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Research Article

Qualitative Study on Solutions of Piecewise Nonlocal Implicit Fractional Differential Equations

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In this paper, we investigate new types of nonlocal implicit problems involving piecewise Caputo fractional operators. The existence and uniqueness results are proved by using some fixed point theorems. Furthermore, we present analogous results involving piecewise Caputo-Fabrizio and Atangana–Baleanu fractional operators. The ensuring of the existence of solutions is shown by Ulam-Hyer's stability. At last, two examples are given to show and approve our outcomes.

1. Introduction

It merits noticing that fractional calculus (FC) has gotten significant thought from scientists and researchers. It is a result of its wide scope of uses in different fields and disciplines. The crucial concepts and definitions of FC have been presented in [1, 2]. In [3, 4], the authors introduced some fundamental history of fractional calculus and its applications to engineering and different areas of science.

Many classes of fractional differential equations (FDEs) have been intensively investigated in the last decades, for instance, theories involving the existence of unique solutions have been notarized [5–7]. Numerical and analytical methods have been evolving with the target to solve such equations [8–10]. These equations have been tracked as useful in modeling some real-world problems with incredible achievement.

The qualitative properties of solutions represent a very important aspect of the theory of FDEs. The formerly aforesaid region has been studied well for classical differential equations. However, for FDEs, there are many aspects that require further studying and reconnoitering. The attention on the existence and uniqueness has been especially focused by applying Riemann-Liouville (R-L), Caputo, Hilfer, and other FDs, see [11–15].

In this regard, Agarwal et al. [16] investigated the existence of solutions of the following Caputo type FDE:

$$\begin{aligned} &{}^{C}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa) = f(\varkappa, \upsilon(\varkappa)), \varkappa \in [0, T], 0 < \vartheta < 1, \\ &\upsilon(0) + g(\upsilon) = \upsilon_0. \end{aligned} \tag{1}$$

The basic theory of implicit FDEs with Caputo FD has been investigated by Kucche et al. [17]. Wahash et al. [18]

considered the following nonlocal implicit FDEs with ψ-Caputo FD

$$\mathbb{D}_{a^+}^{\vartheta;\psi}\upsilon(\varkappa)=f\left(\varkappa,\upsilon(\varkappa),\mathbb{D}_{a^+}^{\vartheta;\psi}\upsilon(\varkappa)\right),\quad \varkappa\in[a,T],0<\vartheta<1$$
$$\upsilon(a)+g(\upsilon)=\upsilon_a. \tag{2}$$

Problem (2) with $\psi(\varkappa)=\varkappa$ has been studied by Benchohra and Bouriah [19].

Motivated by the above works and inspired by [20], we consider the piecewise Caputo implicit FDE (PC-IFDE) of the type:

$$\begin{aligned}{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)&=\Phi\left(\varkappa,\upsilon(\varkappa),{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)\right),\\ \upsilon(0)&=\upsilon_0,\end{aligned}\tag{3}$$

and the following piecewise Caputo nonlocal implicit FDE (PC-NIFDE):

$$\begin{aligned}{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)&=\Phi\left(\varkappa,\upsilon(\varkappa),{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)\right),\\ \upsilon(0)+g(\upsilon)&=\upsilon_0,\end{aligned}\tag{4}$$

where $0<\vartheta\le1,\varkappa\in\mathbb{J}\coloneqq[0,b],\upsilon_0\in\mathbb{R},\Phi\in\mathscr{C}(\mathbb{J}\times\mathbb{R},\mathbb{R})$, $g\in\mathscr{C}(\mathbb{J},\mathbb{R})$, and ${}^{PC}\mathbb{D}_{0^+}^{\vartheta}$ represent the piecewise Caputo FD of order ϑ defined by

$${}^{PC}\mathbb{D}_{0^+}^{\vartheta}f(\varkappa)=\begin{cases}\mathbb{D}f(\varkappa)\text{: if }\varkappa\in[0,\varkappa_1],\\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}f(\varkappa)\text{: if }\varkappa\in[\varkappa_1,b],\end{cases}\tag{5}$$

where $\mathbb{D}f(\varkappa)\coloneqq(d/d\varkappa)f(\varkappa)$ is a classical derivative on $0\le\varkappa\le\varkappa_1$ and ${}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}$ is standard Caputo FD on $\varkappa_1\le\varkappa\le b$.

It is essential to note that the utilization of nonlinear condition $\upsilon(0)+g(\upsilon)=\upsilon_0$ in physical issues yields better impact than the initial condition $\upsilon(0)=\upsilon_0$ (see [21]).

We pay attention to the topic of the novel piecewise operators. As far as we could possibly know, no outcomes in the literature are addressing the qualitative aspects of the aforesaid problems by using the piecewise FC. Consequently, by conquering this gap, we will examine the existence, uniqueness, and Ulam-Hyers stability results of piecewise Caputo problems (3) and (4) based on the standard fixed point theorems due to Banach-type and Schauder-type. Furthermore, we present similar results containing piecewise Caputo-Fabrizio (PCF) type and piecewise Atangana-Baleanu (PAB) type. An open problem with respect to another function is suggested.

Remark 1.

(i) If $g(\upsilon)\equiv0$, then problem (4) reduces to the PC-IFDE (3).

(ii) If ${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)={}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\upsilon(\varkappa)$, then problem (4) has been studied by Benchohra and Bouriah [19], Haoues et al. [22], and Abdo et al. [11] for $\psi(\varkappa)=\varkappa$.

(iii) Our current results for problem (4) stay available on PC-IFDE (3).

The substance of this paper is coordinated as follows: Section 2 presents a few required outcomes and fundamentals about piecewise FC. Our key outcomes for problem (4) are proved in Section 3. Two examples to make sense of the gained outcomes are built in Section 4. Toward the end, we encapsulate our study in the end section.

2. Primitive Results

In this section, we present some concepts of a piecewise FC. Let

$$\mathscr{C}\coloneqq\mathscr{C}(\mathbb{J},\mathbb{R})=\left\{\eta:\mathbb{J}\longrightarrow\mathbb{R};\|\eta\|=\max_{\varkappa\in\mathbb{J}}|\eta(\varkappa)|\right\}.\tag{6}$$

Obviously $\mathscr{C}$ is a Banach space under $\|\eta\|$.

Definition 2 [20]. Let $\vartheta>0$, and $\eta:\mathbb{J}\longrightarrow\mathbb{R}$ be a continuous. Then, the piecewise version of RL integral is given by

$${}^{PRL}\mathbb{I}_{0^+}^{\vartheta}\eta(\varkappa)=\begin{cases}\mathbb{I}\eta(\varkappa),\text{ if }\varkappa\in[0,\varkappa_1],\\ {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta}\eta(\varkappa)\text{ if }\varkappa\in[\varkappa_1,b],\end{cases}\tag{7}$$

where $\mathbb{I}\eta(\varkappa)=\int_0^{\varkappa_1}\eta(\varkappa)d\varkappa$ and ${}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta}\eta(\varkappa)=1/(\Gamma(\vartheta))\int_{\varkappa_1}^{\varkappa}(\varkappa-t))^{\vartheta-1}\eta(t)dt$.

Definition 3 [20]. Let $0<\vartheta\le1$, and $\eta:\mathbb{J}\longrightarrow\mathbb{R}$ be a continuous. Then, the piecewise version of Caputo derivative is given by

$${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\eta(\varkappa)=\begin{cases}\mathbb{D}\eta(\varkappa),\text{ if }\varkappa\in[0,\varkappa_1],\\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\eta(\varkappa)\text{ if }\varkappa\in[\varkappa_1,b],\end{cases}\tag{8}$$

where $\mathbb{D}\eta(\varkappa)=(d/d\varkappa)\eta(\varkappa)$ and ${}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\eta(\varkappa)=1/(\Gamma(1-\vartheta))\int_{\varkappa_1}^{\varkappa}(\varkappa-t))^{-\vartheta}\eta'(t)dt$.

Lemma 4 [20]. *Let $0<\vartheta\le1$, and $f(0)=0$. Then, the following PC-FDE*

$$\begin{aligned}{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\eta(\varkappa)&=f(\varkappa),\\ \eta(0)&=\varkappa_0,\end{aligned}\tag{9}$$

has the following solution

$$\eta(\varkappa)=\begin{cases}\eta(0)+\int_0^{\varkappa_1}\eta(\varkappa)d\varkappa, \text{ if } \varkappa\in[0,\varkappa_1],\\ \eta(\varkappa_1)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\eta(t)dt \text{ if } \varkappa\in[\varkappa_1,b].\end{cases} \tag{10}$$

Lemma 5 [20]. *Let $\vartheta\in(0,1]$, and for a given function, $\eta\in\mathscr{C}$. Then,*

$${}^{PRL}\mathbb{I}_{0^+}^{\vartheta}\,{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\eta(\varkappa)=\begin{cases}\mathbb{I}\mathbb{D}\eta(\varkappa)=\eta(\varkappa)-\eta(0), \text{ if } \varkappa\in[0,\varkappa_1],\\ {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta}\,{}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\eta(\varkappa)=\eta(\varkappa)-\eta(\varkappa_1), \text{ if } \varkappa\in[\varkappa_1,b].\end{cases} \tag{11}$$

For our aim, we need the Banach fixed-point theorem [23] and the Schauder fixed-point theorem [24].

3. Main Results

In this section, we give some qualitative analyses of the PC-IFDE and PC-NIFDE.

Lemma 6. *Let $\Phi(\varkappa,\upsilon,\omega)$: $\mathbb{J}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be continuous.*

Then, PC-NIFDE (4) is equivalent to

$$\upsilon(\varkappa)=\begin{cases}\upsilon_0-g(\upsilon)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt \text{ if } \varkappa\in[0,\varkappa_1],\\ \upsilon(\varkappa_1)-g(\upsilon)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt, \text{ if } \varkappa\in[\varkappa_1,b],\end{cases} \tag{12}$$

where $\Phi_\upsilon\in\mathscr{C}$ satisfies the functional equation

$$\Phi_\upsilon(\varkappa)=\begin{cases}\Phi\left(\varkappa,\upsilon_0-g(\upsilon)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt,\Phi_\upsilon(\varkappa)\right) \text{ if } \varkappa\in[0,\varkappa_1],\\ \Phi\left(\varkappa,\upsilon(\varkappa_1)-g(\upsilon)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt,\Phi_\upsilon(\varkappa)\right), \text{ if } \varkappa\in[\varkappa_1,b].\end{cases} \tag{13}$$

Proof. Let ${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)=\Phi_\upsilon(\varkappa)$.

Then, by applying ${}^{PRL}\mathbb{I}_{0^+}^{\vartheta}$, we obtain

$${}^{PRL}\mathbb{I}_{0^+}^{\vartheta}\,{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)={}^{PRL}\mathbb{I}_{0^+}^{\vartheta}\Phi_\upsilon(\varkappa). \tag{14}$$

□

In view of Lemma 5, we have

Case 1. For $\varkappa\in[0,\varkappa_1]$,

$$\upsilon(\varkappa)=\upsilon(0)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt. \tag{15}$$

Case 2. For $\varkappa\in[\varkappa_1,b]$,

$$\upsilon(\varkappa)=\upsilon(\varkappa_1)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt. \tag{16}$$

Using the nonlocal condition in both cases, we obtain

$$\upsilon(\varkappa)=\begin{cases}\upsilon_0-g(\upsilon)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt, \text{ if } \varkappa\in[0,\varkappa_1],\\ \upsilon(\varkappa_1)-g(\upsilon)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt, \text{ if } \varkappa\in[\varkappa_1,b].\end{cases} \tag{17}$$

So, we get (12). On the other hand, let (13) be satisfied. Set

$$\upsilon(\varkappa)=\begin{cases}\upsilon_0-g(\upsilon)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt \text{ if } \varkappa\in[0,\varkappa_1],\\ \upsilon(\varkappa_1)-g(\upsilon)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt, \text{ if} \varkappa\in[\varkappa_1,b].\end{cases} \tag{18}$$

This implies that

$${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)=\begin{cases}\frac{d}{d\varkappa}\left(\upsilon_0-g(\upsilon)+\int_0^{\varkappa_1}\Phi_\upsilon(t)dt\right) \text{ if } \varkappa\in[0,\varkappa_1],\\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\left(\upsilon(\varkappa_1)-g(\upsilon)+\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa-t)^{\vartheta-1}\Phi_\upsilon(t)dt\right), \text{ if } \varkappa\in[\varkappa_1,b].\end{cases} \tag{19}$$

Since $\mathbb{D}\mathbb{I}\Phi_\upsilon(\varkappa)=(d/d\varkappa)\int_0^{\varkappa_1}\Phi_\upsilon(t)dt=\Phi_\upsilon(\varkappa)$ on $0\le\varkappa\le\varkappa_1$, and ${}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\mathbb{I}_{\varkappa_1}^{\vartheta}\Phi_\upsilon(\varkappa)=\Phi_\upsilon(\varkappa)$ on $\varkappa_1\le\varkappa\le b$, we obtain ${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)=\Phi_\upsilon(\varkappa)$, and hence

$${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)=\Phi\left(\varkappa,\upsilon(\varkappa),{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa)\right), \text{for each } \varkappa\in\mathbb{J}. \tag{20}$$

The next assumptions will be applied in the sequel:

(Assu_1) The functions $\Phi:\mathbb{J}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$, $\Omega:\mathbb{R}^+\longrightarrow(0,\infty)$, and $\varphi,\psi:\mathbb{J}\longrightarrow\mathbb{R}$ are continuous with Ω that is a nondecreasing such that

$$|\Phi(\varkappa,\upsilon,\omega)|\le\varphi(\varkappa)\Omega(|\upsilon|)+\psi(\varkappa)|\omega|, \text{for each } (\varkappa,\upsilon,\omega)\in\mathbb{J}\times\mathbb{R}\times\mathbb{R}. \tag{21}$$

(Assu_2) $g:\mathscr{C}\longrightarrow\mathbb{R}$ is continuous and compact with $|g(\upsilon)|\le a|\upsilon|+b$, for $\upsilon\in\mathscr{C}$,$a,b>0$.

(Assu_3) There exist $\kappa_1,\kappa_2>0$, such that $0<\kappa_1,\kappa_2<1$, and

$$|\Phi(\varkappa,\upsilon,\omega)-\Phi(\varkappa,\bar{\upsilon},\bar{\omega})|\le\kappa_1|\upsilon-\bar{\upsilon}|+\kappa_2|\omega-\bar{\omega}|, \text{for each } \varkappa\in\mathbb{J},\upsilon,\omega,\bar{\upsilon},\bar{\omega}\in\mathbb{R}. \tag{22}$$

(Assu$_4$) There exists $\kappa_3 > 0$, such that $0 < \kappa_3 < 1$ and $|g(v) - g(\omega)| \leq \kappa_3 |v - \omega|$, for $v, \omega \in \mathscr{C}$.

Now, we shall prove the existence theorem for (4) based on Schauder's theorem.

Theorem 7. *Let (Assu$_1$) and (Assu$_2$) hold.*

Then, piecewise Caputo FNIDE (4) has at least one solution on $\mathbb{J}$.

Proof. Consider the operator $Q : \mathscr{C} \longrightarrow \mathscr{C}$, such that $(Qv)(\varkappa) = v(\varkappa)$, i.e.,

$$(Qv)(\varkappa) = \begin{cases} v_0 - g(v) + \int_0^{\varkappa_1} \Phi_v(t)dt, \text{ if } \varkappa \in [0, \varkappa_1], \\ v(\varkappa_1) - g(v) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t) \Big)^{\vartheta-1} \Phi_v(t)dt, \text{ if } \varkappa \in [\varkappa_1, b], \end{cases} \quad (23)$$

where $\Phi_v \in \mathscr{C}$, with $\Phi_v(\varkappa) := \Phi(\varkappa, v(\varkappa), \Phi_v(\varkappa))$. Define the ball

$$\mathscr{S}_\beta = \left\{ v \in \mathscr{C} : \|v\|_{\mathscr{C}} \leq \beta \right\}, \quad (24)$$

where

$$\beta \geq \max \left\{ |v_0| + a\beta + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} b, |v(\varkappa_1)| + a\beta + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \right\}, \quad (25)$$

$\varphi^\star = \sup |\varphi(\varkappa)|$, and $\psi^\star = \sup |\psi(\varkappa)|$, with $0 < \psi^\star < 1$. □

For any $v \in \mathscr{S}_\beta$, and by (Assu $_1$), we have

$$\begin{aligned} |\Phi_v(\varkappa)| &= |\Phi(\varkappa, v(\varkappa), \Phi_v(\varkappa))| \\ &\leq \varphi(\varkappa)\Omega(\|v\|_{\mathscr{C}}) + \psi(\varkappa)|\Phi_v(\varkappa)| \\ &\leq \varphi^\star \Omega(\beta) + \psi^\star \|\Phi_v\|_{\mathscr{C}}. \end{aligned} \quad (26)$$

Since $\psi^\star < 1$, we obtain

$$\|\Phi_v\|_{\mathscr{C}} \leq \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star}. \quad (27)$$

Hence, the proceed is in the following steps:

Step 1. $Q(\mathscr{S}_\beta)$ is bounded.

Case 1. For $\varkappa \in [0, \varkappa_1]$, we have

$$\begin{aligned} |(Qv)(\varkappa)| &\leq |v_0| + \sup_{v \in \mathscr{S}_\beta} |g(v)| + \sup_{\varkappa \in 0, \varkappa_1]} \int_0^{\varkappa_1} |\Phi_v(t)|dt \\ &\leq |v_0| + a\|v\|_{\mathscr{C}} + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} \varkappa_1 \\ &\leq |v_0| + a\beta + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} \varkappa_1 \leq \beta. \end{aligned} \quad (28)$$

Case 2. For $\varkappa \in [\varkappa_1, b]$, we have

$$\begin{aligned} |(Qv)(\varkappa)| &\leq \sup_{\varkappa \in \varkappa_1, b]} |v(\varkappa_1)| + \sup_{v \in \mathscr{S}_\beta} |g(v)| \\ &\quad + \frac{1}{\Gamma(\vartheta)} \sup_{\varkappa \in \varkappa_1, b]} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} |\Phi_v(t)|dt \\ &\leq |v(\varkappa_1)| + a\|v\|_{\mathscr{C}} + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \\ &\leq |v(\varkappa_1)| + a\beta + b + \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star} \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \leq \beta. \end{aligned} \quad (29)$$

From (28) and (29), we conclude that $\|Qv\|_{\mathscr{C}} \leq \beta$. Thus, $Q(\mathscr{S}_\beta) \subset \mathscr{S}_\beta$. Since $\mathscr{S}_\beta$ is bounded, then $Q(\mathscr{S}_\beta)$ is bounded.

Step 2. $Q : \mathscr{S}_\beta \longrightarrow \mathscr{S}_\beta$ is continuous. Let a sequence (v_n) such that $v_n \longrightarrow v$ in $\mathscr{S}_\beta$ as $n \longrightarrow \infty$. Then, for $\varkappa \in [0, \varkappa_1]$, we have

$$|(Qv_n)(\varkappa) - (Qv)(\varkappa)| \leq |g(v_n) - g(v)| + \int_0^{\varkappa_1} \left|\Phi_{v_n}(t) - \Phi_v(t)\right|dt. \quad (30)$$

For $\varkappa \in [\varkappa_1, b]$, we have

$$\begin{aligned} |(Qv_n)(\varkappa) - (Qv)(\varkappa)| &\leq |v_n(\varkappa_1) - v(\varkappa_1)| + |g(v_n) - g(v)| \\ &\quad + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t) \Big)^{\vartheta-1} \left|\Phi_{v_n}(t) - \Phi_v(t)\right|dt, \end{aligned} \quad (31)$$

where $\Phi_v, \Phi_{v_n} \in \mathscr{C}$, with $\Phi_{v_n}(\varkappa) := \Phi(\varkappa, v_n(\varkappa), \Phi_{v_n}(\varkappa))$ and $\Phi_v(\varkappa) := \Phi(\varkappa, v(\varkappa), \Phi_v(\varkappa))$. Since $v_n \longrightarrow v$ as $n \longrightarrow \infty$ and Φ_v, Φ_{v_n}, Φ, and g are continuous, the Lebesgue dominated convergence theorem gives that

$$\|Qv_n - Qv\|_{\mathscr{C}} \longrightarrow 0, \text{as } n \longrightarrow \infty. \quad (32)$$

Step 3. $Q(\mathscr{S}_\beta)$ is equicontinuous. Let $\varkappa \in 0, \varkappa_1]$, then $\varkappa_m < \varkappa_n \in 0, \varkappa_1]$, we have

$$\begin{aligned} |(Qv)(\varkappa_n) - (Qv)(\varkappa_m)| &\leq |g(v(\varkappa_n)) - g(v(\varkappa_m))| \\ &\quad + (\varkappa_n - \varkappa_m) \frac{\varphi^\star \Omega(\beta)}{1 - \psi^\star}. \end{aligned} \quad (33)$$

Let $\varkappa \in [\varkappa_1, b]$, then $\varkappa_m < \varkappa_n \in [\varkappa_1, b]$, we have

$$\begin{aligned}
&|(Q\upsilon)(\varkappa_n) - (Q\upsilon)(\varkappa_m)| \\
&\quad \le |g(\upsilon(\varkappa_n)) - g(\upsilon(\varkappa_m))| + \left| \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_n} (\varkappa_n - t)^{\vartheta-1} \Phi_\upsilon(t) dt \right. \\
&\quad \left. - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_m} (\varkappa_m - t)^{\vartheta-1} \Phi_\upsilon(t) dt \right| \\
&\quad \le |g(\upsilon(\varkappa_n)) - g(\upsilon(\varkappa_m))| + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_n} (\varkappa_n - t)^{\vartheta-1} \\
&\quad - (\varkappa_m - t)^{\vartheta-1} |\Phi_\upsilon(t)| dt + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_n}^{\varkappa_m} (\varkappa_m - t)^{\vartheta-1} |\Phi_\upsilon(t)| dt \\
&\quad \le |g(\upsilon(\varkappa_n)) - g(\upsilon(\varkappa_m))| + \frac{(\varkappa_n - \varkappa_1)^\vartheta}{\Gamma(\vartheta+1)} \frac{\varphi^\star \Omega(\beta)}{1-\psi^\star} \\
&\quad + \left(\frac{(\varkappa_m - \varkappa_n)^\vartheta - (\varkappa_m - \varkappa_1)^\vartheta}{\Gamma(\vartheta+1)} + \frac{(\varkappa_m - \varkappa_n)^\vartheta}{\Gamma(\vartheta+1)} \right) \frac{\varphi^\star \Omega(\beta)}{1-\psi^\star} \\
&\quad \le |g(\upsilon(\varkappa_n)) - g(\upsilon(\varkappa_m))| + \frac{2(\varkappa_m - \varkappa_n)^\vartheta}{\Gamma(\vartheta+1)} \frac{\varphi^\star \Omega(\beta)}{1-\psi^\star}.
\end{aligned} \tag{34}$$

Since g is continuous and compact, (33) and (34) give

$$|(Q\upsilon)(\varkappa_n) - (Q\upsilon)(\varkappa_m)| \longrightarrow 0, \text{as}\, \varkappa_m \longrightarrow \varkappa_n. \tag{35}$$

That means Q is relatively compact on $\mathcal{S}_\beta$. So, Q is completely continuous due to the Arzela–Ascolli theorem. Thus, Schauder's theorem shows that problem (4) has at least one solution.

Next, we prove the uniqueness theorem for (4) based on Banach's theorem.

Theorem 8. *Let (Assu$_3$)-(Assu$_4$) hold.*

If $\max_{\varkappa \in \mathbb{J}} \{\zeta_1, \zeta_2\} = \zeta < 1$, *then PC-NIFDE (4) has a unique solution on* $\mathbb{J}$, *where*

$$\begin{aligned}
\zeta_1 &:= \kappa_3 + \frac{\kappa_1}{1-\kappa_2} \varkappa_1, \\
\zeta_2 &:= \kappa_3 + \frac{\kappa_1}{1-\kappa_2} \frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)}.
\end{aligned} \tag{36}$$

Proof. Consider υ and $\bar{\upsilon}$ in $\mathcal{C}$, then

$$\begin{aligned}
&|\Phi_\upsilon(\varkappa) - \Phi_{\bar{\upsilon}}(\varkappa)| \\
&\quad = |\Phi(\varkappa, \upsilon(\varkappa), \Phi_\upsilon(\varkappa)) - \Phi(\varkappa, \bar{\upsilon}(\varkappa), \Phi_{\bar{\upsilon}}(\varkappa))| \\
&\quad \le \kappa_1 |\upsilon(\varkappa) - \bar{\upsilon}(\varkappa)| + \kappa_2 |\Phi_\upsilon(\varkappa) - \Phi_{\bar{\upsilon}}(\varkappa)|,
\end{aligned} \tag{37}$$

which implies that

$$|\Phi_\upsilon(\varkappa) - \Phi_{\bar{\upsilon}}(\varkappa)| \le \frac{\kappa_1}{1-\kappa_2} |\upsilon(\varkappa) - \bar{\upsilon}(\varkappa)|. \tag{38}$$

□

Hence, we have two cases:

Case 1. For $\varkappa \in [0, \varkappa_1]$,

$$\begin{aligned}
&|(Q\upsilon)(\varkappa) - (Q\bar{\upsilon})(\varkappa)| \\
&\quad \le |g(\upsilon) - g(\bar{\upsilon})| + \int_0^{\varkappa_1} |\Phi_\upsilon(t) - \Phi_{\bar{\upsilon}}(t)| dt \\
&\quad \le \left(\kappa_3 + \frac{\kappa_1 \varkappa_1}{1-\kappa_2} \right) \|\upsilon - \bar{\upsilon}\|_{\mathcal{C}}.
\end{aligned} \tag{39}$$

Case 2. For $\varkappa \in [\varkappa_1, b]$,

$$\begin{aligned}
&|(Q\upsilon)(\varkappa) - (Q\bar{\upsilon})(\varkappa)| \\
&\quad \le |g(\upsilon) - g(\bar{\upsilon})| + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t))^{\vartheta-1} |\Phi_\upsilon(t) - \Phi_{\bar{\upsilon}}(t)| dt \\
&\quad \le \left(\kappa_3 + \frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)} \frac{\kappa_1}{1-\kappa_2} \right) \|\upsilon - \bar{\upsilon}\|_{\mathcal{C}}.
\end{aligned} \tag{40}$$

Consequently,

$$\|Q\upsilon - Q\bar{\upsilon}\|_{\mathcal{C}} \le \zeta \|\upsilon - \bar{\upsilon}\|_{\mathcal{C}}. \tag{41}$$

Since $\zeta < 1$, Q is a contraction. Thus, Banach's theorem shows that PC-NIFDE (4) has a unique solution that exists on $\mathbb{J}$.

3.1. An Analogous Results.

In this part, we show some analogous results according to our preceding outcomes.

3.1.1. Piecewise Caputo-Fabrizio NIFDE (PCF-NIFDE).

Consider the following PCF-NIFDE

$$\begin{aligned}
&{}^{PCF}\mathbb{D}_{0^+}^\vartheta \upsilon(\varkappa) = \Phi\left(\varkappa, \upsilon(\varkappa), {}^{PCF}\mathbb{D}_{0^+}^\vartheta \upsilon(\varkappa)\right), \\
&\upsilon(0) + g(\upsilon) = \upsilon 0,
\end{aligned} \tag{42}$$

where ${}^{PCF}\mathbb{D}_{0^+}^\vartheta$ is the piecewise derivative in the Caputo-Fabrizio sense (see [20]) defined by

$${}^{PCF}\mathbb{D}_{0^+}^\vartheta \upsilon(\varkappa) = \begin{cases} \mathbb{D}\upsilon(\varkappa) = \dfrac{d\upsilon}{dx}, \text{if}\, \varkappa \in [0, \varkappa_1], \\ {}^{CF}\mathbb{D}_{\varkappa_1}^\vartheta \upsilon(\varkappa) = \dfrac{(2-\vartheta)\aleph(\vartheta)}{2(1-\vartheta)} \displaystyle\int_{\varkappa_1}^{\varkappa} \exp\left(\lambda(\varkappa - t)\right) \upsilon'(\varkappa) dt \,\text{if}\, \varkappa \in [\varkappa_1, b], \end{cases} \tag{43}$$

where $\aleph(\vartheta) = (2/2-\vartheta), \lambda = (\vartheta/\vartheta - 1)$, and ${}^{CF}\mathbb{D}_{\varkappa_1}^\vartheta$ are the classical Caputo-Fabrizio FD (see [25]).

Let $\Phi_{\upsilon}(\varkappa) := \Phi(\varkappa, \upsilon(\varkappa), \Phi_{\upsilon}(\varkappa))$; based on PCF-NIFDE (42), the results in Theorems 7 and 8 can be presented by

$$\upsilon(\varkappa) = \begin{cases} \upsilon_0 - g(\upsilon) + \mathbb{I}_{\Phi_{\upsilon}(\varkappa)}, \text{ if } \varkappa \in [0, \varkappa_1] \\ \upsilon(\varkappa_1) - g(\upsilon) + {}^{CF}\mathbb{I}^{\vartheta}_{\varkappa_1}\Phi_{\upsilon}(\varkappa), \text{ if } \varkappa \in [\varkappa_1, b] \end{cases} = \begin{cases} \upsilon_0 - g(\upsilon) + \int_0^{\varkappa_1}\Phi_{\upsilon}(t)dt, \text{ if } \varkappa \in [0, \varkappa_1], \\ \upsilon(\varkappa_1) - g(\upsilon) + \frac{2(1-\vartheta)}{\aleph(\vartheta)(2-\vartheta)}\Phi_{\upsilon}(\varkappa) + \frac{2\vartheta}{\aleph(\vartheta)(2-\vartheta)}\int_{\varkappa_1}^{\varkappa}\Phi_{\upsilon}(t)dt, \text{ if } \varkappa \in [\varkappa_1, b], \end{cases} \tag{44}$$

where $\mathbb{I}_{\Phi_{\upsilon}(\varkappa)} = \int_0^{\varkappa_1}\Phi_{\upsilon}(t)dt$ and ${}^{CF}\mathbb{I}^{\vartheta}_{0^+}$ are a Caputo-Fabrizio integral on $\varkappa_1 \le \varkappa \le b$ (see [25]).

3.1.2. Piecewise Atangana-Baleanu NIFDE (PAB-NIFDE). Consider the following PAB-NIFDE

$$\begin{aligned} &{}^{PAB}\mathbb{D}^{\vartheta}_{0^+}\upsilon(\varkappa) = \Phi\left(\varkappa, \upsilon(\varkappa), {}^{PAB}\mathbb{D}^{\vartheta}_{0^+}\upsilon(\varkappa)\right), \\ &\upsilon(0) + g(\upsilon) = \upsilon0, \end{aligned} \tag{45}$$

where ${}^{PAB}\mathbb{D}^{\vartheta}_{0^+}$ is the piecewise derivative in the Atangana-Baleanu sense defined by (see [20])

$${}^{PAB}\mathbb{D}^{\vartheta}_{0^+}\upsilon(\varkappa) = \begin{cases} \mathbb{D}_{\upsilon(\varkappa)} = \frac{d\upsilon}{dx}, \text{ if } \varkappa \in [0, \varkappa_1], \\ {}^{AB}\mathbb{D}^{\vartheta}_{0^+}\upsilon(\varkappa) = \frac{(2-\vartheta)\aleph(\vartheta)}{2(1-\vartheta)}\int_{\varkappa_1}^{\varkappa} \exp\left(\lambda(\varkappa - t)\right)\upsilon'(\varkappa)dt \text{ if } \varkappa \in [\varkappa_1, b], \end{cases} \tag{46}$$

where $\aleph(\vartheta)$ is the normalization function that satisfies $\aleph(1) = \aleph(0) = 1$; $\lambda = (\vartheta/(\vartheta - 1))$, and ${}^{AB}\mathbb{D}^{\vartheta}_{0^+}$ are the classical Atangana-Baleanu FD ([26]).

Based on PAB-NIFDE (45), the results in Theorems 7 and 8 can be presented by

$$\upsilon(\varkappa) = \begin{cases} \upsilon_0 - g(\upsilon) + \mathbb{I}_{\Phi_{\upsilon}(\varkappa)}, \text{ if } \varkappa \in [0, \varkappa_1] \\ \upsilon(\varkappa_1) - g(\upsilon) + {}^{AB}\mathbb{I}^{\vartheta}_{\varkappa_1}\Phi_{\upsilon}(\varkappa), \text{ if } \varkappa \in [\varkappa_1, b] \end{cases} = \begin{cases} \upsilon_0 - g(\upsilon) + \int_0^{\varkappa_1}\Phi_{\upsilon}(t)dt, \text{ if } \varkappa \in [0, \varkappa_1], \\ \upsilon(\varkappa_1) - g(\upsilon) + \frac{1-\vartheta}{\aleph(\vartheta)}\Phi_{\upsilon}(\varkappa) + \frac{\vartheta}{\aleph(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa - t)^{\vartheta-1}\Phi_{\upsilon}(t)dt, \text{ if } \varkappa \in \{\varkappa_1, b], \end{cases} \tag{47}$$

where ${}^{AB}\mathbb{I}^{\vartheta}_{0^+}$ is the Atangana-Baleanu integral on $\varkappa_1 \le \varkappa \le b$ (see [26]).

Remark 9. Following the strategy of proof utilized in the previous part, we can get the existence results for nonlinear problems (42) and (45).

3.2. UH Stability Analysis. In this portion, we give the UH Stability of problem (4).

Definition 10. PC-NIFDE (4) is UH stable if there exists a $K_f > 0$, such that for all $\varepsilon > 0$ and each solution $\omega \in \mathscr{C}$ of the inequality.

$$\left|{}^{PC}\mathbb{D}^{\vartheta}_{0^+}\omega(\varkappa) - \Phi_{\omega}(\varkappa)\right| \le \varepsilon, \varkappa \in \mathbb{J}, \tag{48}$$

there exists a solution $\upsilon \in \mathscr{C}$ of PC-NIFDE (4) that satisfies

$$|\omega(\varkappa) - \upsilon(\varkappa)| \le K_f\varepsilon, \tag{49}$$

where $\Phi_{\omega}(\varkappa) := {}^{PC}\mathbb{D}^{\vartheta}_{0^+}\omega(\varkappa)$ and $\Phi_{\omega}(\varkappa) = \Phi(\varkappa, \omega(\varkappa), \Phi_{\omega}(\varkappa))$.

Remark 11. $\omega \in \mathscr{C}$ satisfies inequality (48) if there exist function $\sigma \in \mathscr{C}$ with

(i) $|\sigma(\varkappa)| \le \varepsilon, \varkappa \in \mathbb{J}$

(ii) For all $\varkappa \in \mathbb{J}$

$${}^{PC}\mathbb{D}^{\vartheta}_{0^+}\omega(\varkappa) = \Phi_{\omega}(\varkappa) + \sigma(\varkappa). \tag{50}$$

Lemma 12. *Let $0 < \vartheta \le 1$, and $\omega \in \mathscr{C}$ is a solution of inequality (48). Then, ω satisfies*

$$\begin{aligned} &\left|\omega(\varkappa) - \mathscr{W}_0 - \int_0^{\varkappa_1}\Phi_{\omega}(t)dt\right| \le \varkappa_1\varepsilon, \text{ if } \varkappa \in [0, \varkappa_1],. \\ &\left|\omega(\varkappa) - \mathscr{W}_1 - \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}(\varkappa - t)^{\vartheta-1}\Phi_{\omega}(t)dt\right| \\ &\quad \le \frac{(b-\varkappa_1)^{\vartheta}}{\Gamma(\vartheta+1)}\varepsilon, \text{ if } \varkappa \in [\varkappa_1, b], \end{aligned} \tag{51}$$

where $\mathscr{W}_0 = \omega_0 - g(\omega)$ and $\mathscr{W}_1 = \omega(\varkappa_1) - g(\omega)$.

Proof. Let ω be a solution of (48).

By part (ii) of Remark 11, we have

$$\begin{aligned} &{}^{PC}\mathbb{D}_{0^+}^{\vartheta}\omega(\varkappa) = \Phi_\omega(\varkappa) + \sigma(\varkappa), \\ &\omega(0) + g(\omega) = \omega_0. \end{aligned} \tag{52}$$

Then, the solution of problem (52) is

$$\omega(\varkappa) = \begin{cases} \mathscr{W}_0 + \int_0^{\varkappa_1} [\Phi_\omega(t) + \sigma(t)]dt, \text{if } \varkappa \in [0, \varkappa_1], \\ \mathscr{W}_1 + \dfrac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)\Big)^{\vartheta-1} [\Phi_\omega(t) + \sigma(t)]dt, \text{if } \varkappa \in [\varkappa_1, b]. \end{cases} \tag{53}$$

□

Again by (i) of Remark 11, we obtain

$$\begin{aligned} &\left|\omega(\varkappa) - \mathscr{W}_0 - \int_0^{\varkappa_1} \Phi_\omega(t)dt\right| \\ &\quad \leq \int_0^{\varkappa_1} |\sigma(t)|dt \leq \varepsilon\varkappa_1, \text{for } \varkappa \in [0, \varkappa_1], \\ &\left|\omega(\varkappa) - \mathscr{W}_1 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t))^{\vartheta-1} \Phi_\omega(t)dt\right| \\ &\quad \leq \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} |\sigma(t)|dt \\ &\quad \leq \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon, \text{for } \varkappa \in [\varkappa_1, b]. \end{aligned} \tag{54}$$

Theorem 13. *Under the assumptions of Theorem 8. Then, the solution of PC-NIFDE (4) is HU and GHU stable.*

Proof. Let $\omega \in \mathscr{C}$ be a solution of inequality (48), and $\upsilon \in \mathscr{C}$ be a unique solution of the following PC-NIFDE.

$${}^{PC}\mathbb{D}_{0^+}^{\vartheta}\upsilon(\varkappa) = \Phi_\upsilon(\varkappa), \tag{55}$$

□

From Lemma 12, we obtain

$$\upsilon(\varkappa) = \begin{cases} \mathscr{V}_0 + \int_0^{\varkappa_1} \Phi_\upsilon(t)dt, \text{if } \varkappa \in [0, \varkappa_1], \\ \mathscr{V}_1 + \dfrac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} \Phi_\upsilon(t)dt, \text{if } \varkappa \in [\varkappa_1, b], \end{cases} \tag{56}$$

where $\mathscr{V}_0 = \upsilon_0 - g(\upsilon)$ and $\mathscr{V}_1 = \upsilon(\varkappa_1) - g(\upsilon)$. Clearly, if $\upsilon(0) + g(\upsilon) = \omega(0) + g(\omega)$, then $\mathscr{V}_0 = \mathscr{W}_0$, and $\mathscr{V}_1 = \mathscr{W}_1$. Hence, (56) becomes

$$\upsilon(\varkappa) = \begin{cases} \mathscr{W}_0 + \int_0^{\varkappa_1} \Phi_\upsilon(t)dt, \text{if } \varkappa \in [0, \varkappa_1], \\ \mathscr{W}_1 + \dfrac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} \Phi_\upsilon(t)dt, \text{if } \varkappa \in [\varkappa_1, b]. \end{cases} \tag{57}$$

Using Lemma 12 and (Assu_4) for $\varkappa \in 0, \varkappa_1]$, we have

$$\begin{aligned} |\omega(\varkappa) - \upsilon(\varkappa)| &= \left|\omega(\varkappa) - \mathscr{W}_0 - \int_0^{\varkappa_1} \Phi_\upsilon(t)dt\right| \\ &\leq \left|\omega(\varkappa) - \mathscr{W}_0 - \int_0^{\varkappa_1} \Phi_\omega(t)dt\right| \\ &\quad + \int_0^{\varkappa_1} |\Phi_\omega(t) - \Phi_\upsilon(t)|dt \\ &\leq \varepsilon\varkappa_1 + \frac{\kappa_1}{1 - \kappa_2} \int_0^{\varkappa_1} |\omega(t) - \upsilon(t)|dt. \end{aligned} \tag{58}$$

Using classical Gronwall's Lemma [27], we obtain

$$\begin{aligned} |\omega(\varkappa) - \upsilon(\varkappa)| &\leq \varepsilon\varkappa_1 \exp\left(\int_0^{\varkappa_1} \frac{\kappa_1}{1 - \kappa_2}\right)dt \\ &= \varepsilon\varkappa_1 \exp\left(\frac{\kappa_1\varkappa_1}{1 - \kappa_2}\right) \coloneqq \varepsilon K_0. \end{aligned} \tag{59}$$

For $\varkappa \in \varkappa_1, b]$, we have

$$\begin{aligned} |\omega(\varkappa) - \upsilon(\varkappa)| &= \left|\omega(\varkappa) - \mathscr{W}_1 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} \Phi_\upsilon(t)dt\right| \\ &\leq \left|\omega(\varkappa) - \mathscr{W}_1 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} \Phi_\omega(t)dt\right| \\ &\quad + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} |\Phi_\omega(t) - \Phi_\upsilon(t)|dt \\ &\leq \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon + \frac{\kappa_1}{1 - \kappa_2} \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} |\omega(t) - \upsilon(t)|dt. \end{aligned} \tag{60}$$

Using fractional Gronwall's Lemma [27], we obtain

$$\begin{aligned} |\omega(\varkappa) - \upsilon(\varkappa)| &\leq \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon + \frac{\varepsilon}{\Gamma(\vartheta + 1)} \frac{\kappa_1}{1 - \kappa_2} \\ &\quad \times \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t)^{\vartheta-1} (b - \varkappa_1)^\vartheta dt \\ &\leq \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon + \frac{\kappa_1 (b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)(1 - \kappa_2)} \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \varepsilon \\ &= \frac{(b - \varkappa_1)^\vartheta}{\Gamma(\vartheta + 1)} \left(\frac{\kappa_1}{(1 - \kappa_2)} + \frac{1}{\Gamma(\vartheta + 1)}\right) \varepsilon \coloneqq \varepsilon K_{1.} \end{aligned} \tag{61}$$

It follows from (59) and (61) that

$$|\omega(\varkappa) - v(\varkappa)| \le \begin{cases} K_0 \varepsilon, \text{ for } \varkappa \in [0, \varkappa_1], \\ K_1 \varepsilon, \text{ for } \varkappa \in [\varkappa_1, b], \end{cases} \quad (62)$$

where

$$K_0 = \varkappa_1 \exp\left(\frac{\kappa_1 \varkappa_1}{1-\kappa_2}\right), \quad K_1 = \frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)}\left(\frac{\kappa_1}{(1-\kappa_2)} + \frac{1}{\Gamma(\vartheta+1)}\right). \quad (63)$$

Hence, PC-NIFDE (4) is UH stable in $\mathscr{C}$. Moreover, if there exists a nondecreasing function, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that $\varphi(\varepsilon) = \varepsilon$. Then, from (62), we have

$$|\omega(\varkappa) - v(\varkappa)| \le \begin{cases} K_0 \varphi(\varepsilon), \text{ for } \varkappa \in [0, \varkappa_1], \\ K_1 \varphi(\varepsilon), \text{ for } \varkappa \in [\varkappa_1, b], \end{cases} \quad (64)$$

with $\varphi(0) = 0$, which proves PC-NIFDE (4) is GUH stable in $\mathscr{C}$.

4. Examples

In this portion, we present two examples to illustrate the reported results.

Example 1. Consider the following PC-NIFDE

$$\begin{aligned} &{}^{PC}\mathbb{D}_{0^+}^{1/3} v(\varkappa) = \Phi\left(\varkappa, v(\varkappa), {}^{PC}\mathbb{D}_{0^+}^{1/3} v(\varkappa)\right), \quad \varkappa[\in 0, 1], \\ &v(0) + \sum_{i=1}^{n} c_i v(\varkappa i) = \frac{1}{4}, \end{aligned} \quad (65)$$

or

$$\begin{aligned} &v'(\varkappa) = \Phi\left(\varkappa, v(\varkappa), v'(\varkappa)\right), \varkappa \in \left[0, \frac{1}{2}\right], \\ &{}^{C}\mathbb{D}_{1/2^+}^{1/3} v(\varkappa) = \Phi\left(\varkappa, v(\varkappa), {}^{C}\mathbb{D}_{1/2^+}^{1/3} v(\varkappa)\right), \text{if } \varkappa \in \left[\frac{1}{2}, 1\right], \\ &v(0) + \sum_{i=1}^{n} c_i v(\varkappa_i) = \frac{1}{4}, \end{aligned} \quad (66)$$

where $\vartheta = 1/3, v_0 = 1/4, 0 < \varkappa_1 = 1/2 < \cdots < \varkappa_n < 1 = b$, and c_i are positive constants with $\sum_{i=1}^{n} c_i < 1/5$. Set

$$\begin{aligned} &\Phi(\varkappa, v, \omega) = \frac{e^{-\varkappa}}{(8 + e^{\varkappa})(2 + |v| + |\omega|)}, \varkappa \in [0, 1], v, \omega \in [0, \infty), \\ &g(v) = \sum_{i=1}^{n} c_i v(\varkappa_i), v \in [0, \infty). \end{aligned} \quad (67)$$

Let $v, \omega, \bar{v}, \bar{\omega} \in [0, \infty)$, $\varkappa \in [0, 1]$. Then,

$$\begin{aligned} &|f(\varkappa, v, \omega) - f(\varkappa, \bar{v}, \bar{\omega})| \\ &\le \frac{e^{-\varkappa}}{(8 + e^{\varkappa})}\left|\frac{|v - \bar{v}| + |\omega - \bar{\omega}|}{(2 + |v| + |\omega|)(2 + |\bar{v}| + |\bar{\omega}|)}\right| \\ &\le \frac{1}{9}|v - \bar{v}| + \frac{1}{9}|\omega - \bar{\omega}|. \end{aligned} \quad (68)$$

Hence, the condition (Assu_3) holds with $\kappa_1 = \kappa_2 = 1/9$. Also we have

$$\begin{aligned} |g(v) - g(\omega)| &= \left|\sum_{i=1}^{n} c_i v(\varkappa_i) - \sum_{i=1}^{n} c_i \omega(\varkappa_i)\right| \\ &\le \sum_{i=1}^{n} c_i |v - \omega| \le \frac{1}{5}|v - \omega|. \end{aligned} \quad (69)$$

Hence, the condition (Assu_4) holds with $\kappa_3 = 1/5$. Moreover, the following condition

$$\begin{aligned} \max\{\zeta_1, \zeta_2\} &= \max\left\{\kappa_3 + \frac{\kappa_1}{1-\kappa_2}\varkappa_1, \kappa_3 + \frac{\kappa_1}{1-\kappa_2}\frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)}\right\} \\ &= \max\left\{\frac{21}{80}, \frac{1}{5} + \frac{1}{8\sqrt[3]{2}\Gamma(4/3)}\right\} \\ &= \frac{1}{5} + \frac{1}{8\sqrt[3]{2}\Gamma(4/3)} < 1, \end{aligned} \quad (70)$$

is satisfied with $\varkappa_1 = (1/2)$, and $b = 1$. Thus, with the assistance of Theorem 8, problem (65) has a unique solution $[0, 1]$. Further, since $1 - (\kappa_1 \varkappa_1 / 1 - \kappa_2) = (15/16) < 1$, and $1 - ((b - \varkappa_1)^\vartheta / \Gamma(\vartheta + 1))(\kappa_1 / 1 - \kappa_2) = 1 - (1/8\sqrt[3]{2}\Gamma(4/3)) < 1$, then

$$K_0 = \varkappa_1 \exp\left(\frac{\kappa_1 \varkappa_1}{1-\kappa_2}\right) = \frac{1}{2}e^{1/16} > 0, \quad (71)$$

and

$$\begin{aligned} K_1 &= \frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)}\left(\frac{\kappa_1}{(1-\kappa_2)} + \frac{1}{\Gamma(\vartheta+1)}\right) \\ &= \frac{(1/8) + (1/(\Gamma(4/3)))}{\sqrt[3]{2}\Gamma(4/3)} > 0, \end{aligned} \quad (72)$$

which implies that problem (65) is HU stable.

Example 2. Consider the following PC-NIFDE

$$\begin{aligned}&{}^{PC}\mathbb{D}^{1/2}_{1/4^+}\upsilon(\varkappa)=\Phi\left(\varkappa,\upsilon(\varkappa),{}^{PC}\mathbb{D}^{1/2}_{1/4^+}\upsilon(\varkappa)\right),\varkappa\in\left[\frac{1}{4},1\right],\\&\upsilon\left(\frac{1}{4}\right)+\frac{1}{2}\sin\left(\frac{\upsilon(\varkappa)}{3}\right)+\frac{1}{9}=1,\end{aligned}\tag{73}$$

or

$$\begin{aligned}&\upsilon'(\varkappa)=\Phi\left(\varkappa,\upsilon(\varkappa),\upsilon'(\varkappa)\right),\varkappa\in\left[\frac{1}{4},\frac{1}{2}\right],\\&{}^{C}\mathbb{D}^{1/2}_{1/2^+}\upsilon(\varkappa)=\Phi\left(\varkappa,\upsilon(\varkappa),{}^{C}\mathbb{D}^{1/2}_{1/2^+}\upsilon(\varkappa)\right),\text{ if }\varkappa\in\left[\frac{1}{2},1\right],\\&\upsilon\left(\frac{1}{4}\right)+\frac{1}{2}\sin\left(\frac{\upsilon(\varkappa)}{3}\right)+\frac{1}{9}=1,\end{aligned}\tag{74}$$

where $\varkappa_1=(1/2),\vartheta=(1/2),\upsilon_0=1$. Set

$$\Phi(\varkappa,\upsilon(\varkappa),\omega(\varkappa))=\frac{1}{(10+\varkappa^2)}\left(\frac{\upsilon(\varkappa)+\omega(\varkappa)}{(1+|\upsilon(\varkappa)|+|\omega(\varkappa)|)}+\frac{1}{90}\right),\tag{75}$$

for $\varkappa\in[(1/4),1],\upsilon,\omega\in[0,\infty)$, and

$$g(\upsilon)=\frac{1}{2}\sin\left(\frac{\upsilon}{3}\right)+\frac{1}{9},\upsilon\in[0,\infty).\tag{76}$$

Let $\upsilon,\omega\in[0,\infty)$ and $\varkappa\in[(1/4),1]$. Then,

$$\begin{aligned}|\Phi(\varkappa,\upsilon,\omega)|&=\left|\frac{1}{(10+\varkappa^2)}\left(\frac{\upsilon+\omega}{(1+|\upsilon|+|\omega|)}+\frac{1}{90}\right)\right|\\&\le\frac{1}{(10+\varkappa^2)}\left(|\upsilon|+|\omega|+\frac{1}{90}\right).\end{aligned}\tag{77}$$

Putting $\Omega(|\upsilon|)=|\upsilon|+(1/90)$, and $\varphi(\varkappa)=\psi(\varkappa)=(1/(10+\varkappa^2))$ Then, $|\Phi(\varkappa,\upsilon,\omega)|\le\varphi(\varkappa)\Omega(|\upsilon|)+\psi(\varkappa)|\omega|$ valid for any $(\varkappa,\upsilon,\omega)\in[(1/4),1]\times[0,\infty)\times[0,\infty)$, and $\psi^*=(16/161)<1$. Also, $|g(\upsilon)|\le(1/6)|\upsilon|+(1/9)=a|\upsilon|+b$. Hence, $(Assu_1)$ and $(Assu_2)$ hold. Thus, all the assumptions of Theorem 7 are satisfied. Hence, problem (73) has a solution on $[(1/4),1]$.

5. Conclusions

Somewhat recently, numerous methodologies have been proposed to portray behaviors of some complex world problems emerging in numerous scholarly fields. One of these problems is the multistep behavior shown by certain problems. In this regard, Atangana and Araz [20] introduced the concept of piecewise derivative. As an extra contribution to this subject, existence, uniqueness, and UH stability results for PC-NIFDE (4) involving a piecewise Caputo FD have been obtained. Our approach to this work has been based on Banach's and Schaefer's fixed-point theorem and Gronwall's Lemma. In light of our current results, the solution form for analogous problems containing piecewise Caputo-Fabrizio and Atangana–Baleanu operators have been presented. Finally, we have created two examples to validate the results obtained.

As an open problem, it will be very interesting to study the present problems on piecewise fractional operators with another function that is more general; precisely, one has to consider in problem (2) with ${}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+}$ such that

$${}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+}f(\varkappa)=\begin{cases}\mathbb{D}_\psi\ :\text{ if }\varkappa\in[0,\varkappa_1],\\{}^{C}\mathbb{D}^{\vartheta;\psi}_{\varkappa_1}f(\varkappa)\text{: if }\varkappa\in[\varkappa_1,b],\end{cases}\tag{78}$$

where $\mathbb{D}_\psi\coloneqq((1/(\psi'(\varkappa)))(d/d\varkappa))$ and ${}^{C}\mathbb{D}^{\vartheta;\psi}_{0^+}$ are ψ-Caputo FD of order ϑ introduced by Almeida [28].

Data Availability

No real data were used to support this study. The data used in this study are hypothetical.

Conflicts of Interest

No conflicts of interest are related to this work.

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