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Research Article

Study of Generalized q-Close-to-Convex Functions Related to Parabolic Domain

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The concepts of bounded boundary rotation and parabolic domain are used to define certain new classes of analytic functions such as $V_{1,q}(m, g)$, $R_{1,q}(m, g)$, $T_{1,q}(m, g)$, and $Q^{\gamma}_{1,q}(m, g, a)$. The difference of coefficients, necessary conditions, distortion bounds, radius problem, and several other interesting properties of these newly defined classes are also studied.

1. Introduction

Let **A** denote the class of functions f which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and are represented by the series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Two functions f and g, analytic in E, we say f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function $w(z)$ analytic in E with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$. If the function g is univalent in E, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), \quad f(E) \subset g(E). \tag{2}$$

For $f, g \in \mathbf{A}$, f be given by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution (Hadamard product) of f and g is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E. \tag{3}$$

Kanas and Wisniowska [1, 2] defined the subclasses of **A** related to the conic domains Ω_k, where

$$\Omega_k = \left\{u + iv : u^2 > k^2\left\{(u-1)^2 + v^2\right\}\right\}, \quad u > 0, k \geq 0. \tag{4}$$

The domain Ω_0 represents the right half plane, Ω_1 is a domain related to parabola, Ω_k, $0 < k < 1$ represents hyperbola and an ellipse for $k > 1$.

We shall deal with the domain Ω_1 and the function $p_1(z)$ given below as

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \quad (5)$$

plays the role of extremal function for Ω_1.

Let P denote the class of Caratheodory functions p, analytic in E with $p(0) = 1$ and satisfying the property $\mathrm{Re}\, p(z) > 0 (z \in E)$.

Using the concept of q-calculus, Ismail et al. [3] defined a subclass S_q^* of $\mathbf{A}$ consisting of functions called q-starlike in E and established that $\lim_{q \longrightarrow 1^-} S_q^*$ is the well-known class S^* of starlike functions with respect to origin. In [4], a firm foothold of the usage of the q-calculus in the context of geometric function theory was effectively established. For the study of various subclasses of $\mathbf{A}$, the quantum (or q-) calculus has been used as an important tool (see [5–7]).

Jackson [8, 9] first defined, with some applications, the q-derivative and q-integral operators. For more and potentially useful survey-cum-expository review article, refer to [10] for interested researchers and scholars. For convenience, we provide here some basics and details of q-calculus which we use in this study. Throughout this paper, we will assume that q satisfies the condition $0 < q < 1$.

(i) Let $q \in (0, 1)$ and define the q-number

$$[\lambda]_q = \frac{1-q^\lambda}{1-q}, \lambda \in \mathbb{C} = 1 + q + q^2 + \cdots + q^{n-1}, \lambda = n \in \mathbb{N} = 0, \quad \lambda = 0 \quad (6)$$

(ii) The q-derivative (or q-difference) D_q of $f \in \mathbf{A}$ is defined in a given subset of $\mathbb{C}$ by

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \quad (7)$$

provided that $f'(0)$ exists.

From (7), we note that

$$\lim_{q \longrightarrow 1^-} \left(D_q f\right)(z) = \lim_{q \longrightarrow 1^-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z), \quad (8)$$

for a differentiable function f in a given subset of $\mathbb{C}$.

From (1) and (7), we observe that

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \quad (9)$$

By using a special case of general conic domain Ω_k, concept of q-calculus, and principle of subordination, we define as in [11] the class $P_{1,q}$ as follows.

Let p be analytic in E with $p(0) = 1$. Then, $p \in P_{1,q}$ if and ony if

$$p(z) \prec p_{1,q}(z) = \frac{2p_1(z)}{(1+q) + (1-q)p_1(z)}, \quad (10)$$

and $p_1(z)$ is given by (5).

Geometrically, the function $p_1 \in P_{1,q}$ takes all values from the domain $\Omega_{1,q}$, which is defined as

$$\Omega_{1,q} = \left\{ w : \left| \frac{(1+q)w}{2 + (q-1)w} - 1 \right| < \mathrm{Re}\left(\frac{(1+q)w}{2+(q-1)w} \right) \right\}. \quad (11)$$

It can easily be seen that $P_{1,q} \subset P(\alpha)$, with $\alpha = 2/3 + q$, and $p \in P_{1,q}$ implies that $\mathrm{Re}\,(p(z)) > 2/3 + q$. Also, $|\arg p(z)| < \sigma\pi/2$, $\sigma = 2/\pi \tan^{-1}(1 + q/2)$. Also, $\lim_{q \longrightarrow 1^-} P_{1,q} \subset P(1/2)$ and $|\arg p(z)| < \pi/4$.

We generalize the class $P_{1,q}$ as follows.

Definition 1. An analytic function $p : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is in the class $P_{1,q}(m)$ if and only if there exist $p_1, p_2 \in P_{1,q}$ such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z). \quad (12)$$

It is clear that $P_{1,q}(2) = P_{1,q}$.

Using the class $P_{1,q}(m)$, we now define the following classes of analytic function.

Definition 2. Let $V_{1,q}(m, g)$ denote the class of all functions $f \in \mathbf{A}$, with $g \in A$ such that $(g * f)'(z) \neq 0$ in E and $(z((g * f)'))'/(g * f)' \in P_{1,q}(m)(z \in E)$.

With $F = zf'$, $F \in R_{1,q}(m, g)$ if and only if $f \in V_{1,q}(m, g)$ in E. For $g(z) = z/1 - z$, $F \in R_{1,q}(m)$.

Special cases:

(i) $V_{1,q}(m, z/1 - z) = V_{1,q}(m) \subset V_m(\alpha)$, $\alpha = 2/3 + q$ $(1/2 < \alpha < 2/3)$ and V_m is well-known class of functions of bounded boundary rotation (see [12])

(ii) Let

$$g(z) = h_q(z) = \frac{1}{1-q} \log \frac{1-qz}{1-z} = \sum_{n=1}^{\infty} \frac{1-q^n}{1-q} \frac{z^n}{n} \quad (13)$$

be convex univalent in E (see [13]).

We note that

$$zh_q'(z) = \frac{z}{(1-qz)(1+z)} = \sum_{n=1}^{\infty}[n]_q z^n,$$
$$f(z) * zh_q'(z) = z + \sum_{n=2}^{\infty}[n]_q a_n z^n = zD_q f(z). \quad (14)$$

Thus, Definition 2 can be written as

$$\frac{\left(z\left(h_q * f\right)'\right)'}{\left(h_q * f\right)'} = \frac{\left(zh_q' * f\right)'}{\left(zh_q' * f\right)} = \frac{z\left(D_q f\right)'}{D_q f} \in P_{1,q}(m), \quad (15)$$

that is, $D_q f \in R_{1,q}(m)$ in E.

(iii) $V_{1,q}(2, z/1-z) = C_{1,q} \subset C$, where C is the class of convex functions in E. Similarly, $R_{1,q}(2, z/1-z) = R_{1,q}$ is a subclass of the class S^* of starlike functions in E

Definition 3. Let $f, g \in \mathbf{A}$ with $(g * f)(z) \neq 0$ in E. Then, $f \in T_{1,q}(m, g)$, if there exists $\psi \in V_{1,q}(m, g)$ such that

$$\frac{(g * f)'}{(g * \psi)'} \in P_{1,q}, z \in \mathbf{E}. \quad (16)$$

Special cases:

(i) Choose $g(z) = z/1-z$. Then, $T_{1,q}(m, z/1-z)$. Then, $T_{1,q}(m, z/1-z)$ reduces to a subclass of T_m consisting of generalized close-to-convex functions. The class T_m has been introduced and studied in [14], where $T_2 = K$ is the class of close-to-convex functions (see [15])

(ii) Let $g(z) = h_q(z)$, where $h_q(z)$ is given by (44). Then,

$$\frac{\left(h_q * f\right)'}{\left(h_q * \psi\right)'} = \frac{\left(zh_q' * f\right)}{\left(zh_q' * \psi\right)} = \frac{\left(D_q f\right)}{D_q \psi} \in P_{1,q}(m). \quad (17)$$

In this case, we say that $f \in T^*_{1,q}(m)$ in E.

Remark 4.

(i) For $\psi \in \mathbf{A}$, the class C_q *of* q-convex functions is defined as

$$\frac{D_q\left(zD_q\psi\right)}{D_q\psi} \in P, z \in E. \quad (18)$$

As $q \longrightarrow 1^-$, $C_q \longrightarrow C$ and the class S_q^* of q-starlike functions, first introduced and studied in [3]. Also, $f \in C_q$ if and only if $zD_q f \in S_q^*$, and it has been proved [3] that

$$\bigcap_{0<q<1} C_q = C, \bigcap_{0<q<1} S_q^* = S^*. \quad (19)$$

(ii) If $\psi \in C$, then $zD_q\psi$ is starlike of order $\beta = 1 - q/2(1+q)$ for $z \in E$ (see [13]).

Definition 5. Let $f, g \in \mathbf{A}$ with $F(z) = (g * f)(z) \neq 0$, $F(z)F'(z)/z \neq 0$ for $z \in E$. Then, f is said to belong to the class $Q^{\gamma}_{1,q}(m, g, a)$ if and only if there exists a function $G \in T_{1,q}(m, g)$ such that, for $z \in E$, Re $(a) \geq 0$, $0 < \gamma \leq 1$,

$$zF'(z) + aF(z) = (a+1)z\left(G'(z)\right)^{\gamma}. \quad (20)$$

We note that

$$Q^1_{1,q}\left(m, \frac{z}{1-z}, 0\right) = T_{1,q}(m) \quad (21)$$

and $Q^1_{1,q}(2, h_q, 0) = K_{1,q} \subset K_q$, where K_q is the class of q-close-to-convex functions.

2. Preliminaries

Lemma 6 (see [16]). *Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. If $H(z)$ is univalent in E and $H(E)$ is convex, then*

$$|c_n| \leq |C_1|, \quad n \geq 1. \quad (22)$$

Lemma 7 (see [17]). *Suppose that $\beta,\gamma \in \mathbb{C}$ with $\beta \neq 0$, and* Re $(\beta h(z) + \gamma) > 0$, *$z \in E$ and $q \in \mathbf{A}$ such that $q(z) \prec h(z)$, $z \in E$. If $p(z)$ is analytic in E with $p(0) = 1$, then*

$$p(z) + \frac{p(z)}{\beta q(z) + \gamma} \prec h(z), \quad z \in E, \quad (23)$$

implies that $p(z) \prec h(z)$.

Lemma 8 (see [18]). *Let $h \in P$. Then, for $z = re^{i\theta}$*

$$\int_0^{2\pi} \left|h\left(re^{i\theta}\right)\right|^{\lambda} d\theta < c(\lambda)\frac{1}{(1-r)^{\lambda-1}}, \quad z \in E, \lambda > 1, \quad (24)$$

where $c(\lambda)$ is a constant.

Lemma 9 (see [19]). *Let $h \in P$. Then, for $z = re^{i\theta}$*

$$\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2}. \quad (25)$$

Lemma 10 (see [20]). *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in*

E, $\delta \geq 0$, Re $(c) \geq 0$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$. *If*

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{p(z) + \frac{\delta z p'(z)}{c\delta + p(z)}\right\} d\theta > -\pi, \tag{26}$$

then

$$\int_{\theta_1}^{\theta_2} \text{Re}\,\{p(z)\} d\theta > -\pi. \tag{27}$$

3. The Class $V_{1,q}(m, g)$

Using the well-known results, for example, see [21, 22], the following results can easily be proved.

Theorem 11. *Let* $f \in V_{1,q}(m, g)$. *Then, with* $g(z) = z/1 - z$, $m \geq 2$,

$$f'(z) = \frac{\left(f_1'(z)\right)^{(1-\alpha)(m/4+1/2)}}{\left(f_2'(z)\right)^{(1-\alpha)(m/4-1/2)}}, \quad \alpha = \frac{2}{3+q}, \tag{28}$$

and $f_1, f_2 \in V_2 = C$.

When $q \longrightarrow 1^-$, $\alpha \longrightarrow 1/2$, we obtain a known result. Also, for $f \in V_{1,q}(m, h_q)$, (28) can be written as

$$\left(f * h_q\right)' = \frac{\left\{\left(f_1 * h_q\right)'\right\}^{(1-\alpha)(m/4+1/2)}}{\left\{\left(f_2 * h_q\right)'\right\}^{(1-\alpha)(m/4-1/2)}}. \tag{29}$$

That is,

$$D_q f(z) = \frac{\left(D_q f_1\right)^{(1-\alpha)(m/4+1/2)}}{\left(D_q f_2\right)^{(1-\alpha)(m/4-1/2)}}, \quad f_1, f_2 \in C. \tag{30}$$

Theorem 12. *Let* $f \in V_{1,q}(m, g)$ *and let* $F = f * g$. *Then, for* $m \geq 2$, $z = re^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, *we have*

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{\frac{\left(zF'(z)\right)'}{F'(z)}\right\} d\theta > -\left(\frac{m}{2} - 1\right)(1-\alpha)\pi, \tag{31}$$

where $\alpha = 2/3 + q$.

We note that, with $g(z) = z/1 - z$, f is univalent for $(m/2 - 1)(1-\alpha) \leq 1$, $\alpha \in (1/2, 2/3)$. See [23].

Theorem 13. *Let* $f \in V_{1,q}(m, g)$, $(m/2 - 1)(1-\alpha) \leq 1$, $1/2 < \alpha = 2/3 + q < 2/3$. *Then,* $F(E)$, *with* $F = f * g$, *contains the Schlicht disc d:*

$$d = \left\{w : |w| < \frac{4}{8 + m(2-\alpha)}\right\}. \tag{32}$$

Proof. Let

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \tag{33}$$

with $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $p_i(z) = \sum_{n=1}^{\infty} c_{n,i} z^n$.

For $p \in P_{1,q}(m)$, $p_i \in P_{1,q} \subset P(\alpha)$, $\alpha = 2/3 + q$. Then, by Lemma 6, we have

$$|c_n| \leq \frac{m}{2}(2-\alpha), \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right) \quad \text{for all } n \geq 1. \tag{34}$$

From Theorem 12, it follows that $F : F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is univalent in E.

Let $w_0 (w_0 \neq 0)$ be any complex number such that $F(z) \neq w_0$ for $z \in E$. Then, the function

$$F_1(z) = \frac{w_0 F(z)}{w_0 - F(z)} = z + \left(A_2 + \frac{1}{w_0}\right)z^2 + \cdots \tag{35}$$

is analytic and univalent in E.

Now, $(zF'(z))'/F'(z) \in P_{1,q}(m)$, and simple calculations yield that

$$|A_2| \leq \frac{m}{4}(2-\alpha). \tag{36}$$

We use this bound for $|A_2|$ and the well-known sharp bound for the second coefficient of univalent functions and have

$$\frac{1}{|w_0|} - |A_2| \leq \left|A_2 + \frac{1}{w_0}\right| \leq 2\,; \tag{37}$$

this gives us

$$\frac{1}{|w_0|} \leq 2 + |A_2| \leq 2 + \frac{m}{4}(2-\alpha) = \frac{8 + m(2-\alpha)}{4}. \tag{38}$$

The proof is complete. □

For the permissible values of α and m, we obtain several interesting results as special cases of this result.

Theorem 14. *Let* $f \in V_{1,q}(m, h_q)$. *Then,*

$$\left\{\frac{(1+r)^{m/2+1}}{(1-r)^{m/2-1}}\right\}^{(1-\alpha)(1-\beta)} \leq \left|D_q f\right| \leq \left\{\frac{(1-r)^{m/2+1}}{(1+r)^{m/2-1}}\right\}^{(1-\alpha)(1-\beta)}, \tag{39}$$

where $\alpha = 2/3 + q$, $\beta = 1 - q/2(1+q)$, $(r \longrightarrow 1)$.

Proof. From Theorem 11, for $f \in V_{1,q}(m, h_q)$, we have relation (30). Now, for $f_i \in C$, it follows that $zD_q f_i(z) = t_i(z)$ is starlike of order $\beta = 1 - q/2(1+q)$ (see [13]). Also, it is known [24] that there exists $Fi \in S^*$ such that

$$\frac{t_i(z)}{z} = \left(\frac{F_i(z)}{z}\right)^{1-\beta}, \quad i = 1, 2. \tag{40}$$

Thus, we can write (30) as

$$D_q f(z) = \left\{\frac{(F_1(z)/z)^{m/4+1/2}}{(F_2(z)/z)^{m/4-1/2}}\right\}^{(1-\alpha)(1-\beta)}, \quad F_1, F_2 \in S^*. \tag{41}$$

Now, using well-known distortion results for $F_i \in S^*$, we obtain (39) as required. □

Theorem 15. *Let F=(f ∗ g) and* $f \in R_{1,q}(2, g)$, *and let for* $\gamma, \sigma > 0$, F_1 *be defined by*

$$F_1(z) = \left[(\gamma+1)z^{-\gamma}\int_0^z t^{\gamma-1}F^{\sigma}(t)dt\right]^{1/\sigma}, \quad F_1 = f_1 * g. \tag{42}$$

Then, $F_1 \in R_{1,q}(2, g)$ in E.

(i) We note that (42) reduces to the generalized Bernardi operator for $f \in R_{1,q}(2)$, when $\sigma = 1$ and $g(z) = z/1 - z$

(ii) By choosing $g(z) = z/(1-z)^2$, we prove this result for $f \in V_{1,q}(2)$

Proof. From (42), it follows that

$$(z^{\gamma}F_1^{\sigma}(z))' = t^{\gamma-1}F^{\sigma}(z), \tag{43}$$

that is,

$$F_1^{\sigma}(z)[\gamma + \sigma H(z)] = F^{\sigma}(z), H(z) = \frac{zF_1'(z)}{F_1(z)}. \tag{44}$$

With simple computations, we obtain from (44)

$$H(z) + \frac{zH'(z)}{\sigma H(z) + \gamma} = \frac{zF'(z)}{F(z)} \in P_{1,q}, \quad z \in E. \tag{45}$$

Now, using Lemma 7 along with (45), we have $H \in P_{1,q}$ and consequently $F_1 \in R_{1,q}(2, g)$. This completes the proof. □

Theorem 16. *Let* $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_{1,q}(m, g)$ *and* $F = f * g$ *with* $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$. *Then, for* $(1-\alpha)(m+2) > (2-\sigma)$,

$$||A_{n+1}| - |A_n|| \le c(m, q)n^{(1-\alpha)(m/2+1)+\sigma-3}, \tag{46}$$

where c is constant, $\alpha = 2/3 + q$ *and* $\sigma = 2/\pi \tan^{-1}(1 + q/2)$.

Proof. Since $f \in V_{1,q}(m, g)$, it implies that $F \in V_{1,q}(m) \subset V_m(\alpha)$, $\alpha = 2/3 + q$. We can write

$$\begin{aligned}\left(zF'(z)\right)' &= F'(z)h^{\sigma}(z), h \in P_m, \sigma = \frac{2}{\pi}\tan^{-1}\left(\frac{1+q}{2}\right)\\ &= F'(z)\left[\left(\frac{m}{4}+\frac{1}{2}\right)h_1^{\sigma}(z) - \left(\frac{m}{4}-\frac{1}{2}\right)h_2^{\sigma}(z)\right], h_1, h_2 \in P,\\ &= \left(\frac{m}{4}+\frac{1}{2}\right)F'(z)h_1^{\sigma}(z) - \left(\frac{m}{4}-\frac{1}{2}\right)F'(z)h_2^{\sigma}(z).\end{aligned} \tag{47}$$

Then,

$$\begin{aligned}|z-\xi||F'(z)| \le &\left(\frac{m}{4}+\frac{1}{2}\right)|z-\xi||F'(z)||h_1^{\sigma}(z)|\\ &- \left(\frac{m}{4}-\frac{1}{2}\right)|z-\xi||F'(z)||h_2^{\sigma}(z)|.\end{aligned} \tag{48}$$

First, we calculate, for $i = 1, 2$, by using Theorem 11 in the following integral:

$$\int_0^{2\pi}|z-\xi||F'(z)h_i^{\sigma}(z)|d\theta = \int_0^{2\pi}|z-\xi|\left|\frac{(s_1(z)/z)^{m/4+1/2}}{(s_2(z)/z)^{m/4-1/2}}\right|^{(1-\alpha)}|h_i(z)|^{\sigma}d\theta, \tag{49}$$

$s_i \in S^*$, $\xi \in E$ and $z = re^{i\theta}$.

Let $0 < r < 1$. Then, by a result due to Golusin [25], there exists a ξ with $|\xi| = r$ such that for $|z| = r$,

$$|z-\xi||s_1'(z)| \le \frac{2r^2}{1-r^2}. \tag{50}$$

We use (50) and distortion result for $s_2 \in S^*$ and apply Holder's inequality to have

$$\begin{aligned}\int_0^{2\pi}|z-\xi||F'(z)h_i^{\sigma}(z)|d\theta &\le 2\pi c(q)\left(\frac{2r^2}{1-r^2}\right)\\ &\left[\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{(\mu-1)(2/2-\sigma)}d\theta\right] \times \left[\frac{1}{2\pi}\int_0^{2\pi}|h_i(z)|^{\sigma}d\theta\right]^{\sigma/2},\end{aligned} \tag{51}$$

where $\mu = (1-\alpha)(m/4 + 1/2)$.

Now, with subordination for $h_i \in P$, starlike function s_1 and Lemma 8 in (51), it follows that

$$\int_0^{2\pi}|z-\xi||F'(z)||h_i(z)|^{\sigma}d\theta \le c(q)\left(\frac{1}{1-r}\right)^{(1-\alpha)(m/2+1)-1}. \tag{52}$$

From (48), (49), and (52), it follows that

$$|(n+1)^2\xi A_{n+1} - n^2 A_n| \le c(m, q)\left(\frac{1}{1-r}\right)^{(1-\alpha)(m/2+1)+\sigma-1}, \quad (r \longrightarrow 1). \tag{53}$$

Thus, choosing $r = 1 - 1/n (n \longrightarrow \infty)$ in (53), we have

$$||A_{n+1}| - |A_n|| \le c(m, q) n^{(1-\alpha)(m/2+1)+\sigma-3}, \quad (54)$$

where $c(q)$ and $c(m, q)$ are constants. □

As a special case, we note that, for $f \in V_{1,q}(m, z/1-z)$, we have

$$||a_{n+1}| - |a_n|| \le o(1) \cdot n^{(1-\alpha)(m/2+1)+\sigma-3}. \quad (55)$$

When $q \longrightarrow 1^- \alpha = 1/2$, $\sigma = 1$, and in this case, if we choose $m = 6$, then

$$||a_{n+1}| - |a_n|| \le o(1). \quad (56)$$

4. The Class $T_{1,q}(m, g)$

We discuss here some basic properties of $f \in T_{1,q}(m, g)$.

Theorem 17. *Let $f \in T_{1,q}(m, g)$. Then, there exist $f_1, f_2 \in T_{1,q}(2, g)$ if and only if*

$$f'(z) = \frac{\left(f_1'(z)\right)^{m/4+1/2}}{\left(f_2'(z)\right)^{m/4-1/2}}, \quad (z \in E). \quad (57)$$

Proof. From Definition 3, we can write

$$\begin{aligned} f'(z) &= G'(z)h(z), G \in V_{1,q}(m, g), h \in P_{1,q} \\ &= \frac{\left(g_1'(z)\right)^{m/4+1/2}}{\left(g_2'(z)\right)^{m/4-1/2}} h(z), g_1, g_2 \in V_{1,q}(2, g) \\ &= \frac{\left(g_1'(z)h(z)\right)^{m/4+1/2}}{\left(g_2'(z)h(z)\right)^{m/4-1/2}} = \frac{\left(f_1'(z)\right)^{m/4+1/2}}{\left(f_2'(z)\right)^{m/4-1/2}}, f_1, f_2 \in T_{1,q}(2, g). \end{aligned} \quad (58)$$

The converse case follows easily, and the proof is complete. □

Since $P_{1,q} \subset P(\alpha)$, $\alpha = 2/3 + q$, we note that $V_{1,q}(m, z/1 - z) \subset V_m(\alpha)$, and for the class $V_m(\alpha)$, we refer to [22]. From Definition 3, we have

$$T_{1,q}\left(m, \frac{z}{1-z}\right) \subset T_m\left(\frac{2}{3+q}\right). \quad (59)$$

We now prove the following inclusion result.

Theorem 18. *For $0 < \alpha_1 < \alpha_2$, with $\alpha_i = 2/3 + q$,*

$$T_m(\alpha_1) \subset T_m(\alpha_2). \quad (60)$$

Proof. Let $f \in T_m(\alpha_1)$. Then, there exists $G \in V_m(\alpha_1)$ such that

$$\frac{f'(z)}{G'(z)} = (1-\alpha)h(z) + \alpha_1, h \in P. \quad (61)$$

Since $V_m(\alpha_1) \subset V_m(\alpha_2)$, and $P(\alpha_1) \subset P(\alpha_2)$, for $\alpha_1 < \alpha_2$, it follows that $f \in T_m(\alpha_2)$. □

We now discuss a geometrical property for the class $T_{1,q}(m, g)$ and investigate the behaviour of the inclination of the tangent at a point $w = f(re^{i\theta})$ to the image of the circle $C_r = \{z : |z| = r\}$, $0 \le r < 1$, and θ is any number of interval $(0, 2\pi)$ under the mapping of $f \in T_{1,q}(m, g)$.

Let $F = f * g$, $G = \psi * g$. Let

$$\phi(\theta) = \frac{\pi}{2} + \theta + \arg F'\left(re^{i\theta}\right), \quad (62)$$

and for $\theta_2 > \theta_1$, $\theta_1, \theta_2 \in [0, 2\pi]$,

$$\phi(\theta_2) - \phi(\theta_1) = \theta_2 + \arg F'\left(re^{i\theta_2}\right) - \theta_1 - \arg F'\left(re^{i\theta_1}\right). \quad (63)$$

Now, since

$$\theta + \arg F'\left(re^{i\theta}\right) = \theta + \operatorname{Re}\left\{-i \ln F'\left(re^{i\theta}\right)\right\}, \quad (64)$$

we have

$$\frac{\partial}{\partial\theta}\left(\theta + \arg F'\left(re^{i\theta}\right)\right) = \operatorname{Re}\left\{1 + \frac{re^{i\theta}F''\left(re^{i\theta}\right)}{F''\left(re^{i\theta}\right)}\right\}. \quad (65)$$

Therefore,

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\left(\theta + \arg F'\left(re^{i\theta}\right)\right) d\theta = \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta}F''\left(re^{i\theta}\right)}{F''\left(re^{i\theta}\right)}\right\} d\theta. \quad (66)$$

On the other hand,

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\left(\theta + \arg F'\left(re^{i\theta}\right)\right) d\theta &= \theta_2 + \arg F' \\ \left(re^{i\theta_2}\right) - \theta_1 - \arg F'\left(re^{i\theta_1}\right) &= \phi(\theta_2) - \phi(\theta_1). \end{aligned} \quad (67)$$

So, the integral on the left hand side of (67) characterizes the increment of the angle of the tangent to the curve Γ_r between the points $w(\theta_2)$ and $w(\theta_1)$ for $\theta_2 > \theta_1$.

We now prove the following.

Theorem 19. *Let $f \in T_{1,q}(m,g)$ and $F = f * g$. Then, for $0 \le r < 1$, $\theta_2 > \theta_1$, $\theta_1, \theta_2 \in [0, 2\pi]$, we have*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta}F''(re^{i\theta})}{F''(re^{i\theta})}\right\} d\theta > \left[(1-\alpha)\left(\frac{m}{2}-1\right)+\sigma\right]\pi, \tag{68}$$

where $\sigma = 2/\pi \tan^{-1}(1 + q/2)$ and $\alpha = 2/3 + q$.

Proof. From Definition 3, we can write

$$F'(z) = G'(z)h^{\sigma}(z), G \in V_{1,q}(m,g), h \in P, \tag{69}$$

where $\sigma = 2/\pi \tan^{-1}(1 + q/2)$ and $\alpha = 2/3 + q$.

Since, for $h \in P$, it is well known that

$$\left|h(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}. \tag{70}$$

Thus, the values of $h(z)$ are contained in the circle whose diameter is the line segment from $1 - r/1 + r$ to $1 + r/1 - r$. The circle is centered at the point $1 + r^2/1 - r^2$ and has the radius $2r/1 - r^2$. So, $|\arg h(z)|$ attains its maximum at points where a ray from the origin is tangent to the circle, that is, when

$$\arg h(z) = \pm \sin^{-1}\frac{2r}{1-r^2}. \tag{71}$$

Thus, it follows that

$$\max_{h\in P}\left|\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\} d\theta\right| \le 2\sin^{-1}\frac{2r}{1-r^2} = \pi - \cos^{-1}\frac{2r}{1-r^2}. \tag{72}$$

Now, from (69), (72), and Theorem 12, it follows that, for $z = re^{i\theta}$, $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta}F''(re^{i\theta})}{F''(re^{i\theta})}\right\} d\theta > \left[(1-\alpha)\left(\frac{m}{2}-1\right)+\sigma\right]\pi. \tag{73}$$

□

Remark 20. In [23], the class $K(\beta_1)$ of functions $f \in A$ if and only if, for $\theta_1 < \theta_2$, $z = re^{i\theta}$, $\beta_1 \ge 0$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zf''(re^{i\theta})}{f''(re^{i\theta})}\right\} d\theta > -\beta_1\pi. \tag{74}$$

For $0 \le \beta_1 \le 1, f$ is close-to-convex and hence univalent. For $\beta > 1$, f need not to be finitely valent. We note that $T_{1,q}(m,g) \subset K((1-\alpha)(m/2-1)+\sigma)$. It can easily be seen that $T_{1,q}(m,g)$ consists of univalent functions for $((1-\alpha)(m/2-1)+\sigma) \le 1$. Also, it can easily be seen that $T_{1,q}(m,g)$ forms a subset of a linear-invariant family of order $[(1-\alpha)(m/2-1)+\sigma-1]$. For this, we refer to [26].

Theorem 21. *Let $f \in A$ and $(zf'(z))'/G'(z) \in P_{1,q}$, $G = zG_1'$ and $G_1 \in V_{1,q}(m)$. Let h be defined by*

$$h'(z) = \frac{\left(zf'(z)\right)'}{1 + zG_1''(z)/G_1'(z)}. \tag{75}$$

Then, $h \in T_{1,q}(m)$ in E.

Proof. As

$$G'(z) = \left(zG_1'(z)\right)' = G_1'(z)\left[1 + \frac{zG_1''(z)}{G_1'(z)}\right]. \tag{76}$$

Therefore,

$$\frac{\left(zf'(z)\right)'}{G'(z)} = \frac{\left(zf'(z)\right)'}{G_1'(z)\left[1 + zG_1''(z)/G_1'(z)\right]} = \frac{h'(z)}{G_1'(z)} \in P_{1,q}. \tag{77}$$

This proves $h \in T_{1,q}(m)$ in E. □

Theorem 22. *Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to the class $T_{1,q}(m)$. Then, for $m > (2 + 2 - \sigma/1 - \alpha)$, $\alpha = 2/3 + q$, $\sigma = 2/\pi \tan^{-1}(1 + q/2)$,*

$$a_n = o(1)\cdot n^{\{(1-\alpha)(m/2+1)+\sigma-2\}}, \tag{78}$$

where $o(1)$ is a constant.

Proof. For $f \in T_{1,q}(m)$, we can write

$$\begin{aligned} f'(z) &= G'(z)h^{\sigma}(z), G \in V_m(\alpha), \sigma = \frac{2}{\pi}\tan^{-1}\left(\frac{1+q}{2}\right) \\ &= \left(G_1'\right)^{1-\alpha} h^{\sigma}(z), G_1 \in V_m, \alpha = \frac{2}{3+q}. \end{aligned} \tag{79}$$

It is known [27] that, for all $m > 2$, $G_1 \in V_m$, we can write

$$zG_1'(z) = s(z)p^{(m/2-1)}(z), s \in S^*, p \in P. \tag{80}$$

Thus, by using Cauchy theorem and using (79) and (80), we have

$$n|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi} |s(z)|^{1-\alpha}|p(z)|^{(1-\alpha)(m/2-1)}|h(z)|^{\sigma}d\theta \le \frac{1}{r^n} \left(\frac{r}{(1-r)^2}\right)^{(1-\alpha)} \left[\frac{1}{2\pi}\int_0^{2\pi}\left(|p(z)|^{(1-\alpha)(m/2-1)(2/2-\sigma)}\right)d\theta\right]^{2-\sigma/2} \left(\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2 d\theta\right)^{\sigma/2} \le c(m,q)\left(\frac{1}{1-r}\right)^{(1-\alpha)(m/2-1)+\sigma-1}, \tag{81}$$

where we have used Holder's inequality and Lemma 9. Setting $r = 1 - 1/n$, $n \longrightarrow \infty$ in (81), we get

$$a_n = o(1)\cdot n^{(1-\alpha)(m/2-1)+\sigma-2}. \tag{82}$$

□

As a special case, when $q \longrightarrow 1^-$, $f \in T_{1,q}(m) = T_m$, $\alpha = 1/2$, $\sigma = 1/2$, and in this case,

$$a_n = o(1)\cdot n^{(m/4-1)}\ (n \longrightarrow \infty). \tag{83}$$

We now study a radius problem for $f \in T_{1,q}(m, g)$.

Theorem 23. *Let $f \in T_{1,q}(m, g)$, and let $F = f * g$. Then, $f \in V_{1,q}(2, g)$ for $|z| < r_m = m - \sqrt{m^2 - 4}/2$.*

Proof. Let $\psi \in V_{1,q}(m, g)$ and $G = \psi * g$. Then, by Definition 2,

$$\frac{\left(zG'(z)\right)'}{G'(z)} = H(z) \in P_{1,q}(m). \tag{84}$$

Using (12), it can be easily shown that $H \in P_{1,q}$ for $|z| < r_m = m - \sqrt{m^2 - 4}/2$. Now, for $f \in T_{1,q}(m, g)$, we can write

$$\frac{F'(z)}{G'(z)} = p(z), p \in P_{1,q}, G \in V_{1,q}(m, g), \tag{85}$$

and since $G \in V_{1,q}(2, g)$ in $|z| < r_m$, the required result follows at once. □

Special cases:

(i) Let $g(z) = z/1 - z$ and $q \longrightarrow 1^-$. Then, $f \in T_m$, and in this case, f is convex for $|z| < r_m$

(ii) Let $g(z) = h_q(z)$, where $h_q(z)$ is given by (13). Then,

$$\frac{D_q\left(zD_q\psi(z)\right)}{D_q\psi(z)} \in P_{1,q} \text{ for } |z| < r_m, \tag{86}$$

which implies ψ is q-convex in $|z| < r_m$. Consequently, f is q-close-to-convex in $|z| < r_m$.

5. The Class $Q^{\gamma}_{1,q}(m, g, a)$

Using Definition 5, we shall discuss the class $Q^{\gamma}_{1,q}(m, g, a)$ in this section.

Theorem 24 (integral representation). *A function $f \in Q^{\gamma}_{1,q}(m, g, a)$ if and only if there exists some function $F \in Q^{\gamma}_{1,q}(m, g, \infty)$ such that*

$$f(z) = \frac{1+a}{z^a}\int_0^z t^{a-1}F(t)dt. \tag{87}$$

Proof. From Definition 5, it follows that $F \in Q^{\gamma}_{1,q}(m, g, \infty)$ if and only if we can write

$$F(z) = z\left(G'(z)\right)^{\gamma}, G \in T_{1,q}(m, g). \tag{88}$$

Thus,

$$zf'(z) + af(z) = (a+1)F(z), F \in Q^{\gamma}_{1,q}(m, g, \infty), \tag{89}$$

and from this observation, the proof is immediate. □

Theorem 25. *Let $0 < \gamma_1 < \gamma_2 \le 1$. Then,*

$$Q^{\gamma_1}_{1,q}(m, g, a) \subset Q^{\gamma_2}_{1,q}(m, g, a). \tag{90}$$

Proof. Let $f \in Q^{\gamma_1}_{1,q}(m, g, a)$. Then, with $F = f * g$, $G = \psi * g$, we have

$$zF'(z) + aF(z) = (a+1)\cdot z\left(G'(z)\right)^{\gamma_1}, G \in T_{1,q}(m). \tag{91}$$

We can write (91) as

$$zF'(z) + aF(z) = (a+1)z\left(J'(z)\right)^{\gamma_2}, \tag{92}$$

where $J'(z) = (G'(z))^{\gamma_1/\gamma_2}$.

We now show that $J \in T_{1,q}(m)$.

Consider

$$J'(z) = \left(G'(z)\right)^{\gamma_1/\gamma_2}, G \in T_{1,q}(m), \frac{\gamma_1}{\gamma_2} < 1. \tag{93}$$

Let

$$G_1'(z) = \left(g_1'(z)\right)^{\gamma_1/\gamma_2}, g_1 \in V_{1,q}(m). \tag{94}$$

Then,

$$\frac{\left(zG_1'(z)\right)'}{G_1'(z)} = \left(1 - \frac{\gamma_1}{\gamma_2}\right) + \frac{\gamma_1}{\gamma_2}\frac{\left(zg_1'(z)\right)'}{g_1'(z)}. \quad (95)$$

This implies $G_1 \in V_{1,q}(m)$. Now,

$$\frac{J'(z)}{G_1'(z)} = \left(\frac{G'(z)}{g_1'(z)}\right)^{\gamma_1/\gamma_2} = (h(z))^{\gamma_1/\gamma_2}, h \in P_{1,q}. \quad (96)$$

Since $\gamma_1/\gamma_2 < 1$, $(h(z))^{\gamma_1/\gamma_2} \in P_{1,q}$ in E. Thus, from (92), it follows that $F \in Q_{1,q}^{\gamma_2}(m, a)$. □

Theorem 26. *Let $f \in Q_{1,q}^{\gamma}(m, g, a)$, $\operatorname{Re}(a) \geq 0$, $0 < \gamma \leq 1$. Then, for $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$, we have*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{a + p(z)}\right\} d\theta > -\gamma\left[(1-\alpha)\left(\frac{m}{2} - 1\right) + \sigma\right]\pi, \quad (97)$$

where $p(z) = zF'(z)/F(z)$, $F = f * g$.

Proof. Since $F \in Q_{1,q}^{\gamma}(m, a)$, we can write

$$zF'(z) + aF(z) = (a+1)z\left(G'(z)\right)^{\gamma}, G \in T_{1,q}, \operatorname{Re} a \geq 0. \quad (98)$$

That is

$$F(z)[p(z) + a] = (a+1)\left(G'(z)\right)^{\gamma}. \quad (99)$$

Logarithmic differentiation of (99) and using Theorem 19, the required result immediately follows. □

We note the following special cases.

(i) Let $\gamma[(1-\alpha)(m/2 - 1) + \sigma] \leq 1$. Then, for $z = re^{i\theta}$, $\theta_1 < \theta_2$, using Lemma 10, with $\delta = 1$ and $c\delta = a$, we get

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{p(z)\} d\theta > -\pi \quad (100)$$

(ii) If we choose $g(z) = z/(1-z)^2$ in (i), then from (100) and a result due to Kaplan [15], it follows that F is close-to-convex and hence univalent in E

Theorem 27. *Let $f \in Q_{1,q}^{\gamma}(m, z/(1-z)^2, \infty)$. Then, f is starlike of order $\lambda = 1 - \gamma(1-\alpha)$, $\alpha = 2/3 + q$, for $|z| < r_*$, where*

$$r_* = \frac{2}{(m+2) + \sqrt{m^2 + 4m}}. \quad (101)$$

Proof. By Definition 5, we have

$$f(z) = z\left(G'(z)\right)^{\gamma}, G \in T_{1,q}(m). \quad (102)$$

Since $T_{1,q}(m) \subset T_m(\alpha)$, $\alpha = 2/3 + q$, it is known [28] that there exists $G_1 \in T_m$ such that

$$G'(z) = \left(G_1'(z)\right)^{1-\alpha}. \quad (103)$$

Using (103) in (102), with logarithmic differentiation and simple calculations, we obtain (see [14])

$$\frac{1}{1-\lambda}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \lambda\right\} = \operatorname{Re}\left\{\frac{\left(zG_1'(z)\right)'}{G1(z)}\right\},$$
$$\lambda = 1 - \gamma(1-\alpha) \geq \frac{r^2 - (m+2)r + 1}{1 - r^2}. \quad (104)$$

The right-hand side is positive for $|z| < r_*$, where r_* is given by (101). □

If $q \longrightarrow 1^-$ and $\gamma = 1$, then f is starlike of order 1/2 in $|z| < r_*$.

Data Availability

The data used to support the findings of this study are available from the corresponding authors upon request.

Conflicts of Interest

There is no conflict of interest.

Authors' Contributions

The authors read and approved the final manuscript, and they contributed specially as the following; conceptualization was performed by K.I.N., A.M.S., and S.A.S.; methodology was performed by A.A.L. and S.A.K.; validation was performed by K.I.N., A.A.L., and A.M.S.; formal analysis was performed by S.A.S. and S.A.K.; investigation was performed by K.I.N. and A.M.S; writing (original draft preparation) was performed by S.A.S. and A.A.L.; writing (review and editing) was performed by A.A.L., K.I.N., and S.A.S.; and supervision was performed by K.I.N. and A.M.S. The authors agree with the contents of the manuscript.

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