Hindawi
Journal of Function Spaces
Volume 2023, Article ID 2877019, 12 pages
https://doi.org/10.1155/2023/2877019

Research Article

Sehgal-Guseman-Type Fixed Point Theorems in Rectangular b-Metric Spaces and Solvability of Nonlinear Integral Equation

Hongyan Guan, Chen Lang, and Yan Hao

School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, China

Correspondence should be addressed to Yan Hao; haoyan8012@163.com

Received 1 July 2022; Revised 27 October 2022; Accepted 12 May 2023; Published 26 May 2023

Academic Editor: Cristian Chifu

Firstly, the concept of a new triangular α-orbital admissible condition is introduced, and two fixed point theorems for Sehgal-Guseman-type mappings are investigated in the framework of rectangular b-metric spaces. Secondly, some examples are presented to illustrate the availability of our results. At the same time, we furnished the existence and uniqueness of solution of an integral equation.

1. Introduction

In nonlinear analysis, the most famous result is the Banach contraction principle, which is established by Banach [1] in 1922. After that, there are a large number of excellent results for fixed point in metric spaces. On recent development on fixed point theory in metric spaces, one can consult [2] the related references involved. Branciari [3] introduced a new concept, that is, the definition of rectangular metric spaces, and established an analogue of the Banach fixed point theorem in such a space. Then, a lot of fixed point theorems for a wide range of contractions on rectangular metric spaces had emerged in a blowout manner. In such type space, Lakzian and Samet [4] gave some results involving (ψ, ϕ) weakly contraction. Furthermore, several common fixed point results about (ψ, ϕ)-weakly contractions were obtained by Bari and Vetro [5]. In [6], George and Rajagopalan considered common fixed points of a new class of (ψ, ϕ) contractions. By use of C-functions, Budhia et al. furnished several fixed point results in [7].

In [8], Czerwik put forward firstly the definition of b -metric space, an extension of a metric space. Since then, this result has been extended in different angles. In a b-metric space, in [9], Mitrovic provided a new method to prove Czerwik's fixed point theorem. By using of increased range of the Lipschitzian constants, Hussain et al. [10] provided a proof of the Fisher contraction theorem. Mustafa et al. [11] gave several fixed point theorems for some new classes of T-Chatterjea-contraction and T-Kannan-contraction. Recently, also in this type spaces, Mitrovic et al. [12] presented some new versions of existing theorems. Savanović et al. [13] constructed some new results for multivalued quasicontraction. Furthermore, in [14], Aydi et al. obtained the existence of fixed point for α-β_E-Geraghty contractions. In [15], several fixed point theorems of set valued interpolative Hardy-Rogers type contractions were studied. In [16], George et al. put forward the concept of rectangular b-metric mapping. Meanwhile, they gave some fixed point theorems. Lately, Gulyaz-Ozyurt [17], Zheng et al. [18], and Guan et al. [19] also studied fixed point theory in such spaces and obtained some excellent results. In 2021, Hussain [20] presented some fractional symmetric α-η-contractions and built up some new fixed point theorems for these types of contractions in F-metric spaces. Recently, Arif et al. [21] introduced an ordered implicit relation and investigated the existence of the fixed points of contractive mapping dealing with implicit relation in a cone b-metric space. Lately, in [22], some fixed point theorems of two new classes of multivalued almost contractions in a partial b-metric spaces were established by Anwar et al.

On the other hand, in 1969, Sehgal [23] formulated an inequality that can be considered an extension of the renowned Banach fixed point theorem in a metric space. Matkowski [24] generalized some previous results of Khazanchi [25] and Iseki [26]. In 2012, the definition of α-admissible mappings was given by Samet et al. [27]. Later, the notion of triangular α-admissible mappings was introduced by Popescu [28]. Recently, Lang and Guan [29] studied the common fixed point theory of $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction and $\alpha_{i,j}$-φ_{E_N}-Geraghty contractions in a b-metric space.

In this paper, inspired by [30], we established two fixed point theorems for Sehgal-Guseman-type mappings in a rectangular b-metric space. Also, we present two examples to illustrate the usability of established results.

2. Preliminaries

Definition 1 (see [8]). Suppose $\mathbb{G}$ is a nonempty set and $\varsigma : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$. We call ς a b-metric if

(i) $\varsigma(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\varsigma(\epsilon, \varpi) = \varsigma(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\varsigma(\epsilon, \varpi) \leq s[\varsigma(\epsilon, \gamma) + \varsigma(\gamma, \varpi)], \forall \epsilon, \varpi, \gamma \in \mathbb{G}$

where $s \geq 1$ is constant.

It is usual that $(\mathbb{G}, \varsigma)$ is called a b-metric space with parameter $s \geq 1$.

Definition 2 (see [3]). Suppose $\mathbb{G}$ is a nonempty set and $\tau : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$. We call τ a triangular metric if

(i) $\tau(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\tau(\epsilon, \varpi) = \tau(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\tau(\epsilon, \varpi) \leq \tau(\epsilon, \gamma) + \tau(\gamma, \epsilon) + \tau(\epsilon, \varpi), \forall \epsilon, \varpi \in \mathbb{G}, \gamma, \epsilon \in \mathbb{G} - \{\epsilon, \varpi\}$

Usually, $(\mathbb{G}, \tau)$ is called a rectangular metric space.

Definition 3 (see [16]). Suppose $\mathbb{G}$ is a nonempty set and $\upsilon : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$. We call υ a rectangular b-metric if

(i) $\upsilon(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\upsilon(\epsilon, \varpi) = \upsilon(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\upsilon(\epsilon, \varpi) \leq s[\upsilon(\epsilon, \gamma) + \upsilon(\gamma, \varepsilon) + \upsilon(\varepsilon, \varpi)], \forall \epsilon, \varpi \in \mathbb{G}, \gamma, \varepsilon \in \mathbb{G} - \{\epsilon, \varpi\}$

where $s \geq 1$ is constant.

In general, $(\mathbb{G}, \upsilon)$ is called a rectangular b-metric space with parameter $s \geq 1$.

Remark 4. A rectangular metric space is a rectangular b-metric space, so is a b-metric space. Moreover, the converse is not true.

Example 1. Suppose $\mathbb{G} = A \cup B$, where $A = \{0, 2/41, 3/61, 4/81\}$ and $B = \{1/2, 1/3, \cdots, 1/i, \cdots\}$. For $\epsilon, \varpi \in \mathbb{G}$, define $\upsilon : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ with $\upsilon(\epsilon, \varpi) = \upsilon(\varpi, \epsilon)$ and

$$\begin{cases} \upsilon\left(0, \dfrac{2}{41}\right) = \upsilon\left(\dfrac{2}{41}, \dfrac{3}{61}\right) = \upsilon\left(\dfrac{3}{61}, \dfrac{4}{81}\right) = 0.05, \\ \upsilon\left(0, \dfrac{3}{61}\right) = \upsilon\left(\dfrac{2}{41}, \dfrac{4}{81}\right) = 0.08, \\ \upsilon\left(0, \dfrac{4}{81}\right) = 0.3, \\ \upsilon(\epsilon, \varpi) = \max\{\epsilon, \varpi\}, \text{otherwise}. \end{cases} \tag{1}$$

Thus, $(\mathbb{G}, \upsilon)$ is a rectangular b-metric space with $s = 2$. Furthermore, one can obtain the following:

(1) υ is not a b-metric with $s = 2$, since

$$\begin{aligned} \upsilon\left(0, \frac{4}{81}\right) &= 0.3 > 0.26 = 2 \times 0.13 \\ &= 2 \times \left(\upsilon\left(0, \frac{2}{41}\right) + \upsilon\left(\frac{2}{41}, \frac{4}{81}\right)\right). \end{aligned} \tag{2}$$

(2) υ is not a rectangular metric, since

$$\begin{aligned} \upsilon\left(0, \frac{4}{81}\right) = 0.3 > 0.15 = \upsilon\left(0, \frac{2}{41}\right) \\ + \upsilon\left(\frac{2}{41}, \frac{3}{61}\right) + \upsilon\left(\frac{3}{61}, \frac{4}{81}\right). \end{aligned} \tag{3}$$

(3) υ is not a metric, since

$$\upsilon\left(0, \frac{4}{81}\right) = 0.3 > 0.13 = \upsilon\left(0, \frac{2}{41}\right) + \upsilon\left(\frac{2}{41}, \frac{4}{81}\right). \tag{4}$$

Definition 5 (see [16]). Suppose $(\mathbb{G}, \upsilon)$ is a rectangular b-metric space with $s \geq 1$. Assume that $\{\varpi_n\}$ in $\mathbb{G}$ is a sequence and $\varpi \in \mathbb{G}$

(i) $\{\varpi_n\}$ is convergent to ϖ iff $\lim_{n \longrightarrow +\infty} \upsilon(\varpi_n, \varpi) = 0$

(ii) $\{\varpi_n\}$ is Cauchy iff $\upsilon(\varpi_i, \varpi_j) \longrightarrow 0$ as $i, j \longrightarrow +\infty$

(iii) $(\mathbb{G}, \upsilon)$ is complete iff each Cauchy sequence is convergent

Remark 6. In a rectangular b-metric space, a convergent sequence does not possess unique limit and a convergent sequence is not necessarily a Cauchy sequence. However, one can find that the limit of a Cauchy sequence is unique.

In fact, suppose the sequence $\{\varpi_n\}$ is Cauchy and converges to ϖ^* and ϖ^{**} with $\varpi^* \neq \varpi^{**}$. It follows that

$$v(\varpi^*,\varpi^{**}) \le s\left[v(\varpi^*, \varpi_n) + v\left(\varpi_n, \varpi_{n+p}\right) + v\left(\varpi_{n+p},\varpi^{**}\right)\right], \quad (5)$$

for all $p > 0$. Let $n \longrightarrow \infty$; we get that $v(\varpi^*,\varpi^{**}) = 0$. Hence, $\varpi^* = \varpi^{**}$, a contradiction.

Example 2 (see [16]). Let $\mathbb{G} = A \cup B$, where $A = \{1/n : n \in \mathbb{N}\}$ and $B = \mathbb{N}$. Define $v : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ with $v(\epsilon, \varpi) = v(\varpi, \epsilon)$ and

$$v(\epsilon, \varpi) = \begin{cases} 0, & \text{if } \epsilon = \varpi, \\ 2\alpha, & \text{if } \epsilon, \varpi \in A, \\ \dfrac{\alpha}{2n}, & \text{if } \epsilon \in A \text{ and } \varpi \in \{2, 3\}, \\ \alpha, & \text{otherwise.} \end{cases} \quad (6)$$

Here, α is a positive number. Thus, v is a rectangular b-metric with $s = 2$. However, we have that $\{1/n\}$ is convergent to 2 and 3. Moreover, $\lim_{n\longrightarrow\infty} v(1/n, 1/(n + p)) = 2\alpha \neq 0$; therefore, $\{1/n\}$ is not a Cauchy sequence.

Definition 7 (see [28]). Suppose $\mathbb{G}$ is a nonempty set and $T : \mathbb{G} \longrightarrow \mathbb{G}$ and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$ are two mappings. We call $T\alpha$-orbital admissible mapping if

$$\forall \varpi \in \mathbb{G}, \alpha(\varpi, T\varpi) \ge 1 \Rightarrow \alpha\left(T\varpi, T^2\varpi\right) \ge 1. \quad (7)$$

Definition 8 (see [28]). Assume that $T : \mathbb{G} \longrightarrow \mathbb{G}$ and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$. We call T a triangular α-orbital admissible mapping if

(i) $\alpha(\epsilon, \varpi) \ge 1$ and $\alpha(\varpi, T\varpi) \ge 1$ imply $\alpha(\epsilon, T\varpi) \ge 1$, $\forall \epsilon, \varpi \in \mathbb{G}$

(ii) T is α-orbital admissible

Lemma 9 (see [24]). *Assume $\Theta : [0,+\infty) \longrightarrow [0,+\infty)$ is an increasing mapping. Then, $\forall t > 0, \lim_{n\longrightarrow\infty} \Theta^n(t) = 0 \Rightarrow \Theta(t) < t$.*

3. Main Results

In this part, two fixed point results of injective mappings will be presented on rectangular b-metric spaces.

Definition 10. Suppose $\mathbb{G}$ is a nonempty set, $s \ge 1$ and $p > 0$ are two constants, and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$, $T : \mathbb{G} \longrightarrow \mathbb{G}$. We call $T\alpha_{s^p}$ orbital admissible mapping if

$$\forall \varpi \in \mathbb{G}, \alpha(\varpi, T\varpi) \ge s^p \Rightarrow \alpha\left(T\varpi, T^2\varpi\right) \ge s^p. \quad (8)$$

Definition 11. Suppose $\mathbb{G}$ is a nonempty set, $s \ge 1$ and $p > 0$ are two constants, and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$, $T : \mathbb{G} \longrightarrow \mathbb{G}$. We call T triangular α_{s^p} orbital admissible mapping if

(i) $\alpha(\epsilon, \varpi) \ge s^p$ and $\alpha(\varpi, T\varpi) \ge s^p$ imply $\alpha(\epsilon, T\varpi) \ge s^p$, $\forall \epsilon, \varpi \in \mathbb{G}$

(ii) T is α_{s^p} orbital admissible

Lemma 12. *Suppose $\mathbb{G}$ is a nonempty set and $T : \mathbb{G} \longrightarrow \mathbb{G}$, $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ are mappings satisfying T which is triangular α_{s^p} orbital admissible, $s \ge 1, p > 0$. Suppose there has a $\varpi_0 \in \mathbb{G}$ with $\alpha(\varpi_0, T\varpi_0) \ge s^p$. Define $\{\varpi_n\}$ in $\mathbb{G}$ by $\varpi_1 = T^{n(\varpi_0)}\varpi_0, \cdots, \varpi_{n+1} = T^{n(\varpi_n)}\varpi_n, \cdots$. Then, $\forall m \in \mathbb{N} \cup \{0\}$, $\alpha(\varpi_m, T^k\varpi_m) \ge s^p, k = 0, 1, 2, \cdots$.*

Proof. Since $\alpha(\varpi_0, T\varpi_0) \ge s^p$ and T is triangular α_{s^p} orbital admissible, we have

$$\begin{aligned} \alpha(\varpi_0, T\varpi_0) \ge s^p \text{ implies } \alpha\left(T\varpi_0, T^2\varpi_0\right) \\ \ge s^p \text{ and } \alpha\left(\varpi_0, T^2\varpi_0\right) \ge s^p. \end{aligned} \quad (9)$$

Similarly, since $\alpha(T\varpi_0, T^2\varpi_0) \ge s^p$, we get

$$\alpha\left(T^2\varpi_0, T^3\varpi_0\right) \ge s^p, \quad (10)$$

$$\alpha\left(\varpi_0, T^3\varpi_0\right) \ge s^p. \quad (11)$$

Applying the above argument repeatedly, one can deduce that $\alpha(\varpi_0, T^k\varpi_0) \ge s^p$ for all $k \in \mathbb{N} \cup \{0\}$. Since $\alpha(\varpi_0, T\varpi_0) \ge s^p$ implies $\alpha(T\varpi_0, T^2\varpi_0) \ge s^p$ and $\alpha(T\varpi_0, T^2\varpi_0) \ge s^p$ implies $\alpha(T^2\varpi_0, T^3\varpi_0) \ge s^p, \cdots$, we can obtain $\alpha(T^{n(\varpi_0)}\varpi_0, T^{n(\varpi_0)+1}\varpi_0) = \alpha(\varpi_1, T\varpi_1) \ge s^p$. Based on this conclusion, we deduce that $\alpha(\varpi_1, T^k\varpi_1) \ge s^p, k = 0, 1, 2, \cdots$. Repeatedly using the above discussion, we have $\alpha(\varpi_m, T^k\varpi_m) \ge s^p, k = 0, 1, 2, \cdots$ for all $m \in \mathbb{N} \cup \{0\}$. □

Define $\Theta = \{\Phi : \mathbb{R}^{+3} \longrightarrow \mathbb{R}^+$ is increasing and continuous in each coordinate variable$\}$. That is, if $\kappa_1^{(1)}, \kappa_2^{(1)}, \kappa_1^{(2)}, \kappa_2^{(2)}, \kappa_1^{(3)}, \kappa_2^{(3)} \in \mathbb{R}^+$ with $\kappa_1^{(1)} \le \kappa_2^{(1)}, \kappa_1^{(2)} \le \kappa_2^{(2)}, \kappa_1^{(3)} \le \kappa_2^{(3)}$, we have

$$\begin{aligned} \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\le \Phi\left(\kappa_2^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right), \\ \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\le \Phi\left(\kappa_1^{(1)}, \kappa_2^{(2)}, \kappa_1^{(3)}\right), \\ \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\le \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_2^{(3)}\right). \end{aligned} \quad (12)$$

Furthermore, we set $\Phi(\epsilon, \epsilon, \epsilon) = \varphi(\epsilon)$ for $\varepsilon \in \mathbb{R}^+$.

Theorem 13. *Suppose $(\mathbb{G}, v)$ is a complete rectangular b-metric space with $s \ge 1$. Suppose $T : \mathbb{G} \longrightarrow \mathbb{G}$ is a continuous injectivity, $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$ and $p > 0$. Assume that for any $\epsilon \in \mathbb{G}$, there is a positive number $n(\epsilon)$ satisfying*

$$\forall \varpi \in \mathbb{G}, \alpha(\epsilon, \varpi) \geq s^p \Rightarrow \alpha(\epsilon, \varpi) v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \leq \Phi\left(v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right), \tag{13}$$

where $\Phi \in \Theta$ and

(1) $\lim_{\epsilon \longrightarrow \infty}(\epsilon - s\varphi(\epsilon)) = \infty$

(2) $\forall \epsilon > 0, \lim_{m \longrightarrow \infty} \varphi^m(\epsilon) = 0$

Suppose that

(i) there has a ϵ_0 in $\mathbb{G}$ such that $\alpha(\epsilon_0, T\epsilon_0) \geq s^p$

(ii) T is triangular α_{s^p} orbital admissible

(iii) if $\{\varpi_n\}$ in $\mathbb{G}$ satisfies $\alpha(\varpi_n, \varpi_{n+1}) \geq s^p (\forall n \in \mathbb{N})$ and $\varpi_n \longrightarrow \varpi \in \mathbb{G} (n \longrightarrow \infty)$, then one can choose a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ with $\alpha(\varpi_{n_k}, \varpi) \geq s^p, \forall k \in \mathbb{N}$

(iv) $\forall \epsilon \in \mathbb{G}$ with $T^{n(\epsilon)}\epsilon = \epsilon$, we have $\alpha(\epsilon, \varpi) \geq s^p$ for any $\varpi \in \mathbb{G}$

Then, T possesses a unique fixed point $\epsilon^* \in \mathbb{G}$. Further, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ converges to ϵ^*

Proof. By condition (i), one can choose an $\epsilon_0 \in \mathbb{G}$ such that $\alpha(\epsilon_0, T\epsilon_0) \geq s^p$. If ϵ_0 is a fixed point of T and ϖ_0 is the other one, then $\epsilon_0 = T\epsilon_0 = \cdots = T^{n(\epsilon_0)}\epsilon_0 = \cdots$ and $\varpi_0 = T\varpi_0 = \cdots = T^{n(\epsilon_0)}\varpi_0 = \cdots$. From condition (iv), we have $\alpha(\epsilon_0, \varpi_0) \geq s^p$. It follows from (13) that

$$\begin{aligned} v(\epsilon_0, \varpi_0) &\leq \alpha(\epsilon_0, \varpi_0) v\left(T^{n(\epsilon_0)}\epsilon_0, T^{n(\epsilon_0)}\varpi_0\right) \\ &\leq \Phi\left(v(\epsilon_0, \varpi_0), v\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), v\left(\epsilon_0, T^{n(\epsilon_0)}\varpi_0\right)\right) \\ &\leq \varphi(v(\epsilon_0, \varpi_0)). \end{aligned} \tag{14}$$

From Lemma 9, we have $\varphi(v(\epsilon_0, \varpi_0)) < v(\epsilon_0, \varpi_0)$. Thus,

$$v(\epsilon_0, \varpi_0) \leq \varphi(v(\epsilon_0, \varpi_0)) < v(\epsilon_0, \varpi_0), \tag{15}$$

which is contradiction. From this, we get that ϵ_0 is the unique fixed point. After that, in the subsequent discussion, we assume that $T\epsilon_0 \neq \epsilon_0$. Now we define $\{\epsilon_n\}$ in $\mathbb{G}$ by $\epsilon_1 = T^{n(\epsilon_0)}\epsilon_0, \cdots, \epsilon_{n+1} = T^{n(\epsilon_n)}\epsilon_n$.

First, we shall show that the orbit $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded. For this purpose, we fix an integer $\ell, 0 \leq \ell < n(\epsilon_0)$. Let

$$u_j = v\left(\epsilon_0, T^{jn(\epsilon_0)+\ell}\epsilon_0\right), j = 0, 1, 2, \cdots, \tag{16}$$

$$h = \max\left\{u_0, v\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), v\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right), v\left(T^{n(\epsilon_0)}\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right)\right\}. \tag{17}$$

Since $\lim_{\epsilon \longrightarrow \infty}(\epsilon - s\varphi(\epsilon)) = \infty$, there has $c > h$ such that $\epsilon - s\varphi(\epsilon) > 2sh, \epsilon \geq c$. It is obvious that $u_0 \leq h < c$. Assume that there has a positive number j_0 with $u_{j_0} \geq c$. Evidently, one may suppose that $u_i < c, \forall i < j_0$. Let $\epsilon_0, T^{n(\epsilon_0)}\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$ be different from each other. Otherwise, we consider six cases.

Case 1. $\epsilon_0 = T^{n(\epsilon_0)}\epsilon_0$. One can get that

$$\epsilon_0 = T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = \cdots. \tag{18}$$

It follows that $u_j = v(\epsilon_0, T^\ell\epsilon_0)$ is a constant which implies that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 2. $\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0$. We deduce that

$$\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{4n(\epsilon_0)}\epsilon_0 = T^{6n(\epsilon_0)}\epsilon_0 = \cdots, \tag{19}$$

$$T^{n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = T^{5n(\epsilon_0)}\epsilon_0 = \cdots. \tag{20}$$

Hence,

$$u_j = \begin{cases} v\left(\epsilon_0, T^{n(\epsilon_0)+\ell}\epsilon_0\right), & j \text{ is odd}, \\ v\left(\epsilon_0, T^\ell\epsilon_0\right), & j \text{ is even}. \end{cases} \tag{21}$$

It follows that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 3. $T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0$. Obviously,

$$T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = T^{4n(\epsilon_0)}\epsilon_0 = \cdots. \tag{22}$$

As the argument of Case 1, we get that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 4. $\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. In this case, we obtain that $u_{j_0} = 0$, a contradiction.

Case 5. $T^{n(\epsilon_0)}\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. It follows that

$$u_{j_0} = v\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) = v\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right) \leq h < c. \tag{23}$$

It is a contradiction.

Case 6. $T^{2n(\epsilon_0)}\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. It is obvious that

$$u_{j_0} = v\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) = v\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right) \leq h < c, \tag{24}$$

a contradiction.

It is easy to get $\alpha(\epsilon_0, T^k\epsilon_0) \geq s^p, \forall k \in \mathbb{N}$ from Lemma 12. By using triangle inequality and (16), we have

$$\begin{aligned}
&\upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) \\
&\quad \leq s\left[\upsilon\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right) + \upsilon\left(T^{2n(\epsilon_0)}\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right)\right. \\
&\qquad \left. + \upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{j_0(\epsilon_0)+\ell}\epsilon_0\right]\right. \\
&\quad \leq 2sh + s\alpha\left(\epsilon_0, T^{(j_0-1)n(\epsilon_0)+\ell}\epsilon_0\right)\upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) \\
&\quad \leq 2sh + s\Phi\left(\upsilon\left(\epsilon_0, T^{(j_0-1)n(\epsilon_0)+\ell}\epsilon_0\right),\right. \\
&\qquad \left.\upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), \upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right)\right) \\
&\quad \leq 2sh + s\Phi\left(u_{j_0}, u_{j_0}, u_{j_0}\right) = 2sh + s\varphi\left(u_{j_0}\right).
\end{aligned} \tag{25}$$

That is, $u_{j_0} - s\varphi(u_{j_0}) \leq 2sh$, which is impossible. Therefore, $u_j < c$ for $j = 0, 1, 2, \cdots$. It follows that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

If there exists some $n_0 \in \mathbb{N}$ satisfying $\epsilon_{n_0} = \epsilon_{n_0+1} = T^{n(\epsilon_{n_0})}\epsilon_{n_0}$, then ϵ_{n_0} is a fixed point of $T^{n(\epsilon_{n_0})}$. Assume there is $\varpi \in \mathbb{G}$ such that $\varpi = T^{n(\epsilon_{n_0})}\varpi$ and $\varpi \neq \epsilon_{n_0}$, by condition (iv), we have $\alpha(\epsilon_{n_0}, \varpi) \geq s^p$ and

$$\begin{aligned}
\upsilon(\epsilon_{n_0}, \varpi) &\leq \alpha(\epsilon_{n_0}, \varpi)\upsilon\left(T^{n(\epsilon_{n_0})}\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\varpi\right) \\
&\leq \Phi\left(\upsilon(\epsilon_{n_0}, \varpi), \upsilon\left(\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\epsilon_{n_0}\right), \upsilon\left(\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\varpi\right)\right) \\
&\leq \varphi(\upsilon(\epsilon_{n_0}, \varpi)) < \upsilon(\epsilon_{n_0}, \varpi),
\end{aligned} \tag{26}$$

which is contradiction. From this, $T^{n(\epsilon_{n_0})}$ possesses the unique fixed point ϵ_{n_0}. Since $T\epsilon_{n_0} = TT^{n(\epsilon_{n_0})}\epsilon_{n_0} = T^{n(\epsilon_{n_0})}T\epsilon_{n_0}$, we have $T\epsilon_{n_0} = \epsilon_{n_0}$ because of the uniqueness of $T^{n(\epsilon_{n_0})}$. Subsequently, we assume that $\epsilon_n \neq \epsilon_{n+1}, \forall n \in \mathbb{N}$.

Next, we show that $\{\epsilon_n\}$ is Cauchy. Suppose n and i are two positive numbers. It is obvious that $\alpha(\epsilon_{n-1}, T^k\epsilon_{n-1}) \geq s^p, \forall k \in \mathbb{N}$. Then,

$$\begin{aligned}
\upsilon(\epsilon_n, \epsilon_{n+i}) &\leq \alpha\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+n(\epsilon_{n+i-2})+\cdots+n(\epsilon_n)}\epsilon_{n-1}\right) \\
&\quad \cdot \upsilon\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+\cdots+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\
&\leq \Phi\left(\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+n(\epsilon_{n+i-2})+\cdots+n(\epsilon_n)}\epsilon_{n-1}\right),\right. \\
&\quad \left.\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right), \upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+\cdots+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right) \\
&\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right\}\right).
\end{aligned} \tag{27}$$

For each $q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}$, we have

$$\begin{aligned}
\upsilon(\epsilon_{n-1}, q) &= \upsilon(\epsilon_{n-1}, T^m\epsilon_{n-1}) \\
&\leq \alpha(\epsilon_{n-2}, T^m\epsilon_{n-2})\upsilon\left(T^{n(\epsilon_{n-2})}\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right) \\
&\leq \Phi\left(\upsilon(\epsilon_{n-2}, T^m\epsilon_{n-2}), \upsilon\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})}\epsilon_{n-2}\right),\right. \\
&\quad \left.\upsilon\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})+m}\epsilon_{n-2}\right)\right) \\
&\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=0}^{\infty}\right\}.\right.
\end{aligned} \tag{28}$$

According to (27) and (28), we deduce

$$\begin{aligned}
\upsilon(\epsilon_n, \epsilon_{n+i}) &\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right)\right. \\
&\leq \cdots \leq \varphi^n\left(\sup\left\{\upsilon(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=0}^{\infty}\right\}\right) \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{29}$$

That is, $\{\epsilon_n\}$ is Cauchy. In light of the completeness of $(\mathbb{G}, \upsilon)$, one can find an $\epsilon^* \in \mathbb{G}$ with $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon^*$. We might as well let $\epsilon_n \neq \epsilon^*$ and $\epsilon_n \neq T^{n(\epsilon^*)}\epsilon_n$. Otherwise, we have $\epsilon^* = T^{n(\epsilon^*)}\epsilon^*$ according to the continuity of T. In view of triangle inequality, one deduce

$$\begin{aligned}
&\upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right) \\
&\quad \leq s\left[\upsilon(\epsilon^*, \epsilon_n) + \upsilon\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) + \upsilon\left(T^{n(\epsilon^*)}\epsilon_n, T^{n(\epsilon^*)}\epsilon^*\right)\right].
\end{aligned} \tag{30}$$

On the other hand,

$$\begin{aligned}
&\upsilon\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) \\
&\quad \leq \alpha\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right)\upsilon\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\
&\quad \leq \Phi\left(\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right),\right. \\
&\qquad \left.\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right) \\
&\quad \leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right\}\right) \\
&\quad \leq \cdots \leq \varphi^n\left(\sup\left\{\upsilon(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=0}^{\infty}\right\}\right) \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{31}$$

From the continuity of T, $\lim_{n\longrightarrow\infty}\upsilon(T^{n(\epsilon^*)}\epsilon_n, T^{n(\epsilon^*)}\epsilon^*) = 0$. Thereupon, by the use of (30) and (31), one can obtain $\upsilon(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*) = 0$ as $n \longrightarrow \infty$. Assume there exists $\varpi^* \neq \epsilon^*$ satisfying $\varpi^* = T^{n(\epsilon^*)}\varpi^*$ and we have $\alpha(\epsilon^*, \varpi^*) \geq s^p$ according to the condition (iv). Then,

$$\begin{aligned}
\upsilon(\epsilon^*, \varpi^*) &\leq \alpha(\epsilon^*, \varpi^*)\upsilon\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right) \\
&\leq \Phi\left(\upsilon(\epsilon^*, \varpi^*), \upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right)\right) \\
&\leq \varphi(\upsilon(\epsilon^*, \varpi^*)) < \upsilon(\epsilon^*, \varpi^*),
\end{aligned} \tag{32}$$

impossible. After that, $T^{n(\epsilon^*)}$ has the unique fixed point ϵ^*. Since $T\epsilon^* = TT^{n(\epsilon^*)}\epsilon^* = T^{n(\epsilon^*)}T\epsilon^*$, we deduce $T\epsilon^* = \epsilon^*$. That is, T has a fixed point.

Now we show that if condition (iv) is met. So T possesses a unique fixed point. Assume ϖ^* is another one; from condition (iv), one can obtain $\alpha(\epsilon^*, \varpi^*) \geq s^p$. In view of (13), we have

$$\begin{aligned} v(\epsilon^*, \varpi^*) &\leq \alpha(\epsilon^*, \varpi^*)v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right) \\ &\leq \Phi\left(v(\epsilon^*, \varpi^*), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right)\right) \\ &\leq \varphi(v(\epsilon^*, \varpi^*)). \end{aligned} \tag{33}$$

Lemma 9 ensures that $\varphi(v(\epsilon^*, \varpi^*)) < v(\epsilon^*, \varpi^*)$. Thus,

$$v(\epsilon^*, \varpi^*) \leq \varphi(v(\epsilon^*, \varpi^*)) < v(\epsilon^*, \varpi^*), \tag{34}$$

which is impossible. It follows that ϵ^* is the unique fixed point of T.

Finally, we prove the last part. To show this statement, we fix an integer ℓ, $0 \leq \ell < n(\epsilon^*)$, and let $v_k = v(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon)$, $k = 0, 1, 2, \cdots$ for $\epsilon \in \mathbb{G}$. If there exists $k \in \mathbb{N}$ satisfying $v_k = 0$, we have

$$\begin{aligned} v_{k+1} &= v\left(\epsilon^*, T^{(k+1)n(\epsilon^*)+\ell}\epsilon\right) \\ &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \alpha\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right)v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \Phi(v_k, 0, v_{k+1}). \end{aligned} \tag{35}$$

If $v_{k+1} > 0$, one can obtain that $v_{k+1} \leq \Phi(v_{k+1}, v_{k+1}, v_{k+1}) = \varphi(v_{k+1}) < v_{k+1}$, which is a contradiction. Hence, $v_{k+1} = 0$. It follows that $v_{k+2} = v_{k+3} = \cdots = 0$.

Now we suppose that $v_k \neq 0$, $\forall n \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} v\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right) &\leq \alpha\left(\epsilon^*, T^{(k-1)n(\epsilon^*)+\ell}\epsilon\right)v\left(T^{n(\epsilon^*)}\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \Phi\left(v\left(\epsilon^*, T^{(k-1)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right)\right) \\ &= \Phi(v_{k-1}, 0, v_k). \end{aligned} \tag{36}$$

If for some $k \in \mathbb{N}$, $v_k \geq v_{k-1}$, we deduce $v_k \leq \Phi(v_k, v_k, v_k) = \varphi(v_k) < v_k$, which is a contradiction. Hence, we get $v_k \leq \varphi(v_{k-1}) \leq \cdots \leq \varphi^k(v_0) \longrightarrow 0 \quad (k \longrightarrow \infty)$. That is, for ℓ, the sequence $\{T^{kn(\epsilon^*)+\ell}\epsilon\}$ converges to ϵ^* for any $\epsilon \in \mathbb{G}$. Consequently, one can obtain that the sequences $\{T^{kn(\epsilon^*)}\epsilon\}$, $\{T^{kn(\epsilon^*)+1}\epsilon\}, \{T^{kn(\epsilon^*)+2}\epsilon\}, \cdots, \{T^{kn(\epsilon^*)+n(\epsilon^*)-1}\epsilon\}$ are convergent to the point ϵ^*. It follows that we get $\{T^n\epsilon\}$ converges to the point ϵ^* for $\epsilon \in \mathbb{G}$. □

Example 3. Let $(\mathbb{G}, v)$ be the same as it is in Example 1. Define $T : \mathbb{G} \longrightarrow \mathbb{G}$ as

$$T\epsilon = \begin{cases} 0, & \epsilon = 0, \\ \frac{2}{41}, & \epsilon = \frac{1}{2}, \\ \frac{3}{61}, & \epsilon = \frac{1}{3}, \\ \frac{4}{81}, & \epsilon = \frac{1}{4}, \\ \frac{1}{2^2 \cdot 2}, & \epsilon = \frac{2}{41}, \\ \frac{1}{2^2 \cdot 3}, & \epsilon = \frac{3}{61}, \\ \frac{1}{2^2 \cdot 4}, & \epsilon = \frac{4}{81}, \\ \frac{1}{2^2 \cdot \chi}, & \epsilon = \frac{1}{\chi}, \chi \geq 5. \end{cases} \tag{37}$$

Define mapping $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ by

$$\alpha(\epsilon, \varpi) = \begin{cases} s^p, & \epsilon, \varpi \in \{0\} \cup \left\{\frac{1}{\chi}, \chi \geq 5\right\}, \\ 0, & \text{otherwise}. \end{cases} \tag{38}$$

Define $\Phi(\kappa_1, \kappa_2, \kappa_3) = (1/12)(\kappa_1 + \kappa_2 + \kappa_3)$ for all $\kappa_i \in [0, +\infty)(i = 1, 2, 3)$, and it follows that $\varphi(t) = (1/4)t$. Let $n(\epsilon) = 3$ for all $\epsilon \in \mathbb{G}$. For $\epsilon, \varpi \in \mathbb{G}$ such that $\alpha(\epsilon, \varpi) \geq s^p$, we get that $\epsilon, \varpi \in \{0\} \cup \{1/\chi, \chi \geq 5\}$. It follows that we consider the following two cases:

(i) $\epsilon = 0$ and $\varpi \in \{1/\chi, \chi \geq 5\}$

$$\begin{aligned} &\alpha(\epsilon, \varpi)v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\ &\quad = 4 \cdot v\left(T^3(0), T^3\left(\frac{1}{\chi}\right)\right) = \frac{1}{16\chi}, \\ &\Phi\left(v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right) \\ &\quad = \frac{1}{12} \cdot \left[v\left(0, \frac{1}{\chi}\right) + v\left(0, T^3(0)\right) + v\left(0, T^3\left(\frac{1}{\chi}\right)\right)\right] \\ &\quad = \frac{1}{12} \cdot \left(\frac{1}{\chi} + \frac{1}{64\chi}\right) > \frac{1}{12\chi}. \end{aligned} \tag{39}$$

That is, $\alpha(\epsilon, \varpi)v(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi) \leq \Phi(v(\epsilon, \varpi), v(\epsilon, T^{n(\epsilon)}\epsilon), v(\epsilon, T^{n(\epsilon)}\varpi))$.

(ii) $\epsilon, \varpi \in \{1/\chi, r \geq 5\}$. Let $\epsilon = 1/\chi$ and $\varpi = 1/l$ with $l \geq \chi$. One can obtain that

$$\begin{aligned}
&\alpha(\epsilon, \varpi)\upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\
&\quad = 4 \cdot \upsilon\left(T^3\left(\frac{1}{\chi}\right), T^3\left(\frac{1}{l}\right)\right) = \frac{1}{16\chi}, \\
&\Phi\left(\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right) \\
&\quad = \frac{1}{12} \cdot \left[\upsilon\left(\frac{1}{\chi}, \frac{1}{l}\right) + \upsilon\left(\frac{1}{\chi}, T^3\left(\frac{1}{\chi}\right)\right)\right. \\
&\qquad \left. + \upsilon\left(\frac{1}{\chi}, T^3\left(\frac{1}{l}\right)\right)\right] = \frac{1}{4\chi}.
\end{aligned} \tag{40}$$

The above inequalities imply that

$$\begin{aligned}
&\alpha(\epsilon, \varpi)\upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\
&\quad \leq \Phi\left(\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right).
\end{aligned} \tag{41}$$

Thus, all conditions of Theorem 13 are fulfilled with $p = s = 2$. As a result, T possesses a unique fixed point 0. Meanwhile, for each $\epsilon \in \mathbb{G}$, $\{T^n\epsilon\}$ converges to the point 0.

Remark 14.

(1) Since rectangular metric spaces can be seen as rectangular b -metric spaces with parameter $s = 1$, one can get the corresponding conclusions of Sehgal-Guseman-type mappings in rectangular metric spaces

(2) Since b-metric spaces with parameter s can be seen as rectangular b-metric spaces with parameter s^2, one can obtain the corresponding conclusions of Sehgal-Guseman-type mappings in b-metric spaces

(3) If $\alpha(x, y) = s^p$, one can get the generalized Φ-Sehgal-Guseman-type contractive mappings in rectangular b-metric spaces

Theorem 15. *Suppose $(\mathbb{G}, \upsilon)$ is a complete rectangular b -metric space with $s \geq 1$. Suppose $T : \mathbb{G} \longrightarrow \mathbb{G}$ is a continuous injectivity and $\psi : [0, +\infty) \longrightarrow [0, 1/2s)$ satisfying that for any $\epsilon \in \mathbb{G}$; there is a positive number $n(\epsilon)$ satisfying*

$$\upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \leq \psi(M(\epsilon, \varpi))M(\epsilon, \varpi), \forall \varpi \in \mathbb{G}, \tag{42}$$

where

$$M(\epsilon, \varpi) = \max\left\{\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right\}. \tag{43}$$

Then, T possesses a unique fixed point ϵ^*. Furthermore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to ϵ^*.

Proof. Let $\epsilon_0 \in \mathbb{G}$. Consider a sequence $\{\epsilon_n\}$ in $\mathbb{G}$ by $\epsilon_1 = T^{n(\epsilon_0)}\epsilon_0, \cdots, \epsilon_{n+1} = T^{n(\epsilon_n)}\epsilon_n$. If $\epsilon_{n_0} = \epsilon_{n_0+1} = T^{n(\epsilon_{n_0})}\epsilon_{n_0}$ for an $n_0 \in \mathbb{N}$, then ϵ_{n_0} becomes to a fixed point of $T^{n(\epsilon_{n_0})}$. Assume there exists $\varpi \in \mathbb{G}$ with $\varpi = T^{n(\epsilon_{n_0})}\varpi$ and $\varpi \neq \epsilon_{n_0}$; then,

$$\upsilon\left(\epsilon_{n_0}, \varpi\right) = \upsilon\left(T^{n\left(\epsilon_{n_0}\right)}\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\varpi\right) \leq \psi\left(M\left(\epsilon_{n_0}, \varpi\right)\right)M\left(\epsilon_{n_0}, \varpi\right), \tag{44}$$

where

$$\begin{aligned}
M\left(\epsilon_{n_0}, \varpi\right) = \max&\left\{\upsilon\left(\epsilon_{n_0}, \varpi\right), \upsilon\left(\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\epsilon_{n_0}\right),\right. \\
&\left.\upsilon\left(\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\varpi\right)\right\} = \upsilon\left(\epsilon_{n_0}, \varpi\right) > 0.
\end{aligned} \tag{45}$$

From this, we get $\upsilon(\epsilon_{n_0}, \varpi) < (1/2s)\upsilon(\epsilon_{n_0}, \varpi)$ which is impossible. Therefore, ϵ_{n_0} is the unique fixed point of $T^{n(\epsilon_{n_0})}$. Since $T\epsilon_{n_0} = T^{n(\epsilon_{n_0})}T\epsilon_{n_0}$, we have $T\epsilon_{n_0} = \epsilon_{n_0}$ because of the uniqueness of $T^{n(\epsilon_{n_0})}$. Subsequently, we assume that $\epsilon_n \neq \epsilon_{n+1}$, $\forall n \in \mathbb{N}$.

For $\epsilon \in \mathbb{G}$, set $z(\epsilon) = \max\{\upsilon(\epsilon, T^k\epsilon), k = 1, 2, \cdots, n(\epsilon), n(\epsilon) + 1, \cdots, 2n(\epsilon)\}$. We first prove that $r(\epsilon) = \sup \upsilon(\epsilon, T^n\epsilon) < \infty$ for all $n \in \mathbb{N}$. Assume $n > n(\epsilon)$ is a positive number satisfying $n = rn(\epsilon) + \ell, r \geq 1, 0 \leq \ell < n(\epsilon)$ and $\delta_r(\epsilon) = \upsilon(\epsilon, T^{rn(\epsilon)+\ell}\epsilon), r = 0, 1, 2, \cdots$. We suppose that $\epsilon, T^{n(\epsilon)}\epsilon, T^{2n(\epsilon)}\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon$ are four distinct elements. Otherwise, the conclusion is true. Thus,

$$\begin{aligned}
\upsilon(\epsilon, T^n\epsilon) &= \upsilon\left(\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right) \\
&\leq s\left[\upsilon\left(\epsilon, T^{2n(\epsilon)}\epsilon\right) + \upsilon\left(T^{2n(\epsilon)}\epsilon, T^{n(\epsilon)}\epsilon\right)\right. \\
&\quad \left. + \upsilon\left(T^{n(\epsilon)}\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right)\right] \\
&\leq s\left[z(\epsilon) + \psi\left(M\left(\epsilon, T^{n(\epsilon)}\epsilon\right)\right)M\left(\epsilon, T^{n(\epsilon)}\epsilon\right)\right. \\
&\quad \left. + \psi\left(M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right)\right)M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right)\right],
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&M\left(\epsilon, T^{n(\epsilon)}\epsilon\right) \\
&\quad = \max\left\{\upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{2n(\epsilon)}\epsilon\right)\right\} = z(\epsilon),
\end{aligned} \tag{47}$$

$$\begin{aligned}
&M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right) \\
&\quad = \max\left\{\upsilon\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right)\right\} \\
&\quad \leq \max\left\{\delta_{r-1}(\epsilon), z(\epsilon), \delta_r(\epsilon)\right\}.
\end{aligned} \tag{48}$$

By (46), (47), and (48), we deduce

$$\delta_r(\epsilon) \le s\left[z(\epsilon) + \frac{1}{2s}z(\epsilon) + \frac{1}{2s}\max\{\delta_{r-1}(\epsilon), z(\epsilon), \delta_r(\epsilon)\}\right]. \tag{49}$$

Hence, one can conclude that $(1/(1+2s))\delta_r(\epsilon) \le z(\epsilon)$ by induction. Indeed, when $r=1$, we have $\delta_1(\epsilon) \le ((1+2s)/2)z(\epsilon) + (1/2)\max\{z(\epsilon), \delta_1(\epsilon)\}$. If $\delta_1(\epsilon) \ge z(\epsilon)$, we get $\delta_1(\epsilon) \le (1+2s)z(\epsilon)$. If $\delta_1(\epsilon) < z(\epsilon)$, we get $\delta_1(\epsilon) \le (1+s)z(\epsilon) < (1+2s)z(\epsilon)$. We assume $\delta_r(\epsilon) \le (1+2s)z(\epsilon)$; then, $\delta_{r+1}(\epsilon) \le ((1+2s)/2)z(\epsilon) + (1/2)\max\{(1+2s)z(\epsilon), z(\epsilon), \delta_{r+1}(\epsilon)\} \le (1+2s)z(\epsilon)$. Hence, $r(\epsilon) = \sup d(T^n\epsilon, \epsilon) < \infty$.

Next, we prove that $\lim_{n\longrightarrow\infty} v(\epsilon_n, \epsilon_{n+1}) = 0$. By contractive condition (42), we have

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+1}) &= v\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon_n)+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\ &\le \psi\left(M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right)\right)M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right), \end{aligned} \tag{50}$$

where

$$\begin{aligned} &M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right) \\ &\quad = \max\left\{v\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right), v\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right),\right. \\ &\qquad \left. v\left(\epsilon_{n-1}, T^{n(\epsilon_n)+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right\} \\ &\quad \le \sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\}. \end{aligned} \tag{51}$$

It is obvious that $M(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}) > 0$, so

$$v(\epsilon_n, \epsilon_{n+1}) < \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\}. \tag{52}$$

For each $q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}$, we have

$$\begin{aligned} v(\epsilon_{n-1}, q) &= v(\epsilon_{n-1}, T^m\epsilon_{n-1}) \\ &= v\left(T^{n(\epsilon_{n-2})}\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right) \\ &\le \psi(M(\epsilon_{n-2}, T^m\epsilon_{n-2}))M(\epsilon_{n-2}, T^m\epsilon_{n-2}), \end{aligned} \tag{53}$$

where

$$\begin{aligned} &M(\epsilon_{n-2}, T^m\epsilon_{n-2}) \\ &\quad = \max\left\{v(\epsilon_{n-2}, T^m\epsilon_{n-2}), v\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})}\epsilon_{n-2}\right),\right. \\ &\qquad \left. v\left(\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right)\right\} \\ &\quad \le \sup\left\{v(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=1}^{\infty}\right\} > 0. \end{aligned} \tag{54}$$

It means $v(\epsilon_{n-1}, q) < (1/2s)\sup\{v(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=1}^{\infty}\}$. So we deduce

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+1}) &< \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\} \\ &< \cdots < \frac{1}{(2s)^n}\sup\left\{v(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\right\} \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{55}$$

That is, $\lim_{n\longrightarrow\infty} v(\epsilon_n, \epsilon_{n+1}) = 0$.

For the sequence $\{\epsilon_n\}$, we consider $v(\epsilon_n, \epsilon_{n+p})$ by the following cases. For the sake of convenience, set $r_0 = \sup\{v(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\}$.

If p is odd, assume $p = 2m+1$,

$$\begin{aligned} &v(\epsilon_n, \epsilon_{n+2m+1}) \\ &\quad \le s[v(\epsilon_n, \epsilon_{n+1}) + v(\epsilon_{n+1}, \epsilon_{n+2}) + v(\epsilon_{n+2}, \epsilon_{n+2m+1})] \\ &\quad < s\left[\frac{1}{(2s)^n}r_0 + \frac{1}{(2s)^{n+1}}r_0\right] + s^2[v(\epsilon_{n+2}, \epsilon_{n+3}) \\ &\qquad + v(\epsilon_{n+3}, \epsilon_{n+4}) + v(\epsilon_{n+4}, \epsilon_{n+2m+1})] \\ &\quad < \cdots < s\frac{1}{(2s)^n}r_0 + s\frac{1}{(2s)^{n+1}}r_0 + s^2\frac{1}{(2s)^{n+2}}r_0 \\ &\qquad + s^2\frac{1}{(2s)^{n+3}}r_0 + \cdots + s^m\frac{1}{(2s)^{n+2m}}r_0 \\ &\quad \le \frac{s}{(2s)^n}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 + s\frac{1}{(2s)^{n+1}}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 \\ &\quad \le \frac{s}{(2s)^n}\cdot\frac{1+(1/2s)}{1-(1/4s)}r_0 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{56}$$

If p is even, assume $p = 2m$,

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+2m}) &\le s[v(\epsilon_n, \epsilon_{n+1}) + v(\epsilon_{n+1}, \epsilon_{n+2}) + v(\epsilon_{n+2}, \epsilon_{n+2m})] \\ &< s\left[\frac{1}{(2s)^n}r_0 + \frac{1}{(2s)^{n+1}}r_0\right] + s^2\left[\frac{1}{(2s)^{n+2}}r_0 + \frac{1}{(2s)^{n+3}}r_0\right] \\ &\quad + \cdots + s^{m-1}\left[\frac{1}{(2s)^{n+2m-4}}r_0 + \frac{1}{(2s)^{n+2m-3}}r_0\right] \\ &\quad + s^{m-1}v(\epsilon_{n+2m-2}, \epsilon_{n+2m}) \\ &\le s\frac{1}{(2s)^n}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 + s\frac{1}{(2s)^{n+1}}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 \\ &\quad + s^{m-1}\frac{1}{(2s)^{n+2m-2}}r_0 \\ &\le s\frac{1}{(2s)^n}\cdot\frac{1}{2m}\frac{1}{(2s)^{n-2}}r_0 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{57}$$

In view of (56) and (57), one can get that $\{\epsilon_n\}$ is Cauchy. By the completeness of $(\mathbb{G}, v)$, one can choose a point $\epsilon^* \in \mathbb{G}$ with $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon^*$. We might as well let $\epsilon_n \ne \epsilon^*$ and ϵ_n

$\neq T^{n(\epsilon^*)}\epsilon_n$. Otherwise, we have $\epsilon^* = T^{n(\epsilon^*)}\epsilon^*$ according to the continuity of T. And from that, one can deduce

$$\begin{aligned} v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) &= v\left(T^{\epsilon_{n-1}}\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\ &\le \psi\left(M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right)\right)M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \end{aligned} \tag{58}$$

where

$$\begin{aligned} M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right) = \max\Big\{& v\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \\ & v\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right), v\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_n\right)\Big\} > 0. \end{aligned} \tag{59}$$

It follows that

$$\begin{aligned} v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) &< \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) \mid q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\} \\ &< \cdots < \frac{1}{(2s)^n}\sup\left\{v(\epsilon_0, q) \mid q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\right\} \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{60}$$

Since T is a continuous mapping, $\lim_{n\longrightarrow\infty} d(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\epsilon_n) = 0$. Therefore,

$$\begin{aligned} v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right) &\le s\Big[v(\epsilon^*, \epsilon_n) + v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) \\ &\quad + v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\epsilon_n\right)\Big] \longrightarrow 0\ (n \longrightarrow \infty). \end{aligned} \tag{61}$$

This means that $\epsilon^* = T^{n(\epsilon^*)\epsilon^*}$. Now,

$$\begin{aligned} v(\epsilon^*, T\epsilon^*) &= v\left(T^{n(\epsilon^*)}\epsilon^*, TT^{n(\epsilon^*)}\epsilon^*\right) \\ &\le \psi(M(\epsilon^*, T\epsilon^*))M(\epsilon^*, T\epsilon^*), \end{aligned} \tag{62}$$

where

$$\begin{aligned} M(\epsilon^*, T\epsilon^*) = \max\Big\{& v(\epsilon^*, T\epsilon^*), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \\ & v\left(\epsilon^*, T^{n(\epsilon^*)}T\epsilon^*\right)\Big\} = v(\epsilon^*, T\epsilon^*). \end{aligned} \tag{63}$$

Hence, we get $v(\epsilon^*, T\epsilon^*) \le (1/2s)v(\epsilon^*, T\epsilon^*)$, i.e., $\epsilon^* = T\epsilon^*$. Assume there has a ϖ^* satisfying $\varpi^* = T\varpi^*$ and $\epsilon^* \neq \varpi^*$; then, $\varpi^* = T\varpi^* = \cdots = T^{n(\epsilon^*)}\varpi^*$ and

$$\begin{aligned} v(\epsilon^*, \varpi^*) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right) \\ &\le \psi(M(\epsilon^*, \varpi^*))M(\epsilon^*, \varpi^*) < \frac{1}{2s}d(\epsilon^*, \varpi^*), \end{aligned} \tag{64}$$

which is impossible. So T possesses the unique fixed point ε^*.

At the end, we prove the last part. To do this, we fix an integer ℓ, $0 \le \ell < n(\epsilon^*)$, and $\forall n > n(\epsilon^*)$; we put $n = in(\epsilon^*) + \ell$, $i \ge 1$. Then, $\forall \epsilon \in \mathbb{G}$; we have

$$\begin{aligned} v(\epsilon^*, T^n\epsilon) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{in(\epsilon^*)+\ell}\epsilon\right) \\ &\le \psi\left(M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right)\right)M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \end{aligned} \tag{65}$$

where

$$\begin{aligned} & M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \\ &\quad = \max\left\{v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v(\epsilon^*, T^n\epsilon)\right\}. \end{aligned} \tag{66}$$

If $v(\epsilon^*, T^n\epsilon) \ge v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon)$, then $M(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) = v(\epsilon^*, T^n\epsilon)$. According to (65), we have

$$v(\epsilon^*, T^n\epsilon) \le \frac{1}{2s}v(\epsilon^*, T^n\epsilon), i.e., \epsilon^* = T^n\epsilon. \tag{67}$$

It follows that $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$. If $v(\epsilon^*, T^n\epsilon) < v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon)$, one can get that

$$v(\epsilon^*, T^n\epsilon) \le \frac{1}{2s}v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right). \tag{68}$$

Similarly,

$$\begin{aligned} v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \\ &\le \psi\left(M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right)\right)M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right), \end{aligned} \tag{69}$$

where

$$\begin{aligned} & M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right) \\ &\quad = \max\Big\{v\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \\ &\qquad v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right)\Big\}. \end{aligned} \tag{70}$$

If $v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) \ge v(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon)$, then

$$M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right) = v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \tag{71}$$

that is,

$$\begin{aligned} v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) &\le \frac{1}{2s}v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \text{i.e.,} \epsilon^* \\ &= T^{(i-1)n(\epsilon^*)+\ell}\epsilon. \end{aligned} \tag{72}$$

Since ϵ^* is a fixed point of T, one get $\epsilon^* = T^{n(\epsilon^*)}\epsilon^* = T^{n(\epsilon^*)}T^{(i-1)n(\epsilon^*)+\ell}\epsilon$. Consequently, $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$.

If $\upsilon(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) < \upsilon(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon)$, then

$$\upsilon\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \leq \frac{1}{2s}\upsilon\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right). \quad (73)$$

We continue to calculate according to this method; if there exists $i_0 \leq i$ satisfying $\epsilon^* = T^{(i-i_0)n(\epsilon^*)+\ell}\epsilon$, then $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$. Otherwise, one can conclude that

$$\upsilon(\epsilon^*, T^n\epsilon) \leq \cdots \leq \frac{1}{(2s)^i}\upsilon\left(\epsilon^*, T^\ell\epsilon\right) \longrightarrow 0(i \longrightarrow \infty). \quad (74)$$

Therefore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to ϵ^*. □

Example 4. Let $\mathbb{G} = [0,+\infty)$ and $\upsilon(\epsilon, \varpi) = (\epsilon - \varpi)^2$. Obviously, $(\mathbb{G}, \upsilon)$ is a complete rectangular b -metric space with $s = 3$. Define $T : \mathbb{G} \longrightarrow \mathbb{G}$ with

$$T\epsilon = \frac{\epsilon}{2}, \quad \epsilon \in [0,+\infty). \quad (75)$$

Define mappings $\psi(\epsilon) = 1/3s$ and $n(\epsilon) = 3$, $\forall\epsilon \in [0,+\infty)$. One has

$$\upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) = \upsilon\left(T^3\epsilon, T^3\varpi\right) = \frac{1}{64}(\epsilon - \varpi)^2, \quad (76)$$

$$\begin{aligned}&\psi(M(\epsilon, \varpi))M(\epsilon, \varpi)\\ &\quad = \frac{1}{9}\max\left\{\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^3\epsilon\right), \upsilon\left(\epsilon, T^3\varpi\right)\right\}\\ &\quad \geq \frac{1}{9}\upsilon(\epsilon, \varpi) = \frac{1}{9}(\epsilon - \varpi)^2.\end{aligned} \quad (77)$$

That is, $\upsilon(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi) \leq \psi(M(\epsilon, \varpi))M(\epsilon, \varpi)$.

Thus, all hypotheses of Theorem 15 are fulfilled. So T possesses the unique common fixed point 0. Furthermore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to 0.

4. Application

In this part, we will prove the solvability of this initial value problem:

$$\begin{cases} m\dfrac{d^2\epsilon}{d\varepsilon^2} + c\dfrac{d\epsilon}{d\varepsilon} - mF(\varepsilon, \epsilon(\varepsilon)) = 0, \\ \epsilon(0) = 0, \\ \epsilon'(0) = 0, \end{cases} \quad (78)$$

where m and $c > 0$ are constants and $F : [0, H] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous mapping.

Obviously, problem (78) is related to the integral equation:

$$\epsilon(\varepsilon) = \int_0^H Y(\varepsilon, \nu)F(\nu, \epsilon(\nu))d\nu, \varepsilon \in [0, H], \quad (79)$$

where $Y(\varepsilon, r)$ is defined as

$$Y(\varepsilon, \rho) = \begin{cases} \dfrac{1 - e^{\omega(\varepsilon-\nu)}}{\omega}, & 0 \leq \varrho \leq \varepsilon \leq H, \\ 0, & 0 \leq \varepsilon \leq \varrho \leq H, \end{cases} \quad (80)$$

where $\omega = c/m$ is a constant.

Next, by using Theorem 13 and Theorem 15, we shall present the solvability of the integral equation:

$$\epsilon(\varepsilon) = \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho))d\varrho. \quad (81)$$

Let $\mathbb{G} = C([0, H])$. For $p \geq 2, \varepsilon, \varpi \in \mathbb{G}$, define

$$\upsilon(\epsilon, \varpi) = \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - \varpi(\varepsilon)|^p. \quad (82)$$

Hence, $(\mathbb{G}, \upsilon)$ is a complete rectangular b-metric space with $s = 3^{p-1}$.

In the following, define $T : \mathbb{G} \longrightarrow \mathbb{G}$ by

$$T\epsilon(\varepsilon) = \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho))d\varrho. \quad (83)$$

Suppose $\Xi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function that satisfies the following condition:

$$\begin{aligned}\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \geq 0 \text{ and } \Xi(\varpi(\varepsilon), T\varpi(\varepsilon))\\ \geq 0 \text{ implies } \Xi(\epsilon(\varepsilon), T\varpi(\varepsilon))\\ \geq 0, \forall\epsilon, \varpi \in \mathbb{G}.\end{aligned} \quad (84)$$

Theorem 16. *Assume that*

(i) $\Gamma : [0, H] \times [0, H] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous

(ii) there has an $\epsilon_0 \in \mathbb{G}$ satisfying $\Xi(\epsilon_0(\varepsilon), T\epsilon_0(\varepsilon)) \geq 0$ for all $\varepsilon \in [0, H]$

(iii) $\forall\varepsilon \in [0, H]$ and $\epsilon, y \in \mathbb{G}$, $\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \geq 0$ imply $\Xi(T\epsilon(\varepsilon), T\varpi(\varepsilon)) \geq 0$

(iv) if $\{\epsilon_n\} \subset \mathbb{G}$ satisfies $\Xi(\epsilon_n(\varepsilon), \epsilon_{n+1}(\varepsilon)) \geq 0$, $\forall n \in \mathbb{N}$, and $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon$, then we can choose a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that $\Xi(\epsilon_{n_k}(\varepsilon), \epsilon(\varepsilon)) \geq 0$, $\forall k \in \mathbb{N}$

(v) for each $\epsilon \in \mathbb{G}$ with $T^{n(\epsilon)}\epsilon = \epsilon$, we have $\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \geq 0$ for any $\varpi \in \mathbb{G}$

(vi) there is a continuous mapping $Y : [0, H] \times [0, H] \longrightarrow \mathbb{R}^+$ satisfying

$$\sup_{\varepsilon\in[0,H]}\int_0^H Y(\varepsilon, \varrho)d\varrho \leq \sqrt[p]{\frac{1}{3^{p^2+1}}}, \quad (85)$$

$$|\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \rho, \varpi(\varrho))| \le Y(\varepsilon, \varrho)|\epsilon(\varrho) - \varpi(\varrho)|. \tag{86}$$

Then, (81) possesses a unique solution $\epsilon \in \mathbb{G}$.

Proof. Set $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ by

$$\alpha(\epsilon, \varpi) = \begin{cases} s^p, & \text{if } \Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \ge 0, \\ 0, & \text{otherwise.} \end{cases} \tag{87}$$

One can check that T is triangular α_{s^p} orbital admissible. In view of (i)-(vi), for $\epsilon, \varpi \in \mathbb{G}$, we obtain

$$\begin{aligned} & s^p \upsilon(T\epsilon(\varepsilon), T\varpi(\varepsilon)) \\ &\quad = s^p \sup_{\varepsilon\in[0,H]} |T\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p \\ &\quad = s^p \sup_{\varepsilon\in[0,H]} \left| \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho - \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right|^p \\ &\quad \le s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H |\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \varrho, \varpi(\varrho))| d\varrho \right)^p \\ &\quad \le s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho)|\epsilon(\varrho) - \varpi(\varrho)| d\varrho \right)^p \\ &\quad \le s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) d\varrho \right)^p \sup_{\varepsilon\in[0,H]} |\epsilon(t) - \varpi(\varepsilon)|^p \\ &\quad \le s^p \cdot \frac{1}{3^{p^2+1}} \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - \varpi(\varepsilon)|^p \\ &\quad \le \frac{\upsilon(\epsilon(\varepsilon), \varpi(\varepsilon))}{3^{p+1}}, \end{aligned} \tag{88}$$

which implies that

$$\begin{aligned} & \alpha(\epsilon(\varepsilon), \varpi(\varepsilon)) \upsilon\left(T^{n(\epsilon)}\epsilon(\varepsilon), T^{n(\epsilon)}\varpi(\varepsilon)\right) \\ &\quad \le \Phi\left(\upsilon(\epsilon(\varepsilon), \varpi(\varepsilon)), \upsilon\left(\epsilon(\varepsilon), T^{n(\epsilon)}\epsilon(\varepsilon)\right), \upsilon\left(\epsilon(\varepsilon), T^{n(\epsilon)}\varpi(\varepsilon)\right)\right), \end{aligned} \tag{89}$$

where $\Phi(\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon_1 + \epsilon_2 + \epsilon_3)/3^{p+1}$, $s = 3^{p-1}$, and $n(\epsilon) = 1$. After that, all hypotheses of Theorem 13 are fulfilled. Hence, T has a unique fixed point $\epsilon \in \mathbb{G}$. That is, ϵ is the unique solution of integral equation (81). □

Remark 17. If $\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) = Y(\varepsilon, \varrho)F(\varrho, \epsilon(\varrho))$, $|F(\varrho, \epsilon(\varrho)) - F(\varrho, \varpi(\varrho))| \le |\epsilon(\varrho) - \varpi(\varrho)|$; then, (78) has a unique solution by Theorem 16.

Theorem 18. *Suppose that*

(i) $\Gamma : [0, H] \times [0, H] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous

(ii) there is a continuous mapping $Y : [0, H] \times [0, H] \longrightarrow \mathbb{R}^+$ satisfying

$$\begin{aligned} & |\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \varrho, \varpi(\varrho))| \\ &\quad \le Y(\varepsilon, \varrho) \left| \epsilon(\varepsilon) + \varpi(\varepsilon) - \left(\int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho + \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right) \right|, \end{aligned} \tag{90}$$

$$\sup_{\varepsilon\in[0,H]} \int_0^H Y(\varepsilon, \varrho) d\varrho \le \frac{1}{3^2}. \tag{91}$$

Then, (81) possesses a unique solution $\epsilon \in \mathbb{G}$.

Proof. For $\epsilon, \varpi \in \mathbb{G}$, according to the conditions (i)-(ii), one can get

$$\begin{aligned} & \upsilon(T\epsilon(\varepsilon), T\varpi(\varepsilon)) \\ &\quad = \sup_{\varepsilon\in[0,H]} |T\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p \\ &\quad = \sup_{\varepsilon\in[0,H]} \left| \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho - \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right|^p \\ &\quad \le \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) \left| \epsilon(\varepsilon) + \varpi(\varepsilon) \right.\right. \\ &\qquad \left.\left. - \left(\int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho + \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right) \right| d\varrho \right)^p \\ &\quad \le \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) (|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - T\epsilon(\varepsilon)|) d\varrho \right)^p \\ &\quad \le \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) (|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - \epsilon(\varepsilon)| + |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|) d\varrho \right)^p \\ &\quad \le \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) d\varrho \right)^p \cdot \sup_{\varepsilon\in[0,H]} (|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - \epsilon(\varepsilon)| + |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|)^p \\ &\quad \le \frac{1}{3^{2p}} \cdot 3^p \cdot \frac{\sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p + \sup_{\varepsilon\in[0,H]} |\varpi(\varepsilon) - \epsilon(\varepsilon)|^p + \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|^p}{3} \\ &\quad \le \frac{1}{3s} M(\epsilon, \varpi), \end{aligned} \tag{92}$$

where $M(\epsilon, \varpi)$ is the same as in Theorem 15. Thus, all the hypotheses of Theorem 15 are fulfilled with $\psi(\varepsilon) = 1/3s$ and $n(\varepsilon) = 1$. It follows that T possesses a unique fixed point $\epsilon \in \mathbb{G}$, and so is a solution of (81). □

5. Conclusions

In rectangular b-metric spaces, we introduced a new triangular α-orbital admissible condition and established two fixed point results for mappings with a contractive iterate at a point. Further, we provided two examples that elaborated the usability of presented results. At the same time, we proved the existence and uniqueness of solution of an integral equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

No potential conflicts of interest are declared by the authors.

Authors' Contributions

All authors contributed equally in writing this article. All authors approved the final manuscript.

Acknowledgments

This work was financially supported by the Science and Research Project Foundation of Liaoning Province Education Department (No. LJC202003).

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 3, pp. 133–181, 1922.

[2] P. Agarwal, M. Jleli, and B. Samet, *Fixed Point Theory in Metric Spaces*, Springer, Berlin, 2018.

[3] A. Branciari, "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematiques*, vol. 57, no. 1-2, pp. 31–37, 2000.

[4] H. Lakzian and B. Samet, "Fixed points for(ψ, φ)-weakly contractive mappings in generalized metric spaces," *Applied Mathematics Letters*, vol. 25, no. 5, pp. 902–906, 2012.

[5] C. D. Bari and P. Vetro, "Common fixed points in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7322–7325, 2012.

[6] R. George and R. Rajagopalan, "Common fixed point results for $\psi-\varphi$ contractions in rectangular metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 5, no. 1, pp. 44–52, 2013.

[7] L. B. Budhia, H. Aydi, A. H. Ansari, and D. Gopal, "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations," *Nonlinear Analysis: Modelling and Control*, vol. 25, no. 4, pp. 580–597, 2020.

[8] S. Czerwik, "Contraction mappings in b – metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[9] Z. D. Mitrovic, "Fixed point results in b – metric space," *Fixed Point Theory*, vol. 20, no. 2, pp. 559–566, 2019.

[10] N. Hussain, Z. D. Mitrović, and S. Radenović, "A common fixed point theorem of Fisher in b-metric spaces," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 949–956, 2019.

[11] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Fixed point theorems for weakly T – Chatterjea and weakly T – Kannan contractions in b – metric spaces," *Journal of Inequalities and Applications*, vol. 2014, no. 1, Article ID 46, 2014.

[12] Z. D. Mitrovic, A. Bodaghi, A. Aloqaily, N. Mlaiki, and R. George, "New versions of some results on fixed points in b -metric spaces," *Mathematics*, vol. 11, no. 5, p. 1118, 2023.

[13] N. Savanović, I. D. Aranđelović, and Z. D. Mitrović, "The results on coincidence and common fixed points for a new type multivalued mappings in b-metric spaces," *Mathematics*, vol. 10, no. 6, p. 856, 2022.

[14] H. Aydi, A. Felhi, E. Karapinar, H. AlRubaish, and M. AlShammari, "Fixed points for $\alpha-\beta_E$ Geraghty contractions on b – metric spaces and applications to matrix equations," *Filomat*, vol. 33, no. 12, pp. 3737–3750, 2019.

[15] M. U. Ali, H. Aydi, and M. Alansari, "New generalizations of set valued interpolative Hardy-Rogers type contractions in b – metric spaces," *Journal of Function Spaces*, vol. 2021, Article ID 6641342, 8 pages, 2021.

[16] R. George, S. Radenovic, and K. P. Reshma, "Rectangular b-metric space and contraction principles," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1005–1013, 2015.

[17] S. Gulyaz-Ozyurt, "On some alpha-admissible contraction mappings on Branciari b-metric spaces," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 1, no. 1, pp. 1–13, 2017.

[18] D. Zheng, G. Ye, and D. Liu, "Sehgal–Guseman-type fixed point theorem in b-rectangular metric spaces," *Mathematics*, vol. 9, no. 24, p. 3149, 2021.

[19] H. Guan, J. Li, and Y. Hao, "Common fixed point theorems for weakly contractions in rectangular b – metric spaces with supportive applications," *Journal of Function Spaces*, vol. 2022, Article ID 8476040, 16 pages, 2022.

[20] A. Hussain, "Solution of fractional differential equations utilizing symmetric contraction," *Journal of Mathematics*, vol. 2021, Article ID 5510971, 17 pages, 2021.

[21] A. Arif, M. Nazam, A. Hussain, and M. Abbas, "The ordered implicit relations and related fixed point problems in the cone b – metric spaces," *AIMS Mathematics*, vol. 7, no. 4, pp. 5199–5219, 2022.

[22] S. Anwar, M. Nazam, H. H. Al Sulami, A. Hussain, K. Javed, and M. Arshad, "Existence fixed-point theorems in the partial b – metric spaces and an application to the boundary value problem," *AIMS Mathematics*, vol. 7, no. 5, pp. 8188–8205, 2022.

[23] V. M. Sehgal, "A fixed point theorem for mappings with a contractive iterate," *Proceedings of the American Mathematical Society*, vol. 23, no. 3, pp. 631–634, 1969.

[24] J. Matkowski, "Fixed point theorems for mappings with a contractive iterate at a point," *Proceedings of the American Mathematical Society*, vol. 62, no. 2, pp. 344–348, 1977.

[25] L. Khazanchi, "Results on fixed points in complete metric space," *Mathematica Japonica*, vol. 19, no. 3, pp. 283–289, 1974.

[26] K. Iseki, "A generalization of Sehgal-Khazanchi's fixed point theorems," *Mathematics Seminar Notes*, vol. 2, pp. 89–95, 1974.

[27] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha-\psi-$contractive type mappings," *Nonlinear Analysis: Theory Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[28] O. Popescu, "Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 190, 12 pages, 2014.

[29] C. Lang and H. Guan, "Common fixed point and coincidence point results for generalized $\alpha-\varphi E$-Geraghty contraction mappings in *b*-metric spaces," *AIMS Mathematics*, vol. 7, no. 8, pp. 14513–14531, 2022.

[30] M. Ming, J. C. Saut, and P. Zhang, "Long-time existence of solutions to Boussinesq systems," *SIAM Journal on Mathematical Analysis*, vol. 44, no. 6, pp. 4078–4100, 2012.