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Research Article

The Solution of Two-Dimensional Coupled Burgers' Equation by *G*-Double Laplace Transform

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The two-dimensional coupled Burgers' equation, a foundational partial differential equation, boasts widespread relevance across numerous scientific domains. Attaining precise solutions to this equation stands as a pivotal endeavor, fostering a comprehensive understanding of both physical phenomena and mathematical models. In this article, we underscore the paramount significance of the *G*-double Laplace transform, a transformative mathematical tool. Leveraging this innovative technique, we furnish dependable and exact solutions, addressing both homogeneous and nonhomogeneous variants of the coupled Burgers' equations. This approach not only delivers reliability but also serves as an invaluable instrument for delving deeper into the equation's intricate behavior and its profound implications across diverse disciplinary landscapes.

1. Introduction

Burgers' equation is a fundamental partial differential equation and convection-diffusion equation that finds applications in various fields of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. Solving this equation holds great significance in the context of mathematical models and understanding physical phenomena. Over the years, numerous scientists and mathematicians have proposed analytical solutions for the one-dimensional coupled Burgers' equation. Various analytical methods have been developed to tackle Burger's equation, as evidenced in [1–3].

In recent years, a substantial endeavor has been directed toward employing the Laplace decomposition method (LDM) and its modified variants to investigate physical model equations, as discussed in detail in [4]. Furthermore, the scientific community has witnessed remarkable advancements in analytical approaches for tackling differential equations. Researchers have explored a myriad of techniques in this field, including the modified Adomian-Rach decomposition method (MDM) [5], which has been effectively utilized to obtain solutions for scenarios such as the isentropic flow of an inviscid gas model (IFIG) through the modified decomposition method (MDM), elaborated in [6]. The Adomian decomposition method (ADM) [7] has proven valuable in addressing systems of conservation laws featuring a mixed hyperbolic-elliptic character. Additionally, the reduced differential transform method (RDTM) [8] has found practical applications.

In a separate study, as documented in [9], an author enhanced the Laplace decomposition method to derive approximate analytical solutions for both linear and nonlinear differential equations and systems. Furthermore, researchers have ventured into exploring the application of the double Laplace-Sumudu transform [10] to tackle a diverse array of partial differential equations.

To delve into specifics, the Adomian decomposition method has been effectively employed to obtain exact solutions for Burgers' equation [11]. A modified expanded tanh-function method was introduced to achieve exact solutions, as highlighted in references [12, 13]. Additionally, the homotopy perturbation method has been suggested for addressing the nonlinear Burgers' equation [14]. An innovative approach combining the Laplace transform and the new homotopy perturbation method (NHPM) was proposed to derive closed-form solutions for coupled viscous Burgers' equations [15]. This approach holds significant implications in understanding polydispersity and its relationship with gravitational effects [16].

The study of coupled Burgers' equations is of paramount importance as it elucidates the precipitation of polydispersity in the presence of gravity effects [16]. Furthermore, the solution of time-fractional two-mode coupled Burgers' equations has been explored [17, 18].

In this context, the G transform, initially introduced in [19] and later applied to solve nonlinear dynamical models with noninteger order [20], offers a promising avenue for tackling complex equations.

The core focus of this article revolves around harnessing the formidable G-double Laplace transform to attain precise solutions of exceptional reliability. Our primary aim is to unravel exact solutions for both homogeneous and nonhomogeneous instances of the two-dimensional coupled Burgers' equations. This transformative mathematical approach not only promises exactitude but also offers an unprecedented opportunity to gain profound insights into the intricate behavior of these equations. Furthermore, it opens up new horizons for their application across a multitude of scientific domains.

2. An Introduction to G_α-Laplace Transforms: Fundamental Concepts

In this study, we employ the G_α transform and Laplace transform to assist us in solving a set of partial differential equations.

Definition 1 (see [21]). Consider an integrable function $f(\nu)$ defined for all $t \geq 0$. The generalized integral transform denoted as G_α of the function $f(\nu)$ is defined as follows:

$$F(s) = G_\alpha(f) = s^\alpha \int_0^\infty f(\nu) e^{-t/s} d\nu, \tag{1}$$

where s belongs to the complex numbers $\mathbb{C}$ and α is an integer represented by Z.

To provide a practical example, if we substitute $\alpha = 0$ and $p = 1/s$ into Eq. (1), we can express the result as

$$G_0(f) = \int_0^\infty f(\nu) e^{-t/s} d\nu. \tag{2}$$

This showcases that the G_α transform serves as a comprehensive extension of not only the Laplace transform but also various other transforms. Its scope encompasses a broader and more fundamental range compared to existing transforms. For a deeper insight into this concept, we recommend referring to [21].

Definition 2. The Laplace transform of the function $f(\nu)$ is defined as

$$L[f(\nu)] = F(s) = \int_0^\infty f(\nu) e^{-st} d\nu, \tag{3}$$

where s belongs to the complex numbers $\mathbb{C}$.

Definition 3. The G_α-double Laplace transform of the function $f(x, y, t)$ is a well-behaved and integrable function defined for all nonnegative values of x, y, and ν. This transform is denoted as $L_x L_y G_\alpha[f(x, y, \nu)] = F_\alpha(p, q, s)$, with $F_\alpha(p, q, s)$ expressed as

$$F_\alpha(p, q, s) = s^\alpha \int_0^\infty \int_0^\infty \int_0^\infty f(x, y, \nu) e^{-px-qy-(\nu/s)} d\nu dy dx. \tag{4}$$

Here, the notation $L_x L_y G_\alpha$ signifies the G_α-double Laplace transform, and p, q, and s belong to the complex number set $\mathbb{C}$.

Definition 4. The inverse G_α-double Laplace transform of $L_p^{-1} L_q^{-1} G_s^{-1}[F_\alpha(p, q, s)] = f(x, y, \nu)$ is expressed as

$$f(x, y, \nu) = \frac{1}{(2\pi i)^3} \int_{\tau-i\infty}^{\tau+i\infty} \int_{\varsigma-i\infty}^{\varsigma+i\infty} \int_{\eta-i\infty}^{\eta+i\infty} e^{px+qy+(1/s)\nu} F_\alpha(p, q, s) ds\, dp dq. \tag{5}$$

Example 1. $L_x L_y G_\alpha$ transform of the $f(x, y, t) = e^{i(ax+by+ct)}$ is given by

$$\begin{aligned} L_x L_y G_\alpha\left[e^{i(ax+by+c\nu)}\right] &= \frac{s^{\alpha+1}}{(p-ai)(q-bi)(1-csi)} \\ &= \frac{s^{\alpha+1}(p+ai)(q+bi)(1+csi)}{(p-ai)(q-bi)(1-csi)(p+ai)(q+bi)(1+csi)} \\ &= \frac{s^{\alpha+1}(pq-ab-cbps-acqs)}{(p^2+a^2)(q^2+b^2)(1+c^2s^2)} \\ &\quad + \frac{s^{\alpha+1}(aq+pb-cbps-acqs)i}{(p^2+a^2)(q^2+b^2)(1+c^2s^2)}, \end{aligned} \tag{6}$$

where $L_x L_y G_\alpha$ indicate to G_α-double Laplace transform; consequently,

$$\begin{aligned} L_x L_y G_\alpha[\cos(ax+by+c\nu)] &= \frac{s^{\alpha+1}(pq-ab-cbps-acqs)}{(p^2+a^2)(q^2+b^2)(1+c^2s^2)}, \\ L_x L_y G_\alpha[\sin(ax+by+c\nu)] &= \frac{s^{\alpha+1}(aq+pb-cbps-acqs)}{(p^2+a^2)(q^2+b^2)(1+c^2s^2)}. \end{aligned} \tag{7}$$

Example 2. The $L_x L_y G_\alpha$ transform of $f(x,y,\nu)=(xy\nu)^n$ is determined by

$$L_\mu G_\alpha[(xy\nu)^n]=\frac{(n!)^3 s^{\alpha+n+1}}{p^{n+1}q^{n+1}}, \tag{8}$$

where n is nonnegative integer. If $\theta(>-1)$ and $\beta(>-1)\in\mathbb{R}$, then

$$L_x L_y G_\alpha\left[x^\theta y^\delta \nu^\beta\right]=\frac{\Gamma(\theta+1)\Gamma(\delta+1)\Gamma(\beta+1)s^{\alpha+\beta+1}}{p^{\theta+1}q^{\delta+1}} \tag{9}$$

can be derived from the definition of G_α-double Laplace transform; we have

$$\begin{aligned}L_x L_y G_\alpha\left[x^\theta y^\delta \nu^\beta\right]&=s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty x^\theta y^\delta \nu^\beta e^{-px-qy-(\nu/s)}d\nu dy dx\\&=\left(\int_0^\infty x^\theta e^{-px}dx\right)\left(\int_0^\infty y^\delta e^{-qy}dx\right)\\&\quad\cdot\left(s^\alpha\int_0^\infty \nu^\beta e^{-\nu/s}d\nu\right).\end{aligned} \tag{10}$$

By applying the definition of the double Laplace transform to the integral within the first bracket of Eq. (10), we arrive at the following result:

$$L_x L_y G_\alpha\left[x^\theta y^\delta \nu^\beta\right]=\frac{\Gamma(\theta+1)}{p^{\theta+1}}\frac{\Gamma(\delta+1)}{q^{\delta+1}}\left(s^\alpha\int_0^\infty \nu^\beta e^{-\nu/s}d\nu\right). \tag{11}$$

In Eq. (11) put $r=\nu/s, \nu=rs$, we get

$$\begin{aligned}L_x L_y G_\alpha\left[x^\theta y^\delta \nu^\beta\right]&=\frac{\Gamma(\theta+1)}{p^{\theta+1}}\frac{\Gamma(\delta+1)}{q^{\delta+1}}\left(s^{\alpha+\beta+1}\int_0^\infty r^\beta e^{-r}dr\right)\\&=\frac{\Gamma(\theta+1)\Gamma(\delta+1)\Gamma(\beta+1)s^{\alpha+\beta+1}}{p^{\theta+1}q^{\delta+1}},\end{aligned} \tag{12}$$

where the gamma function of $\beta+1$ is defined through the convergent integral expression.

$$\Gamma(\beta+1)=\int_0^\infty e^{-r}r^\beta dr, \beta>0. \tag{13}$$

Example 3. G_α-Laplace transform for the function

$$f(x,\nu)=H(\mu)\otimes H(\nu)\ln x\ln \nu \tag{14}$$

is expressed as

$$L_x G_\alpha[H(x)\otimes H(\nu)\ln x\ln \nu]=s^\alpha\int_0^\infty\int_0^\infty \ln x\ln \nu e^{-px-(\nu/s)}d\nu dx, \tag{15}$$

where $H(x,\nu)=H(\nu)\otimes H(x)$ represents the dimensional Heaviside function and $\otimes$ denotes a tensor product (as described in [22]).

By introducing the substitutions $\zeta=px$ and $\eta=(1/s)\nu$, the integral transforms into the following form:

$$\begin{aligned}L_x G_\alpha[H(x)\otimes H(\nu)\ln x\ln \nu]&=\frac{s^{\alpha+1}}{p}\int_0^\infty e^{-\eta}\ln(s\eta)\\&\quad\cdot\left(\int_0^\infty e^{-\zeta}\ln\left(\frac{1}{p}\zeta\right)d\zeta\right)d\eta\\&=\frac{s^{\alpha+1}}{p}(\gamma+\ln p)(-\gamma+\ln s),\end{aligned} \tag{16}$$

where $\gamma=\int_0^\infty e^{-\eta}\ln(\eta)d\eta=0.5772$ is Euler's constant.

2.1. Existence Criteria for the G_α-Double Laplace Transform.

In the forthcoming theorem, we establish the prerequisites for the existence of the G_α-Laplace transform of $f(\mu,\nu)$. Consider $f(x,y,\nu)$ which exhibits exponential behavior with positive orders $a(>0)$, $b(>0)$, and $c(>0)$ over the domain $0\le x,y,\nu<\infty$. We say that $f(x,y,\nu)$ belongs to this class if there exists a nonnegative constant M such that, for all $x>X$, $y>Y$, and $t>T$, the inequality

$$|f(x,y,\nu)|\le Me^{ax+by+c\nu} \tag{17}$$

holds. In this scenario, we can express $f(x,y,\nu)$ as

$$f(\mu,\nu)=O\left(e^{ax+by+c\nu}\right), \tag{18}$$

as $x\longrightarrow\infty$, $y\longrightarrow\infty$, and $\nu\longrightarrow\infty$. Equivalently, we have

$$\lim_{\substack{x\longrightarrow\infty\\y\longrightarrow\infty\\\nu\longrightarrow\infty}} e^{-px-qy-(\nu/s)}|f(x,y,\nu)|=M\lim_{\substack{x\longrightarrow\infty\\y\longrightarrow\infty\\\nu\longrightarrow\infty}} e^{-(p-a)x-(q-b)y-((1/s)-c)\nu}=0, \tag{19}$$

for $p>a$, $q>b$, and $s>(1/c)$. We refer to the function $f(x,y,\nu)$ as having exponential order a as $x\longrightarrow\infty$, $y\longrightarrow\infty$, and $\nu\longrightarrow\infty$. It is evident that it does not grow at a rate exceeding $Me^{ax+by+c\nu}$.

Theorem 5. *If $f(x,y,\nu)$ is a continous function in every bounded interval $(0,x)$, $(0,y)$, and $(0,\nu)$ and of exponential order $e^{ax+by+c\nu}$, then the $L_x L_y G_\alpha$ transform of $f(x,y,\nu)$ exists for all p, q, and s provided $p>a$, $q>b$, and $(1/s)>b$.*

Proof. Combining Eq. (17) and Eq. (19) within Eq. (4) yields

$$\begin{aligned}|F_\alpha(p,q,s)| &= \left| s^\alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-px-qy-(\nu/s)} f(x,y\nu) d\mu d\nu \right| \\ &\le M s^\alpha \int_0^\infty \int_0^\infty e^{-(p-a)x-(q-b)y-((1/s)-c)\nu} dx dy d\nu \\ &= \frac{M s^{\alpha+1}}{(p-a)(q-b)(1-cs)},\end{aligned} \tag{20}$$

for $p > a$, $q > b$, and $(1/s) > b$. □

Lemma 6. *If $f(x,y,\nu)$ is a piecewise continuous function defined on $[0,\infty) \times [0,\infty) \times [0,\infty)$ and has an exponential order at infinity with $Me^{ax+by+c\nu}$ for $x \ge a, y \ge b$ and $\nu \ge c$, where a, b, and c are constant, then for any real number $\rho \ge 0$, $\tau \ge 0$, and $\sigma \ge 0$, we have*

$$\begin{aligned}&L_x L_y G_\alpha[f(x-\rho, y-\tau, \nu-\sigma) H(x-\rho, y-\tau, \nu-\sigma)] \\ &\quad = e^{-p\rho-q\tau-(\sigma/s)} F_\alpha(p,q,s),\end{aligned} \tag{21}$$

where $L_x L_y G_\alpha(f(x,y,\nu)) = F_\alpha(p,q,s)$ and $H(x-\rho, y-\tau, \nu-\sigma)$ is the Heaviside function, defined by

$$H(x-\rho, y-\tau, \nu-\sigma) = \begin{cases} 1 & \text{if } x > \rho, y > \tau, \nu > \sigma, \\ 0 & \text{if } x < \rho, y < \tau, \nu < \sigma. \end{cases} \tag{22}$$

Proof. Utilizing the definition of the $L_x L_y G_\alpha$ transform,

$$\begin{aligned}&L_x L_y G_\alpha[f(x-\rho, y-\tau, \nu-\sigma) H(x-\rho, y-\tau, \nu-\sigma)] \\ &\quad = s^\alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-px-qy-(\nu/s)} f(x-\rho, y-\tau, \nu-\sigma) H(x-\rho, y-\tau, \nu-\sigma) dx dy d\nu \\ &\quad = s^\alpha \int_\rho^\infty \int_\tau^\infty \int_\sigma^\infty e^{-p\mu-(\nu/s)} f(x-\rho, y-\tau, \nu-\sigma) dx dy d\nu.\end{aligned} \tag{23}$$

Let $z = \mu - \rho$, $k = y - \tau$, and $r = \nu - \sigma$; Eq. (23) becomes

$$\begin{aligned}&L_x L_y G_\alpha[f(x-\rho, y-\tau, \nu-\sigma) H(x-\rho, y-\tau, \nu-\sigma)] \\ &\quad = e^{-p\rho-q\tau-(\sigma/s)} \left[s^\alpha \int_0^\infty \int_0^\infty e^{-pz-qk-(r/s)} f(z,k,r) dz dk dr \right] \\ &\quad = e^{-p\rho-q\tau-(\sigma/s)} F_\alpha(p,q,s).\end{aligned} \tag{24}$$

□

Theorem 7. *If $f(x,y,\nu)$ is a periodic function with periods λ, τ, and μ to satisfy this periodicity condition, we must have*

$$f(x+\lambda, y+\tau, \nu+\upsilon) = f(x,y,\nu), \text{for all } \lambda, \tau, \upsilon. \tag{25}$$

In that case, the G_α-double Laplace transform of $f(x,y,\nu)$ can be expressed as follows:

$$F_\alpha(p,q,s) = \left(1 - e^{-p\lambda-q\tau-(\upsilon/\nu)}\right) s^\alpha \int_0^\lambda \int_0^\tau \int_0^\upsilon e^{-px-qy-(\nu/s)} f(x,y,\nu) dx dy d\nu. \tag{26}$$

Proof. By utilizing the definition of the G_α-double Laplace transform,

$$\begin{aligned}L_x L_y G_\alpha[f(x,y,\nu)] &= F_\alpha(p,q,s) \\ &= s^\alpha \int_0^\infty \int_0^\infty \int_0^\infty f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx.\end{aligned} \tag{27}$$

Applying the property of improper integrals to Eq. (27),

$$\begin{aligned}F_\alpha(p,q,s) = s^\alpha \Bigg[&\int_0^\lambda \int_0^\tau \int_0^\upsilon f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx \\ &+ \int_\lambda^\infty \int_\tau^\infty \int_\upsilon^\infty f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx \Bigg].\end{aligned} \tag{28}$$

Substituting $x = \lambda + \gamma$, $y = \tau + \varpi$, and $\nu = \upsilon + \delta$ in the second part of the integral in Eq. (28), we obtain

$$\begin{aligned}F_\alpha(p,q,s) = s^\alpha \Bigg[&\int_0^\lambda \int_0^\tau \int_0^\upsilon f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx \\ &+ \int_0^\infty \int_0^\infty \int_0^\infty f(x,y,\nu) e^{-p(\lambda+\gamma)-q(\tau+\varpi)-((\upsilon+\delta)/s)} d\delta d\varpi d\gamma \Bigg].\end{aligned} \tag{29}$$

Equation (29) can be rewritten as follows:

$$\begin{aligned}F_\alpha(p,s) = &s^\alpha \int_0^\lambda \int_0^\tau \int_0^\upsilon f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx \\ &+ e^{-p\lambda-q\tau-(\mu/s)} s^\alpha \int_0^\infty \int_0^\infty f(x,y,\nu) e^{-p\gamma-q\varpi-(\delta/s)} d\delta d\varpi d\gamma.\end{aligned} \tag{30}$$

The second integral in Eq. (30), given the definition of the G_α-double Laplace transform, leads to

$$\begin{aligned}F_\alpha(p,q,s) = &s^\alpha \int_0^\lambda \int_0^\tau \int_0^\upsilon f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx \\ &+ e^{-p\lambda-q\tau-(\mu/s)} F_\alpha(p,q,s).\end{aligned} \tag{31}$$

Hence,

$$F_\alpha(p,q,s) = \left(1 - e^{-p\lambda-q\tau-(\mu/\nu)}\right) s^\alpha \int_0^\lambda \int_0^\tau \int_0^\upsilon f(x,y,\nu) e^{-px-qy-(\nu/s)} d\nu dy dx. \tag{32}$$

□

Theorem 8 (convolution theorem). *Let $L_xL_yG_\alpha[\phi(x,y,\nu)]$ and $L_xL_yG_\alpha[\varphi(x,y,\nu)]$ exist and $L_xL_yG_\alpha[\phi(x,y,\nu)]=\phi_\alpha(p,q,s)$, and $L_xL_yG_\alpha[\varphi(x,y,\nu)]=\varphi_\alpha(p,q,s)$; then,*

$$s^\alpha L_xL_yG_\alpha[\phi(x,y,\nu)**\varphi(x,y,\nu)]=\phi_\alpha(p,q,s)\varphi_\alpha(p,q,s), \tag{33}$$

where

$$\phi(x,y,\nu)**\varphi(x,y,\nu)=\int_0^x\int_0^y\int_0^\nu\phi(x-\zeta,y-\gamma,\nu-\eta)\varphi(\zeta,\gamma,\eta)d\zeta d\gamma d\eta, \tag{34}$$

where the symbol $**$ denotes the double convolution with respect to x and t.

Proof. When we apply the definition of the G_α-double Laplace transform, we obtain

$$\begin{aligned}&L_xL_yG_\alpha[\phi(x,y,\nu)**\varphi(x,y,\nu)]\\&=s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-(\nu/s)}\phi(x,y,\nu)**\varphi(x,y,\nu)dxdyd\nu\\&=s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-(\nu/s)}\left({}_{000}^{xy\nu}\phi(x-\zeta,y-\gamma,\nu-\eta)\varphi(\zeta,\gamma,\eta)d\zeta d\gamma d\eta\right)dxdyd\nu.\end{aligned} \tag{35}$$

By setting $\rho=x-\zeta$, $\tau=y-\gamma$, and $\sigma=\nu-\eta$ and employing the appropriate expansion for the upper bounds of the integrals as $x\longrightarrow\infty$, $y\longrightarrow\infty$, and $\nu\longrightarrow\infty$, Eq. (35) can be expressed as

$$\begin{aligned}&L_xL_yG_\alpha[\phi(x,y,\nu)**\varphi(x,y,\nu)]\\&=s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-\zeta p-\gamma q-(\eta/s)}\varphi\\&\quad\times(x-\rho,y-\tau,\nu-\sigma)d\zeta d\gamma d\eta\int_{-\zeta}^\infty\int_{-\gamma}^\infty\int_{-\eta}^\infty e^{-p\rho-q\tau-(\sigma/s)}\phi\\&\quad\times(\rho,\tau,\sigma)d\rho d\tau d\sigma.\end{aligned} \tag{36}$$

The functions $\phi(x,y,\nu)$ and $\varphi(x,y,\nu)$ are both zero for $x<0$, $y<0$, and $\nu<0$. Therefore, considering the lower limits of integration, we have

$$\begin{aligned}&L_xL_yG_\alpha[\phi(x,y,\nu)**\varphi(x,y,\nu)]=\frac{1}{s^\alpha}\\&\cdot\left(s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-\zeta p-\gamma q-(\eta/s)}\varphi(\zeta,\gamma,\eta)d\zeta d\gamma d\eta\right)\\&\cdot\left(s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-p\rho-q\tau-(\sigma/s)}\phi(\rho,\tau,\sigma)d\rho d\tau d\sigma\right).\end{aligned} \tag{37}$$

It is evident that

$$s^\alpha L_xL_yG_\alpha[\phi(x,y,\nu)**\varphi(x,y,\nu)]=\phi_\alpha(p,q,s)\varphi_\alpha(p,q,s). \tag{38}$$

□

Theorem 9. *If the G_α-double Laplace transform of the function $f(x,y,\nu)$ is denoted as $L_xL_yG_\alpha[f(x,y,\nu)]=F_\alpha(p,q,s)$, then the G_α-double Laplace transform of $\partial^nf(x,y,\nu)/\partial x^n$ and $\partial^nf(x,y,\nu)/\partial\nu^n$ is as follows:*

For $\partial^nf(x,y,\nu)/\partial x^n$:

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^nf(x,y,\nu)}{\partial x^n}\right]&=p^nF_\alpha(p,q,s)\\&\quad-\sum_{k=1}^np^{n-k}L_yG_\alpha\left[\frac{\partial^{k-1}f(0,y,\nu)}{\partial x^{k-1}}\right].\end{aligned} \tag{39}$$

For $\partial^nf(x,y,\nu)/\partial\nu^n$:

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^nf(x,y,\nu)}{\partial\nu^n}\right]&=\frac{F_\alpha(p,q,s)}{s^n}\\&\quad-s^\alpha\sum_{k=1}^n\frac{1}{s^{n-k}}L_xL_y\left[\frac{\partial^{k-1}f(x,y,0)}{\partial\nu^{k-1}}\right].\end{aligned} \tag{40}$$

Proof. Substituting $n=1$ into Eq. (39), we get

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial f(x,y,\nu)}{\partial x}\right]&=s^\alpha\int_0^\infty\int_0^\infty e^{-qy-(\nu/s)}\\&\quad\cdot\left[{}_0^\infty e^{-px}\frac{\partial f(x,y,\nu)}{\partial x}dx\right]dyd\nu.\end{aligned} \tag{41}$$

Now, let us calculate the integral inside the bracket:

$$\begin{aligned}\int_0^\infty e^{-px}\frac{\partial f(x,y,\nu)}{\partial x}dx&=[e^{-px}f(x,y,\nu)]_0^\infty+p\int_0^\infty e^{-px}f(x,y,\nu)dx\\&=pF(p,y,\nu)-f(0,y,\nu).\end{aligned} \tag{42}$$

Substituting this result back into Eq. (96), we obtain

$$L_xL_yG_\alpha\left[\frac{\partial f(x,y,\nu)}{\partial x}\right]=s^\alpha\int_0^\infty\int_0^\infty e^{-qy-(\nu/s)}[pF(p,y,\nu)-f(0,y,\nu)]dyd\nu. \tag{43}$$

Then,

$$L_xL_yG_\alpha=pF_\alpha(p,q,s)-G_\alpha[f(0,q,\nu)]. \tag{44}$$

At $n = 2$, in a similar manner, one can readily observe that

$$L_xL_yG_\alpha\left[\frac{\partial^2 f(x,y,\nu)}{\partial x^2}\right] = p^2F_\alpha(p,q,s) - pL_yG_\alpha[f(0,y,\nu)] - L_yG_\alpha\left[\frac{\partial f(0,y,\nu)}{\partial x}\right]. \tag{45}$$

Let us assume that $n = m$ holds for some m. Therefore,

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial x^m}\right] &= p^mF_\alpha(p,q,s) - p^{m-1}L_yG_\alpha[f(0,y,\nu)] \\ &- p^{m-2}L_yG_\alpha\left[\frac{\partial f(0,y,\nu)}{\partial x}\right] \\ &-\cdots-pL_yG_\alpha\left[\frac{\partial^{m-2} f(0,y,\nu)}{\partial x^{m-2}}\right] \\ &- L_yG_\alpha\left[\frac{\partial^{m-1} f(0,y,\nu)}{\partial x^{m-1}}\right].\end{aligned} \tag{46}$$

Hang on for $(\partial^0/\partial) = 1$, now we show that

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^{m+1} f(x,y,\nu)}{\partial x^{m+1}}\right] &= p^{m+1}F_\alpha(p,q,s) - p^mL_yG_\alpha[f(0,y,\nu)] \\ &- p^{m-1}L_yG_\alpha\left[\frac{\partial f(0,y,\nu)}{\partial x}\right] \\ &-\cdots-pL_yG_\alpha\left[\frac{\partial^{m-1} f(0,y,\nu)}{\partial x^{m-1}}\right] \\ &- L_yG_\alpha\left[\frac{\partial^{m} f(0,y,\nu)}{\partial x^{m}}\right].\end{aligned} \tag{47}$$

Using the concept of $n = 1$, we have

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^{m+1} f(x,y,\nu)}{\partial x^{m+1}}\right] &= p\left[p^mF_\alpha(p,q,s) - p^{m-1}L_yG_\alpha[f(0,y,\nu)]\right. \\ &- p^{m-2}L_yG_\alpha\left[\frac{\partial f(0,y,\nu)}{\partial x}\right]\cdots \\ &\left.- L_yG_\alpha\left[\frac{\partial^{m-1} f(0,y,\nu)}{\partial x^{m-1}}\right]\right] \\ &- L_yG_\alpha\left[\frac{\partial^{m} f(0,y,\nu)}{\partial x^{m}}\right].\end{aligned} \tag{48}$$

The formula inside bracket

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial x^m}\right] &= p^mF_\alpha(p,q,s) - p^{m-1}G_\alpha[f(0,y,\nu)] \\ &- p^{m-2}L_yG_\alpha\left[\frac{\partial f(0,y,\nu)}{\partial x}\right]\cdots \\ &- L_yG_\alpha\left[\frac{\partial^{m-1} f(0,y,\nu)}{\partial x^{m-1}}\right].\end{aligned} \tag{49}$$

Therefore,

$$L_xL_yG_\alpha\left[\frac{\partial^{m+1} f(x,y,\nu)}{\partial x^{m+1}}\right] = pL_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial x^m}\right] - L_yG_\alpha\left[\frac{\partial^m f(0,y,\nu)}{\partial x^m}\right]. \tag{50}$$

Hence, Eq. (39) can be written as follows:

$$L_xL_yG_\alpha\left[\frac{\partial^n f(x,y,\nu)}{\partial x^n}\right] = p^nF_\alpha(p,q,s) - \sum_{k=1}^{n} p^{n-k}L_yG_\alpha\left[\frac{\partial^{k-1} f(0,y,\nu)}{\partial x^{k-1}}\right]. \tag{51}$$

For Eq. (40), substituting $n = 1$ in Eq. (39)

$$L_xL_yG_\alpha\left[\frac{\partial f(x,y,\nu)}{\partial \nu}\right] = \int_0^\infty\int_0^\infty e^{-px-qy}\left[s^\alpha\int_0^\infty e^{-\nu/s}\frac{\partial f(x,y,\nu)}{\partial \nu}d\nu\right]dydx, \tag{52}$$

we calculate the integral inside bracket

$$\begin{aligned}s^\alpha\int_0^\infty e^{-\nu/s}\frac{\partial f(x,y,\nu)}{\partial \nu}d\nu &= s^\alpha\left[e^{-\nu/s}f(x,y,\nu)\right]_0^\infty \\ &+ s^\alpha\int_0^\infty e^{-\nu/s}f(x,y,\nu)d\nu \\ &= -s^\alpha f(x,y,0) + \frac{1}{s}F_\alpha(x,y,s).\end{aligned} \tag{53}$$

Therefore,

$$\begin{aligned}L_xL_yG_\alpha\left[\frac{\partial f(x,y,\nu)}{\partial \nu}\right] &= -s^\alpha\int_0^\infty\int_0^\infty e^{-px-qy}f(x,y,0)dydx \\ &+ \frac{1}{s}\int_0^\infty\int_0^\infty e^{-px-qy}F_\alpha(x,y,s)dydx, \\ L_xL_yG_\alpha\left[\frac{\partial f(x,y,\nu)}{\partial \nu}\right] &= \frac{1}{s}F_\alpha(p,q,s) - s^\alpha f(p,q,0).\end{aligned} \tag{54}$$

Now, assume that $n = m$, Eq. (40) is correct for some m. Thus,

$$L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial\nu^m}\right] = \frac{F_\alpha(p,q,s)}{s^m} - s^\alpha\sum_{k=1}^{m}\frac{1}{s^{m-k}}L_xL_y\left[\frac{\partial^{k-1}f(x,y,0)}{\partial\nu^{k-1}}\right],$$
$$L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial\nu^m}\right] = \frac{F_\alpha(p,q,s)}{s^m} - \frac{s^\alpha}{s^{m-1}}L_xL_y[f(x,y,0)] - \frac{s^\alpha}{s^{m-2}}L_xL_y\left[\frac{\partial f(x,y,0)}{\partial\nu}\right] - \cdots - s^{\alpha-1}L_xL_y\left[\frac{\partial^{m-2}f(x,y,0)}{\partial\nu^{m-2}}\right] - s^\alpha L_xL_y\left[\frac{\partial^{m-1}f(x,y,0)}{\partial\nu^{m-1}}\right]. \tag{55}$$

Let us indicate

$$L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial\nu^m}\right] = \frac{F_\alpha(p,q,s)}{s^{m+1}} - \frac{s^\alpha}{s^m}L_xL_y[f(x,y,0)] - \frac{s^\alpha}{s^{m-1}}L_xL_y\left[\frac{\partial f(x,y,0)}{\partial\nu}\right] - \cdots - s^{\alpha-1}L_xL_y\left[\frac{\partial^{m-1}f(x,y,0)}{\partial\nu^{m-1}}\right] - s^\alpha L_xL_y\left[\frac{\partial^m f(x,y,0)}{\partial\nu^m}\right]. \tag{56}$$

By the notion of $n = 1$, we have

$$L_xL_yG_\alpha\left[\frac{\partial^{m+1}f(x,y,\nu)}{\partial\nu^{m+1}}\right] = \frac{1}{s}\left[\frac{F_\alpha(p,q,s)}{s^m} - \frac{s^\alpha}{s^{m-1}}L_xL_y[f(x,y,0)] - \frac{s^\alpha}{s^{m-2}}L_xL_y\left[\frac{\partial f(x,y,0)}{\partial\nu}\right] - \cdots - s^{\alpha-1}L_xL_y\left[\frac{\partial^{m-2}f(x,y,0)}{\partial\nu^{m-2}}\right] - s^\alpha L_xL_y\left[\frac{\partial^{m-1}f(x,y,0)}{\partial\nu^{m-1}}\right]\right] - s^\alpha L_xL_y\left[\frac{\partial^m f(x,y,0)}{\partial\nu^m}\right],$$
$$L_xL_yG_\alpha\left[\frac{\partial^{m+1}f(x,y,\nu)}{\partial\nu^{m+1}}\right] = \frac{1}{s}L_xL_yG_\alpha\left[\frac{\partial^m f(x,y,\nu)}{\partial\nu^m}\right] - s^\alpha L_xL_y\left[\frac{\partial^m f(x,y,0)}{\partial\nu^m}\right]. \tag{57}$$

Thus, the theorem is correct at an arbitrary natural number k. Hence, Eq. (40) is correct. □

3. G_α-Application of the Laplace Transform Decomposition Method to Coupled Burgers' Equation

In this section, we investigate the solutions of two problems using the G_α-double Laplace transform decomposition method: the first problem is the standard Burgers' equation represented as follows:

$$\phi_\nu - \phi_{xx} - \phi_{yy} + \phi\phi_x + \phi\phi_y = f(x,y,\nu), \tag{58}$$

with the boundary condition:

$$\phi(x,y,0) = f_1(x,y), \tag{59}$$

where $f(x,y,\nu)$ and $f_1(x,y)$ are given functions. Applying the G_α-double Laplace transform to both sides of Eq. (58) and the double Laplace transform to Eq. (59), we obtain

$$\Psi_\alpha(p,q,s) = s^{\alpha+1}F_1(p,q) + sF_\alpha(p,q,s) + sL_xL_yG_\alpha\left[\phi_{xx} + \phi_{yy} - \phi\phi_x - \phi\phi_y\right]. \tag{60}$$

Now, by utilizing the inverse G_α-double Laplace transform for Eq. (60), we have

$$\phi(x,y,\nu) = f_1(x,y) + L_p^{-1}L_q^{-1}G_\alpha^{-1}[sF_\alpha(p,q,s)] + L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{xx} + \phi_{yy}\right]\right] - L_p^{-1}G_\alpha^{-1}\left[sL_\mu G_\alpha\left[\phi\phi_x + \phi\phi_y\right]\right]. \tag{61}$$

The G_α-double Laplace transform decomposition method (GDLTDM) assumes that the solution $\phi(x,y,\nu)$ can be expanded into an infinite series as

$$\phi(x,y,\nu) = \sum_{n=0}^{\infty}\phi_n(x,y,\nu). \tag{62}$$

Adomian's polynomials A_n and B_n are defined as follows:

$$N(\phi) = A_n = \sum_{n=0}^{\infty}\phi_n\phi_{nx}, \quad N(\phi) = B_n = \sum_{n=0}^{\infty}\phi_n\phi_{ny}. \tag{63}$$

The Adomian polynomials for the nonlinear terms $\phi\phi_x$ and $\phi\phi_y$ are given by

$$A_0 = \phi_0\phi_{0x}, \tag{64}$$

$$A_1 = \phi_0\phi_{1x} + \phi_1\phi_{0x}, \tag{65}$$

$$A_2 = \phi_0\phi_{2x} + \phi_1\phi_{1x} + \phi_2\phi_{0x}, \quad (66)$$

$$A_3 = \phi_0\phi_{3x} + \phi_1\phi_{2x} + \phi_2\phi_{1x} + \phi_3\phi_{0x}, \quad (67)$$

$$A_4 = \phi_0\phi_{4x} + \phi_1\phi_{3x} + \phi_2\phi_{2x} + \phi_3\phi_{1x} + \phi_4\phi_{0x}, \quad (68)$$

$$B_0 = \phi_0\phi_{0y}, \quad (69)$$

$$B_1 = \phi_0\phi_{1y} + \phi_1\phi_{0y}, \quad (70)$$

$$B_2 = \phi_0\phi_{2y} + \phi_1\phi_{1y} + \phi_2\phi_{0y}, \quad (71)$$

$$B_3 = \phi_0\phi_{3y} + \phi_1\phi_{2y} + \phi_2\phi_{1\mu y} + \phi_3\phi_{0y}, \quad (72)$$

$$B_4 = \phi_0\phi_{4y} + \phi_1\phi_{3y} + \phi_2\phi_{2y} + \phi_3\phi_{1y} + \phi_4\phi_{0y}. \quad (73)$$

By substituting Eq. (62) into Eq. (61), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n(x,y,\nu) &= f_1(x,y) + L_p^{-1}L_q^{-1}G_\alpha^{-1}[sF_\alpha(p,q,s)] \\ &\quad + L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}\left(\phi_{nxx}+\phi_{nyy}\right)\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}A_n + B_n\right]\right].\end{aligned} \quad (74)$$

Upon comparing both sides of Eq. (74) and Eq. (62), we derive the following iterative algorithm:

$$\phi_0 = f_1(x,y) + L_p^{-1}L_q^{-1}G_\alpha^{-1}[sF_\alpha(p,q,s)]. \quad (75)$$

The remaining component, represented as ϕ_{n+1}, for $n \geq 0$, is determined by applying the following relationship:

$$\begin{aligned}\phi_{n+1} &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nxx}+\phi_{nyy}\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_n + B_n]\right].\end{aligned} \quad (76)$$

To illustrate the application of this method to two-dimensional Burgers' equations, we consider the following example.

Example 4. Let us consider the one-dimensional Burgers' equation given by

$$\phi_\nu - \phi_{xx} - \phi_{yy} + \phi\phi_x + \phi\phi_y = 0, \quad (77)$$

subject to the initial condition

$$\phi(x,y,0) = 2x + 2y. \quad (78)$$

By applying the G_α-double Laplace transform to both sides of Eq. (77) and the Laplace transform to Eq. (78), we obtain

$$\begin{aligned}\Psi_\alpha(p,q,s) &= \left(\frac{2}{p^2} + \frac{2}{q^2}\right)s^{\alpha+1} + sL_xL_yG_\alpha\left[\phi_{xx}+\phi_{yy}\right] \\ &\quad - sL_\mu G_\alpha\left[\phi\phi_x + \phi\phi_y\right].\end{aligned} \quad (79)$$

By using Eq. (74), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n(x,y,\nu) &= 2x + 2y + L_p^{-1}L_q^{-1}G_\alpha^{-1} \\ &\quad \cdot \left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}\left(\phi_{nxx}+\phi_{nyy}\right)\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}A_n + B_n\right]\right],\end{aligned} \quad (80)$$

where A_n and B_n are given by Eq. (64) and Eq. (69). By equating both sides of Eq. (74) and Eq. (80), we get

$$\phi_0 = 2x + 2y. \quad (81)$$

Overall, the recursive link is given by

$$\begin{aligned}\phi_{n+1} &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\left(\phi_{nxx}+\phi_{nyy}\right)\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_n + B_n]\right],\end{aligned} \quad (82)$$

where $n \geq 0$. At $n = 0$,

$$\begin{aligned}\phi_1 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0xx}+\phi_{0yy}\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_0 + B_0]\right] \\ &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0xx}+\phi_{0yy}\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_0\phi_{0\mu}\right]\right] \\ &= -L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_\mu G_\alpha[4x+4y]\right] \\ &= -4(x+y)\nu.\end{aligned} \quad (83)$$

Similarly, for $n = 1$, we have

$$\begin{aligned}\phi_2 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{1xx}+\phi_{1yy}\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_1 + B_1]\right] \\ &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{1xx}+\phi_{1yy}\right]\right] \\ &\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\begin{array}{c}\phi_0\phi_{1x}+\phi_1\phi_{0x} \\ +\phi_0\phi_{1y}+\phi_1\phi_{0y}\end{array}\right]\right] \\ &= -L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[16(x+y)\nu]\right] = 8(x+y)\nu^2.\end{aligned} \quad (84)$$

Similarly, when $n = 2$, we obtain

$$\begin{aligned}\phi_3 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{2xx}+\phi_{2yy}\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_2+B_2]\right]\\&= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{2xx}+\phi_{2yy}\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\begin{array}{c}\phi_0\phi_{2x}+\phi_1\phi_{1x}+\phi_2\phi_{0x}\\+\phi_0\phi_{2y}+\phi_1\phi_{1y}+\phi_2\phi_{0y}\end{array}\right]\right]\\&= -L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[48(x+y)\nu^2\right]\right]\\&= -L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[\left(\frac{96}{p^2}+\frac{96}{q^2}\right)s^{\alpha+4}\right]\\&= -16(x+y)\nu^3.\end{aligned} \tag{85}$$

Utilizing Equation (62), we can determine the specific convergent solutions as follows:

$$\begin{aligned}\phi(x,y,\nu) &= \phi_0+\phi_1+\phi_2+\phi_3+\cdots\\&= (2x+2y)\left(1-2\nu+(2\nu)^2-(2\nu)^3+(2\nu)^4-\cdots\right).\end{aligned} \tag{86}$$

Hence, the intricate solution transforms into

$$\phi(\mu,\nu) = \frac{2x+2y}{1+2\nu}. \tag{87}$$

Problem 10. Consider the next two-dimensional Burgers' equations

$$\begin{aligned}\phi_\nu &= \phi_{xx}+\phi_{xx}-\phi\phi_x-(\phi\varphi)_x-\phi\phi_y-(\phi\varphi)_y+f(x,y,\nu),\\\varphi_\nu &= \varphi_{xx}+\varphi_{yy}-\varphi\varphi_x-(\phi\varphi)_x-\varphi\varphi_y-(\phi\varphi)_y+h(x,y,\nu),\end{aligned} \tag{88}$$

subject to

$$\phi(x,y,0) = f_1(x,y), \tag{89}$$

$$\varphi(x,y,0) = h_1(x,y). \tag{90}$$

On using G_α-double Laplace transform and characteristic of differentiation double Laplace transform, we have

$$\begin{aligned}\Psi_\alpha(p,q,s) &= s^{\alpha+1}F_1(p,q)+sF_\alpha(p,q,s)\\&\quad + sL_xL_yG_\alpha\left[\phi_{xx}+\phi_{yy}\right]\\&\quad - sL_xL_yG_\alpha\left[\phi\phi_x+(\phi\varphi)_x+\phi\phi_y+(\phi\varphi)_y\right],\end{aligned} \tag{91}$$

$$\begin{aligned}\Phi_\alpha(p,q,s) &= s^{\alpha+1}H_1(p,q)+sH_\alpha(p,q,s)\\&\quad + sL_xL_yG_\alpha\left[\varphi_{xx}+\varphi_{yy}\right]\\&\quad - sL_xL_yG_\alpha\left[\varphi\varphi_x+(\phi\varphi)_x+\varphi\varphi_y+(\phi\varphi)_y\right].\end{aligned} \tag{92}$$

The next stage in the double Laplace transform decomposition method within the context of G_α involves expressing the solution of the following series:

$$\phi(x,y,\nu) = \sum_{n=0}^{\infty}\phi_n(x.y,\nu), \varphi(x,y,\nu) = \sum_{n=0}^{\infty}\varphi_n(x,y,\nu). \tag{93}$$

The nonlinear operators are defined or characterized by

$$\begin{aligned}B_n &= \sum_{n=0}^{\infty}\varphi_n\varphi_{nx},\\C_n &= \sum_{n=0}^{\infty}\varphi_n\phi_{nx},\\D_n &= \sum_{n=0}^{\infty}\phi_n\varphi_{nx},\end{aligned} \tag{94}$$

$$\begin{aligned}E_n &= \sum_{n=0}^{\infty}\varphi_n\varphi_{ny},\\F_n &= \sum_{n=0}^{\infty}\varphi_n\phi_{ny},\\G_n &= \sum_{n=0}^{\infty}\phi_n\varphi_{ny},\end{aligned} \tag{95}$$

where terms $\varphi\varphi_x$, $\varphi\phi_x$, and $\phi\varphi_x$ are determined by

$$K_0 = \varphi_0\varphi_{0x}, \tag{96}$$

$$K_1 = \varphi_0\varphi_{1x}+\varphi_1\varphi_{0x}, \tag{97}$$

$$K_2 = \varphi_0\varphi_{2x}+\varphi_1\varphi_{1x}+\varphi_2\varphi_{0x}, \tag{98}$$

$$K_3 = \varphi_0\varphi_{3x}+\varphi_1\varphi_{2x}+\varphi_2\varphi_{1x}+\varphi_3\varphi_{0x}, \tag{99}$$

$$C_0 = \varphi_0\phi_{0x}, \tag{100}$$

$$C_1 = \varphi_0\phi_{1x}+\varphi_1\phi_{0x}, \tag{101}$$

$$C_2 = \varphi_0\phi_{2x}+\varphi_1\phi_{1x}+\varphi_2\phi_{0x}, \tag{102}$$

$$C_3 = \varphi_0\phi_{3x}+\varphi_1\phi_{2x}+\varphi_2\phi_{1x}+\varphi_3\phi_{0x}, \tag{103}$$

$$D_0 = \phi_0\varphi_{0x}, \tag{104}$$

$$D_1 = \phi_0\varphi_{1x}+\phi_1\varphi_{0x}, \tag{105}$$

$$D_2 = \phi_0\varphi_{2x}+\phi_1\varphi_{1x}+\phi_2\varphi_{0x}, \tag{106}$$

$$D_3 = \phi_0\varphi_{3x}+\phi_1\varphi_{2x}+\phi_2\varphi_{1x}+\phi_3\varphi_{0x}, \tag{107}$$

where the definitions of E_n, F_n, and G_n are analogous to those in Equations (96), (100), and (104). Apply the inverse G_α-double Laplace transform to Equation (91), and then utilize Equations (89), (94), and (??), yielding

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n(x,y,\nu)&=f_1(x,y)+L_p^{-1}L_q^{-1}G_\alpha^{-1}[sF_\alpha(p,q,s)]\\&+L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}\phi_{nxx}+\phi_{nyy}\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}A_n+B_n\right.\right.\\&\left.\left.+C_n+D_n+F_n+G_n\right]\right],\\\sum_{n=0}^{\infty}\varphi_n(x,y,\nu)&=h_1(x,y)+L_p^{-1}L_q^{-1}G_\alpha^{-1}[sH_\alpha(p,q,s)]\\&+L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}\varphi_{nxx}+\varphi_{nyy}\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\sum_{n=0}^{\infty}K_n+C_n+D_n\right.\right.\\&\left.\left.+E_n+F_n+G_n\right]\right].\end{aligned}\tag{108}$$

This yields the desired recursive relation as follows:

$$\begin{aligned}\phi_0&=f_1(x,y)+L_p^{-1}L_q^{-1}G_\alpha^{-1}[sF_\alpha(p,q,s)],\\\varphi_0&=h_1(x,y)+L_p^{-1}L_q^{-1}G_\alpha^{-1}[sH_\alpha(p,q,s)],\end{aligned}\tag{109}$$

and the remaining terms are represented by

$$\begin{aligned}\phi_{n+1}&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nxx}+\phi_{nyy}\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[A_n+B_n+C_n+D_n+F_n+G_n]\right],\\\varphi_{n+1}&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{nxx}+\varphi_{nyy}\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[K_n+C_n+D_n+E_n+F_n+G_n]\right].\end{aligned}\tag{110}$$

To illustrate this method for the two-dimensional coupled Burgers' equation, we examine the following examples.

Example 5 (see [23]). Consider the following two-dimensional homogeneous Burgers' equations:

$$\begin{aligned}\phi_\nu&=\phi_{xx}+\phi_{yy}+2\phi\phi_x-\phi_x\varphi-\phi\varphi_x+2\phi\phi_y-\phi_y\varphi-\phi\varphi_y,\\\varphi_\nu&=\varphi_{xx}+\varphi_{yy}+2\varphi\varphi_x-\phi_x\varphi-\phi\varphi_x+2\varphi\varphi_y-\phi_y\varphi-\phi\varphi_y,\end{aligned}\tag{111}$$

subject to

$$\begin{aligned}\phi(x,y,0)&=\sin(x+y),\\\varphi(x,y,0)&=\sin(x+y).\end{aligned}\tag{112}$$

By employing the aforementioned method, we obtain

$$\begin{aligned}\phi_0&=\sin(x+y),\\\varphi_0&=\sin(x+y),\\\phi_{n+1}&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nxx}+\phi_{nyy}+2\phi_{nx}\phi_n+2\phi_{ny}\phi_n\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{ny}\varphi_n+\phi_n\varphi_{ny}+\phi_{nx}\varphi_n+\phi_n\varphi_{nx}\right]\right],\\\varphi_{n+1}&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{nxx}+\varphi_{nyy}+2\varphi_{nx}\varphi_n+2\varphi_{ny}\varphi_n\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nx}\varphi_n+\phi_n\varphi_{nx}+\phi_{ny}\varphi_n+\phi_n\varphi_{ny}\right]\right].\end{aligned}\tag{113}$$

The subsequent terms are presented as follows:
Where n. At $n=0$,

$$\begin{aligned}\phi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0xx}+\phi_{0yy}+2\phi_{0x}\phi_0+2\phi_{0y}\phi_0\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0y}\varphi_0+\phi_0\varphi_{0y}+\phi_{0x}\varphi_0+\phi_0\varphi_{0x}\right]\right]\\&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\begin{array}{c}-\sin(x+y)+2\sin(x+y)\cos(x+y)\\-2\sin(x+y)\cos(x+y)\end{array}\right]\right],\\\phi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[-\sin(x+y)]\right],\\\phi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[-\frac{s^{\alpha+2}(p+q)}{(p^2+1)(q^2+1)}\right]=-\nu\sin(x+y),\\\varphi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{0nxx}+\varphi_{0yy}+2\varphi_{0x}\varphi_n+2\varphi_{0y}\varphi_0\right]\right]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0x}\varphi_0+\phi_0\varphi_{0x}+\phi_{0y}\varphi_0+\phi_0\varphi_{0y}\right]\right]\\&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\begin{array}{c}-\sin(x+y)+2\sin(x+y)\cos(x+y)\\-2\sin(x+y)\cos(x+y)\end{array}\right]\right],\\\varphi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[-\sin(x+y)]\right],\\\varphi_1&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[-\frac{s^{\alpha+2}(p+q)}{(p^2+1)(q^2+1)}\right]=-\nu\sin(x+y).\end{aligned}\tag{114}$$

Similarly, at $n=1$, we obtain

$$\begin{aligned}\phi_2&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[\phi_{1xx}+\phi_{1xx}\right.\\&\left.+2\left(\phi_0\phi_{1x}+\phi_1\phi_{0x}+\phi_0\phi_{1y}+\phi_1\phi_{0y}\right)\right]\Big]\\&-L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[(\varphi_0\phi_{1x}+\varphi_1\phi_{0x})\right.\\&\left.+(\phi_0\varphi_{1x}+\phi_1\varphi_{0x})]\right]-L_p^{-1}L_q^{-1}G_\alpha^{-1}\\&\cdot\left[sL_xL_yG_\alpha\left[\phi_{0y}\varphi_1+\phi_{1y}\varphi_0+\left(\phi_0\varphi_{1y}+\phi_1\varphi_{0y}\right)\right]\right]\\&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[\nu\sin(x+y)]\right],\\\phi_2&=L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[\frac{s^{\alpha+3}(p+q)}{(p^2+1)(q^2+1)}\right]=\frac{\nu^2}{2!}\sin(x+y),\end{aligned}$$

$$\begin{aligned}\varphi_2 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{1xx}+\varphi_{1yy}+2(\varphi_0\varphi_{1x}+\varphi_1\varphi_{0x}\right.\right.\\&\quad\left.\left.+\varphi_0\varphi_{1y}+\varphi_1\varphi_{0y}\right)\right]\Big] - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[(\varphi_0\phi_{1x}\right.\\&\quad\left.+\varphi_1\phi_{0x}+\varphi_0\phi_{1y}+\varphi_1\phi_{0y})+(\phi_0\varphi_{1x}+\phi_1\varphi_{0x})]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_0\varphi_{1x}+\phi_1\varphi_{0x}+\phi_0\varphi_{1y}+\phi_1\varphi_{0y}\right]\right]\\&= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha[\nu\sin(x+y)]\right],\end{aligned}$$

$$\varphi_2 = L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[\frac{s^{\alpha+3}(p+q)}{(p^2+1)(q^2+1)}\right] = \frac{\nu^2}{2!}\sin(x+y). \tag{115}$$

Similarly, we obtain the remaining terms as follows:

$$\begin{aligned}\phi_3 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[-\frac{s^{\alpha+4}(p+q)}{(p^2+1)(q^2+1)}\right] = -\frac{\nu^3}{3!}\sin(x+y),\\ \varphi_3 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[-\frac{s^{\alpha+4}(p+q)}{(p^2+1)(q^2+1)}\right] = -\frac{\nu^3}{3!}\sin(x+y).\end{aligned} \tag{116}$$

We keep the same style to obtain the approximate solutions:

$$\begin{aligned}\phi(x,y,\nu) &= \phi_0+\phi_2+\phi_3+\cdots = \left(1-\nu+\frac{\nu^2}{2!}-\frac{\nu^3}{3!}+\cdots\right)\sin(x+y),\\ \varphi(x,y,\nu) &= \varphi_0+\varphi_2+\varphi_3+\cdots = \left(1-\nu+\frac{\nu^2}{2!}-\frac{\nu^3}{3!}+\cdots\right)\sin(x+y).\end{aligned} \tag{117}$$

Hence, the solutions become ideal.

$$\phi(x,y,\nu) = e^{-\nu}\sin(x+y). \tag{118}$$

In the following example, we apply our method to solve the inhomogeneous coupled system of Burgers' equation.

Example 6. Examine the following set of two-dimensional Burgers' equations:

$$\begin{aligned}\phi_\nu &= \phi_{xx}+\phi_{yy}+2\phi\phi_x-\phi_x\varphi-\phi\varphi_x+2\phi\phi_y-\phi_y\varphi-\phi\varphi_y-(x+y)e^{-\nu},\\ \varphi_\nu &= \varphi_{xx}+\varphi_{yy}+2\varphi\varphi_x-\phi_x\varphi-\phi\varphi_x+2\varphi\varphi_y-\phi_y\varphi-\phi\varphi_y-(x+y)e^{-\nu},\end{aligned} \tag{119}$$

subject to

$$\begin{aligned}\phi(x,y,0) &= (x+y),\\ \varphi(x,y,0) &= (x+y).\end{aligned} \tag{120}$$

Using the method mentioned above, we obtain

$$\begin{aligned}\phi_0 &= (x+y)e^{-\nu},\\ \varphi_0 &= (x+y)e^{-\nu},\\ \phi_{n+1} &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nxx}+\phi_{nyy}+2\phi_{nx}\phi_n+2\phi_{ny}\phi_n\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{ny}\varphi_n+\phi_n\varphi_{ny}+\phi_{nx}\varphi_n+\phi_n\varphi_{nx}\right]\right],\\ \varphi_{n+1} &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{nxx}+\varphi_{nyy}+2\varphi_{nx}\varphi_n+2\varphi_{ny}\varphi_n\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{nx}\varphi_n+\phi_n\varphi_{nx}+\phi_{ny}\varphi_n+\phi_n\varphi_{ny}\right]\right].\end{aligned} \tag{121}$$

Furthermore, the subsequent terms are represented by

$$\begin{aligned}\phi_1 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0xx}+\phi_{0yy}+2\phi_{0x}\phi_0+2\phi_{0y}\phi_0\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0y}\varphi_0+\phi_0\varphi_{0y}+\phi_{0x}\varphi_0+\phi_0\varphi_{0x}\right]\right]\\&= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[0+2(x+y)e^{-2\nu}-2(x+y)e^{-2\nu}\right]\right],\\ \phi_1 &= 0,\\ \varphi_1 &= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\varphi_{0nxx}+\varphi_{0yy}+2\varphi_{0x}\varphi_n+2\varphi_{0y}\varphi_0\right]\right]\\&\quad - L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[\phi_{0x}\varphi_0+\phi_0\varphi_{0x}+\phi_{0y}\varphi_0+\phi_0\varphi_{0y}\right]\right]\\&= L_p^{-1}L_q^{-1}G_\alpha^{-1}\left[sL_xL_yG_\alpha\left[0+2(x+y)e^{-2\nu}-2(x+y)e^{-2\nu}\right]\right],\\ \varphi_1 &= 0.\end{aligned} \tag{122}$$

Continuing in a similar fashion, we have

$$\begin{aligned}\phi_2 &= 0, \quad \phi_3 = 0, \phi_4 = 0, \cdots,\\ \varphi_2 &= 0, \quad \varphi_3 = 0, \varphi_4 = 0, \cdots\end{aligned} \tag{123}$$

Hence, the solution series is given by

$$\begin{aligned}\phi(x,y,\nu) &= (x+y)e^{-\nu},\\ \varphi(x,y,\nu) &= (x+y)e^{-\nu}.\end{aligned} \tag{124}$$

4. Conclusion

In the course of this study, we introduced the innovative G -double Laplace transform, meticulously elaborating on its diverse definitions, theorems, existence conditions, partial derivatives, and the double convolution theorems. Leveraging these novel insights, we successfully uncovered the exact solutions for both Burgers' equation and its coupled counterpart. To validate the efficacy of our technique, we presented three illustrative examples. As a result, we strongly recommend the adoption of this method for future endeavors in addressing equations commonly encountered in the realms of physics and engineering. The G-double Laplace transform, with its profound potential, stands poised to make significant contributions to the field, promising enhanced problem-solving capabilities for complex mathematical models in these domains.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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