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Research Article

The Existence Result for a p-Kirchhoff-Type Problem Involving Critical Sobolev Exponent

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In this paper, by using the mountain pass theorem and the concentration compactness principle, we prove the existence of a positive solution for a p-Kirchhoff-type problem with critical Sobolev exponent.

1. Introduction and Main Result

In this article, we study the existence of a positive solution for the following p-Kirchhoff-type problem:

$$\begin{cases} -\mathscr{M}_{a,b}(u)\,\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, & \text{in}\, \Omega, \\ u = 0, & \text{on}\partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < N$, $\mathscr{M}_{a,b}(u) = a\|u\|^{p(r-1)} + b$, $a > 0$, $b \geq 0, \lambda > 0$, $p < rp < q < p^*$, $\|.\|$ is the usual norm in $W_0^{1,p}(\Omega)$ given by $\|u\|^p = \int_\Omega |\nabla u|^p dx$, and $p^* = pN/(N-p)$ is the critical Sobolev exponent corresponding to the noncompact embedding of $W_0^{1,p}(\Omega)$ into $L_{p^*}(\Omega)$. This problem contains an integral over Ω, and it is no longer a pointwise identity; therefore, it is often called nonlocal problem. It is also called nondegenerate if $b > 0$ and $a \geq 0$, while it is named degenerate if $b = 0$ and $a > 0$.

In the past several decades, much attention has been paid to the Kirchhoff-type problem which is closely related to the stationary analog of the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

proposed by Kirchhoff in [1] as an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations, where ρ, ρ_0, h, E, and L are constants. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations. These problems also serve to model other physical phenomena as biological systems where u describes a process which depends on the average of itself (for example, population density). The presence of the nonlocal term makes the theoretical study of these problems so difficult; then, they have attracted the attention of many researchers in particular after the work of Lions [2], where a functional analysis approach was proposed to attack them.

In the last few years, great attention has been paid to the study of Kirchhoff problems involving critical nonlinearities. These problems create many difficulties in applying variational methods because of the lack of the compactness of the Sobolev embedding. It is worth mentioning that the first work on the Kirchhoff-type problem with critical Sobolev exponent is Alves et al. in [3]. After that, many researchers dedicated to the study of several kinds of elliptic Kirchhoff equations with critical exponent of Sobolev in bounded domain or in the whole space $\mathbb{R}^N$; some interesting studies can be found in [4–9] and the references therein. More precisely, Naimen in [8] generalized the results of [10] to the semilinear Kirchhoff problem:

$$\begin{cases} -\left(b + a\int_{\Omega}|\nabla u|^2 dx\right)\Delta u = u^5 + \lambda f(x,u), u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, and $\lambda \in \mathbb{R}$. Under several conditions on f and λ, he proved the existence and nonexistence of solutions. For larger dimensional case, Figueiredo in [5] considers the case $N \geq 3$ if $\lambda > 0$ is sufficiently large. Matallah et al. in [7] studied the existence and nonexistence of solutions for the following p-Kirchhoff problem:

$$\begin{cases} -\mathscr{M}(u)\Delta_p u = |u|^{p^*-2}u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $M : R^+ \longrightarrow R^+$ is a continuous function satisfying some extra assumptions and $f : \Omega \times R \longrightarrow R$ is a continuous function satisfying some conditions. Benaissa and Matallah in [4] discussed the problem

$$-\left(a\int_{\mathbb{R}^N}|\nabla u|^p dx + b\right)\Delta_p u = |u|^{p^*-2}u + \lambda f(x) \quad \text{in } \mathbb{R}^N, \tag{5}$$

where f satisfies some conditions. Very recently, Benchira et al. in [11] have generalized the results of [12] to the nonlocal problem (1) with $q = p$, $\lambda \in (0, b\lambda_1)$, $b > 0$, $N \geq p^2$, $a > 0$ if $r < N/(N-p)$, and $0 < a < S^{-r}$ if $r = N/(N-p)$ (S is the best Sobolev constant for the imbedding $W_0^{1,p}((\Omega) \hookrightarrow L^{p^*}(\Omega))$.

Inspired by the above works, especially by [8, 11], we are devoted to studying the existence of positive solutions for problem (1) for all λ positive. In our problem, a typical difficulty occurs in proving the existence of solutions because of the lack of the compactness of the Sobolev embedding $W_0^{1,p}(\Omega) \longrightarrow L_{p^*}(\Omega)$. Furthermore, in view of the corresponding energy, the interaction between the Kirchhoff-type perturbation $\|u\|^{p(r-1)}$ and the critical nonlinearity $\int_\Omega |u|^{p^*} dx$ is crucial.

The main result of this paper is the following.

Theorem 1. *Assume that $a > 0$, $b \geq 0$, $1 < p < N$, and* max $\{rp, (N(p-1))/(N-p)\} < q < p^*$. *Then, problem (1) has a positive solution for all $\lambda > 0$.*

Remark 2. If $N \geq p^2$, then $\max\{rp, (N(p-1))/(N-p)\} = rp$. In the case where $q \leq rp$, it is difficult to show that a Palais-Smale sequence of the corresponding energy is bounded; in this case, the authors in [7, 9] used the truncation method to show the existence of solution under the condition " sufficiently large." Our objective in this paper is the existence of solution for all $\lambda > 0$.

Let us simply give a sketch of the Proof of Theorem 1. The main tool is variational methods; more precisely, by using the mountain pass theorem [13], we obtain a critical point of the corresponding energy. The main difficulties appear in the fact that problem (1) contains the critical Sobolev exponent; then, the functional energy does not satisfy the Palais-Smale condition in all range; to overcome the lack of compactness, we need to determine a good level of the Palais-Smale condition, and we must verify that the critical value is contained in the range of this level. This is the key point to obtain the existence of a mountain pass solution.

This paper is composed of two sections in addition to the introduction. In Section 2, we give some preliminary results which we will use later. Section 3 is devoted to the proof of main result.

2. Preliminary Results

In this paper, we use the following notations: $\longrightarrow$(resp $\rightharpoonup$) denotes strong (resp., weak) convergence, $o_n(1)$ denotes $o_n(1) \longrightarrow 0$ as $n \longrightarrow +\infty$, $B_R(x_0)$ is the ball centred at x_0 and of radius R, $u^- = \max\{-u, 0\}$, and $C, C_1, C_2, \cdots$, denote various positive constants. We define the best Sobolev constant for the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ as

$$S \coloneqq \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}}. \tag{6}$$

Recall that the infimum S is attained in $\mathbb{R}^N$ by the functions of the form

$$u_\varepsilon(x) \coloneqq \left(N\varepsilon\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{(N-p)/p^2} \left(\varepsilon + |x|^{p/(p-1)}\right)^{(p-N)/p}, \varepsilon > 0. \tag{7}$$

Moreover, u_ε satisfies

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx = S^{p^*/(p^*-p)}. \tag{8}$$

Let R be a positive constant and set $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ for $|x| \leq R$ and $\varphi(x) \equiv 1$ for $|x| \leq R/2$ and $B_R(0) \subset \Omega$. Set $v_\varepsilon(x) = \varphi(x)u_\varepsilon(x)$ and take $z_\varepsilon(x) = v_\varepsilon(x)\left(\int_\Omega |v_\varepsilon(x)|^{p^*} dx\right)^{-1/p^*}$ so that $\int_\Omega |z_\varepsilon|^{p^*} dx = 1$. Then, we have the well-known estimates as $\varepsilon \longrightarrow 0$:

$$\begin{cases} \|z_\varepsilon\|^p = S + O\left(\varepsilon^{N-p/p}\right), \\ \int_\Omega |z_\varepsilon|^q dx = \begin{cases} O\left(\varepsilon^{\left(((N-p)(p-1))/p^2\right)(p^*-q)}\right) & \text{if } q > \dfrac{N(p-1)}{N-p}, \\ O\left(\varepsilon^{\left((N-p)/p^2\right)q}|\ln \varepsilon|\right) & \text{if } q = \dfrac{N(p-1)}{N-p}, \\ O\left(\varepsilon^{\left((N-p)/p^2\right)q}\right) & \text{if } q < \dfrac{N(p-1)}{N-p}. \end{cases} \end{cases} \quad (9)$$

(See [14, 15]).

The energy function corresponding to problem (1) is given by

$$E(u) = \frac{a}{rp}\|u\|^{rp} + \frac{b}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}dx - \frac{\lambda}{q}\int_\Omega |u|^q dx, \forall u \in W_0^{1,p}(\Omega). \quad (10)$$

Notice that E is well defined in $W_0^{1,p}(\Omega)$ and belongs to $C^1(W_0^{1,p}(\Omega), \mathbb{R})$. We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a weak solution of (1), if for any $v \in W_0^{1,p}(\Omega)$ there holds

$$\mathscr{M}_{a,b}(u)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \int_\Omega |u|^{p^*-2}uv dx - \lambda\int_\Omega |u|^{q-2}uv dx = 0. \quad (11)$$

Hence, a critical point of functional E is a weak solution of problem (1).

Definition 3. Let $c \in \mathbb{R}$; a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ is called a $(\mathrm{PS})_c$ sequence (Palais-Smale sequence at level c) if

$$E(u_n) \longrightarrow c \text{ and } E'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \quad (12)$$

Let $c \in \mathbb{R}$. We say that E satisfies the Palais-Smale condition at level c, if any $(PS)_c$ sequence contains a convergent subsequence in $W_0^{1,p}(\Omega)$.

By [11], we have the following result.

Lemma 4. *Let $a > 0$, $b \geq 0$, $r, \theta > 1$, and $\tilde{y} = ((a/\theta)S^r)^{1/(\theta-1)}$. For $y \geq 0$, we consider the function $f_\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^*$, given by*

$$f_\theta(y) = S^{-1}y^\theta - aS^{r-1}y - b. \quad (13)$$

Then,

(1) when $b \neq 0$, the equation $f_\theta(y) = 0$ has a unique positive solution $y_0 > \tilde{y}$ and $f_\theta(y) \geq 0$ for all $y \geq y_0$

(2) when $b = 0$, the equation $f_\theta(y) = 0$ has a unique positive solution $y_1 = (aS^r)^{1/(\theta-1)}$ and $f_\theta(y) \geq 0$ for all $y \geq y_1$

3. Proof of Main Result

To prove our main result, we use the mountain pass theorem. First, we will verify that the functional E exhibits the mountain pass geometry.

Lemma 5. *Suppose that $a > 0$, $b \geq 0$, $1 < p < N$, and $rp < q < p^*$. Then, there exists $e \in W_0^{1,p}(\Omega) \setminus \{0\}$ and positive numbers δ_1 and ρ_1 such that*

(a) $E(u) \geq \delta_1 > 0$, with $\|u\| = \rho_1$

(b) $\|e\| > \rho_1$ and $E(e) < 0$

Proof.

(1) Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$; by Sobolev and Young inequalities, we have

$$E(u) = \frac{a}{rp}\|u\|^{rp} + \frac{b}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}dx - \frac{\lambda}{q}\int_\Omega |u|^q dx \geq -\frac{1}{p^*}S^{-p^*/p}\|u\|^{p^*} + \frac{a}{rp}\|u\|^{rp} + \frac{b}{p}\|u\|^p - \frac{\lambda}{q}C\|u\|^q. \quad (14)$$

Let $\rho = \|u\|$, from (14), one has

$$E(u) \geq \frac{a}{rp}\rho^{rp} - \frac{1}{p^*}S^{-p^*/p}\rho^{p^*} - \frac{\lambda}{q}C\rho^q. \quad (15)$$

As $rp < q < p^*$ and $a > 0$ there exists a sufficiently small positive numbers ρ_1 and δ_1 such that

$$E(u) \geq \delta_1 > 0, \text{with } \|u\| = \rho_1. \quad (16)$$

(2) Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$; as $p^* > rp$, it holds that $E(tu) \longrightarrow -\infty$ as $t \longrightarrow +\infty$, so we can easily find $e \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $\|e\| > \rho_1$, such that $E(e) < 0$. The proof is complete

□

We define

$$\Gamma = \left\{\xi \in C\left([0,1], W_0^{1,p}(\Omega)\right), \xi(0) = 0, \xi(1) = e\right\}, \quad c = \inf_{\xi \in \Gamma}\max_{t \in [0,1]} E(\xi(t)), \quad (17)$$

where e is taken from Lemma 5.

Now, we prove the following lemma which is important to ensure the local compactness of (PS) sequences for E.

Let y_0 and y_1 be defined in Lemma 4 and define

$$C^* := \frac{p^* - rp}{rpp^*} aS^r y_*^{r/(r-1)} + \frac{p^* - p}{pp^*} bSy_*^{1/(r-1)}, \quad (18)$$

with

$$y_* = \begin{cases} y_0 \text{ if } b \neq 0, \\ y_1 \text{ if } b = 0. \end{cases} \quad (19)$$

Lemma 6. *Assume that $a > 0$, $b \geq 0$, $1 < p < N$, $rp < q < p^*$, and $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a $(PS)_c$ sequence for E with*

$$c < C^*. \quad (20)$$

Then, $\{u_n\}$ contains a subsequence converging strongly in $W_0^{1,p}(\Omega)$.

Proof. As $n \longrightarrow \infty$ and $rp < q < p^*$, we have

$$\begin{aligned} c + o_n(1) - \frac{1}{q} o_n(1)\|u_n\| &= E(u_n) - \frac{1}{q}\left\langle E'(u_n), u_n \right\rangle \\ &\geq a\frac{q - rp}{rpq}\|u_n\|^{rp} + b\frac{q-p}{pq}\|u_n\|^p. \end{aligned} \quad (21)$$

Then, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by the concentration compactness principle due to Lions (see [6, 16]), there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{cases} |\nabla u_n|^p \rightharpoonup d\eta \geq |\nabla u|^p + \sum_{i\in I} \eta_i \tilde{\delta}_{x_i}, \\ |u_n|^{p^*} \rightharpoonup d\gamma = |u|^{p^*} + \sum_{i\in I} \gamma_i \tilde{\delta}_{x_i}, \end{cases} \quad (22)$$

where I is an at most countable index set, η_i, γ_i are nonnegative numbers, and $\tilde{\delta}_{x_i}$ is the Dirac mass at x_i. Moreover, by the Sobolev inequality, we infer that

$$\eta_i \geq S\gamma_i^{p/p^*} \text{ for all } i \in I. \quad (23)$$

We now claim that $I = \emptyset$. To this end, by contradiction, suppose that $I \neq \emptyset$; then, there exists $j \in I$. For $\varepsilon > 0$, let $\phi_{\varepsilon,j}$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j} \leq 1, \phi_{\varepsilon,j}|_{B_\varepsilon(x_j)} = 1$, $\phi_{\varepsilon,j}|_{\Omega\setminus B_{2\varepsilon}(x_j)} = 0$, and $|\nabla\phi_{\varepsilon,j}(x)| \leq 2/\varepsilon$. Clearly, $\{\phi_{\varepsilon,j}u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. It follows from $E'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ that

$$\begin{aligned} 0 = \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty} \Bigg(&\mathscr{M}_{a,b}(\|u_n\|)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\left(\phi_{\varepsilon,j}u_n\right)dx \\ &- \int_\Omega |u_n|^{p^*-2}u_n\left(\phi_{\varepsilon,j}u_n\right)dx - \lambda\int_\Omega |u_n|^{q-2}u_n\left(\phi_{\varepsilon,j}u_n\right)dx\Bigg). \end{aligned} \quad (24)$$

On the one hand, by Hölder's inequality and (6), we have

$$\begin{aligned} \left|\int_\Omega |u_n|^q \phi_{\varepsilon,j}\,dx\right| &\leq \left|\int_{B_{2\varepsilon}(x_j)} |u_n|^q \phi_{\varepsilon,j}dx\right| \\ &\leq \left(\int_{B_{2\varepsilon}(x_j)} |u_n|^{p^*}dx\right)^{q/p^*}\left(\int_{B_{2\varepsilon}(x_j)} dx\right)^{(p^*-q)/p^*} \\ &\leq C\|u_n\|^q \varepsilon^{N((p^*-q)/p^*)}. \end{aligned} \quad (25)$$

Since $N((p^*-q)/p^*) > 0$ and $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, then

$$\lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_\Omega |u_n|^q\phi_{\varepsilon,j}\,dx = 0. \quad (26)$$

Moreover, by using Hölder's inequality, we find

$$\begin{aligned} &\lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_{\varepsilon,j}dx\right| \\ &\leq \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_\Omega \left(|\nabla u_n|^p\right)^{(p-1)/p}\left|u_n\nabla\phi_{\varepsilon,j}\right|dx \\ &\leq \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left[\int_\Omega |\nabla u_n|^p\right]^{(p-1)/p}\left[\int_\Omega |u_n|^p\left|\nabla\phi_{\varepsilon,j}\right|^p dx\right]^{1/p} \\ &\leq C_1\lim_{\varepsilon\longrightarrow 0}\left[\int_\Omega |u|^p\left|\nabla\phi_{\varepsilon,j}\right|^p dx\right]^{1/p} \\ &\leq C_1\lim_{\varepsilon\longrightarrow 0}\left(\int_{B_{2\varepsilon}(x_j)} |u|^{p^*}dx\right)^{1/p^*}\left(\int_{B_{2\varepsilon}(x_j)}\left|\nabla\phi_{\varepsilon,j}\right|^{pp^*/(p^*-p)}dx\right)^{(p^*-p)/pp^*} \\ &\leq C_1\lim_{\varepsilon\longrightarrow 0}\left(\int_{B_{2\varepsilon}(x_j)} |u|^{p^*}dx\right)^{1/p^*}\left(\int_{B_{2\varepsilon}(x_j)}\left|\nabla\phi_{\varepsilon,j}\right|^N dx\right)^{1/N} \\ &\leq C_2\lim_{\varepsilon\longrightarrow 0}\left(\int_{B_{2\varepsilon}(x_j)} |u|^{p^*}dx\right)^{1/p^*} = 0. \end{aligned} \quad (27)$$

So, as $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we deduce that

$$\lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\mathscr{M}_{a,b}(\|u_n\|)\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_{\varepsilon,j}dx = 0. \quad (28)$$

By (22), (26), (28), and Hölder's inequality, we obtain

$$0 = \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left\langle E'(u_n), \phi_{\varepsilon,j}u_n\right\rangle \geq \left(b + a\eta_j^{r-1}\right)\eta_j - \gamma_j, \quad (29)$$

that is,

$$\gamma_j \geq b\eta_j + a\eta_j^r. \quad (30)$$

Then, by (23), we obtain

$$\gamma_j = 0 \text{ or } S^{-1}\left(\gamma_j\right)^{(p^*-p)/p^*} - aS^{r-1}\left(\gamma_j\right)^{(p/p^*)(r-1)} - b \geq 0. \quad (31)$$

Now, we assume by contradiction that $\gamma_j \neq 0$. Set $y = (\gamma_j)^{(p(r-1))/p^*}$ and $\theta = (p^* - p)/(p(r-1))$; then by (31), we get

$$S^{-1}y^{\theta} - aS^{r-1}y - b \geq 0. \tag{32}$$

It is clear that $\theta > 1$ thanks to $p^* > rp$. So, from (32) and the definition of f_θ in Lemma 4, we get

$$f_\theta(y) = S^{-1}y^{(p^*-p)/(p(r-1))} - aS^{r-1}y - b \geq 0. \tag{33}$$

According to Lemma 4, there exist $y_0 > (((ap(r-1))/(p^*-p))S^r)^{(p(r-1))/(p^*-rp)}$ and $y_1 = (aS^r)^{(p(r-1))/(p^*-rp)}$ such that $f_\theta(y_*) = 0$ and $f_\theta(y) \geq 0$ if $y \geq y_*$ with

$$y_* = \begin{cases} y_0 \text{ if } b \neq 0, \\ y_1 \text{ if } b = 0, \end{cases} \tag{34}$$

which implies that

$$S\left(\gamma_j\right)^{p/p^*} \geq Sy_*^{1/(r-1)}. \tag{35}$$

Moreover, using (23), we conclude that

$$\eta_j \geq S\gamma_j^{p/p^*} \geq Sy_*^{1/(r-1)}. \tag{36}$$

On the other hand, by the fact $p < rp < q < p^*$, one can get

$$\begin{aligned} c + o_n(1) &= E(u_n) - \frac{1}{q}\left\langle E'(u_n), u_n\right\rangle \\ &= b\frac{q-p}{qp}\|u_n\|^p + \frac{p^*-q}{qp^*}\int_\Omega |u_n|^{p^*}dx + a\frac{q-rp}{rqp}\|u_n\|^{rp} \\ &\geq b\frac{q-p}{qp}\left(\|u\|^p + \eta_j\right) + \frac{p^*-q}{qp^*}\left(\int_\Omega |u|^{p^*} + \gamma_j\right) \\ &\quad + a\frac{q-rp}{rqp}\left(\|u\|^p + \eta_j\right)^r \\ &\geq b\frac{q-p}{qp}\|u\|^p + \frac{p^*-q}{qp^*}\int_\Omega |u|^{p^*} + a\frac{q-rp}{rqp}\|u\|^{rp} \\ &\quad + b\frac{q-p}{qp}\eta_j + \frac{p^*-q}{qp^*}\gamma_j + a\frac{q-rp}{rqp}\eta_j^r, \end{aligned} \tag{37}$$

which implies that

$$\begin{aligned} c &\geq b\frac{q-p}{qp}Sy_*^{1/(r-1)} + \frac{p^*-q}{qp^*}\left(Sy_*^{1/(r-1)}\right)^{p^*/p}S^{-p^*/p} + a\frac{q-rp}{rqp}\left(Sy_*^{1/(r-1)}\right)^r \\ &\geq a\frac{p^*-rp}{rpp^*}\left(Sy_*^{1/(r-1)}\right)^r + b\frac{p^*-p}{pp^*}Sy_*^{1/(r-1)} - a\frac{p^*-rp}{rpp^*}\left(Sy_*^{1/(r-1)}\right)^r \\ &\quad - b\frac{p^*-p}{pp^*}Sy_*^{1/(r-1)} + b\frac{q-p}{qp}Sy_*^{1/(r-1)} \\ &\quad + \frac{p^*-q}{qp^*}\left(Sy_*^{1/(r-1)}\right)^{p^*/p}S^{-p^*/p} + a\frac{q-rp}{rqp}\left(Sy_*^{1/(r-1)}\right)^r \\ &\geq a\frac{p^*-rp}{rpp^*}\left(Sy_*^{1/(r-1)}\right)^r + b\frac{p^*-p}{pp^*}Sy_*^{1/(r-1)} \\ &\quad + \frac{p^*-q}{qp^*}\left(Sy_*^{1/(r-1)}\right)^{p^*/p}S^{-p^*/p} + -a\frac{p^*-q}{qp^*}\left(Sy_*^{1/(r-1)}\right)^r - b\frac{p^*-q}{qp^*}Sy_*^{1/(r-1)} \\ &\geq a\frac{p^*-rp}{rpp^*}S^r y_*^{r/(r-1)} + b\frac{p^*-p}{pp^*}Sy_*^{1/(r-1)} \\ &\quad + \frac{p^*-q}{qp^*}Sy_*^{1/(r-1)}\left(S^{-1}y_*^{(p^*-p)/(p(r-1))} - aS^{r-1}y_* - b\right) \\ &= C^* + \frac{p^*-q}{qp^*}Sy_*^{1/(r-1)}f_\theta(y_*) = C^*. \end{aligned} \tag{38}$$

This is a contradiction. Hence, I is empty and so

$$\int_\Omega |u_n|^{p^*}dx \longrightarrow \int_\Omega |u|^{p^*}dx. \tag{39}$$

On the other hand, we have

$$\begin{aligned} \left\langle E'(u_n), u_n\right\rangle &= \mathscr{M}_{a,b}(u)\|u_n\|^p \\ &\quad - \int_\Omega |u_n|^{p^*}dx - \lambda\int_\Omega |u_n|^q dx = o_n(1), \end{aligned} \tag{40}$$

$$\begin{aligned} \left\langle E'(u_n), v\right\rangle &= \mathscr{M}_{a,b}(u)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx \\ &\quad - \int_\Omega |u_n|^{p^*-2}u_n v dx - \lambda\int_\Omega |u_n|^{q-2}u_n v dx = o_n(1), \end{aligned} \tag{41}$$

for any $v \in W_0^{1,p}(\Omega)$. Set $l = \lim\|u_n\|$ as $n \longrightarrow +\infty$; then, from (40) and (41), we deduce that

$$\left(al^{p(r-1)} + b\right)l^p - \int_\Omega |u|^{p^*}dx - \lambda\int_\Omega |u|^q dx = 0, \tag{42}$$

$$\begin{aligned} &\left(al^{p(r-1)} + b\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \\ &\quad - \int_\Omega |u|^{p^*-2}uv dx - \lambda\int_\Omega |u|^{q-2}uv dx = 0. \end{aligned} \tag{43}$$

Taking the test function $v = u$ in (43), we get

$$\left(al^{p(r-1)} + b\right)\|u\|^p - \int_\Omega |u|^{p^*}dx - \lambda\int_\Omega |u|^q dx = 0. \tag{44}$$

Therefore, the equalities (42) and (44) imply that $\|u\| = l$. Consequently, $\{u_n\}$ converges strongly in $W_0^{1,p}(\Omega)$, which is the desired result.

The energy functional E satisfies the Palais-Smale condition at level c for any $c < C^*$. So, the existence of the solution follows immediately from the following lemma. □

Lemma 7. *Let $a > 0$, $b \geq 0$, $1 < p < N$, and*

$$\max\left\{rp, \frac{N(p-1)}{N-p}\right\} < q < p^*. \quad (45)$$

Then,

$$\sup_{t\geq 0} E(tz_\varepsilon) < C^*. \quad (46)$$

Proof. We define the functions g and h such that

$$g(t) = E(tz_\varepsilon) = \frac{a}{rp} t^{rp}\|z_\varepsilon\|^{rp} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{1}{p^*} t^{p^*} - \frac{\lambda}{q} t^q \int_\Omega |z_\varepsilon|^q dx,$$
$$h(t) = \frac{a}{rp} t^{rp}\|z_\varepsilon\|^{rp} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{1}{p^*} S^{-p^*/p} \|z_\varepsilon\|^{p^*} t^{p^*}. \quad (47)$$

Then,

$$g(t) = h(t) - \frac{t^{p^*}}{p^*}\left(1 - S^{-p^*/p}\|z_\varepsilon\|^{p^*}\right) - \lambda \frac{t^q}{q}\int_\Omega |z_\varepsilon|^q dx. \quad (48)$$

Note that $\lim_{t\longrightarrow+\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0, so $\sup_{t\geq 0} g(t)$ is attained for some $T_\varepsilon > 0$. Furthermore, from $g'(T_\varepsilon) = 0$, it follows that

$$-T_\varepsilon^{p^*-1} + aT_\varepsilon^{rp-1}\|z_\varepsilon\|^{rp} + bT_\varepsilon^{p-1}\|z_\varepsilon\|^p = \lambda T_\varepsilon^{q-1}\int_\Omega |z_\varepsilon|^q dx, \quad (49)$$

$$-\lambda T_\varepsilon^{q-1}\int_\Omega |z_\varepsilon|^q dx + aT_\varepsilon^{rp-1}\|z_\varepsilon\|^{rp} + bT_\varepsilon^{p-1}\|z_\varepsilon\|^p = T_\varepsilon^{p^*-1}. \quad (50)$$

By multiplying the equation in (49) by T_ε^{1-p}, we obtain

$$-T_\varepsilon^{p^*-p} + aT_\varepsilon^{p(r-1)}\|z_\varepsilon\|^{rp} + b\|z_\varepsilon\|^p > 0. \quad (51)$$

Easy computations show that

$$T_\varepsilon \leq \left(\frac{ap(r-1)}{p^*-p}\|z_\varepsilon\|^{rp}\right)^{1/(p^*-rp)}. \quad (52)$$

By applying (9), we have for ε small enough

$$T_\varepsilon \leq \left(\frac{ap(r-1)}{p^*-p} S^r\right)^{1/(p^*-rp)} =: \tau_0. \quad (53)$$

On the other hand, we multiply the equation in (50) by T_ε^{1-rp} and by recalling (53), we obtain

$$\begin{aligned} T_\varepsilon^{p^*-rp} &\geq a\|z_\varepsilon\|^{rp} - \lambda T_\varepsilon^{q-rp}\int_\Omega |z_\varepsilon|^q dx \\ &\geq a\|z_\varepsilon\|^{rp} - \lambda(\tau_0)^{q-rp}\int_\Omega |z_\varepsilon|^q dx. \end{aligned} \quad (54)$$

By applying (9), we have for ε small enough

$$T_\varepsilon \geq \left(aS^r\right)^{1/(p^*-rp)} =: \tau_1. \quad (55)$$

Now, we estimate $g(T_\varepsilon)$.

It follows from $h'(t) = 0$ that

$$-S^{-p^*/p}\|z_\varepsilon\|^{p^*} t^{p^*-1} + at^{rp-1}\|z_\varepsilon\|^{rp} + bt^{p-1}\|z_\varepsilon\|^p = 0, \quad (56)$$

that is,

$$-\left[S^{-p^*/p}\|z_\varepsilon\|^{p^*-p} t^{p^*-p} - at^{p(r-1)}\|z_\varepsilon\|^{p(r-1)} - b\right] = 0. \quad (57)$$

Set

$$y = t^{p(r-1)} S^{1-r}\|z_\varepsilon\|^{p(r-1)}, \quad (58)$$

and $\theta = (p^* - p)/(p(r-1))$. As $\theta > 1$, then by (57), we get

$$-\left[S^{-1}y^\theta - aS^{r-1}y - b\right] = -f_\theta(y) = 0, \quad (59)$$

which implies from Lemma 4 that $f_\theta(y_*) = 0$ with y_* defined in (34). Therefore, $h'(t_*) = 0$, where

$$t_* = S^{1/p}\|z_\varepsilon\|^{-1}(y_*)^{1/(p(r-1))}. \quad (60)$$

As $f_\theta(y)$ is concave, then $h'(t)$ is convex and so

$$\max_{t\geq 0} h(t) = h(t_*) = -\frac{1}{p^*} S^{-p^*/p}\|z_\varepsilon\|^{p^*} t_*^{p^*} + \frac{a}{rp}\|z_\varepsilon\|^{rp} t_*^{rp} + \frac{b}{p}\|z_\varepsilon\|^p t_*^p. \quad (61)$$

Since $h'(t_*) = 0$, one has

$$S^{-p^*/p}\|z_\varepsilon\|^{p^*} t_*^{p^*} = a\|z_\varepsilon\|^{rp} t_*^{rp} + b\|z_\varepsilon\|^p t_*^p. \quad (62)$$

So, we deduce that

$$\begin{aligned} \max_{t\geq 0} h(t) &= -\frac{1}{p^*}\left(a\|z_\varepsilon\|^{rp} t_*^{rp} + b\|z_\varepsilon\|^p t_*^p\right) + \frac{a}{rp}\|z_\varepsilon\|^{rp} t_*^{rp} + \frac{b}{p}\|z_\varepsilon\|^p t_*^p \\ &= a\left(\frac{1}{rp} - \frac{1}{p^*}\right) t_*^{rp}\|z_\varepsilon\|^{rp} + b\left(\frac{1}{p} - \frac{1}{p^*}\right) t_*^p \|z_\varepsilon\|^p \\ &= a\left(\frac{1}{rp} - \frac{1}{p^*}\right) S^r y_*^{r/(r-1)} + b\left(\frac{1}{p} - \frac{1}{p^*}\right) S y_*^{1/(r-1)} = C^*. \end{aligned} \quad (63)$$

Consequently, by (9) and as $q > (N(p-1))/(N-p)$, we have

$$\begin{aligned}\sup_{t\geq 0} E(tz_\varepsilon) &= g(T_\varepsilon) = h(T_\varepsilon) + \frac{1}{p^*}\left(S^{-p^*/p}\|z_\varepsilon\|^{p^*} - 1\right)T_\varepsilon^{p^*} - \frac{\lambda}{q}T_\varepsilon^q\int_\Omega |z_\varepsilon|^q dx \\ &\leq C^* + \frac{1}{p^*}\left(S^{-p^*/p}\|z_\varepsilon\|^{p^*} - 1\right)T_\varepsilon^{p^*} - \frac{\lambda}{q}T_\varepsilon^q\int_\Omega |z_\varepsilon|^q dx \\ &\leq C^* + \frac{1}{p^*}O\left(\varepsilon^{(N-p)/p}\right)(\tau_1)^{p^*} - \frac{\lambda}{q}(\tau_0)^q O\left(\varepsilon^{\left(((N-p)(p-1))/p^2\right)(p^*-q)}\right) \\ &\leq C^* + O\left(\varepsilon^{(N-p)/p}\right) - O\left(\varepsilon^{\left(((N-p)(p-1))/p^2\right)(p^*-q)}\right).\end{aligned} \tag{64}$$

Taking ε small enough, we obtain $\sup_{t\geq 0} E(tz_\varepsilon) < C^*$. Thus, the proof of this lemma is completed. □

Now, we can proof the existence of a mountain pass-type solution.

Proof of Theorem 1. Applying Lemma 5, we get that E possesses a mountain pass geometry. Then, from the mountain pass theorem [13], there exists a $(\mathrm{PS})_c$ sequences $\{u_n\} \subset W_0^{1,p}(\Omega)$ of E. According to Lemmas 6 and 7, $\{u_n\}$ has a subsequence (still denoted by $\{u_n\}$) such that $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$. Hence, u is a critical point of E and therefore a solution of (1). □

Now, we show that $u > 0$. To obtain a contradiction assume that $u = u^-$. We have

$$\begin{aligned}0 = \left\langle E'(u), u^-\right\rangle &= \left(a\|u\|^{p(r-1)} + b\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u^- dx \\ &\quad - \int_\Omega |u|^{p^*-2}uu^- dx - \lambda\int_\Omega |u|^{q-2}uu^- dx \\ &\geq \left(a\|u\|^{p(r-1)} + b\right)\|u^-\|^p + \int_\Omega |u|^{p^*}dx + \lambda\int_\Omega |u|^q dx \geq b\|u^-\|^p.\end{aligned} \tag{65}$$

Then, $u^- = 0$. By the strong maximum principle [17], one has $u > 0$. Theorem 1 can be concluded.

Data Availability

Data from functional analysis and variational methods used to support the results of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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