beta-derivative and contributes to the literature. In our further work, we will implement the new version generalized F-expansion method to other complex fractional systems defined by Atangana's beta-derivative. Also, the method offered in this paper can be generalized in future work for advanced definitions like the paper [43].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final manuscript.

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