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Research Article

Global Well-Posedness and Large-Time Behavior for the Equilibrium Diffusion Model in Radiation Hydrodynamics

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In this paper, we investigate the global well-posedness of the equilibrium diffusion model in radiation hydrodynamics. The model consists of the compressible Navier-Stokes equations coupled with radiation effect terms described by the fourth power of temperature. The global existence of classical solutions to the Cauchy problem in the whole space is established when initial data is a small smooth perturbation of a constant equilibrium state: moreover, an algebraic rate of convergence of solutions toward equilibrium is obtained under additional conditions on initial data. The proof is based on the refined energy method and Fourier's analysis.

1. Introduction

Radiation hydrodynamics is concerned with the propagation of thermal radiation through a fluid or gas and the effect of this radiation on the dynamics. The importance of thermal radiation in physical problems increases as the temperature rises. Hydrodynamics with an explicit account of radiation energy and momentum contribution constitutes the character of radiation hydrodynamics. Such consideration finds their practical applications in the understanding of certain reentry of space vehicles, astrophysical, supernova explosions, laser fusion, and so on (cf. [1, 2]). As in classical fluid mechanics, the equations of motion in radiation hydrodynamics are derived from the conservation laws for macroscopic quantities. However, due to the presence of radiation, the classical "material" flow has to be coupled with radiation which is an assembly of photons and needs a priori relativistic treatment. Hence, the whole problem to be considered is then a coupling between the standard hydrodynamics for the matter and a radiative transfer equation for the photon distribution. Moreover, if the specific intensity of radiation is isotropic (cf. [3, 4]) and assumes that the photons emitted by the gas have a high probability of reabsorption within the optically thick regions, then the equilibrium diffusion model can be derived [5, 6]. This model is widely used in radiation hydrodynamics research (cf. [7, 8]).

In equilibrium diffusion theory, the quantities of primary interest (i.e., the radiative energy density, pressure, and flux) in radiation hydrodynamic problems can be explicitly calculated via the Planck formula once the temperature distribution is known. The radiation energy density associated with a Planck distribution varies as much as the fourth power of temperature, and the radiative transfer equation can be omitted because the radiation field is isotropic. Therefore, based on the standard hydrodynamics, the system consists of the compressible Navier-Stokes equations coupled with radiation effect terms as described above in the following form (see [4, 9]):

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(P_F + P_R) = \operatorname{div} \mathbb{S}, \tag{2}$$

$$\left(\rho(e_F + e_R) + \frac{1}{2}\rho u^2\right)_t + \operatorname{div}\left(\left(\rho(e_F + e_R) + \frac{1}{2}\rho u^2 + (P_F + P_R)\right)u\right) + \operatorname{div}(q_F + q_R) = \operatorname{div} \mathbb{S}u, \tag{3}$$

with initial data

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3. \tag{4}$$

Here, $\rho = \rho(x, t) > 0$, $\theta = \theta(x, t) > 0$, $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$, for $t \geq 0, x \in \mathbb{R}^3$ denote the mass density, temperature, and velocity field of the fluid, respectively. $P_F = R\rho\theta$, $e_F = c_v\theta$, and $q_F = -\kappa_F \nabla\theta$ are the pressure, internal energy, and heat flux of the fluid, respectively; R, c_v, and κ_F are positive constants; $P_R = (a/3)\theta^4$, $e_R = (a/\rho)\theta^4$, and $q_R = -\kappa_R\theta^3\nabla\theta$ represent radiation pressure, energy, and heat flux, respectively, and a, κ_R are positive constants. The symbol $\mathbb{S}$ stands for the viscous stress tensor $\mathbb{S} = \lambda'(\mathrm{div})\mathbb{I} + \mu(\nabla u + (\nabla u)^T)$, λ' and μ are the constant viscosity coefficients of the fluid satisfying $3\lambda' + 2\mu > 0$, and $\mathbb{I}$ is the 3×3 identity matrix.

Let us introduce some related mathematical results of the equilibrium diffusion model. For the one-dimensional case, the global existence of strong solutions for the Cauchy problem was proved in [10]. The existence of a global smooth solution for an equilibrium diffusion model with the magnetic field to the Dirichlet boundary problem was addressed by Zhang and Xie [11]; similar results were obtained in [12]. Ducomet and Zlotnik [13] studied the stabilization properties and the existence of global solutions on the bounded domain for 1D radiative and reactive viscous gas dynamics by constructing the global Lyapunov functionals under certain growth assumption on heat-conductivity κ_F. The global existence and exponential stability of strong solutions were established in [14], see also [15] on the global solution for a self-gravitating viscous radiative and reactive gas for free boundary problem. The existence of a global strong solution and large-time behavior to the Cauchy problem was studied in [16]. For the multidimensional case, Ducomet and Feireisl [17, 18] investigated the global existence of weak variational solutions for this model with gravitational force and magnetic field in a bounded domain. A similar result can be obtained under more general constitutive assumptions [19]. The existence of the global weak solution to the equilibrium diffusion model in the unbounded domain was studied by Poul [20].

The purpose of this paper is to construct global classical solutions to the system (Equation (1)) near an equilibrium state $(1, 0, 1)$ and investigate the large-time behavior of this solution. Therefore, it is natural to introduce the transforms

$$\rho = 1 + \varrho, \quad \theta = 1 + \Theta \tag{5}$$

to rewrite the system (Equation (1)) as

$$\varrho_t + (1 + \varrho)\,\mathrm{div}\, u + \nabla\varrho \cdot u = 0,$$

$$u_t + u \cdot \nabla u + R\nabla\Theta + \frac{R(1+\Theta)\nabla\varrho}{1+\varrho} + \frac{(4a/3)(1+\Theta)^3\nabla\Theta}{1+\varrho} = \frac{\lambda'\Delta u}{1+\varrho} + \frac{\left(\lambda' + \mu\right)\nabla\mathrm{div}\, u}{1+\varrho},$$

$$\begin{aligned}
&\Theta_t + \frac{(1+\varrho)u\cdot\nabla\Theta}{F(\varrho,\Theta)} + \frac{R(1+\varrho)(1+\Theta) + (4a/3)(1+\Theta)^4}{F(\varrho,\Theta)}\,\mathrm{div}\, u \\
&\quad + \frac{4a(1+\Theta)^3\nabla\Theta\cdot u}{F(\varrho,\Theta)} = \frac{\kappa_F\Delta\Theta}{F(\varrho,\Theta)} + \frac{\kappa_R(1+\Theta)^3\Delta\Theta}{F(\varrho,\Theta)} \\
&\quad + \frac{3\kappa_R(1+\Theta)^2|\nabla\Theta|^2}{F(\varrho,\Theta)} + \frac{\lambda'(\mathrm{div}\, u)^2}{F(\varrho,\Theta)} + \frac{2\mu D\cdot D}{F(\varrho,\Theta)},
\end{aligned} \tag{6}$$

where $F(\varrho,\Theta) = c_v(1+\varrho) + 4a(1+\Theta)^3$, and $D = D(u)$ is the deformation tensor,

$$D_{\mathrm{ij}} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ and } D\cdot D := \sum_{i,j=1}^{3} D_{\mathrm{ij}}^2. \tag{7}$$

For simplicity of the presentation and without loss of generality, we assume the positive constants $R = \mu = \lambda' = \kappa_F = \kappa_R \equiv 1$, $c_v = 3$, $a = 3/4$. We can further simplify system (6) into

$$\varrho_t + (1+\varrho)\,\mathrm{div}\, u + \nabla\varrho\cdot u = 0, \tag{8}$$

$$u_t + u\cdot\nabla u + \nabla\Theta + \frac{(1+\Theta)\nabla\rho}{1+\varrho} + \frac{(1+\Theta)^3\nabla\Theta}{1+\varrho} = \frac{\Delta u}{1+\varrho} + \frac{2\nabla\mathrm{div}\, u}{1+\varrho}, \tag{9}$$

$$\Theta_t + \frac{(1+\varrho)u\cdot\nabla\Theta}{F(\varrho,\Theta)} + \frac{1}{3}(1+\Theta)\,\mathrm{div}\, u + \frac{3(1+\Theta)^3\nabla\Theta\cdot u}{F(\varrho,\Theta)}, \tag{10}$$

$$= \frac{\Delta\Theta}{F(\varrho,\Theta)} + \frac{(1+\Theta)^3\Delta\Theta}{F(\varrho,\Theta)} + \frac{3(1+\Theta)^2|\nabla\Theta|^2}{F(\varrho,\Theta)} + \frac{(\mathrm{div}\, u)^2}{F(\varrho,\Theta)} + \frac{2D\cdot D}{F(\varrho,\Theta)}, \tag{11}$$

with initial data

$$\begin{aligned}
(\varrho, u, \Theta)|_{t=0} &= (\varrho_0(x), u_0(x), \Theta_0(x)) \\
&= (\rho_0(x) - 1, u_0(x), \theta_0(x) - 1).
\end{aligned} \tag{12}$$

Remark 1. The purpose of taking special values of parameters is to simplify the equation form and facilitate calculation. But for (6) with general parameters, we can also achieve the goal by the following processing:

$$\begin{aligned}
&\frac{R(1+\varrho)(1+\Theta) + (4a/3)(1+\Theta)^4}{F(\varrho,\Theta)} \\
&\quad\cdot\ \mathrm{div}\, u = \left(\frac{R(1+\varrho)(1+\Theta) + (4a/3)(1+\Theta)^4}{F(\varrho,\Theta)} - \frac{R + (4a/3)}{c_v + 4a}\right) \\
&\quad\cdot\ \mathrm{div}\, u + \frac{R + (4a/3)}{c_v + 4a}\,\mathrm{div}\, u.
\end{aligned} \tag{13}$$

After this processing, the left side of the above equation can be decomposed into nonlinear and linear parts, and how to control the linear terms is the main difficulty that needs to be overcome in the proof process.

For the later use in this paper, we give some notations. The norm of L^2 is denoted by $\|\cdot\|$. C denotes a positive (generally large) constant and λ a positive (generally small) constant and takes a different value from line to line. $A \sim B$ means $CA \le B \le (1/C)A$ for a generic constant $C > 0$. For an integrable function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx, \quad x\cdot\xi = \sum_{j=1}^{3} x_j \xi_j, \quad \xi \in \mathbb{R}^3, \quad (14)$$

where $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit. For two complex numbers or vectors g and h, $(g|h) := g \cdot \bar{h}$ denotes the dot product of g with the complex conjugate of h.

With the above preparations, the main result can be stated as follows.

Theorem 2. *Suppose that $\|(\rho_0, u_0, \Theta_0)\|_{H^4}$ is small enough. Then, the Cauchy problem (8)–(12) admits a unique global solution $(\rho(x,t), u(x,t), \Theta(x,t))$, which satisfies*

$$\rho, u, \Theta \in C\left([0,\infty]; H^4(\mathbb{R}^3)\right), \quad \sup_{t\ge 0} \|(\rho, u, \Theta)\|_{H^4} \le C\|(\rho_0, u_0, \Theta_0)\|_{H^4}. \quad (15)$$

Theorem 3. *Moreover, assume that $\|(\rho_0, u_0, \Theta_0)\|_{H^4\cap L^1}$ is sufficiently small, under the assumptions of Theorem 2, then for all $t \ge 0$, we have the following time decay*

$$\|(\rho, u, \Theta)\|_{L^2} \le C(1+t)^{-3/4}, \quad (16)$$

$$\|\nabla(\rho, u, \Theta)\|_{H^3} \le C(1+t)^{-5/4}. \quad (17)$$

The existence of global classical solutions to the problem (Equations (8)–(12)) will be proved by extending the local solutions with respect to time based on the global a priori estimates and continuum arguments. We mainly use refined energy methods to establish these estimates in high-order Sobolev's spaces. In particular, for the nonlinear term containing $1/F(\varrho, \Theta)$ and $(1+\Theta)^3$, we need to decompose the linear part of it as in Remark 1 and modify some methods motivated by Matsumura and Nishida [21, 22] to obtain the corresponding estimates. For algebra decay in time, we first get the time-decay property for the linearized system (Equations (43) and (44)) by using the Fourier multiplier technique. Then, the time-decay rate can be given by combining the global a priori estimate obtained in Theorem 2, and the above time-decay property applies the energy estimate technique to the nonlinear problem (Equations (8)–(12)), whose solutions can be represented by the solution-semigroup operator for the linearized system (Equations (43) and (44)) by using the Duhamel principle. The key point is to obtain a Lyapunov-type inequality for $\mathscr{E}_1(t)$ (see Equation (56)).

We organize the rest of the paper as follows: in Section 2, we derive the uniform-in-time a priori estimates and combine local existence results to establish the existence of global classical solutions. In Section 3, we investigate the decay rates of solutions in the whole space.

2. Global Existence

In what follows, our analysis is based on the reformulated Cauchy problem (Equations (8)–(12)). To obtain the global existence, the most important point is to obtain the uniform-in-time a priori estimates.

2.1. A Priori Estimates. In this subsection, we will establish the uniform-in-time a priori estimates in the whole space $\mathbb{R}^3$. First, let us assume that (ϱ, u, Θ) is the smooth solution to the Cauchy problem (Equations (8)–(12)) on $0 \le t < T$ for $T > 0$ and satisfies

$$\sup_{t\ge 0} \|(\varrho, u, \Theta)\|_{H^4} \le \delta, \quad (18)$$

where $0 < \delta < 1$ is a generic constant small enough. Next, we introduce a technical lemma which are useful in the subsequent estimates:

Lemma 4 (see [23, 24]). *There exists a positive constant C, such that for any $f, g \in H^4(\mathbb{R}^3)$ and any multi-index k with $1 \le |k| \le 4$,*

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}^3)} &\le C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{1/2} \left\|\nabla^2 f\right\|_{L^2(\mathbb{R}^3)}^{1/2}, \\ \|fg\|_{H^3(\mathbb{R}^3)} &\le C\|f\|_{H^3(\mathbb{R}^3)} \|\nabla g\|_{H^3(\mathbb{R}^3)}, \\ \left\|\partial^k (fg)\right\|_{L^2(\mathbb{R}^3)} &\le C\|\nabla f\|_{H^3(\mathbb{R}^3)} \|\nabla g\|_{H^3(\mathbb{R}^3)}. \end{aligned} \quad (19)$$

Then, we begin to give the priori estimate of ϱ, u, Θ.

Lemma 5. *Let ϱ, u, Θ be the smooth solution to (8)–(12). Then, for all $0 \le t \le T$ with any $T > 0$, it holds*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\left(\varrho, u, \sqrt{6}\Theta\right)\right\|^2 + \lambda\left(\|\nabla(u,\Theta)\|^2 + \|\operatorname{div} u\|^2\right) \\ &\quad \le C\left(\|(\varrho, u, \Theta)\|_{H^2} + \|(\varrho, u, \Theta)\|_{H^2}^2\right) \\ &\quad\quad \times \left(\|\nabla(\varrho, u, \Theta)\|_{H^1}^2 + \|\operatorname{div} u\|^2\right). \end{aligned} \quad (20)$$

Proof. Multiplying (8)–(11) by $\varrho, u, 6\Theta$, respectively, and then taking integration and summation, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varrho\|^2+\|u\|^2+\left\|\sqrt{6}\Theta\right\|^2\right)+\int\frac{|\nabla u|^2}{1+\varrho}dx\\
&\quad+\int\frac{2(\operatorname{div} u)^2}{1+\varrho}dx+\int\frac{12|\nabla\Theta|^2}{F(\varrho,\Theta)}dx\\
&=-\int\frac{1}{2}\varrho^2\operatorname{div} u dx-\int(u\cdot\nabla u)\cdot u dx+\int\frac{(\varrho-\Theta)}{1+\varrho}\nabla\varrho\cdot u dx\\
&\quad+\int\frac{\varrho-\left(3\Theta+3\Theta^2+\Theta^3\right)}{1+\varrho}\nabla\Theta\cdot u dx-\int\nabla\left(\frac{1}{1+\varrho}\right)\cdot\nabla u\cdot u dx\\
&\quad-\int 2\nabla\left(\frac{1}{1+\varrho}\right)\cdot u\operatorname{div} u dx-\int 2\Theta^2\operatorname{div} u dx\\
&\quad-\int 12\Theta\nabla\frac{1}{F(\varrho,\Theta)}\cdot\nabla\Theta dx-\int\frac{6(1+\varrho)\Theta+18(1+\Theta)^3\Theta}{F(\varrho,\Theta)}\nabla\Theta\cdot u dx\\
&\quad+\int\frac{6\left(3\Theta^2+3\Theta^3+\Theta^4\right)\Delta\Theta}{F(\varrho,\Theta)}dx+\int\frac{18(1+\Theta)^2\Theta|\nabla\Theta|^2}{F(\varrho,\Theta)}dx\\
&\quad+\int\frac{6\Theta(\operatorname{div} u)^2}{F(\varrho,\Theta)}dx+\int\frac{12\Theta D\cdot D}{F(\varrho,\Theta)}dx\equiv\sum_{j=1}^{13}I_j
\end{aligned}\tag{21}$$

Using Hölder and Sobolev's inequalities, for $I_1, I_2, I_3, I_5, I_6, I_7, I_{12}, I_{13}$, we have

$$\begin{aligned}
I_1+I_2+I_3&\le C\|\varrho\|_{L^3}\|\operatorname{div} u\|_{L^2}\|\varrho\|_{L^6}+C\|u\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6}\\
&\quad+C(\|\varrho\|_{L^3}+\|\Theta\|_{L^3})\|\nabla\varrho\|_{L^2}\|u\|_{L^6}\\
&\le C\|\varrho\|_{H^1}\|\operatorname{div} u\|_{L^2}\|\nabla\varrho\|_{L^2}+C\|u\|_{H^1}\|\nabla u\|_{L^2}^2\\
&\quad+C(\|\varrho\|_{H^1}+\|\Theta\|_{H^1})\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2},\\
&\le C\|(\varrho,u,\Theta)\|_{H^1}\left(\|\nabla(\varrho,u)\|^2+\|\operatorname{div} u\|^2\right),\\
I_5+I_6+I_7&\le C(\|\nabla u\|_{L^3}+\|\operatorname{div} u\|_{L^3})\left\|\nabla\left(\frac{1}{1+\varrho}\right)\right\|_{L^2}\|u\|_{L^6}\\
&\quad+C\|\Theta\|_{L^3}\|\operatorname{div} u\|_{L^2}\|\Theta\|_{L^6}\\
&\le C\|u\|_{H^2}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2}+C\|\Theta\|_{H^1}\|\operatorname{div} u\|_{L^2}\|\nabla\Theta\|_{L^2},\\
I_{12}+I_{13}&\le C\|\Theta\|_{L^\infty}\|\operatorname{div} u\|_{L^2}^2+C\sum_{i,j=1}^{3}\left\|u_{x_j}^i\right\|_{L^2}\left\|u_{x_i}^j\right\|_{L^2}\|\Theta\|_{L^\infty}\\
&\le C\|\Theta\|_{H^2}\left(\|\operatorname{div} u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right),
\end{aligned}\tag{22}$$

For the rest terms, under the assumption (18), we have

$$\begin{aligned}
I_8&\le\left\|\nabla\left(\frac{1}{F(\varrho,\Theta)}\right)\right\|_{L^2}\|\nabla\Theta\|_{L^3}\|\Theta\|_{L^6}\\
&\le C\left(\|\nabla\varrho\|_{L^2}+\|\nabla\Theta\|_{L^2}+\|\Theta\|_{L^\infty}\|\nabla\Theta\|_{L^2}+\|\Theta\|_{L^\infty}^2\|\nabla\Theta\|_{L^2}\right)\|\nabla\Theta\|_{L^3}\|\Theta\|_{L^6}\\
&\le C(\|\nabla\varrho\|_{L^2}+\|\nabla\Theta\|_{L^2}+\|\Theta\|_{H^2}\|\nabla\Theta\|_{L^2})\|\Theta\|_{H^1}\|\nabla\Theta\|_{L^2},
\end{aligned}\tag{23}$$

and similarly, we get

$$\begin{aligned}
I_4&\le C\left(\|\varrho\|_{H^1}+\|\Theta\|_{H^1}+\|\Theta\|_{H^1}^2\right)\|\nabla\Theta\|_{L^2}\|\nabla u\|_{L^2},\\
I_9&\le C(1+\|\varrho\|_{H^2}+\|\Theta\|_{H^2})\|\Theta\|_{H^1}\|\nabla\Theta\|_{L^2}\|\nabla u\|_{L^2},\\
I_{10}+I_{11}&\le C\left(\|\Theta\|_{H^1}+\|\Theta\|_{H^2}^2\right)\|\nabla\Theta\|_{H^1}^2.
\end{aligned}\tag{24}$$

Then, (20) follows by plugging all estimates above into (21), and hence Lemma 5 is proved. □

Lemma 6. *Let ϱ, u, Θ be the smooth solution to (8)–(12). Then, for all $0\le t\le T$ with any $T>0$, it holds*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\le|k|\le4}\left\|\left(\partial^k\varrho,\partial^k u,\sqrt{6}\partial^k\Theta\right)\right\|^2\\
&\quad+\lambda\sum_{1\le|k|\le4}\left(\left\|\partial^k\nabla u\right\|^2+\left\|\partial^k\operatorname{div} u\right\|^2+\left\|\partial^k\nabla\Theta\right\|^2\right)\\
&\le C\left(\|(\varrho,u,\Theta)\|_{H^4}+\|(\varrho,u,\Theta)\|_{H^4}^2\right)\\
&\quad\times\left(\|\nabla\varrho\|_{H^3}^2+\|\nabla(u,\Theta)\|_{H^4}^2+\|\operatorname{div} u\|_{H^4}^2\right).
\end{aligned}\tag{25}$$

Proof. Applying ∂^k with $1\le|k|\le4$ to (8)–(11), multiplying by $\partial^k\varrho$, $\partial^k u$, and $6\partial^k\Theta$, respectively, and then taking integration and summation, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\partial^k\varrho\right\|^2+\left\|\partial^k u\right\|^2+\left\|\sqrt{6}\partial^k\Theta\right\|^2\right)+\int\frac{\left|\nabla\partial^k u\right|^2}{1+\varrho}dx\\
&\quad+\int\frac{2\left(\operatorname{div}\partial^k u\right)^2}{1+\varrho}dx+\int\frac{12\left|\nabla\partial^k\Theta\right|^2}{F(\varrho,\Theta)}dx\\
&=-\int\partial^k(\varrho\operatorname{div} u)\partial^k\varrho dx+\int\left[-\partial^k,u\cdot\nabla\right]\varrho\partial^k\varrho dx\\
&\quad+\int\frac{1}{2}\left|\partial^k\varrho\right|^2\operatorname{div} u dx-\int\partial^k(u\cdot\nabla u)\partial^k u dx\\
&\quad+\int\partial^k\left(\frac{\varrho-\left(3\Theta+3\Theta^2+\Theta^3\right)}{1+\varrho}\nabla\Theta\right)\partial^k u dx\\
&\quad+\int\frac{(\Theta-\varrho)}{1+\varrho}\partial^k\varrho\operatorname{div}\partial^k u dx+\int\partial^k\varrho\nabla\left(\frac{1+\Theta}{1+\varrho}\right)\cdot\partial^k u dx\\
&\quad+\int\left[-\partial^k,\frac{1+\Theta}{1+\varrho}\nabla\right]\varrho\partial^k u dx-\int\nabla\left(\frac{1}{1+\varrho}\right)\cdot\nabla\partial^k u\cdot\partial^k u dx\\
&\quad-\int 2\nabla\left(\frac{1}{1+\varrho}\right)\cdot\partial^k u\operatorname{div}\partial^k u dx\\
&\quad+\sum_{0\le k'<k}C_{k,k'}\int\partial^{k-k'}\left(\frac{1}{1+\varrho}\right)\partial^{k'}\Delta u\cdot\partial^k u dx\\
&\quad+\sum_{0\le k'<\alpha}C_{k,k'}\int 2\partial^{k-k'}\left(\frac{1}{1+\varrho}\right)\partial^{k'}\nabla\operatorname{div} u\cdot\partial^k u dx\\
&\quad-\int\partial^k\left(\frac{6(1+\varrho)+18(1+\Theta)^3}{F(\varrho,\Theta)}\nabla\Theta\cdot u\right)\partial^k\Theta dx\\
&\quad-\int 2\partial^k(\Theta\operatorname{div} u)\partial^k\Theta dx-\int 12\nabla\left(\frac{1}{F(\varrho,\Theta)}\right)\cdot\nabla\partial^k\Theta\partial^k\Theta dx\\
&\quad+\sum_{0\le k'<k}C_{k,k'}\int 12\partial^{k-k'}\left(\frac{1}{F(\varrho,\Theta)}\right)\partial^{k'}\Delta\Theta\partial^k\Theta dx\\
&\quad+\int 6\partial^k\left(\frac{\left(3\Theta+3\Theta^2+\Theta^3\right)\Delta\Theta}{F(\varrho,\Theta)}\right)\partial^k\Theta dx\\
&\quad+\int 18\partial^k\left(\frac{(1+\Theta)^2|\nabla\Theta|^2}{F(\varrho,\Theta)}\right)\partial^k\Theta dx+\int 6\partial^k\left(\frac{(\operatorname{div} u)^2\Theta}{F(\varrho,\Theta)}\right)\partial^k\Theta dx\\
&\quad+\int 12\partial^k\left(\frac{D\cdot D\Theta}{F(\varrho,\Theta)}\right)\partial^k\Theta dx\equiv\sum_{j=1}^{20}I_j,
\end{aligned}\tag{26}$$

where $[\cdot,\cdot]$ stands for $[A,B]=AB-BA$ for two operators A and B, $C_{k,k'}$ is constant depending only on k and k'. Each term can be estimated as follows. Using Hölder, Sobolev, and Young's inequalities, (18) and Lemma 4, we easily get the following bounds

$$I_1+I_2\le C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}(\|\operatorname{div} u\|_{H^4}+\|\nabla u\|_{H^3}),$$

$$I_3+I_4\le C\|u\|_{H^3}\left(\|\nabla\varrho\|_{H^3}^2+\|\nabla u\|_{H^4}^2\right),$$

$$I_6\le C\|\varrho\|_{H^4}(\|\nabla\varrho\|_{H^1}+\|\nabla\Theta\|_{H^1})\|\operatorname{div} u\|_{H^4},$$

$$I_7\le C\|\varrho\|_{H^4}\|u\|_{H^4}(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2}+\|\nabla\varrho\|_{H^2}\|\nabla\Theta\|_{H^1}),$$

$$\begin{aligned}I_8&\le C\|u\|_{H^4}\left(\|\nabla\varrho\|_{H^3}^2+\|\nabla\varrho\|_{H^2}\|\nabla\Theta\|_{H^2}\right)\\&\quad+C\|\varrho\|_{H^4}\|u\|_{H^4}\|\nabla\varrho\|_{H^2}\|\nabla\Theta\|_{H^2},\end{aligned}$$

$$\begin{aligned}I_9+I_{10}&\le C\|u\|_{H^4}\|\nabla\varrho\|_{H^2}(\|\nabla u\|_{H^4}+\|\operatorname{div} u\|_{H^4}),\\I_{14}&\le C\|\Theta\|_{H^4}\|\operatorname{div} u\|_{H^4}\|\nabla\Theta\|_{H^3},\\I_{15}&\le C\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2})\|\nabla\Theta\|_{H^4}.\end{aligned}\tag{27}$$

For I_{11}, we have

$$I_{11}\le C\|u\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla u\|_{H^4},\tag{28}$$

where the first inequality follows that for $k'<k$,

$$\int\partial^{k-k'}\left(\frac{1}{1+\varrho}\right)\partial^{k'}\Delta u\partial^k u dx\le\begin{cases}\left\|\partial^k\left(\frac{1}{1+\varrho}\right)\right\|\|\Delta u\|_{L^\infty}\left\|\partial^k u\right\|, & \left(|k'|=0\right),\\ \left\|\partial^{k-k'}\left(\frac{1}{1+\varrho}\right)\right\|_{L^3}\left\|\partial^{k'}\Delta u\right\|_{L^6}\left\|\partial^k u\right\|, & \left(|k'|=1\right),\\ \left\|\partial^{k-k'}\left(\frac{1}{1+\varrho}\right)\right\|_{L^\infty}\left\|\partial^{k'}\Delta u\right\|_{L^2}\left\|\partial^k u\right\|, & \left(|k'|\ge 2\right),\end{cases}\tag{29}$$

and Sobolev and Young's inequalities were further used.

Similarly, by (18), we have

$$\begin{aligned}I_5&=\sum_{0\le k'\le k}C_{k,k'}\int\partial^{k-k'}\left(\frac{\varrho-\left(3\Theta+3\Theta^2+\Theta^3\right)}{1+\varrho}\right)\partial^{k'}\nabla\Theta\partial^k u dx\\&\le C(1+\|\nabla\varrho\|_{H^3})\left(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3}+\|\nabla\Theta\|_{H^3}^2+\|\nabla\Theta\|_{H^3}^3\right)\\&\quad\cdot\|\nabla\Theta\|_{H^4}\|u\|_{H^4}+C(\|\varrho\|_{H^4}+\|\Theta\|_{H^4})(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\\&\quad\cdot\|\nabla\Theta\|_{H^4}\|u\|_{H^4}\le C(1+\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\\&\quad\cdot\|u\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\|\nabla\Theta\|_{H^4},\\I_{12}&\le C\|u\|_{H^4}\|\nabla\varrho\|_{H^3}\|\operatorname{div} u\|_{H^4},\\I_{13}&\le C\|u\|_{H^4}\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\\&\quad\cdot\|\nabla\Theta\|_{H^4}+C\|\Theta\|_{H^4}\|\nabla u\|_{H^3}\|\nabla\Theta\|_{H^4},\\I_{16}&\le C(1+\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\|\nabla\Theta\|_{H^4},\\I_{18}&\le C(\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\\&\quad\cdot\|\nabla\Theta\|_{H^4}+C\|\Theta\|_{H^4}^2\|\nabla\Theta\|_{H^4}^2,\\I_{19}&\le C(\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\\&\quad\cdot\|\operatorname{div} u\|_{H^4}+C\|\Theta\|_{H^4}^2\|\operatorname{div} u\|_{H^4}^2,\\I_{20}&\le C(\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\\&\quad\cdot\|\nabla u\|_{H^4}+C\|\Theta\|_{H^4}^2\|\nabla u\|_{H^4}^2.\end{aligned}\tag{30}$$

Finally, for I_{17}, we divided into three parts to deal with

$$\begin{aligned}I_{17}&=6\int\nabla\left(\frac{3\Theta+3\Theta^2+\Theta^3}{F(\varrho,\Theta)}\right)\cdot\nabla\partial^k\Theta\partial^k\Theta dx\\&\quad+6\int\frac{\left(3\Theta+3\Theta^2+\Theta^3\right)\left|\nabla\partial^k\Theta\right|^2}{F(\varrho,\Theta)}dx\\&\quad+6\int\left[\partial^k,\frac{3\Theta+3\Theta^2+\Theta^3}{F(\varrho,\Theta)}\Delta\right]\Theta\partial^k\Theta dx\equiv\sum_{j=1}^{3}I_{17}^j.\end{aligned}\tag{31}$$

By Lemma 4 and (18), we have

$$\begin{aligned}I_{17}^1&\le\left\|\nabla\left(\frac{3\Theta+3\Theta^2+\Theta^3}{F(\varrho,\Theta)}\right)\right\|_{L^\infty}\|\nabla\partial^\alpha\Theta\|\|\partial^\alpha\Theta\|\\&\le C(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2})\|\Theta\|_{H^2}\|\nabla\Theta\|_{H^4}\|\Theta\|_{H^4}\\&\quad+C\|\nabla\Theta\|_{H^2}\|\nabla\Theta\|_{H^4}\|\Theta\|_{H^4}\le C\|\Theta\|_{H^4}^2\\&\quad\cdot(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2})\|\nabla\Theta\|_{H^4}+C\|\Theta\|_{H^4}\|\nabla\Theta\|_{H^4}^2,\\I_{17}^2&\le C\left(\|\Theta\|_{H^2}+\|\Theta\|_{H^2}^2\right)\|\nabla\Theta\|_{H^4}^2,\\I_{17}^3&\le C\|\Theta\|_{H^4}\|\nabla\Theta\|_{H^4}^2+C(\|\varrho\|_{H^4}+\|\Theta\|_{H^4})\|\Theta\|_{H^4}\\&\quad\cdot(\|\nabla\varrho\|_{H^3}+\|\nabla\Theta\|_{H^3})\|\nabla\Theta\|_{H^4}.\end{aligned}\tag{32}$$

Therefore,

$$I_{17}\le C\left(\|(\varrho,\Theta)\|_{H^4}+\|(\varrho,\Theta)\|_{H^4}^2\right)\left(\|\nabla\varrho\|_{H^3}^2+\|\nabla\Theta\|_{H^4}^2\right).\tag{33}$$

Plugging these estimates into (26) and taking the sum over $1\le|\alpha|\le 3$, (25) follows, and, thus, Lemma 6 is proved. □

At last, we give the estimate of $\nabla\rho$.

Lemma 7. *Let (ϱ,u,Θ) be the smooth solution to (8)–(12). Then for all $0\le t\le T$ with any $T>0$, it holds*

$$\begin{aligned}&\frac{d}{dt}\sum_{|k|\le 3}\int\nabla\partial^k\varrho\cdot\partial^k u dx+\lambda\|\nabla\varrho\|_{H^3}^2\\&\quad\le C\left(\|\nabla u\|_{H^4}^2+\|\operatorname{div} u\|_{H^4}^2+\|\nabla\Theta\|_{H^3}^2\right)\\&\qquad+C\left(\|(\varrho,u,\Theta)\|_{H^4}+\|(\varrho,u,\Theta)\|_{H^4}^2\right)\|\nabla(\varrho,u,\Theta)\|_{H^3}^2.\end{aligned}\tag{34}$$

Proof. Taking differentiation $\partial^k(|k|\le 3)$ to (9), and carrying an direct calculation, we get

$$\int \left|\nabla\partial^k \varrho\right|^2 dx = -\int \nabla\partial^k\varrho\cdot\partial^k u_t dx - \int \nabla\partial^k\varrho\partial^k(u\cdot\nabla u)dx - \int \nabla\varrho\cdot\nabla\partial^k\Theta dx - \int \nabla\partial^k\varrho\cdot\partial^k\left(\left(\frac{\Theta-\varrho}{1+\varrho}\right)\nabla\varrho\right)dx - \int \nabla\partial^k\varrho\cdot\partial^k\left(\frac{(1+\Theta)^3}{1+\varrho}\nabla\Theta\right) + \int \nabla\partial^k\varrho\cdot\partial^k\left(\frac{\Delta u}{1+\varrho}\right)dx + \int 2\nabla\partial^k\varrho\cdot\partial^k\left(\frac{\nabla \mathrm{div}\, u}{1+\varrho}\right)dx \equiv \sum_{j=1}^{7} I_j. \tag{35}$$

For I_1, applying (8), we have

$$I_1 = -\frac{d}{dt}\int \nabla\partial^k\varrho\cdot\partial^k u dx + \int \partial^k \,\mathrm{div}\, u\partial^k[(1+\varrho)\,\mathrm{div}\, u + \nabla\varrho\cdot u]dx \le -\frac{d}{dt}\int \nabla\partial^k\varrho\cdot\partial^k u dx + C\|\nabla u\|_{H^3}^2 + C\|\varrho\|_{H^4}\|\nabla u\|_{H^3}^2. \tag{36}$$

Using Hölder, Soblev, and Young's inequalities and (18), we obtain

$$\begin{aligned} I_2 &\le C\|\varrho\|_{H^4}\|\nabla u\|_{H^3}^2,\\ I_3 &\le \varepsilon\left\|\nabla\partial^k\varrho\right\|^2 + C_\varepsilon\|\nabla\Theta\|_{H^3}^2,\\ I_4 &\le C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2}) + C\|\varrho\|_{H^4}^2\|\nabla\varrho\|_{H^2}\|\nabla\Theta\|_{H^2},\\ I_5 &\le C\|\varrho\|_{H^4}\|\nabla\Theta\|_{H^3}(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2}) + C\|\varrho\|_{H^4}\|\Theta\|_{H^4}(\|\nabla\varrho\|_{H^2}+\|\nabla\Theta\|_{H^2})^2 + \varepsilon\left\|\nabla\partial^k\varrho\right\|^2 + C_\varepsilon\|\nabla\Theta\|_{H^3}^2,\\ I_6 + I_7 &\le C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla u\|_{H^4} + C\left\|\nabla\partial^k\varrho\right\|\|\nabla u\|_{H^4} + C\left\|\nabla\partial^k\varrho\right\|\|\mathrm{div}\, u\|_{H^4} \le \varepsilon\left\|\nabla\partial^k\varrho\right\|^2 + C_\varepsilon\left(\|\nabla u\|_{H^4}^2 + \|\mathrm{div}\, u\|_{H^4}^2\right) + C\|\varrho\|_{H^4}\|\nabla\varrho\|_{H^3}\|\nabla u\|_{H^3}. \end{aligned} \tag{37}$$

Putting these estimates into (35) and taking the sum over $|\alpha| \le 3$ gives (34), Lemma 7 is proved. □

2.2. Global Existence. In this subsection, we will show that there exists a unique global-in-time solution to the problem Equations (8)–(12). Firstly, define a total temporal energy functional $\mathscr{E}(t)$ and corresponding dissipation rate functional $\mathscr{D}(t)$ by

$$\begin{aligned} \mathscr{E}(t) &= \|\varrho\|^2 + \|u\|^2 + \left\|\sqrt{6}\Theta\right\|^2 + \sum_{1\le|k|\le4}\left(\left\|\partial^k\varrho\right\|^2 + \left\|\partial^k u\right\|^2 + \left\|\sqrt{6}\partial^k\Theta\right\|^2\right) + \tau_1\sum_{|k|\le3}\int \nabla\partial^k\varrho\cdot\partial^k u dx,\\ \mathscr{D}(t) &= \|\nabla\varrho\|_{H^3}^2 + \|\nabla(u,\Theta)\|_{H^4}^2 + \|\mathrm{div}\, u\|_{H^4}^2, \end{aligned} \tag{38}$$

where $0 < \tau_1 \ll 1$ are constant. Then, it follows from (18) that

$$\mathscr{E}(t) \sim \|(\varrho, u, \Theta)(t)\|_{H^4}^2. \tag{39}$$

Now, the sum of Equations (20), (25), and (34) $\tau_1 \times$ gives

$$\frac{d}{dt}\mathscr{E}(t) + \lambda\mathscr{D}(t) \le C\left[\mathscr{E}^{1/2}(t) + \mathscr{E}(t)\right]\mathscr{D}(t), \tag{40}$$

for all $0 \le t < T$. From Equation (18), the time integration of Equation (40) yields

$$\mathscr{E}(t) + \lambda\int_0^t \mathscr{D}(s)ds \le \mathscr{E}(0), \tag{41}$$

for all $0 \le t < T$. Besides, Equation (18) can be justified by choosing

$$\mathscr{E}(0) \sim \|(\varrho_0, u_0, \Theta_0)\|_{H^4}^2 \tag{42}$$

sufficiently small. So that the a priori estimate is closed. For brevity, the proof for local existence of smooth solutions is omitted. By Equation (41) and the result on the local existence, the standard continuity arguments give the global existence and uniqueness of solutions to the Cauchy problem (Equation (8)–(12)). Thus, we complete the proofs of Theorem 2.

3. Time Decay of Solutions

In this section, so as to obtain the time-decay rates of solutions to the nonlinear system (Equations (8)–(12)), we consider the following Cauchy problem of the linearized system:

$$\begin{aligned} \varrho_t + \mathrm{div}\, u &= 0,\\ u_t + \nabla\varrho + 2\nabla\Theta - \Delta u - 2\nabla \mathrm{div}\, u &= 0,\\ \Theta_t + \frac{1}{3}\,\mathrm{div}\, u - 2\Delta\Theta &= 0, \end{aligned} \tag{43}$$

with initial data

$$(\varrho, u, \Theta)|_{t=0} = (\varrho_0, u_0, \Theta_0)(x), \quad x \in \mathbb{R}^3. \tag{44}$$

For simplicity of later presentation, we denote that $U(t) = (\varrho(t), u(t), \Theta(t))$ and $U_0 = (\varrho_0, u_0, \Theta_0)$, then the solution to Equations (43) and (44) can be presented as follows:

$$U(t) = \Lambda(t)U_0, \tag{45}$$

where $\Lambda(t)$ is the solution operator of Equations (43) and (44). With the above preparations, we have the following decay result.

Lemma 8. *Let $1 \leq q \leq 2$. For any k, k' with $k' \leq k$ and $m = |k-k'|$,*

$$\left\|\partial^k \Lambda(t) U_0\right\|_{L^2} \leq C(1+t)^{-3/2(1/q-1/2)-m/2}\left(\left\|\partial^{k'} U_0\right\|_{L^q}+\left\|\partial^k U_0\right\|_{L^2}\right), \tag{46}$$

hold for all $t \geq 0$.

Proof. By Fourier, transforming (43) with respect to x, one has

$$\widehat{\varrho}_t + i\xi \cdot \widehat{u} = 0, \tag{47}$$

$$\widehat{u}_t + i\xi\widehat{\varrho} + 2i\xi\widehat{\Theta} + |\xi|^2\widehat{u} + 2\xi(\xi\cdot\widehat{u}) = 0, \tag{48}$$

$$\widehat{\Theta}_t + 2|\xi|^2\widehat{\Theta} + \frac{1}{3} i\xi \cdot \widehat{u} = 0. \tag{49}$$

Firstly, one can acquire from the system (47)–(49) that

$$\partial_t\left|\left(\widehat{\varrho}, \widehat{u}, \sqrt{6}\widehat{\Theta}\right)\right|^2 + |\xi|^2|\widehat{u}|^2 + 2|\xi\cdot\widehat{u}|^2 + 12|\xi|^2\left|\widehat{\Theta}\right|^2 = 0. \tag{50}$$

Multiplying (48) by $i\xi\bar{\widehat{\varrho}}$, utilizing integration by parts in t, and replacing $\partial_t\widehat{\varrho}$ by (47), we have

$$\partial_t(\widehat{u}|i\xi\widehat{\varrho}) + |\xi|^2|\widehat{\varrho}|^2 = |\xi\cdot\widehat{u}|^2 + 3|\xi|^2 i\xi\cdot\widehat{u}\bar{\widehat{\varrho}} - 2|\xi|^2\widehat{\Theta}\bar{\widehat{\varrho}}. \tag{51}$$

Then, taking the real part of (51) and utilizing the Cauchy-Schwarz inequality, one has

$$\begin{aligned}\partial_t \operatorname{Re}(\widehat{u}|i\xi\widehat{\varrho}) + |\xi|^2|\widehat{\varrho}|^2 &\leq |\xi\cdot\widehat{u}|^2 + \varepsilon|\xi|^2|\widehat{\varrho}|^2 \\ &+ C_\varepsilon|\xi|^2|\xi\cdot\widehat{u}|^2 + \varepsilon|\xi|^2|\widehat{\varrho}|^2 + C_\varepsilon|\xi|^2\left|\widehat{\Theta}\right|^2\end{aligned} \tag{52}$$

with $\varepsilon > 0$ a small constant. Multiplying it by $1/(1+|\xi|^2)$, we conclude that there exists $\lambda > 0$ such that

$$\frac{\partial_t \operatorname{Re}(\widehat{u}|i\xi\widehat{\varrho})}{1+|\xi|^2} + \frac{\lambda|\xi|^2|\widehat{\varrho}|^2}{1+|\xi|^2} \leq C|\xi\cdot\widehat{u}|^2 + \frac{C|\xi|^2\left|\widehat{\Theta}\right|^2}{1+|\xi|^2}. \tag{53}$$

Lastly, we define the time-frequency Lyapunov functional as

$$\mathscr{E}\left(\widehat{U}(t,\xi)\right) = \left|\left[\widehat{\rho}, \widehat{u}, \sqrt{6}\widehat{\Theta}\right]\right|^2 + \tau_2\frac{\operatorname{Re}(\widehat{u}|i\xi\widehat{\rho})}{1+|\xi|^2}, \tag{54}$$

where $0 < \tau_2 \ll 1$ are constant. It is also immediate to verify that $\mathscr{E}(\widehat{U}) \sim |\widehat{U}|^2$. Moreover, by suitably choosing constant τ_2, the sum of Equations (50) and (53), $\tau_2 \times$ gives

$$\partial_t\mathscr{E}\left(\widehat{U}(t,\xi)\right) + \frac{\lambda|\xi|^2}{1+|\xi|^2}\mathscr{E}\left(\widehat{U}(t,\xi)\right) \leq 0. \tag{55}$$

The conclusions of Lemma 8 directly follows from the above estimate, and the detailed proof is omitted for brevity. □

Now, we continue to proof the rate of convergence. First, define

$$\begin{aligned}\mathscr{E}_1(t) &= \sum_{1\leq|k|\leq4}\left(\left\|\partial^k\varrho\right\|^2 + \left\|\partial^k u\right\|^2 + \left\|\sqrt{6}\partial^k\Theta\right\|^2\right) \\ &+ \tau_1\sum_{|k|\leq3}\int\nabla\partial^k\varrho\cdot\partial^k u dx,\end{aligned} \tag{56}$$

$$\mathscr{D}_1(t) = \sum_{1\leq|k|\leq3}\left\|\partial^k\nabla\varrho\right\|^2 + \sum_{1\leq|k|\leq4}\left\|\partial^k\nabla(u,\Theta)\right\|^2 + \sum_{1\leq|k|\leq4}\left\|\partial^k \operatorname{div} u\right\|^2. \tag{57}$$

By using Lemma 4 and similar arguments to those in the proof of Lemma 5–Lemma 7, we can deduce

$$\frac{d}{dt}\mathscr{E}_1(t) + \lambda\mathscr{D}_1(t) \leq C\left(\mathscr{E}_1^{1/2}(t) + \mathscr{E}_1(t)\right)\mathscr{D}_1(t). \tag{58}$$

Thus,

$$\frac{d}{dt}\mathscr{E}_1(t) + \lambda\mathscr{D}_1(t) \leq 0, \tag{59}$$

if $\mathscr{E}_1(t)$ is small enough.

On the other hand, by the definitions of $\mathscr{E}_1(t)$ and $\mathscr{D}_1(t)$, we have

$$\mathscr{E}_1(t) \leq C\left(\mathscr{D}_1(t) + \|\nabla U\|_{L^2}^2\right). \tag{60}$$

From (59), we have

$$\frac{d}{dt}\mathscr{E}_1(t) + \left(\frac{\lambda}{C}\mathscr{E}_1(t) - \lambda\|\nabla U\|_{L^2}^2\right) \leq \frac{d}{dt}\mathscr{E}_1(t) + \lambda\mathscr{D}_1(t) \leq 0, \tag{61}$$

which implies

$$\frac{d}{dt}\mathscr{E}_1(t) + \lambda\mathscr{E}_1(t) \leq C\|\nabla U\|_{L^2}^2. \tag{62}$$

Next, we give the estimate of $\|\nabla U\|_{L^2}^2$. For this purpose, we can rewrite the nonlinear Cauchy problem (Equations (8)–(12)) as

$$U(t) = \Lambda(t)U_0 + \int_0^t \Lambda(t-s)(G_1, G_2, G_3)ds \equiv \sum_{j=1}^4 I_i(t) \tag{63}$$

with

$$\begin{aligned} G_1 &= -\varrho \operatorname{div} u - \nabla\varrho \cdot u, \\ G_2 &= -u\cdot\nabla u - \frac{\Theta-\varrho}{1+\varrho}\nabla\varrho - \frac{\varrho-\left(3\Theta+3\Theta^2+\Theta^3\right)}{1+\varrho}\nabla\Theta \\ &\quad - \frac{\varrho}{1+\varrho}\Delta u - \frac{2\varrho}{1+\varrho}\nabla\operatorname{div} u, \\ G_3 &= -\frac{\left(1+\varrho+3(1+\Theta)^3\right)u\cdot\nabla\Theta}{F(\varrho,\Theta)} - \frac{1}{3}\Theta \operatorname{div} u \\ &\quad + \frac{\left(2-2F(\varrho,\Theta)+\Theta+3\Theta+3\Theta^2+\Theta^3\right)\Delta\Theta}{F(\varrho,\Theta)} \\ &\quad + \frac{3(1+\Theta)^2|\nabla\Theta|^2}{F(\varrho,\Theta)} + \frac{(\operatorname{div} u)^2}{F(\varrho,\Theta)} + \frac{2D\cdot D}{F(\varrho,\Theta)}, \\ I_1(t) &= \Lambda(t)U_0, \\ I_2(t) &= \int_0^t \Lambda(t-s)(G_1,0,0)ds, \\ I_3(t) &= \int_0^t \Lambda(t-s)(0,G_2,0)ds, \\ I_4(t) &= \int_0^t \Lambda(t-s)(0,0,G_3)ds. \end{aligned} \tag{64}$$

Define

$$\mathscr{E}_\infty(t) = \sup_{0\le s\le t}(1+s)^{5/2}\mathscr{E}_1(s). \tag{65}$$

One has that from Lemma 8,

$$\begin{aligned} \|\nabla I_1(t)\|_{L^2} &\le C(1+t)^{-5/4}\left(\|\nabla U_0\|_{L^2} + \|U_0\|_{L^1}\right), \\ \|\nabla I_2(t)\|_{L^2} &\le C\int_0^t (1+t-s)^{-5/4}\left(\|\nabla G_1\|_{L^2} + \|G_1\|_{L^1}\right)ds \\ &\le C\int_0^t (1+t-s)^{-5/4}\mathscr{E}_1(s)ds \\ &\le C\int_0^t (1+t-s)^{-5/4}(1+s)^{-5/2}ds\,\mathscr{E}_\infty(t) \\ &\le C(1+t)^{-5/4}\mathscr{E}_\infty(t), \\ \|\nabla I_4(t)\|_{L^2} + \|\nabla I_5(t)\|_{L^2} &\le C\int_0^t (1+t-s)^{-5/4} \\ &\quad \cdot\left(\|\nabla G_2\|_{L^2} + \|\nabla G_3\|_{L^2} + \|G_2\|_{L^1} + \|G_3\|_{L^1}\right)ds \\ &\le C\int_0^t (1+t-s)^{-5/4}\left(\mathscr{E}_1(s) + \mathscr{E}_1^2(s)\right)ds \\ &\le C(1+t)^{-5/4}\left(\mathscr{E}_\infty(t) + \mathscr{E}_\infty^2(t)\right). \end{aligned} \tag{66}$$

Therefore, it follows that

$$\|\nabla U\|_{L^2}^2 \le C(1+t)^{-5/2}\left\{\|\nabla U_0\|_{L^2}^2 + \|U_0\|_{L^1}^2 + \mathscr{E}_\infty^2(t) + \mathscr{E}_\infty^4(t)\right\}. \tag{67}$$

Then, by Equations (62) and (67) and Gronwall's inequality, we obtain

$$\begin{aligned} \mathscr{E}_1(t) &\le e^{-\lambda t}\mathscr{E}_1(0) + C(1+t)^{-5/2} \\ &\quad \cdot\left(\|\nabla U_0\|_{L^2}^2 + \|U_0\|_{L^1}^2 + \mathscr{E}_\infty^2(t) + \mathscr{E}_\infty^4(t)\right), \end{aligned} \tag{68}$$

and hence,

$$\mathscr{E}_\infty(t) \le C\left(\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2 + \mathscr{E}_\infty^2(t) + \mathscr{E}_\infty^4(t)\right). \tag{69}$$

Thus, since $\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2$ can be small enough, one has

$$\mathscr{E}_\infty(t) \le C\left(\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2\right), \tag{70}$$

for all $t \ge 0$, that is,

$$\mathscr{E}_1(t) \le C(1+t)^{-5/2}\left(\|\nabla U_0\|_{H^3}^2 + \|U_0\|_{L^1}^2\right), \tag{71}$$

which means

$$\|\nabla(\varrho,u,\Theta)\|_{H^3} \le C(1+t)^{-5/4}, \tag{72}$$

for all $t \ge 0$, this obtain Equation (17). We can use the similar way to prove Equation (16) and omit here. This completes the proof of Theorem 3.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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