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Research Article

Exponentially Convex Functions on the Coordinates and Novel Estimations via Riemann-Liouville Fractional Operator

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In this study, the modification of the concept of exponentially convex function, which is a general version of convex functions, given on the coordinates, is recalled. With the help of an integral identity which includes the Riemann-Liouville (RL) fractional integral operator, new Hadamard-type inequalities are proved for exponentially convex functions on the coordinates. Many special cases of the results are discussed.

1. Introduction

The Hermite-Hadamard (HH) inequality, which has an important place in the inequality theory, which produces lower and upper bounds for the mean value of convex functions, has been the focus of attention of researchers working in this field. This famous inequality is presented as follows.

Let $\Psi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $\epsilon_1 < \epsilon_2$. The following double inequality

$$\Psi\left(\frac{\epsilon_1 + \epsilon_2}{2}\right) \leq \frac{1}{\epsilon_2 - \epsilon_1}\int_{\epsilon_1}^{\epsilon_2} \Psi(x)dx \leq \frac{\Psi(\epsilon_1) + \Psi(\epsilon_2)}{2}, \quad (1)$$

is called HH inequality. In the case of Ψ is concave, one has the above inequalities in the reversed direction.

Generalizations of the HH inequality for different kinds of convex functions, new versions, and variants with the potential to produce new limits have been derived. Essentially, researchers have not only obtained numerous new findings in the theory of inequality thanks to the concept of a convex function, which is the basis of this inequality, but also mentioned the applications of these findings in numerical integration, statistics, and other branches of mathematics. Undoubtedly, the contribution of different kinds of convex function classes to these efforts is incredible. The exponentially convex function definition, which is a product of these efforts, is given by Awan et al. as follows.

Definition 1 (See [1]). A function $\Psi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function, if

$$\Psi((1-\zeta)u_1 + \zeta u_2) \leq (1-\zeta)\frac{\Psi(u_1)}{e^{\alpha u_1}} + \zeta\frac{\Psi(u_2)}{e^{\alpha u_2}}, \quad (2)$$

for all $u_1, u_2 \in I, \zeta \in [0, 1]$ and $\alpha \in \mathbb{R}$.

In [2], the author has given some general classes of functions which are called strongly preinvex functions. Several extensions and generalizations can be found in the literature. In [3], Dragomir mentioned another important modification of convexity that causes the field of inequalities to expand into multidimensional spaces as follows.

Definition 2. Let us consider the bidimensional interval $\Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2, \epsilon_3 < \epsilon_4$. Recall that the mapping $\Psi : \Delta \longrightarrow \mathbb{R}$ is convex on Δ if the following inequality holds

$$\Psi(\nu\epsilon_1 + (1-\nu)\epsilon_2, \nu\epsilon_3 + (1-\nu)\epsilon_4) \le \nu\Psi(\epsilon_1, \epsilon_2) + (1-\nu)\Psi(\epsilon_3, \epsilon_4), \tag{3}$$

for all $(\epsilon_1, \epsilon_2), (\epsilon_3, \epsilon_4) \in \Delta$ and $\nu \in [0, 1]$.

In [3], the expansion of the HH inequality for convex functions on the coordinates on a rectangle from the plane $\mathbb{R}^2$ has been proved by Dragomir.

Theorem 3. *Suppose that $\Psi : \Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4] \longrightarrow \mathbb{R}$ is convex on the coordinates on Δ. Then, the following inequalities are valid;*

$$\begin{aligned}\Psi\left(\frac{\epsilon_1+\epsilon_2}{2}, \frac{\epsilon_3+\epsilon_4}{2}\right) &\le \frac{1}{2}\left[\frac{1}{\epsilon_2-\epsilon_1}\int_{\epsilon_1}^{\epsilon_2}\Psi\left(x, \frac{\epsilon_3+\epsilon_4}{2}\right)du_1 + \frac{1}{\epsilon_4-\epsilon_3}\int_{\epsilon_3}^{\epsilon_4}\Psi\left(\frac{\epsilon_1+\epsilon_2}{2}, u_2\right)du_2\right] \\ &\le \frac{1}{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}\Psi(u_1, u_2)du_1du_2 \\ &\le \frac{1}{4}\left[\frac{1}{(\epsilon_2-\epsilon_1)}\int_{\epsilon_1}^{\epsilon_2}\Psi(u_1, \epsilon_3)du_1 + \frac{1}{(\epsilon_2-\epsilon_1)}\int_{\epsilon_1}^{\epsilon_2}\Psi(u_1, \epsilon_4)du_1 + \frac{1}{(\epsilon_4-\epsilon_3)}\int_{\epsilon_3}^{\epsilon_4}\Psi(\epsilon_1, u_2)du_2 + \frac{1}{(\epsilon_4-\epsilon_3)}\int_{\epsilon_3}^{\epsilon_4}\Psi(\epsilon_2, u_2)du_2\right] \\ &\le \frac{\Psi(\epsilon_1, \epsilon_3) + \Psi(\epsilon_1, \epsilon_4) + \Psi(\epsilon_2, \epsilon_3) + \Psi(\epsilon_2, \epsilon_4)}{4}.\end{aligned} \tag{4}$$

The above inequalities are sharp.

In [4], Alomari and Darus have extended the concept of log-convexity to the coordinates and given some inequalities of Hadamard type for two variables. The modification of Jensen inequality on the coordinates has been given by Bakula and Pečarić in [5]. They have also proved several new variants of Jensen inequality on the coordinates. In [6], Özdemir et al. have established the notion $m-$ and $(\alpha, m)-$ convex functions on the coordinates and given several novel inequalities based on these new definitions. In [7], the authors have obtained some new integral inequalities for two variables by using partial differentiable convex and $s-$convex functions. In the same paper, they have given a new integral identity to provide the main findings. In [8], the authors have used a different method and proved some new inequalities of the Pachpatte type for the product of two $s-$convex functions on the coordinates.

A new variant of the HH inequality in multidimensional spaces has been obtained for convex functions by Sarikaya et al. as follows:

Theorem 4 (See [9]). *Let $\Psi : \Delta \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2$ and $\epsilon_3 < \epsilon_4$. If $|\partial^2\Psi/\partial\zeta\partial\xi|$ is a convex function on the coordinates on Δ, then one has the inequalities:*

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1, \epsilon_3) + \Psi(\epsilon_1, \epsilon_4) + \Psi(\epsilon_2, \epsilon_3) + \Psi(\epsilon_2, \epsilon_4)}{4}\frac{1}{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}\Psi(x, y)dxdy - A\right| \\ &\quad\le \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{16}\left(\frac{|\partial^2\Psi/\partial\zeta\partial\xi|(\epsilon_1, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|(\epsilon_1, \epsilon_4) + |\partial^2\Psi/\partial\zeta\partial\xi|(\epsilon_2, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|(\epsilon_2, \epsilon_4)}{4}\right),\end{aligned} \tag{5}$$

where

$$A = \frac{1}{2}\left[\frac{1}{(\epsilon_2-\epsilon_1)}\int_{\epsilon_1}^{\epsilon_2}[\Psi(x, \epsilon_3) + \Psi(x, \epsilon_4)]dx + \frac{1}{(\epsilon_4-\epsilon_3)}\int_{\epsilon_3}^{\epsilon_4}[\Psi(\epsilon_1, y)dy + \Psi(\epsilon_2, y)]dy\right]. \tag{6}$$

Theorem 5. *Let $\Psi : \Delta \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2$ and $\epsilon_3 < \epsilon_4$. If $|\partial^2\Psi/\partial\zeta\partial\xi|^q, q > 1$, is a convex function on the coordinates on Δ, then the following inequalities are valid:*

$$\begin{aligned} &\left|\frac{\Psi(\epsilon_1, \epsilon_3) + \Psi(\epsilon_1, \epsilon_4) + \Psi(\epsilon_2, \epsilon_3) + \Psi(\epsilon_2, \epsilon_4)}{4} \frac{1}{(\epsilon_2 - \epsilon_1)(\epsilon_4 - \epsilon_3)} \int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4} \Psi(x, y)dxdy - A\right| \\ &\quad \leq \frac{(\epsilon_2 - \epsilon_1)(\epsilon_4 - \epsilon_3)}{4(p+1)^{2/p}} \left(\frac{|\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_1, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_1, \epsilon_4) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_2, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_2, \epsilon_4)}{4}\right)^{1/q}, \end{aligned} \tag{7}$$

where

$$A = \frac{1}{2}\left[\frac{1}{(\epsilon_2 - \epsilon_1)} \int_{\epsilon_1}^{\epsilon_2} [\Psi(x, \epsilon_3) + \Psi(x, \epsilon_4)]dx + \frac{1}{(\epsilon_4 - \epsilon_3)} \int_{\epsilon_3}^{\epsilon_4} [\Psi(\epsilon_1, y)dy + \Psi(\epsilon_2, y)]dy\right], \tag{8}$$

and $1/p + 1/q = 1$.

Theorem 6. *Let $\Psi : \Delta \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2$ and $\epsilon_3 < \epsilon_4$. If $|\partial^2\Psi/\partial\zeta\partial\xi|^q, q \geq 1$, is a convex function on the coordinates on Δ, then the following inequalities are valid:*

$$\begin{aligned} &\left|\frac{\Psi(\epsilon_1, \epsilon_3) + \Psi(\epsilon_1, \epsilon_4) + \Psi(\epsilon_2, \epsilon_3) + \Psi(\epsilon_2, \epsilon_4)}{4} \frac{1}{(\epsilon_2 - \epsilon_1)(\epsilon_4 - \epsilon_3)} \int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4} \Psi(x, y)dxdy - A\right| \\ &\quad \leq \frac{(\epsilon_2 - \epsilon_1)(\epsilon_4 - \epsilon_3)}{16} \left(\frac{|\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_1, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_1, \epsilon_4) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_2, \epsilon_3) + |\partial^2\Psi/\partial\zeta\partial\xi|^q(\epsilon_2, \epsilon_4)}{4}\right)^{1/q}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} A = \frac{1}{2}\bigg[&\frac{1}{(\epsilon_2 - \epsilon_1)} \int_{\epsilon_1}^{\epsilon_2} [\Psi(x, \epsilon_3) + \Psi(x, \epsilon_4)]dx \\ &+ \frac{1}{(\epsilon_4 - \epsilon_3)} \int_{\epsilon_3}^{\epsilon_4} [\Psi(\epsilon_1, y)dy + \Psi(\epsilon_2, y)]dy\bigg]. \end{aligned} \tag{10}$$

The concept of exponentially convex function on the coordinates and the associated results are presented as the following:

Definition 7 (See [10]). Let us consider the interval such as $\Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2, \epsilon_3 < \epsilon_4$. The function $\Psi : \Delta \longrightarrow \mathbb{R}$ is exponentially convex on Δ if

$$\Psi((1-\zeta)u_1 + \zeta u_3, (1-\zeta)u_2 + \zeta u_4) \leq (1-\zeta)\frac{\Psi(u_1, u_2)}{e^{\alpha(u_1+u_2)}} + \zeta\frac{\Psi(u_3, u_4)}{e^{\alpha(u_3+u_4)}}, \tag{11}$$

for all $(u_1, u_2), (u_3, u_4) \in \Delta, \alpha \in \mathbb{R}$, and $\zeta \in [0, 1]$.

An equivalent definition of the exponentially convex function definition in coordinates can be done as follows:

Definition 8 (See [10]). The mapping $\Psi : \Delta \longrightarrow \mathbb{R}$ is exponentially convex function on the coordinates on Δ, if

$$\begin{aligned} \Psi(\zeta\epsilon_1 + (1-\zeta)\epsilon_2, \xi\epsilon_3 + (1-\xi)\epsilon_4) &\leq \zeta\xi\frac{\Psi(\epsilon_1, \epsilon_3)}{e^{\alpha(\epsilon_1+\epsilon_3)}} \\ &+ \zeta(1-\xi)\frac{\Psi(\epsilon_1, \epsilon_4)}{e^{\alpha(\epsilon_1+\epsilon_4)}} + (1-\zeta)\xi\frac{\Psi(\epsilon_2, \epsilon_3)}{e^{\alpha(\epsilon_2+\epsilon_3)}} \\ &+ (1-\zeta)(1-\xi)\frac{\Psi(\epsilon_2, \epsilon_4)}{e^{\alpha(\epsilon_2+\epsilon_4)}}, \end{aligned} \tag{12}$$

for all $(\epsilon_1, \epsilon_3), (\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_3), (\epsilon_2, \epsilon_4) \in \Delta, \alpha \in \mathbb{R}$, and $\zeta, \xi \in [0, 1]$.

Lemma 9 (See [10]). *A function $\Psi : \Delta \longrightarrow \mathbb{R}$ will be called exponentially convex on the coordinates if the partial mappings $\Psi_{u_2} : [\epsilon_1, \epsilon_2] \longrightarrow \mathbb{R}, \Psi_{u_2}(u) = e^{\alpha u_2}\Psi(u, u_2)$, and $\Psi_{u_1} : [\epsilon_3, \epsilon_4] \longrightarrow \mathbb{R}, \Psi_{u_1}(v) = e^{\alpha u_1}\Psi(u_1, v)$ are exponentially convex where defined for all $u_2 \in [\epsilon_3, \epsilon_4]$ and $u_1 \in [\epsilon_1, \epsilon_2]$.*

Based on the definition of exponentially convex functions on the coordinates the following HH-type inequality is valid:

Theorem 10 (See [10]). *Let $\Psi : \Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4] \longrightarrow \mathbb{R}$ be partial differentiable mapping on $\Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$, and $\Psi \in L(\Delta)$,$\alpha \in \mathbb{R}$. If Ψ is exponentially convex function on the coordinates on Δ, then one has*

$$\begin{aligned}&\frac{1}{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}\Psi(x,y)dxdy\\&\quad\leq\frac{1}{4}\left[\frac{\Psi(\epsilon_1,\epsilon_3)}{e^{\alpha(\epsilon_1+\epsilon_3)}}+\frac{\Psi(\epsilon_1,\epsilon_4)}{e^{\alpha(\epsilon_1+\epsilon_4)}}+\frac{\Psi(\epsilon_2,\epsilon_3)}{e^{\alpha(\epsilon_2+\epsilon_3)}}+\frac{\Psi(\epsilon_2,\epsilon_4)}{e^{\alpha(\epsilon_2+\epsilon_4)}}\right].\end{aligned} \tag{13}$$

In the field of inequality theory, numerous new results have been produced on the concept of convexity with the help of classical derivatives and integral operators. However, fractional analysis, which has rapidly increased its influence in mathematics and many applied sciences, has also become very popular in the field of inequalities in recent years. Fractional derivative and integral operator, which differ in terms of their kernel structures and offer new approaches to many problems with these differences, are the only reason for the rapid development of fractional analysis. Because researchers have made comparisons by examining each new operator in terms of innovation and differences offered to problem solutions. In particular, differences such as time memory effect, singularity, and locality reveal the efficiency and functionality of the operators. We will now continue by introducing the RL fractional operators, one of the cornerstones of fractional analysis.

Definition 11 (See [11]). Let $\Psi \in L_1[\epsilon_1, \epsilon_2]$. The RL integrals $J^{\alpha}_{\epsilon_1^+}\Psi$ and $J^{\alpha}_{\epsilon_2^-}\Psi$ of order $\alpha > 0$ with $\epsilon_1 \geq 0$ are defined by

$$\begin{aligned}J^{\alpha}_{\epsilon_1^+}\Psi(u_1)&=\frac{1}{\Gamma(\alpha)}\int_{\epsilon_1}^{u_1}(u_1-\zeta)^{\alpha-1}\Psi(\zeta)d\zeta,\quad u_1>\epsilon_1,\\J^{\alpha}_{\epsilon_2^-}\Psi(u_1)&=\frac{1}{\Gamma(\alpha)}\int_{u_1}^{\epsilon_2}(\zeta-u_1)^{\alpha-1}\Psi(\zeta)d\zeta,\quad u_1<\epsilon_2,\end{aligned} \tag{14}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-\zeta} u^{\alpha-1} du$, here is $J^0_{\epsilon_1^+}\Psi(u_1) = J^0_{\epsilon_2^-}\Psi(u_1) = \Psi(u_1)$.

In the above definition, if we set $\alpha = 1$, the definition overlaps with the classical integral.

Several researchers have studied fractional integral operators to obtain new and more general results in inequality theory. In [12], Dahmani has proved some novel inequalities via fractional integral operators. Also, in [13], he proved Minkowski and Hadamard-type inequalities by using some basic properties of the functions. In [14, 15], Dahmani et al. gave some new results of Grüss and Chebyshev type by using Riemann-Liouville fractional integral operators. These papers have extended the discussion of inequality theory to fractional inequalities and several general forms of the earlier works have been provided. In [16], Sarikaya and Ogunmez have proved some Hadamard-type inequalities by using the Riemann-Liouville fractional integral operators using the definitions of convex functions. Interested readers can find definitions, extensions, refinements, and properties of fractional integral operators in [17]. In [11], Miller and Ross have made efforts to establish a survey for relationships between fractional calculus and differential equations. In [18], the authors have given the refinement of the famous Hermite-Hadamard inequality via Riemann-Liouville fractional integrals. Also, in the same paper, they have obtained a new integral identity to produce some fractional inequalities. In [19], Agarwal et al. have proved many new inequalities of the Hadamard type by using the generalized $k-$fractional integral operator. Recently, several new integral inequalities of Jensen, Jensen-Mercer, and Hadamard type have been provided via generalized fractional integral operators in [20, 21] and [22].

Definition 12 (See [23]). Let $\Psi \in L_1([\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4])$. The RL integrals $J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_3^+}$, $J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_4^-}$, $J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_3^+}$, and $J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_4^-}$ of order $\alpha, \beta > 0$ with $\epsilon_1, \epsilon_3 \geq 0$ are defined by

$$\begin{aligned}J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_3^+}\Psi(u_1,u_2)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{u_1}\int_{\epsilon_3}^{u_2}(u_1-\zeta)^{\alpha-1}(u_2-\xi)^{\beta-1}\Psi(\zeta,\xi)d\xi d\zeta,\quad u_1>\epsilon_1,u_2>\epsilon_3,\\J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_4^-}\Psi(u_1,u_2)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{u_1}\int_{u_2}^{\epsilon_4}(u_1-\zeta)^{\alpha-1}(\xi-u_2)^{\beta-1}\Psi(\zeta,\xi)d\xi d\zeta,\quad u_1>\epsilon_1,u_2<\epsilon_4,\\J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_3^+}\Psi(u_1,u_2)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{u_1}^{\epsilon_2}\int_{\epsilon_3}^{u_2}(\zeta-u_1)^{\alpha-1}(u_2-\xi)^{\beta-1}\Psi(\zeta,\xi)d\xi d\zeta,\quad u_1<\epsilon_2,u_2>\epsilon_3,\\J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_4^-}\Psi(u_1,u_2)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{u_1}^{\epsilon_2}\int_{u_2}^{\epsilon_4}(\zeta-u_1)^{\alpha-1}(\xi-u_2)^{\beta-1}\Psi(\zeta,\xi)d\xi d\zeta,\quad u_1<\epsilon_2,u_2<\epsilon_4,\end{aligned} \tag{15}$$

respectively. Here, Γ is the Gamma function,

$$\begin{aligned} J^{0,0}_{\epsilon_1^+,\epsilon_3^+}\Psi(u_1,u_2) &= J^{0,0}_{\epsilon_1^+,\epsilon_4^-}\Psi(u_1,u_2) = J^{0,0}_{\epsilon_2^-,\epsilon_3^+}\Psi(u_1,u_2) \\ &= J^{0,0}_{\epsilon_2^-,\epsilon_4^-}\Psi(u_1,u_2) = \Psi(u_1,u_2), \\ J^{1,1}_{\epsilon_1^+,\epsilon_3^+}\Psi(u_1,u_2) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{u_1}\int_{\epsilon_3}^{u_2}\Psi(t,s)dsdt. \end{aligned} \tag{16}$$

To make the paper more readable, it will be effective to use the following notations in the main findings:

$$\begin{aligned} A &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\epsilon_2-\epsilon_1)^\alpha(\epsilon_4-\epsilon_3)^\beta}\left[J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_4^-}\Psi(x,y) + J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_4^-}\Psi(x,y)\right. \\ &\quad \left. + J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_3^+}\Psi(x,y) + J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_3^+}\Psi(x,y)\right] - \frac{\Gamma(\beta+1)}{4(\epsilon_4-\epsilon_3)^\beta} \\ &\quad \cdot\left[J^{\beta}_{\epsilon_4^-}\Psi(\epsilon_1,\epsilon_3) + J^{\beta}_{\epsilon_4^-}\Psi(\epsilon_2,\epsilon_3) + J^{\beta}_{\epsilon_3^+}\Psi(\epsilon_2,\epsilon_4) + J^{\beta}_{\epsilon_3^+}\Psi(\epsilon_1,\epsilon_4)\right] \\ &\quad - \frac{\Gamma(\alpha+1)}{4(\epsilon_2-\epsilon_1)^\alpha}\left[J^{\alpha}_{\epsilon_2^-}\Psi(\epsilon_1,\epsilon_3) + J^{\alpha}_{\epsilon_2^-}\Psi(\epsilon_2,\epsilon_3)\right. \\ &\quad \left. + J^{\alpha}_{\epsilon_1^+}\Psi(\epsilon_2,\epsilon_4) + J^{\alpha}_{\epsilon_1^+}\Psi(\epsilon_1,\epsilon_4)\right], \end{aligned} \tag{17}$$

where

$$\begin{aligned} J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_4^-}\Psi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}(x-\epsilon_1)^{\alpha-1}(y-\epsilon_3)^{\beta-1}\Psi(x,y)dydx, \\ J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_4^-}\Psi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}(x-\epsilon_1)^{\alpha-1}(\epsilon_4-y)^{\beta-1}\Psi(x,y)dydx, \\ J^{\alpha,\beta}_{\epsilon_2^-,\epsilon_3^+}\Psi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}(\epsilon_2-x)^{\alpha-1}(y-\epsilon_3)^{\beta-1}\Psi(x,y)dydx, \\ J^{\alpha,\beta}_{\epsilon_1^+,\epsilon_3^+}\Psi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\epsilon_1}^{\epsilon_2}\int_{\epsilon_3}^{\epsilon_4}(\epsilon_2-x)^{\alpha-1}(\epsilon_4-y)^{\beta-1}\Psi(x,y)dydx. \end{aligned} \tag{18}$$

The main motivation of the paper is to provide some novel integral inequalities for partial differentiable exponentially convex functions on the coordinates and to give some special cases to demonstrate that the main findings are general forms of the earlier results.

2. Main Results

The following integral identity will play a key role to provide new estimations for partial differentiable exponentially convex functions on the coordinates.

Lemma 13 (See [23]). *Let $\Psi:\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]\longrightarrow\mathbb{R}$ be a twice partial differentiable mapping on $\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]$. If $\partial^2\Psi/\partial\zeta\partial\xi\in L(\Delta)$ and $\alpha,\beta>0,\epsilon_1,\epsilon_3\geq 0$, then, one has*

$$\begin{aligned} &\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \\ &= \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\int_0^1\int_0^1\zeta^\alpha\xi^\beta\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3\right. \\ &\quad +(1-\xi)\epsilon_4)d\xi d\zeta - \int_0^1\int_0^1(1-\zeta)^\alpha\xi^\beta\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3 \\ &\quad +(1-\xi)\epsilon_4)d\xi d\zeta - \int_0^1\int_0^1\zeta^\alpha(1-\xi)^\beta\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3 \\ &\quad +(1-\xi)\epsilon_4)d\xi d\zeta + \int_0^1\int_0^1(1-\zeta)^\alpha(1-\xi)^\beta\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3 \\ &\quad \left. +(1-\xi)\epsilon_4)d\xi d\zeta\right]. \end{aligned} \tag{19}$$

Theorem 14. *Let $\Psi:\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]\longrightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]$ and $\partial^2\Psi/\partial\zeta\partial\xi\in L(\Delta)$, $\alpha,\beta>0,\alpha_1\in\mathbb{R}$. If $|\partial^2\Psi/\partial\zeta\partial\xi|$ is exponentially convex function on the coordinates on Δ, then one has*

$$\begin{aligned} &\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \\ &\leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4(\alpha+1)(\beta+1)}\times\left[\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)|}{e^{\alpha_1(\epsilon_1+\epsilon_3)}}\right. \\ &\quad + \frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)|}{e^{\alpha_1(\epsilon_1+\epsilon_4)}} + \frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)|}{e^{\alpha_1(\epsilon_2+\epsilon_3)}} \\ &\quad \left. + \frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)|}{e^{\alpha_1(\epsilon_2+\epsilon_4)}}\right]. \end{aligned} \tag{20}$$

Proof. By considering Lemma 13 and by using the integral form of the triangle inequality, we can write

$$\begin{aligned} &\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \\ &\leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\int_0^1\int_0^1\zeta^\alpha s^\beta\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3\right.\right. \\ &\quad \left.+(1-\xi)\epsilon_4)\right|d\xi d\zeta + \int_0^1\int_0^1(1-\zeta)^\alpha\xi^\beta\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1\right. \\ &\quad \left.+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|d\xi d\zeta + \int_0^1\int_0^1\zeta^\alpha(1-\xi)^\beta\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1\right. \\ &\quad \left.+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|d\xi d\zeta + \int_0^1\int_0^1(1-\zeta)^\alpha(1-\xi)^\beta\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1\right. \\ &\quad \left.\left.+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|d\xi d\zeta\right]. \end{aligned} \tag{21}$$

Since $|\partial^2\Psi/\partial\zeta\partial\xi|$ is exponentially convex functions on the coordinates, we can write

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\int_0^1\int_0^1\left[\zeta^\alpha\xi^\beta+(1-\zeta)^\alpha\xi^\beta+\zeta^\alpha(1-\xi)^\beta\right.\right.\\&\quad\left.+(1-\zeta)^\alpha(1-\xi)^\beta\right]\left[\zeta\xi\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)\right|}{e^{\alpha(\epsilon_1+\epsilon_3)}}\right.\\&\quad+\zeta(1-\xi)\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)\right|}{e^{\alpha(\epsilon_1+\epsilon_4)}}+(1-\zeta)\xi\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)\right|}{e^{\alpha(\epsilon_2+\epsilon_3)}}\\&\quad\left.\left.+(1-\zeta)(1-\xi)\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)\right|}{e^{\alpha(\epsilon_2+\epsilon_4)}}\right]d\xi d\zeta\right].\end{aligned}\tag{22}$$

After the necessary basic computations, the proof is completed. □

Remark 15. If we assume that all the conditions of Theorem 14 are valid with $\alpha_1=0$, we obtain Theorem 5 in [23].

Corollary 16. *If we assume that all the conditions of Theorem 14 are valid with $\alpha_1=1$, one has*

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4(\alpha+1)(\beta+1)}\times\left[\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)\right|}{e^{\epsilon_1+\epsilon_3}}\right.\\&\quad+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)\right|}{e^{\epsilon_1+\epsilon_4}}+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)\right|}{e^{\epsilon_2+\epsilon_3}}\\&\quad\left.+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)\right|}{e^{\epsilon_2+\epsilon_4}}\right].\end{aligned}\tag{23}$$

Corollary 17. *If we assume that all the conditions of Theorem 14 are valid with $|\partial^2\Psi/\partial\zeta\partial\xi|$ is bounded, i.e.,*

$$\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty=\sup_{(\zeta,\xi)\in(\epsilon_1,\epsilon_2)\times(\epsilon_3,\epsilon_4)}\left|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right|<\infty,\tag{24}$$

we get

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4(\alpha+1)(\beta+1)}\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty\\&\quad\times\left[\frac{1}{e^{\alpha_1(\epsilon_1+\epsilon_3)}}+\frac{1}{e^{\alpha_1(\epsilon_1+\epsilon_4)}}+\frac{1}{e^{\alpha_1(\epsilon_2+\epsilon_3)}}+\frac{1}{e^{\alpha_1(\epsilon_2+\epsilon_4)}}\right].\end{aligned}\tag{25}$$

Theorem 18. *Let $\Psi:\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]\longrightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]$ and $\partial^2\Psi/\partial\zeta\partial\xi\in L(\Delta)$, $\alpha,\beta\in(0,1]$. If $|\partial^2\Psi/\partial\zeta\partial\xi|^q$, $q>1$, $\alpha_1\in\mathbb{R}$, is exponentially convex function on the coordinates on Δ, then one has*

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4^{1+1/q}(\alpha p+1)^{1/p}(\beta p+1)^{1/p}}\times\left(\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_1+\epsilon_3)}}\right.\\&\quad+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)\right|^q}{e^{\alpha_1(\epsilon_1+\epsilon_4)}}+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_2+\epsilon_3)}}\\&\quad\left.+\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)\right|^q}{e^{\alpha_1(\epsilon_2+\epsilon_4)}}\right)^{1/q},\end{aligned}\tag{26}$$

where $p^{-1}+q^{-1}=1$.

Proof. Taking into account the integral identity that is given in Lemma 13 and by considering the Hölder inequality for double integrals, then one has

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\times\left[\left(\int_0^1\int_0^1\left[\zeta^\alpha\xi^\beta\right]^p dsdt\right)^{1/p}\right.\\&\quad+\left(\int_0^1\int_0^1\left[(1-\zeta)^\alpha\xi^\beta\right]^p dsdt\right)^{1/p}+\left(\int_0^1\int_0^1\left[\zeta^\alpha(1-\xi)^\beta\right]^p dsdt\right)^{1/p}\\&\quad+\left(\int_0^1\int_0^1\left[(1-\zeta)^\alpha(1-\xi)^\beta\right]^p d\xi d\zeta\right)^{1/p}\\&\quad\left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|^q d\xi d\zeta\right)^{1/q}\right].\end{aligned}\tag{27}$$

By making use of necessary computations, we obtain

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4(\alpha p+1)^{1/p}(\beta p+1)^{1/p}}\\&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_2+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)d)\right|^q d\xi d\zeta\right)^{1/q}\Bigg].\end{aligned}\tag{28}$$

Since $|\partial^2\Psi/\partial\zeta\partial\xi|^q$ is coordinated convex, we have

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4(\alpha p+1)^{1/p}(\beta p+1)^{1/p}}\times\left(\int_0^1\int_0^1\left[\zeta\xi\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_1+\epsilon_3)}}\right.\right.\\&\quad\left.+\zeta(1-\xi)\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)\right|^q}{e^{\alpha_1(\epsilon_1+\epsilon_4)}}\right]+(1-\zeta)\xi\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_2+\epsilon_3)}}\\&\quad\left.+(1-\zeta)(1-\xi)\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)\right|^q}{e^{\alpha_1(\epsilon_2+\epsilon_4)}}d\xi d\zeta\right)^{1/q}.\end{aligned}\tag{29}$$

By computing these integrals, we obtain

$$\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4^{1+1/q}(\alpha p+1)^{1/p}(\beta p+1)^{1/p}} \times \left(\frac{\left|\partial^2\Psi/\partial t\partial s(\epsilon_1,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_1+\epsilon_3)}} + \frac{\left|\partial^2\Psi/\partial t\partial s(\epsilon_1,d)\right|^q}{e^{\alpha_1(\epsilon_1+d)}} + \frac{\left|\partial^2\Psi/\partial t\partial s(\epsilon_2,\epsilon_3)\right|^q}{e^{\alpha_1(\epsilon_2+\epsilon_3)}} + \frac{\left|\partial^2\Psi/\partial t\partial s(\epsilon_2,d)\right|^q}{e^{\alpha_1(\epsilon_2+d)}}\right)^{1/q}. \tag{30}$$

This completes the proof of the theorem. □

Corollary 19. *If we assume that all the conditions of Theorem 18 are valid with $\alpha_1=1$, we have the following inequality:*

$$\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4^{1+1/q}(\alpha p+1)^{1/p}(\beta p+1)^{1/p}} \times \left(\frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)\right|^q}{e^{(\epsilon_1+\epsilon_3)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)\right|^q}{e^{(\epsilon_1+\epsilon_4)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)\right|^q}{e^{(\epsilon_2+\epsilon_3)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)\right|^q}{e^{(\epsilon_2+\epsilon_4)}}\right)^{1/q}. \tag{31}$$

Corollary 20. *If we assume that all the conditions of Theorem 18 are valid with $|\partial^2\Psi/\partial\zeta\partial\xi|$ is bounded, i.e.,*

$$\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty = \sup_{(\zeta,\xi)\in(\epsilon_1,\epsilon_2)\times(\epsilon_3,\epsilon_4)}\left|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right| < \infty, \tag{32}$$

we get

$$\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4^{1+1/q}(\alpha p+1)^{1/p}(\beta p+1)^{1/p}}\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty \times\left(\frac{1}{e^{\alpha_1(\epsilon_1+\epsilon_3)}}+\frac{1}{e^{\alpha_1(\epsilon_1+\epsilon_4)}}+\frac{1}{e^{\alpha_1(\epsilon_2+\epsilon_3)}}+\frac{1}{e^{\alpha_1(\epsilon_2+\epsilon_4)}}\right)^{1/q}. \tag{33}$$

Theorem 21. *Let $\Psi:\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]\longrightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[\epsilon_1,\epsilon_2]\times[\epsilon_3,\epsilon_4]$ and $\partial^2\Psi/\partial\zeta\partial\xi\in L(\Delta)$, $\alpha,\beta\in(0,1]$, $\alpha_1\in\mathbb{R}$. If $|\partial^2\Psi/\partial\zeta\partial\xi|^q$, $q>1$, is exponentially convex function on the coordinates on Δ, then one has*

$$\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| \leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,c)\right|^q}{qe^{\alpha_1(\epsilon_1+c)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,d)\right|^q}{qe^{\alpha_1(\epsilon_1+d)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,c)\right|^q}{qe^{\alpha_1(\epsilon_2+c)}} + \frac{\left|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,c)\right|^q}{qe^{\alpha_1(\epsilon_2+d)}}\right], \tag{34}$$

where $p^{-1}+q^{-1}=1$.

Proof. We will start with the integral identity that is given in Lemma 13 and by considering Young inequality, then we can write

$$\begin{aligned}\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right| &\leq \frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\\ &\times\left[\frac{1}{p}\int_0^1\int_0^1\left(\zeta^\alpha\xi^\beta\right)^p d\xi d\zeta + \frac{1}{q}\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|^q d\xi d\zeta\right.\\ &+\frac{1}{p}\int_0^1\int_0^1\left((1-\zeta)^\alpha\xi^\beta\right)^p d\xi d\zeta + \frac{1}{q}\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|^q d\xi d\zeta\\ &+\frac{1}{p}\int_0^1\int_0^1\left(\zeta^\alpha(1-\xi)^\beta\right)^p d\xi d\zeta + \frac{1}{q}\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|^q d\xi d\zeta\\ &\left.+\frac{1}{p}\int_0^1\int_0^1\left((1-\zeta)^\alpha(1-\xi)^\beta\right)^p d\xi d\zeta + \frac{1}{q}\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial\zeta\partial\xi}(\zeta\epsilon_1+(1-\zeta)\epsilon_2,\xi\epsilon_3+(1-\xi)\epsilon_4)\right|^q d\xi d\zeta\right].\end{aligned} \tag{35}$$

Since $|\partial^2\Psi/\partial t\partial s|^q$ is coordinated convex, we can provide

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\times\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p}\right.\\&\qquad+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)|^q}{qe^{\alpha_1(\epsilon_1+\epsilon_3)}}+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)|^q}{qe^{\alpha_1(\epsilon_1+\epsilon_4)}}\\&\qquad\left.+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)|^q}{qe^{\alpha_1(\epsilon_2+\epsilon_3)}}+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)|^q}{qe^{\alpha_1(\epsilon_2+\epsilon_4)}}\right].\end{aligned}\tag{36}$$

Which completes the proof. □

Corollary 22. *If we assume that all the conditions of Theorem 21 are valid with $\alpha_1=1$, we have the following inequality:*

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\times\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p}\right.\\&\qquad+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)|^q}{qe^{(\epsilon_1+\epsilon_3)}}+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)|^q}{qe^{(\epsilon_1+\epsilon_4)}}\\&\qquad\left.+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)|^q}{qe^{(\epsilon_2+\epsilon_3)}}+\frac{|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)|^q}{qe^{(\epsilon_2+\epsilon_4)}}\right].\end{aligned}\tag{37}$$

Corollary 23. *If we assume that all the conditions of Theorem 21 are valid with $\alpha_1=0$, we have the following inequality:*

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p}\right.\\&\qquad+\frac{1}{q}\left(|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_3)|^q+|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_1,\epsilon_4)|^q\right.\\&\qquad\left.\left.+|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_3)|^q+|\partial^2\Psi/\partial\zeta\partial\xi(\epsilon_2,\epsilon_4)|^q\right)\right].\end{aligned}\tag{38}$$

Corollary 24. *If we assume that all the conditions of Theorem 21 are valid with $|\partial^2\Psi/\partial\zeta\partial\xi|$ is bounded, i.e.,*

$$\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty=\sup_{(\zeta,\xi)\in(\epsilon_1,\epsilon_2)\times(\epsilon_3,\epsilon_4)}\left|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right|<\infty,\tag{39}$$

we get

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le\frac{(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)}{4}\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p}+\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty\right.\\&\qquad\left.\times\left(\frac{1}{qe^{\alpha_1(\epsilon_1+\epsilon_3)}}+\frac{1}{qe^{\alpha_1(\epsilon_1+\epsilon_4)}}+\frac{1}{qe^{\alpha_1(\epsilon_2+\epsilon_3)}}+\frac{1}{qe^{\alpha_1(\epsilon_2+\epsilon_4)}}\right)\right].\end{aligned}\tag{40}$$

Corollary 25. *If we assume that all the conditions of Theorem 21 are valid with $|\partial^2\Psi/\partial\zeta\partial\xi|$ is bounded, i.e.,*

$$\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty=\sup_{(\zeta,\xi)\in(\epsilon_1,\epsilon_2)\times(\epsilon_3,\epsilon_4)}\left|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right|<\infty,\tag{41}$$

and $\alpha_1=0$, we get

$$\begin{aligned}&\left|\frac{\Psi(\epsilon_1,\epsilon_3)+\Psi(\epsilon_1,\epsilon_4)+\Psi(\epsilon_2,\epsilon_3)+\Psi(\epsilon_2,\epsilon_4)}{4}+A\right|\\&\quad\le(\epsilon_2-\epsilon_1)(\epsilon_4-\epsilon_3)\left[\frac{1}{p(\alpha+1)^p(\beta+1)^p}+\left\|\frac{\partial^2\Psi(\zeta,\xi)}{\partial\zeta\partial\xi}\right\|_\infty\left(\frac{1}{q}\right)\right].\end{aligned}\tag{42}$$

3. Conclusions

For the inequality theory, which is one of the important subjects of mathematical analysis, producing new and original integral inequalities is the main motivation point. For this purpose, researchers sometimes use new function classes, sometimes new integral operators, and sometimes try to obtain modifications of some famous inequalities in different spaces.

Two main innovations of the study are the usage of bivariate versions of exponentially convex functions that produce more optimal results than the concept of convexity and the usage of RL fractional integral operators. The fractional analysis is a subject that produces effective solutions to real-world problems, helps explain various concepts in physics and engineering, and brings innovation to fields such as mathematics, statistics, economics, disease models, mathematical biology, and modelling. Considering this mission, fractional analysis, which has been widely used by researchers in the field of inequality theory, has brought a unique orientation to the literature.

In this study, new and original findings were obtained by using RL fractional integral operators and exponentially convex functions on the coordinates and basic analysis proof techniques. It has been confirmed that the main findings have general forms, which have been reduced to the results that were obtained in the literature by giving many special cases.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

These authors contributed equally to this work.

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